Discrete-Time Market Models

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Stochastic Calculus I

Geneviève Gauthier

HEC Montréal

Introduction I

- We will study in depth Section 2: "The Finite Theory" in the article *Martingales, Stochastic Integrals and Continuous Trading* by J. M. Harrison and S. R. Pliska.
- The market model here is rather general. Indeed, a finite number (which may however be enormous) of securities are modelled during a finite number (which, again, may however be very large) of time periods.
- The only restriction on the distributions of the unit prices of the securities at each time is that such distributions must be discrete and positive, i.e. for each security and at each time, the unit price of the security at that time can only take a finite number of strictly positive values.

Introduction II

- The first important result in that Section is Proposal 2.6 on page 227.
  - It is proven that, if there exists a probability measure under which all discounted price processes are martingales, then we can build, based on such a measure, a price system for attainable contingent claims that is consistent with the market model.
  - On the other hand, if there exists such a price system, then we can build, based on the latter, a probability measure that transforms our discounted price processes into martingales.
  - Such a proposal thus establishes a bijective correspondence between the set of martingale measures and the consistent price systems.

Introduction III

- It should be noted that such a proposal is silent on whether at least one martingale measure exists, or whether at least one price system exists. It only states that, if either one exists, then both exist, and it makes the link between them explicit.

Introduction IV

- Theorem 2.7 on page 228 sets the necessary and sufficient condition for at least one martingale measure, and as a consequence, at least one consistent price system, to exist. That condition is that the market model contains no arbitrage opportunities.
- Since there may be several price systems that are consistent with the market, we need to ensure that, whatever the price system that is used, the price associated with any given attainable conditional claim X is always the same, i.e. if π_1 and π_2 are two consistent price systems, then for any attainable contingent claim X, $\pi_1(X) = \pi_2(X)$. Such a result is provided by a corollary on page 228.

Introduction V

- How can it be verified that our market model contains no arbitrage opportunities? A necessary and sufficient condition for no arbitrage is established by a lemma on page 228.
- Proposal 2.8, on page 230, proves that the discounted market value of an admissible strategy is a martingale under all martingale measures in a market model. Such a result is used in proving Proposal 2.9 (page 230), which shows how to determine the market value of an attainable contingent claim at any time.
- The last part addresses the notion of market completeness.

The probability space

- We are working with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.
- The sample space Ω has a finite number of elements, each of which is interpreted as a possible state of the world. We assume that the market participants agree on the fact that Ω represents all the possible states of the world.
- Thus, since every state of the world $\omega \in \Omega$ is possible, the probability measure P applying on the measurable space $(\Omega, \mathcal{F})$ must be such that $\forall \omega \in \Omega, P(\omega) > 0$.
- The measure P, that associates a probability to each state of the world, represents a particular investor's vision, i.e. two investors may associate different probabilities to the same state of the world ω.

The filtration

Information structure

- We choose to observe the system until a time horizon T, which is a terminal date for all economic activity under consideration.
- Since time is considered discretely, the filtration

$$\mathbb{F} = \{\mathcal{F}_t : t \in \{0, 1, ..., T\}\}$$

  represents information available at each time.
- We set $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the sigma-algebra that contains no information, and $\mathcal{F}_T =$ the set of all events in Ω, i.e. $\mathcal{F}_T$ allows us to distinguish each possible state of the world.
- Since $Card\,(\Omega) < \infty$, for all $t \in \{0, 1, ..., T\}$, there exists a finite partition

$$\mathcal{P}_t = \{A_1^t, ..., A_{n_t}^t\}$$

  that generates the sub-sigma-algebra $\mathcal{F}_t$.

The securities I

- We model $K+1$ securities.

Definition

The multidimensional stochastic process

$$\overrightarrow{S} = \left\{ \overrightarrow{S}_t : t \in \{0, 1, ..., T\} \right\}$$

representing the evolution of the individual unit prices of all securities is made up of the random column vectors $\overrightarrow{S}_t = \left(S_t^0, S_t^1, ...S_t^K\right)'$ where

$$S_t^k = \text{ the unit price of the security } k \text{ at time } t.$$

The securities II

- Each of the components $S^k = \left\{ S_t^k : t \in \{0, 1, ..., T\} \right\}$ of the price process $\overrightarrow{S}$ is a stochastic process adapted to the filtration $\mathbb{F}$ and it takes strictly positive values.
- Since $Card\left(\Omega\right) < \infty$, we have that $\forall t \in \{0, 1, ..., T\}$ and $\forall k \in \{0, 1, ..., K\}$, S_t^k discretely distributed random variable, i.e. it can only take a finite number of values.

The securities III

Definition

The 0 th security plays a part a bit peculiar, in that it is a **riskless security** (we can think, for example, of a bank account or a bond). It implies that

$$\forall \omega \in \Omega \text{ and } \forall t \in \{0, 1, ..., T-1\}, S_t^0(\omega) \leq S_{t+1}^0(\omega).$$

- It can also be assumed, without loss of generality, that

$$\forall \omega \in \Omega, S_0^0(\omega) = 1.$$

- The unidimensional stochastic process

$$\beta_t = \frac{1}{S_t^0}$$

will be used as discount factor.

The securities IV

- The filtration $\mathbb{F}$ describes how information is revealed to the investors at each time.
  - The fact that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ implies that the components of $\overrightarrow{S}_0$ are constants, i.e. at time $t = 0$, all security prices are known with certainty.
  - Since $\mathcal{F}_t$ is generated by the finite partition $\mathcal{P}_t = \left\{A_1^t, ..., A_{n_t}^t\right\}$, then, at time t, every investor knows with certainty which of the cells of $\mathcal{P}_t$ has occurred, but the investor is not able to distinguish between the elements in that cell.
  - Since $\mathcal{F}_T$ is the sigma-algebra made up of all possible events of Ω, investors can, at time T, determine with certainty which state of the world ω has occurred.
  - This means that we group under a single name, say ω_i, all the states of the world that have the same effect on the market. This is one the reasons why it is warranted to restrict the sample space to a finite set $(Card\,(\Omega) < \infty)$.

Trading strategy I

Definition

A **trading strategy** is a predictable vector stochastic process

$$\overrightarrow{\phi} = \left\{ \overrightarrow{\phi}_t : t \in \{1, ..., T\} \right\}$$

where the random row vector $\overrightarrow{\phi}_t = \left(\phi_t^0, \phi_t^1, ...\phi_t^K\right)$ represents the investor's portfolio at time t :

$$\phi_t^k = \text{ the number of shares of security } k \text{ held at time } t.$$

Trading strategy II

- By requiring that the strategy be a predictable process, we are allowing the investor to select his time t portfolio right after the share prices at time $t-1$ are announced. As a result, the random variable ϕ_t^k represents the number of shares of security k held during the time period $(t-1, t]$ (except for ϕ_1^k that represents the number of shares of security k held during the time period $[0, 1]$).
- The market value of portfolio $\overrightarrow{\phi}_t$, right after the security prices at time $t-1$ are available, is $\overrightarrow{\phi}_t \overrightarrow{S}_{t-1}$, whereas the market value of the same portfolio, but at time t, is $\overrightarrow{\phi}_t \overrightarrow{S}_t$.

Trading strategy III

Definition

We say a strategy is **self-financing** if no funds are added to or withdrawn from the value of the portfolio after time $t = 0$, i.e.

$$\forall t \in \{1, ..., T-1\}, \overrightarrow{\phi}_t \overrightarrow{S}_t = \overrightarrow{\phi}_{t+1} \overrightarrow{S}_t.$$

- So, $\overrightarrow{\phi}_t \overrightarrow{S}_t$ represents the amount we receive at time $t + \varepsilon$ (ε represents a very small positive quantity) when the portfolio is liquidated $\overrightarrow{\phi}_t$ and $\overrightarrow{\phi}_{t+1} \overrightarrow{S}_t$ is the amount we must pay, again at time $t + \varepsilon$, to purchase portfolio $\overrightarrow{\phi}_{t+1}$.
- Note that there are no transaction costs.

Trading strategy IV

- The process $V\left(\overrightarrow{\phi}\right) = \left\{V_t\left(\overrightarrow{\phi}\right) : t \in \{0, 1, ..., T\}\right\}$ represents the market value of the strategy at any time.

$$V_t\left(\overrightarrow{\phi}\right) = \begin{cases} \overrightarrow{\phi}_1 \overrightarrow{S}_0 & \text{if } t = 0 \\ \overrightarrow{\phi}_t \overrightarrow{S}_t & \text{if } t \in \{1, ..., T\}. \end{cases}$$

Trading strategy V

Definition

A trading strategy is called **admissible** if it is self-financing and its market value is never negative.

- So, an admissible strategy is such that an investor is never put into a situation of debt. This doesn't mean however that short sales are prohibited.

$$\begin{aligned}\Phi &= \text{the set of admissible strategies} \\ &= \left\{ \overrightarrow{\phi} \;\middle|\; \begin{array}{c} \text{each of the components } \phi^k \\ \text{of } \overrightarrow{\phi} \text{ is predictable,} \\ \overrightarrow{\phi} \text{ is self-financing} \\ \text{and } \forall t \in \{0, ..., T\},\ V_t\left(\overrightarrow{\phi}\right) \geq 0 \end{array} \right\}.\end{aligned}$$

Arbitrage opportunity I

Definition

An admissible strategy $\overrightarrow{\phi}$ is an **arbitrage opportunity** if

$$V_0\left(\overrightarrow{\phi}\right) = 0 \text{ and if } \mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] > 0.$$

- Note that the latter condition implies that there exists $\omega \in \Omega$ for which $V_T\left(\overrightarrow{\phi}, \omega\right) > 0$.
  - Indeed,

$$\mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] = \sum_{\omega\in\Omega} \underbrace{V_T\left(\overrightarrow{\phi}, \omega\right)}_{\geq 0} \underbrace{P(\omega)}_{>0}$$

So, in order to have $\mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] > 0$, there must be at least one ω for which $V_T\left(\overrightarrow{\phi}, \omega\right) > 0$.

Arbitrage opportunity II

- So, the strategy $\overrightarrow{\phi}$ is an arbitrage opportunity when, while investing nothing $\left(V_0\left(\overrightarrow{\phi}\right)=0\right)$, we are assured not to lose any money ($V_T\left(\overrightarrow{\phi},\omega\right)\geq 0$ since $\overrightarrow{\phi}$ is admissible) and we have a positive probability to make a gain ($\exists\omega\in\Omega$ for which $V_T\left(\overrightarrow{\phi},\omega\right)>0$).

Contingent claims I

Definition

A **contingent claim** X is a $(\Omega, \mathcal{F}_T)$ −non-negative random variable. It can be thought of as a contract between two parties, such that $X(\omega)$ will be paid by one party to the other if state ω pertains.

Definition

$$\begin{aligned} \mathbf{X} &= \text{the set of contingent claims} \\ &= \left\{ X \,\middle|\, \begin{array}{c} X \text{ is a } (\Omega, \mathcal{F}_T) - \text{random variable} \\ \text{such that } \forall \omega \in \Omega,\ X(\omega) \geq 0 \end{array} \right\}. \end{aligned}$$

Contingent claims II

Definition

A contingent claim X is said to be **attainable** if there exists an admissible trading strategy $\overrightarrow{\phi}$ that can replicate the cash flow generated by X, i.e. $V_T\left(\overrightarrow{\phi}\right) = X$.

Price systems I

Definition

A price system π is a linear operator on $\mathbf{X}$ that returns non-negative values $\pi : \mathbf{X} \to [0, \infty)$ and that satisfies

$$\pi(X) = 0 \Leftrightarrow X = 0$$

and

$$\forall a, b \geq 0 \text{ and } \forall X_1 X_2 \in \mathbf{X},\ \pi(aX_1 + bX_2) = a\pi(X_1) + b\pi(X_2).$$

- As its name indicates, a price system is intended to associate with each of the contingent claims, a price.
- Since, for any admissible strategy $\overrightarrow{\phi}$, $V_T\left(\overrightarrow{\phi}\right)$ is a $(\Omega, \mathcal{F}_T)$ −random variable taking non-negative values, $V_T\left(\overrightarrow{\phi}\right) \in \mathbf{X}$.

Price systems II

Definition

A price system is said to be **consistent** with the market model if the price associated with the contingent claim $V_T\left(\overrightarrow{\phi}\right)$ is its market value at time $t = 0$, $V_0\left(\overrightarrow{\phi}\right)$, i.e.
$\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right)$.

Price systems III

Definition

$$
\begin{aligned}
& \Pi \\
= \; & \text{the set of price systems consistent with the market model} \\
= \; & \left\{ \pi : \mathbf{X} \to [0, \infty) \;\middle|\; \begin{array}{c} \pi(X) = 0 \Leftrightarrow X = 0 \\ \\ \forall a, b \geq 0 \text{ and } \forall X_1, X_2 \in \mathbf{X}, \\ \pi(aX_1 + bX_2) = a\pi(X_1) + b\pi(X_2) \\ \\ \forall \overrightarrow{\phi} \in \Phi,\ \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right) \end{array} \right\}.
\end{aligned}
$$

Risk-neutral measures I

- We will build, if possible, a set of probability measures on $(\Omega, \mathcal{F})$ that we call equivalent martingale measures (also know as risk-neutral measures).
- Such measures bear, *a priori*, no relationship to the actual probability that an event occurs, no more than they reflect the market knowledge of an investor.
- They are nothing more than a very convenient calculation tool in pricing contingent claims.

Definition

$$\begin{aligned} \mathbb{P} &= \text{the set of martingale measures equivalent to } P \\ &= \left\{ Q \,\middle|\, \begin{array}{c} Q \text{ is a probability measure on } (\Omega, \mathcal{F}), \\ \forall \omega \in \Omega,\ Q(\omega) > 0 \text{ and} \\ \forall k \in \{0, 1, ..., K\},\ \beta S^k \text{ is a } Q-\text{martingale.} \end{array} \right\} \end{aligned}$$

Indicator functions I

Definition

Throughout this text, indicator functions will be denoted by $\mathbb{I}$, i.e. $\forall A \in \mathcal{F}$, $\mathbb{I}_A : \Omega \to \{0, 1\}$ is defined by

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}.$$

To become familiar with such functions, the reader can verify that:

(i) If $A \in \mathcal{F}_t$ then $\mathbb{I}_A$ is $\mathcal{F}_t -$ measurable.

(ii) If $A, B \in \mathcal{F}$ are disjoint, then $\mathbb{I}_{A \cup B} = \mathbb{I}_A + \mathbb{I}_B$.

(iii) If Y is a random variable, then $Y = \sum_{\omega \in \Omega} Y(\omega) \mathbb{I}_\omega$.

Statement

Proposal 2.6

Theorem

Proposal. *There is bijective correspondence between the set* Π *of price systems that are consistent with the market model and the set* $\mathbb{P}$ *of martingale measures equivalent to* P. *Such as correspondence is*

$$(i) \quad \pi(X) = \mathrm{E}^{Q}\left[\beta_T X\right], \; X \in \mathbf{X}$$

$$(ii) \quad Q(A) = \pi\left(S_T^0 \mathbb{I}_A\right), \; A \in \mathcal{F}.$$

Interpretation I

Proposal 2.6

- If we know a martingale measure Q equivalent to P, then we can build a consistent price system π by setting $\forall X \in \mathbf{X}$, $\pi(X) = \mathrm{E}^Q[\beta_T X]$.
- On the other hand, if a price system π consistent with the market model is available, then we can build a martingale measure Q equivalent to P by defining $\forall A \in \mathcal{F}$, $Q(A) = \pi\left(S_T^0 \mathbb{I}_A\right)$.
- Such a proposal tells us that, if there exists a martingale measure equivalent to P or if there exists a price system consistent with the market model, then both exist, and the proposal establishes the link between them.
- However, nothing allows us to show that either of such two objects exist.

Interpretation II

Proposal 2.6

- The necessary and sufficient condition for a martingale measure, and, as a consequence, for the price system to exist, is addressed by Theorem 2.7.

Definition

Proof

Definition

Let P and Q be two probability measures that exist on the measurable space $(\Omega, \mathcal{F})$. The measures P and Q are said to be **equivalent** if and only if the impossible events are the same under both measures, i.e.

$$\forall A \in \mathcal{F},\ P(A) = 0 \Leftrightarrow Q(A) = 0.$$

In our case, since $\forall \omega \in \Omega,\ P(\omega) > 0$, all measures Q equivalent to P shall satisfy the condition

$$\forall \omega \in \Omega, Q(\omega) > 0.$$

First part I

Proof

To be shown. *Let Q be a martingale measure equivalent to P. Then, the function π defined on the set of contingent claims X by $\pi(X) = \mathrm{E}^Q[\beta_T X]$ is a price system consistent with the market model.*

Since $Q \in \mathbb{P}$ then

1. Q is equivalent to P, i.e. $\forall \omega \in \Omega, Q(\omega) > 0$,
2. and the components of the discounted security prices, $\beta \overrightarrow{S}$, are martingale under the measure Q.

First part II

Proof

We want to show that π is a consistent price system. We therefore need to verify that

(i) $\pi(X) = 0 \Leftrightarrow X = 0$,

(ii) $\forall a, b \geq 0$ and $\forall X_1, X_2 \in \mathbf{X}$,

$$\pi(aX_1 + bX_2) = a\pi(X_1) + b\pi(X_2),$$

(iii) $\forall \overrightarrow{\phi} \in \Phi$, $\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right)$.

First part - Point (i)

Proof

Starting from the definition of π, we get

$$\pi(X) = \mathrm{E}^{Q}[\beta_T X] = \sum_{\omega\in\Omega} \underbrace{\beta_T(\omega)}_{>0} \underbrace{X(\omega)}_{\geq 0} \underbrace{Q(\omega)}_{>0}.$$

Note that $\beta_T(\omega) > 0$ since $\beta_T = \frac{1}{S_T^0}$ and the market model assumes that security prices can only be strictly positive. Therefore,

$$\begin{aligned} \pi(X) = 0 \quad &\Leftrightarrow \quad \sum_{\omega\in\Omega} \underbrace{\beta_T(\omega)}_{>0} \underbrace{X(\omega)}_{\geq 0} \underbrace{Q(\omega)}_{>0} = 0 \\ &\Leftrightarrow \quad \forall \omega \in \Omega,\ X(\omega) = 0. \ \blacksquare \end{aligned}$$

First part - Point (ii)

Proof

$\forall a, b \geq 0$ and $\forall X_1, X_2 \in \mathbf{X}$,

$$\begin{aligned} \pi\left(aX_1 + bX_2\right) &= \mathrm{E}^Q\left[\beta_T\left(aX_1 + bX_2\right)\right] \\ &= a\mathrm{E}^Q\left[\beta_T X_1\right] + b\mathrm{E}^Q\left[\beta_T X_2\right] \\ &= a\pi\left(X_1\right) + b\pi\left(X_2\right). \blacksquare \end{aligned}$$

First part - Point (iii) I

Proof

We want to show that $\forall \overrightarrow{\phi} \in \Phi$,

$$\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right).$$

But, from the definition of π,

$$\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = \mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right], \tag{1}$$

so we determine, in a first step, $\beta_T V_T\left(\overrightarrow{\phi}\right)$:

First part - Point (iii) II

Proof

$$
\begin{aligned}
& \beta_T V_T\left(\overrightarrow{\phi}\right) \\
= \quad & \beta_T \overrightarrow{\phi}_T \overrightarrow{S}_T \text{ from the definition of } V_T, \text{ (ref: eq (2.4), p. 226).} \\
= \quad & \beta_T \overrightarrow{\phi}_T \overrightarrow{S}_T + \sum_{i=1}^{T-1} \beta_i \underbrace{\left(\overrightarrow{\phi}_i \overrightarrow{S}_i - \overrightarrow{\phi}_{i+1} \overrightarrow{S}_i\right)}_{=0} \text{ since } \overrightarrow{\phi} \text{ is self-financing.} \\
= \quad & \beta_T \overrightarrow{\phi}_T \overrightarrow{S}_T + \sum_{i=1}^{T-1} \beta_i \overrightarrow{\phi}_i \overrightarrow{S}_i - \sum_{i=1}^{T-1} \beta_i \overrightarrow{\phi}_{i+1} \overrightarrow{S}_i \\
= \quad & \beta_T \overrightarrow{\phi}_T \overrightarrow{S}_T + \sum_{i=1}^{T-1} \beta_i \overrightarrow{\phi}_i \overrightarrow{S}_i - \sum_{i=2}^{T} \beta_{i-1} \overrightarrow{\phi}_i \overrightarrow{S}_{i-1} \\
= \quad & \beta_T \overrightarrow{\phi}_T \overrightarrow{S}_T + \beta_1 \overrightarrow{\phi}_1 \overrightarrow{S}_1 + \sum_{i=2}^{T-1} \overrightarrow{\phi}_i \left(\beta_i \overrightarrow{S}_i - \beta_{i-1} \overrightarrow{S}_{i-1}\right) - \beta_{T-1} \overrightarrow{\phi}_T \overrightarrow{S}_{T-1} \\
= \quad & \beta_1 \overrightarrow{\phi}_1 \overrightarrow{S}_1 + \sum_{i=2}^{T} \overrightarrow{\phi}_i \left(\beta_i \overrightarrow{S}_i - \beta_{i-1} \overrightarrow{S}_{i-1}\right)
\end{aligned}
$$

First part - Point (iii) III

Proof

Therefore, substituting the expression above into the equation (1),

$$
\begin{aligned}
& \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) \\
= \quad & \mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] \\
= \quad & \mathrm{E}^Q\left[\beta_1 \overrightarrow{\phi}_1 \overrightarrow{S}_1 + \sum_{i=2}^{T} \overrightarrow{\phi}_i \left(\beta_i \overrightarrow{S}_i - \beta_{i-1} \overrightarrow{S}_{i-1}\right)\right] \\
= \quad & \mathrm{E}^Q\left[\beta_1 \overrightarrow{\phi}_1 \overrightarrow{S}_1\right] + \sum_{i=2}^{T} \mathrm{E}^Q\left[\overrightarrow{\phi}_i \left(\beta_i \overrightarrow{S}_i - \beta_{i-1} \overrightarrow{S}_{i-1}\right)\right] \\
= \quad & \mathrm{E}^Q\left[\mathrm{E}^Q\left[\beta_1 \overrightarrow{\phi}_1 \overrightarrow{S}_1 \,|\mathcal{F}_0\right]\right] + \sum_{i=2}^{T} \mathrm{E}^Q\left[\mathrm{E}^Q\left[\overrightarrow{\phi}_i \left(\beta_i \overrightarrow{S}_i - \beta_{i-1} \overrightarrow{S}_{i-1}\right) |\mathcal{F}_{i-1}\right]\right] \\
= \quad & \mathrm{E}^Q\left[\overrightarrow{\phi}_1 \mathrm{E}^Q\left[\beta_1 \overrightarrow{S}_1 \,|\mathcal{F}_0\right]\right] + \sum_{i=2}^{T} \mathrm{E}^Q\left[\overrightarrow{\phi}_i \mathrm{E}^Q\left[\left(\beta_i \overrightarrow{S}_i - \beta_{i-1} \overrightarrow{S}_{i-1}\right) |\mathcal{F}_{i-1}\right]\right]
\end{aligned}
$$

since $\overrightarrow{\phi}$ is predictable,

First part - Point (iii) IV

Proof

$$= \mathrm{E}^Q\left[\overrightarrow{\phi}_1 \underbrace{\mathrm{E}^Q\left[\beta_1 \overrightarrow{S}_1 \,|\mathcal{F}_0\right]}_{\beta_0 \overrightarrow{S}_0}\right] + \sum_{i=2}^{T} \mathrm{E}^Q\left[\overrightarrow{\phi}_i \underbrace{\mathrm{E}^Q\left[\left(\beta_i \overrightarrow{S}_i - \beta_{i-1}\overrightarrow{S}_{i-1}\right)|\mathcal{F}_{i-1}\right]}_{=\mathrm{E}^Q\left[\beta_i \overrightarrow{S}_i|\mathcal{F}_{i-1}\right]-\beta_{i-1}\overrightarrow{S}_{i-1}=\overrightarrow{0}}\right]$$

since the components of $\beta\overrightarrow{S}$ are martingales under the measure Q,

$$= \mathrm{E}^Q\left[\overrightarrow{\phi}_1 \beta_0 \overrightarrow{S}_0\right] = \beta_0 \overrightarrow{\phi}_1 \overrightarrow{S}_0$$

since β_0, $\overrightarrow{\phi}_1$ and $\overrightarrow{S}_0$ are $\mathcal{F}_0-$measurable, therefore constant,

$$= \overrightarrow{\phi}_1 \overrightarrow{S}_0 \text{ since } \beta_0 = \frac{1}{S_0^0} = 1$$

$$= V_0\left(\overrightarrow{\phi}\right) \text{ from the definition of } V_0 \text{ (ref: eq (2.4), p. 226).}\blacksquare$$

Proof I

Second part

To be shown. *Let π be a price system consistent with the market model. Then, the function Q, defined on the set of events $\mathcal{F}$ by $Q(A) = \pi\left(S_T^0 \mathbb{I}_A\right)$ martingale measure equivalent to P.*

Since $\pi \in \prod$ then

1. $\pi(X) = 0 \Leftrightarrow X = 0$;
2. $\forall a, b \geq 0$ and $\forall X_1, X_2 \in \mathbf{X}$, $\pi(aX_1 + bX_2) = a\pi(X_1) + b\pi(X_2)$;
3. $\forall \overrightarrow{\phi} \in \Phi$, $\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right)$.

Proof II

Second part

We want to show that $Q \in \mathbb{P}$, i.e.

(i) Q is a probability measure;

(ii) Q is equivalent to P, i.e. $\forall \omega \in \Omega, Q(\omega) > 0$;

(iii) The components of the discount price process for the securities $\beta \overrightarrow{S}$, are martingales under the measure Q.

Second part - Point (i) I

Proof

- Let's start by building a strategy that will be subsequently of use :

$$\forall t \in \{1, ..., T\}, \text{ let } \overrightarrow{\phi}_t = (1, 0, ..., 0).$$

  - This trading strategy is such that, at any time, we hold nothing more than a single share of the riskless security.
  - Note that at any time $\overrightarrow{\phi}_t$ is constant, i.e. $\forall \omega \in \Omega, \overrightarrow{\phi}_t(\omega) = (1, 0, ..., 0)$.
  - A a consequence, $\overrightarrow{\phi}_t$ is $\mathcal{F}_0-$measurable, therefore $\mathcal{F}_{t-1}-$measurable, which implies that $\overrightarrow{\phi}$ is predictable.
  - It is also easy to show that $\overrightarrow{\phi}$ is self-financing since we keep the same portfolio.

Second part - Point (i) II

Proof

- Now,

$$\begin{aligned} V_T\left(\overrightarrow{\phi}\right) &= \sum_{k=0}^{K} \phi_T^k S_T^k \text{ from the definition of } V\left(\overrightarrow{\phi}\right) \\ &= S_T^0 \text{ from the definition of } \overrightarrow{\phi}. \end{aligned} \tag{2}$$

and

$$\begin{aligned} V_0\left(\overrightarrow{\phi}\right) &= \sum_{k=0}^{K} \phi_1^k S_0^k \text{ from the definition of } V\left(\overrightarrow{\phi}\right) \\ &= S_0^0 \text{ from the definition of } \overrightarrow{\phi}. \\ &= 1 \text{ since, by hypothesis, } S_0^0 = 1. \end{aligned} \tag{3}$$

Second part - Point (i) III

Proof

- Let's now show that $Q(\Omega) = 1$.
  - Starting from the definition Q,

$$
\begin{aligned}
& Q(\Omega) \\
= \; & \pi\left(S_T^0 \mathbb{I}_\Omega\right) = \pi\left(S_T^0\right) \\
= \; & \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) \text{ from Equality (2).} \\
= \; & V_0\left(\overrightarrow{\phi}\right) \text{ since } \pi \text{ is consistent with the market model.} \\
= \; & 1 \text{ from Equality (3).} \blacksquare
\end{aligned}
$$

Second part - Point (i) IV

Proof

- We must also ensure that for all disjoint events A_1, ..., $A_n \in \mathcal{F}$,

$$Q\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} Q\left(A_i\right).$$

Second part - Point (i) V

Proof

But

$$
\begin{aligned}
Q\left(\bigcup_{i=1}^{n} A_i\right) &= \pi\left(S_T^0 \mathbb{I}_{\bigcup_{i=1}^{n} A_i}\right) \\
&= \pi\left(S_T^0 \sum_{i=1}^{n} \mathbb{I}_{A_i}\right) \\
&= \sum_{i=1}^{n} \pi\left(S_T^0 \mathbb{I}_{A_i}\right) \text{ from Property (2.5b), p. 226.} \\
&= \sum_{i=1}^{n} Q\left(A_i\right) \text{ from the very definition of } Q. \blacksquare
\end{aligned}
$$

Second part - Point (ii)

Proof

- We want to prove that $\forall \omega \in \Omega, Q(\omega) > 0$.
- Let's assume there exists at least one $\omega^* \in \Omega$ for which $Q(\omega^*) = 0$ and let's show this leads to a contradiction.

From the equivalence relation $\pi(X) = 0 \Leftrightarrow X = 0$, we have

$$\begin{aligned} & 0 = Q(\omega^*) = \pi\left(S_T^0 \mathbb{I}_{\omega^*}\right) \\ \Leftrightarrow \quad & \forall \omega \in \Omega, \ S_T^0(\omega) \mathbb{I}_{\omega^*}(\omega) = 0 \\ \Rightarrow \quad & S_T^0(\omega^*) = S_T^0(\omega^*) \mathbb{I}_{\omega^*}(\omega^*) = 0. \end{aligned}$$

Since the prices are strictly positive, we have that $\forall \omega \in \Omega$, $S_T^0(\omega) > 0$. As a consequence

$$S_T^0(\omega^*) \mathbb{I}_{\omega^*}(\omega^*) = S_T^0(\omega^*) > 0.$$

Contradiction! So, there cannot exist $\omega \in \Omega$ for which $Q(\omega) = 0$. ■

Second part - Point (iii) I

Proof

- The objective is to show that each of the components in the vector of discounted security prices, $\beta \overrightarrow{S}$, is a $Q-$martingale.
- Let's choose one of the components arbitrarily: let $k \in \{0, 1, ..., K\}$.
- By the *Optional Stopping Theorem*, it will be sufficient to show that
$$\mathrm{E}^Q \left[\beta_\tau S_\tau^k\right] = \mathrm{E}^Q \left[\beta_0 S_0^k\right]$$
for all $(\Omega, \mathcal{F}, \mathbb{F})$ $-$bounded stopping time τ $(\forall \omega \in \Omega, \tau(\omega) \leq T)$.
- So let's choose arbitrarily a bounded stopping time τ.

Second part - Point (iii) II

Proof

Let's consider the trading strategy defined as follows:

$$\phi_t^j = \begin{cases} \mathbb{I}_{\{t \leq \tau\}} & \text{if } j = k \\ \frac{S_\tau^k}{S_\tau^0} \mathbb{I}_{\{t > \tau\}} = S_\tau^k \beta_\tau \mathbb{I}_{\{t > \tau\}} & \text{if } j = 0 \\ 0 & \text{if } j \in \{1, ..., K\}, j \neq k. \end{cases}$$

- Such a strategy consists in holding a share of security k until the random time τ and then in exchanging it for a random quantity $S_\tau^k \beta_\tau$ of bonds (riskless security) until maturity $(t = T)$.

Second part - Point (iii) III

Proof

- Let's verify that such as strategy is admissible, i.e. $\overrightarrow{\phi}$ is a predictable process, that strategy is self-financing and its market value, $V_t\left(\overrightarrow{\phi}\right)$ is non-negative at all times.

Second part - Point (iii) IV

Proof

- Let's begin by verifying that $\overrightarrow{\phi}$ is a predictable process :

$\forall t \in \{1, ..., T\}$,

1. $\phi_t^0 = S_\tau^k \beta_\tau \mathbb{I}_{\{\tau<t\}} = \sum_{i=0}^{t-1} S_\tau^k \beta_\tau \mathbb{I}_{\{\tau=i\}} = \sum_{i=0}^{t-1} \underbrace{S_i^k \beta_i \mathbb{I}_{\{\tau=i\}}}_{\mathcal{F}_i-\text{measurable}}$ is $\mathcal{F}_{t-1}-$measurable,
2. $\phi_t^k = \mathbb{I}_{\{\tau\geq t\}}$ is $\mathcal{F}_{t-1}-$measurable since
$$\{\tau \geq t\} = \{\tau < t\}^c = \left(\bigcup_{i=0}^{t-1} \underbrace{\{\tau = i\}}_{\in \mathcal{F}_i \subseteq \mathcal{F}_{t-1}}\right)^c \in \mathcal{F}_{t-1} \text{ and}$$
3. $\phi_t^j = 0$ is $\mathcal{F}_0-$measurable therefore $\mathcal{F}_{t-1}-$measurable. ■

Second part - Point (iii) V

Proof

- For $\overrightarrow{\phi}$ to be self-financing, it must be that $\forall t \in \{1, ..., T-1\}$,

$$\overrightarrow{\phi}_t \overrightarrow{S}_t = \overrightarrow{\phi}_{t+1} \overrightarrow{S}_t.$$

But

$$\overrightarrow{\phi}_t \overrightarrow{S}_t = \begin{cases} S_t^k & \text{if } t \leq \tau \\ S_\tau^k \beta_\tau S_t^0 & \text{if } t > \tau \end{cases} = \begin{cases} S_t^k & t < \tau \\ S_t^k & t = \tau \\ S_\tau^k \beta_\tau S_t^0 & t > \tau \end{cases}$$

and since $\forall t \in \{0, ..., T\}$, $\beta_t S_t^0 = \frac{1}{S_t^0} S_t^0 = 1$, then

$$\overrightarrow{\phi}_{t+1} \overrightarrow{S}_t = \begin{cases} S_t^k & \text{if } t+1 \leq \tau \\ S_\tau^k \beta_\tau S_t^0 & \text{if } t+1 > \tau \end{cases} = \begin{cases} S_t^k & t < \tau \\ S_t^k & t = \tau \\ S_\tau^k \beta_\tau S_t^0 & t > \tau \end{cases}$$

therefore $\overrightarrow{\phi}$ truly is self-financing. ■

Second part - Point (iii) VI

Proof

- Now, let's verify that $\forall t \in \{0, ..., T\}$, $V_t\left(\overrightarrow{\phi}\right) \geq 0$.

$$V_0\left(\overrightarrow{\phi}\right) = \begin{cases} S_0^k & \text{if } \tau > 0 \\ S_\tau^k \beta_\tau S_0^0 & \text{if } \tau = 0 \end{cases}$$

$$= \begin{cases} S_0^k & \text{if } \tau > 0 \\ S_0^k \beta_0 S_0^0 & \text{if } \tau = 0 \end{cases} = S_0^k > 0$$

and $\forall t \in \{1, ..., T\}$,

$$V_t\left(\overrightarrow{\phi}\right) = \overrightarrow{\phi}_t \overrightarrow{S}_t = \begin{cases} S_t^k & \text{if } t \leq \tau \\ S_\tau^k \beta_\tau S_t^0 & \text{if } t > \tau \end{cases} > 0.$$

Therefore the trading strategy $\overrightarrow{\phi}$ truly is admissible. ■

Second part - Point (iii) VII

Proof

- Let's now show that βS^k is a martingale under the measure Q.

From its definition, we already know that the process βS^k is adapted to the filtration $\mathbb{F}$. Recall that we want to show that

$$\mathrm{E}^Q\left[\beta_\tau S_\tau^k\right] = \mathrm{E}^Q\left[\beta_0 S_0^k\right].$$

Recall also that

$$V_T\left(\overrightarrow{\phi}\right) = S_\tau^k \beta_\tau S_T^0$$

and

$$V_0\left(\overrightarrow{\phi}\right) = S_0^k.$$

Second part - Point (iii) VIII

Proof

Then

$$
\begin{aligned}
& \mathrm{E}^{Q}\left[\beta_{\tau} S_{\tau}^{k}\right] \\
= & \sum_{\omega \in \Omega} \beta_{\tau(\omega)}(\omega) S_{\tau(\omega)}^{k}(\omega) Q(\omega) \\
= & \sum_{\omega \in \Omega} \underbrace{\beta_{\tau(\omega)}(\omega) S_{\tau(\omega)}^{k}(\omega)}_{>0} \pi\left(S_{T}^{0} \mathbb{I}_{\omega}\right) \text{ from the definition of } Q. \\
= & \pi\left(\sum_{\omega \in \Omega} \beta_{\tau(\omega)}(\omega) S_{\tau(\omega)}^{k}(\omega) S_{T}^{0} \mathbb{I}_{\omega}\right) \\
& \qquad \text{(ref : l'eq (2.5b), p. 226).} \\
= & \pi\left(\beta_{\tau} S_{\tau}^{k} S_{T}^{0}\right)
\end{aligned}
$$

Second part - Point (iii) IX

Proof

$$
\begin{aligned}
&= \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) \text{ see equality as recalled above.} \\
&= V_0\left(\overrightarrow{\phi}\right) \text{ because } \pi \text{ is consistent and } \overrightarrow{\phi} \text{ is admissible} \\
&= S_0^k \text{ see equality as recalled above.} \\
&= \beta_0 S_0^k \text{ simce } \beta_0 = 1. \\
&= \mathrm{E}^Q\left[\beta_0 S_0^k\right] \text{ since } \beta_0 S_0^k \text{ is } \mathcal{F}_0 - \text{measurable, hence constant.}
\end{aligned}
$$

The proof is now complete. ■

Statement

Lemma, p.228

Theorem

Lemma on page 228. *If there exists a self-financing strategy* $\overrightarrow{\phi}$ *(not necessarily admissible) such that* $V_0\left(\overrightarrow{\phi}\right) = 0$, $V_T\left(\overrightarrow{\phi}\right) \geq 0$ *and* $\mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] > 0$, *then there exists an arbitrage opportunity.*

Interpretation I

Lemma, p.228

- This lemma provides us with a necessary and sufficient condition for the existence of an arbitrage opportunity.
- The advantage of this result is that, from now on, we don't have to verify any more whether the strategy that is a candidate for an arbitrage opportunity is admissible.

Interpretation II

Lemma, p.228

- We also want to emphasize a comment made on page 229 in the article : a close look at the proof of this lemma reveals that the definition of admissibility can be relaxed so that self-financing strategies for which the market value may turn negative become admissible, provided that the market value at maturity, i.e. at time $t = T$, be non-negative.

Interpretation III

Lemma, p.228

- Based on such a new definition of admissibility, we can redefine the arbitrage opportunity as being a self-financing trading strategy the market value of which at time $t = 0$ is nil,

$$V_0\left(\overrightarrow{\phi}\right) = 0,$$

the market value at maturity is non-negative,

$$V_T\left(\overrightarrow{\phi}\right) \geq 0,$$

and the expected market value, again at maturity, is positive,

$$\mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] > 0.$$

- Under these new definitions of admissibility and arbitrage, Theorem 2.7 remains valid.

Proof I

- If $\forall t \in \{0, 1, ..., T\}$, $V_t\left(\overrightarrow{\phi}\right) \geq 0$, then the strategy $\overrightarrow{\phi}$ is admissible the hypotheses of the lemma are such that it is itself an arbitrage opportunity.

Proof II

- So let's assume there exists a $t \in \{0, 1, ..., T\}$ and a $\omega \in \Omega$ for which

$$V_t\left(\overrightarrow{\phi}, \omega\right) < 0. \tag{4}$$

- Let

$$a = \min_{\substack{t \in \{0,1,...,T\} \\ \omega \in \Omega}} V_t\left(\overrightarrow{\phi}, \omega\right), \tag{5}$$

be the smallest market value that can ever be reached by the value process of strategy $\overrightarrow{\phi}$ (note that due to inequality (4), $a < 0$);

Proof III

- The first time when the market value process of strategy $\overrightarrow{\phi}$, $V\left(\overrightarrow{\phi}\right)$, may ever reach that quantity a is

$$t_0 = \min\left\{t \in \{0, 1, ..., T\} \middle| \exists \omega \in \Omega \text{ such that } V_t\left(\overrightarrow{\phi}, \omega\right) = a\right\},$$

- $A = \left\{\omega \in \Omega \middle| V_{t_0}\left(\overrightarrow{\phi}, \omega\right) = a\right\}$.
- Note that $\forall \omega \in \Omega$, $P(\omega) > 0$ implies that $P(A) > 0$.

Proof IV

- We build, from strategy $\overrightarrow{\phi}$, a strategy $\overrightarrow{\psi}$ which is, for its part, admissible:

$$\psi_t^k(\omega) = \begin{cases} 0 & \text{if } \omega \in A^c \\ 0 & \text{if } \omega \in A \text{ and } t \in \{0, ..., t_0\} \\ \phi_t^0(\omega) - \frac{a}{S_{t_0}^0(\omega)} & \text{if } \omega \in A,\ t \in \{t_0+1, , ..., T\} \text{ and } k = 0 \\ \phi_t^k(\omega) & \text{if } \omega \in A,\ t \in \{t_0+1, , ..., T\} \text{ and } k \in \{1, ..., K\}. \end{cases}$$

- We probability $1 - P(A)$, we are holding a portfolio empty at all times: no gain, no loss!
- But, with probability $P(A)$, the trading strategy $\overrightarrow{\psi}$ consists in holding nothing until time t_0, then in acquiring portfolio $\overrightarrow{\phi}_{t_0+1}$ plus some quantity of bonds (riskless security). It is due to the latter that the market value of the strategy $\overrightarrow{\psi}$ is non-negative at all times.
- Recall that, on the set A, $V_{t_0}\left(\overrightarrow{\phi}, \omega\right) = a < 0$, which implies that, to acquire the portfolio $\overrightarrow{\phi}_{t_0+1}(\omega)$, we will receive an amount $-a$. That is the amount we invest into buying bonds.

Proof V

To ensure that $\overrightarrow{\psi}$ is an admissible strategy, we must verify that :

(i) $\overrightarrow{\psi}$ is predictable,

(ii) $\overrightarrow{\psi}$ is self-financing,

(iii) $\forall t \in \{0, 1, ..., T\}$ and $\forall \omega \in \Omega$, $V_t\left(\overrightarrow{\psi}, \omega\right) \geq 0$.

Point (i) I

Proof

By construction, $\overrightarrow{\psi}$ is predictable and we leave it to the reader to convince himself that this is so.

Point (ii) I

Proof

We want to prove that strategy $\overrightarrow{\psi}$ is self-financing, i.e.

$$\forall t \in \{1, ..., T-1\}, \overrightarrow{\psi}_t \overrightarrow{S}_t = \overrightarrow{\psi}_{t+1} \overrightarrow{S}_t.$$

If $\omega \in A^c$, then

$$\forall t \in \{1, ..., T-1\}, \overrightarrow{\psi}_t(\omega) \overrightarrow{S}_t(\omega) = 0 = \overrightarrow{\psi}_{t+1}(\omega) \overrightarrow{S}_t(\omega).$$

So let's assume that $\omega \in A$.

$$\forall t \in \{1, ..., t_0 - 1\}, \overrightarrow{\psi}_t(\omega) \overrightarrow{S}_t(\omega) = 0 = \overrightarrow{\psi}_{t+1}(\omega) \overrightarrow{S}_t(\omega).$$

Point (ii) II

Proof

Since $\overrightarrow{\phi}$ is self-financing, then $\forall t \in \{t_0, ..., T-1\}$,

$$\begin{aligned}
& \overrightarrow{\psi}_{t+1}(\omega)\, \overrightarrow{S}_t(\omega) \\
= & \sum_{k=0}^{K} \psi_{t+1}^k(\omega)\, S_t^k(\omega) \\
= & \left(\phi_{t+1}^0(\omega) - \frac{a}{S_{t_0}^0(\omega)}\right) S_t^0(\omega) + \sum_{k=1}^{K} \phi_{t+1}^k(\omega)\, S_t^k(\omega) \\
= & -a\frac{S_t^0(\omega)}{S_{t_0}^0(\omega)} + \sum_{k=0}^{K} \phi_{t+1}^k(\omega)\, S_t^k(\omega) \\
= & -a\frac{S_t^0(\omega)}{S_{t_0}^0(\omega)} + \sum_{k=0}^{K} \phi_t^k(\omega)\, S_t^k(\omega).
\end{aligned}$$

Point (ii) III

Proof

So if $t = t_0$,

$$\begin{aligned}
& \overrightarrow{\psi}_{t_0+1}(\omega)\,\overrightarrow{S}_{t_0}(\omega) \\
= \quad & -a\frac{S^0_{t_0}(\omega)}{S^0_{t_0}(\omega)} + \underbrace{\sum_{k=0}^{K} \phi^k_{t_0}(\omega)\, S^k_{t_0}(\omega)}_{=V_{t_0}\left(\overrightarrow{\phi},\omega\right)=a} \\
= \quad & 0 \\
= \quad & \overrightarrow{\psi}_{t_0}(\omega)\,\overrightarrow{S}_{t_0}(\omega)
\end{aligned}$$

Point (ii) IV

Proof

and if $t > t_0$,

$$\begin{aligned}
& \overrightarrow{\psi}_{t+1}(\omega)\,\overrightarrow{S}_t(\omega) \\
= & -a\frac{S_t^0(\omega)}{S_{t_0}^0(\omega)} + \sum_{k=0}^{K}\phi_t^k(\omega)\,S_t^k(\omega) \\
= & \left(\phi_t^0(\omega) - \frac{a}{S_{t_0}^0(\omega)}\right)S_t^0(\omega) + \sum_{k=1}^{K}\phi_t^k(\omega)\,S_t^k(\omega) \\
= & \overrightarrow{\psi}_t(\omega)\,\overrightarrow{S}_t(\omega).
\end{aligned}$$

So, as we had announced, strategy $\overrightarrow{\psi}$ is self-financing. ■

Point (iii) I

Proof

- The goal, this time, is to prove that for all time $t \in \{0, 1, ..., T\}$ and for all $\omega \in \Omega$, $V_t\left(\overrightarrow{\psi}, \omega\right) \geq 0$.
- If $\omega \in A^c$, then

$$\begin{aligned} V_0\left(\overrightarrow{\psi}, \omega\right) &= \overrightarrow{\psi}_1(\omega)\,\overrightarrow{S}_0(\omega) = 0 \\ \forall t \in \{1, ..., T\},\ V_t\left(\overrightarrow{\psi}, \omega\right) &= \overrightarrow{\psi}_t(\omega)\,\overrightarrow{S}_t(\omega) = 0. \end{aligned}$$

- So, let's assume $\omega \in A$.

$$\begin{aligned} V_0\left(\overrightarrow{\psi}, \omega\right) &= \overrightarrow{\psi}_1(\omega)\,\overrightarrow{S}_0(\omega) = 0 \\ \forall t \in \{1, ..., t_0\},\ V_t\left(\overrightarrow{\psi}, \omega\right) &= \overrightarrow{\psi}_t(\omega)\,\overrightarrow{S}_t(\omega) = 0 \end{aligned}$$

Obviously, if $t_0 = 0$, the line above does not apply.

Point (iii) II

Proof

- $\forall \omega \in A$ and $\forall t \in \{t_0 + 1, ..., T\}$,

$$
\begin{aligned}
V_t\left(\overrightarrow{\psi}, \omega\right) &= \overrightarrow{\psi}_t(\omega)\, \overrightarrow{S}_t(\omega) \\
&= \left(\phi_t^0(\omega) - \frac{a}{S_{t_0}^0(\omega)}\right) S_t^0(\omega) + \sum_{k=1}^{K} \phi_t^k(\omega)\, S_t^k(\omega) \\
&= -a\frac{S_t^0(\omega)}{S_{t_0}^0(\omega)} + \sum_{k=0}^{K} \phi_t^k(\omega)\, S_t^k(\omega) \\
&= -a\frac{S_t^0(\omega)}{S_{t_0}^0(\omega)} + \underbrace{V_t\left(\overrightarrow{\phi}, \omega\right)}_{\geq a \text{ from the choice of } a} \quad (\text{ref : eq } (5)). \\
&\geq \underbrace{-a\left(\frac{S_t^0(\omega)}{S_{t_0}^0(\omega)} - 1\right)}_{\geq 0} \geq 0
\end{aligned}
$$

since the 0 th security is the riskless investment, therefore $\forall t \in \{0, ..., T-1\}$ and $\forall \omega \in \Omega$, $S_{t+1}^0(\omega) \geq S_t^0(\omega)$. ■

Point (iii) III

Proof

- According to the arbitrage definition provided on page 226 in the article, we must show that $V_0\left(\overrightarrow{\psi}\right) = 0$ and that $\mathrm{E}^P\left[V_T\left(\overrightarrow{\psi}\right)\right] > 0$. But, as it has been previously established,

$$V_0\left(\overrightarrow{\psi}\right) = \overrightarrow{\psi}_1 \overrightarrow{S}_0 = 0$$

Point (iii) IV

Proof

and

$$
\begin{aligned}
\mathrm{E}^P\left[V_T\left(\overrightarrow{\psi}\right)\right] &= \sum_{\omega\in\Omega} V_T\left(\overrightarrow{\psi},\omega\right)P(\omega) \\
&= \sum_{\omega\in A} V_T\left(\overrightarrow{\psi},\omega\right)P(\omega) + \sum_{\omega\in A^c} \underbrace{V_T\left(\overrightarrow{\psi},\omega\right)}_{=0} P(\omega) \\
&= \sum_{\omega\in A}\left(\underbrace{V_T\left(\overrightarrow{\phi},\omega\right)}_{\geq 0 \text{ (by hypothesis)}} - a\frac{S_T^0(\omega)}{S_{t_0}^0(\omega)}\right)P(\omega) \\
&\geq \underbrace{-a}_{>0} \sum_{\omega\in A} \underbrace{\frac{S_T^0(\omega)}{S_{t_0}^0(\omega)}}_{>0} \underbrace{P(\omega)}_{>0} \\
&> 0.
\end{aligned}
$$

The proof is therefore complete. ■

Statement

Theorem 2.7

Theorem

Theorem 2.7. *The market model contains no arbitrage opportunities if and only if there exists at least one martingale measure equivalent to P.*

Proof

Theorem 2.7

The proof is broken down into two parts:

1. **To be shown**. *if there exists at least one martingale measure equivalent to P, then the market model contains no arbitrage opportunities.*
2. **To be shown**. *If the market model contains no arbitrage opportunities, then there exists at least one martingale measure equivalent to P.*

First part I

Proof

To be shown. *If there exists at least one martingale measure equivalent to P, then the market model contains no arbitrage opportunities.*

First part II

Proof

- In order to show that the model contains no arbitrage opportunities, we must ensure that all admissible strategy $\overrightarrow{\phi} \in \Phi$ satisfying $V_0\left(\overrightarrow{\phi}\right) = 0$ also verifies the equality $\mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] = 0$ (ref.: definition of arbitrage, page 226).
- Let $\overrightarrow{\phi} \in \Phi$, be an admissible strategy such that $V_0\left(\overrightarrow{\phi}\right) = 0$.
- We want to show that, in such as case, $\mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] = 0$.

First part III

Proof

- We claim that it is sufficient to show that $\mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] = 0$ since

$$\mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] = 0 \Leftrightarrow \mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] = 0.$$

Indeed,

$$0 = \mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] = \sum_{\omega\in\Omega} \underbrace{V_T\left(\overrightarrow{\phi},\omega\right)}_{\geq 0} \underbrace{P(\omega)}_{>0}$$

$$\Leftrightarrow \quad \forall\omega\in\Omega,\ V_T\left(\overrightarrow{\phi},\omega\right) = 0$$

$$\Leftrightarrow \quad \mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] = \sum_{\omega\in\Omega} \underbrace{\beta_T(\omega)}_{>0} \underbrace{V_T\left(\overrightarrow{\phi},\omega\right)}_{\geq 0} \underbrace{Q(\omega)}_{>0} = 0. \ \blacksquare$$

First part IV

Proof

- On the other hand, using Proposal 2.6, we know there exists a price system π consistent with the market model: $\pi(X) = \mathrm{E}^Q[\beta_T X]$.
- Thus

$$\mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] = \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right) = 0. \quad (6)$$

where the second equality is warranted by the consistency of the price system. The last equality, that derives from the choice of the strategy $\overrightarrow{\phi}$, completes the proof. ■

Second part

Proof of Theorem 2.7

To be shown. *If the market model contains no arbitrage opportunities, then there exists at least one martingale measure equivalent to P.*

Remarks about random variables I

Proof of Theorem 2.7

- Since $Card\left(\Omega\right) < \infty$, we can represent the random variables as points in the space $\mathbb{R}^{Card(\Omega)}$, i.e. if Y is a $\left(\Omega, \mathcal{F}\right)$ −random variable and

$$y_i = Y\left(\omega_i\right), \ i \in \left\{1, ..., Card\left(\Omega\right)\right\}$$

then

$$\left(Y\left(\omega_1\right), ...Y\left(\omega_{Card(\Omega)}\right)\right) = \left(y_1, ..., y_{Card(\Omega)}\right) \in \mathbb{R}^{Card(\Omega)}.$$

Remarks about random variables II

Proof of Theorem 2.7

- An alternative representation of the random variable Y will be most useful throughout the proof: since

$$\forall \omega \in \Omega, \ Y(\omega) = \sum_{i=1}^{Card(\Omega)} y_i \mathbb{I}_{\omega_i}(\omega)$$

we can write

$$Y = \sum_{i=1}^{Card(\Omega)} y_i \mathbb{I}_{\omega_i}.$$

Notation and definitions I

Proof of Theorem 2.7

Let

1. $\mathbf{Y}$ = the set of random variables built on $(\Omega, \mathcal{F})$;
2. the set of random variables on $(\Omega, \mathcal{F})$ (not necessarily with non-negative values) such that there exists a self-financing strategy $\overrightarrow{\phi}$ (not necessarily admissible) for which $V_T\left(\overrightarrow{\phi}\right) = X$ and $V_0\left(\overrightarrow{\phi}\right) = 0$:

$$\mathbf{X}_0 = \left\{ X \in \mathbf{Y} \left| \begin{array}{c} \exists \overrightarrow{\phi} \text{ self-financing such that} \\ V_T\left(\overrightarrow{\phi}\right) = X \text{ and } V_0\left(\overrightarrow{\phi}\right) = 0 \end{array} \right. \right\},$$

3. and the set of contingent claims with positive values the expectation of which, under the measure P, is strictly positive

$$\mathbf{X}^+ = \left\{ X \in \mathbf{X} \left| \forall \omega \in \Omega,\ X(\omega) \geq 0 \text{ and } \mathrm{E}^P[X] > 0 \right. \right\}.$$

Idea of the proof I

Proof of theorem 2.7

- The proof of this theorem is based on a corollary of the Hahn-Banach Theorem, in functional analysis, which is called the separating hyperplane theorem.
- We will state that corollary, and using a simple example, illustrate its purpose. We will establish the links between such a result and the topic of interest in order to finally complete the proof.

The separating hyperplane I

Idea of the proof of Theorem 2.7

Theorem

Separating hyperplane theorem. *Let $\mathcal{V}$, a topological vector space, $\mathcal{C} \subset \mathcal{V}$, a non-empty convex cone $\mathcal{S} \subset \mathcal{V}$, a vector subspace such that $\mathcal{C} \cap \mathcal{S} = 0$. Then there exists a continuous linear form L on $\mathcal{V}$ such that*

$$L(v) = 0 \ \forall v \in \mathcal{S} \text{ and } L(v) > 0 \ \forall v \in \mathcal{C}, v \neq 0.$$

Definition

$\mathcal{V}$ is a **vector space** if, among other things, $\forall a, b \in \mathbb{R}$ and $\forall v_1, v_2 \in \mathcal{V}$, $av_1 + bv_2 \in \mathcal{V}$ (see the appendix for the other characteristics). For example, the space $\mathbb{R}^n$ is a vector space.

The separating hyperplane II

Idea of the proof of Theorem 2.7

Definition

A subset $\mathcal{C}$ of $\mathcal{V}$ is a **cone** if for all point $v \in \mathcal{C}$, $cv \in \mathcal{C}$ for all $c \in [0, \infty)$.

Definition

A subset A of a vector space is said to be **convex** if, for any two points $v_1, v_2 \in A$, every point $\alpha v_1 + (1 - \alpha) v_2$, $0 < \alpha < 1$, also belongs to the set A.

Definition

$\mathcal{S}$ is a **vector subspace** of $\mathcal{V}$ if $\mathcal{S}$ is a vector space included in $\mathcal{V}$.

The separating hyperplane III

Idea of the proof of Theorem 2.7

Definition

$L : \mathcal{V} \to \mathbb{R}$ is a **linear form** on $\mathcal{V}$ if $\forall a, b \in \mathbb{R}$ and $\forall v_1, v_2 \in \mathcal{V}$,

$$L(av_1 + bv_2) = aL(v_1) + bL(v_2).$$

The separating hyperplane IV

Idea of the proof of Theorem 2.7

- We will be able to show that the set $\mathbf{Y}$ of the random variables built on the measurable space $(\Omega, \mathcal{F})$ is a vector space.
- We will prove that the set $\mathbf{X}_0$ is a vector subspace of $\mathbf{Y}$, whereas $\mathbf{X} = \mathbf{X}^+ \cup \{0\}$ is a convex cone in the same space.
- We will use the hypothesis of absence of arbitrage to show that $\mathbf{X}_0 \cap \mathbf{X}^+ = \varnothing$.
- We will then be able to apply the Hahn-Banach theorem, hence the existence of a linear form L on the space $\mathbf{Y}$ such that

$$L(X) = 0 \ \forall X \in \mathbf{X}_0 \text{ and } L(X) > 0 \ \forall X \in \mathbf{X}^+. \tag{7}$$

The separating hyperplane V

Idea of the proof of Theorem 2.7

- Let's define for all contingent claim $X \in \mathbf{X}$,

$$\pi(X) = \frac{L(X)}{L(S_T^0)}. \tag{8}$$

- We will show that the application $\pi : \mathbf{X} \to \mathbb{R}$, so conveniently chosen, is a price system consistent with the market model.
- Using Proposal 2.6, we will then be able to build a martingale measure Q equivalent to P from the price system.

Structure of the proof I

Proof of Theorem 2.7

We must show that

(i) the set $\mathbf{Y}$ of the random variables built on $(\Omega, \mathcal{F})$ is a vector space,

(ii) $\mathbf{X}_0$ is a vector subspace of $\mathbf{Y}$,

(iii) $\mathbf{X} = \mathbf{X}^+ \cup \{0\}$ is a convex cone of $\mathbf{Y}$,

(iv) $\mathbf{X}_0 \cap \mathbf{X}^+ = \varnothing$,

(v) π, defined on line (8), is a price system consistent with the market model.

We will prove the points (iv) and (v) before the first three ones.

Point (iv) I

Proof of Theorem 2.7

- We want to show that $\mathbf{X}_0 \cap \mathbf{X}^+ = \varnothing$.
- Let's assume there exists a random variable $X \in \mathbf{X}_0 \cap \mathbf{X}^+$ and let's show that, in such as case, there exist arbitrage opportunities, which directly contradicts our basic hypothesis.
- If $X \in \mathbf{X}_0$, then there exists a self-financing strategy $\overrightarrow{\phi}$ (not necessarily admissible) for which

$$V_T\left(\overrightarrow{\phi}\right) = X \text{ and } V_0\left(\overrightarrow{\phi}\right) = 0.$$

- On the other hand, if $X \in \mathbf{X}^+$, then X is a contingent claim $(X \geq 0)$ such that

$$\mathrm{E}^P[X] > 0.$$

Point (iv) II

Proof of Theorem 2.7

- So, if $X \in \mathbf{X}_0 \cap \mathbf{X}^+$, then

$$\begin{aligned} V_0\left(\overrightarrow{\phi}\right) &= 0; \\ V_T\left(\overrightarrow{\phi}\right) &= X \geq 0 \text{ and} \\ \mathrm{E}^P\left[V_T\left(\overrightarrow{\phi}\right)\right] &= \mathrm{E}^P\left[X\right] > 0. \end{aligned}$$

- From the lemma on page 228, we know that, in such a case, it is possible to build an arbitrage opportunity. Since, by hypothesis, the market model contains no arbitrage opportunities, then the set $\mathbf{X}_0 \cap \mathbf{X}^+$ must be empty. ■

Point (v) I

Proof of Theorem 2.7

- We want to show that $\pi(X) = L(X) / L(S_T^0)$ is a price system consistent with our market model.
- Since S_T^0 is a random variable strictly positive, $S_T^0 \in \mathbf{X}^+$ then from choosing the linear form (7), $L(S_T^0) > 0$.
- It implies that, when defining π, on line (8), we did not make the silly mistake of dividing by zero.
- Moreover, $\forall \omega^* \in \Omega$, $\mathbb{I}_{\omega^*} \in \mathbf{X}^+$ implies that $L(\mathbb{I}_{\omega^*}) > 0$.

Point (v) II

Proof of Theorem 2.7

Theorem

π is a price system.

Point (v) III

Proof of Theorem 2.7

Proof.

- First, we will show that $\forall X \in \mathbf{X}$, $\pi(X) \geq 0$.

Indeed, $X \in \mathbf{X} \Rightarrow X = \sum_{i=1}^{Card(\Omega)} x_i \mathbb{I}_{\omega_i}$ where $\forall i \in \{1, ..., Card(\Omega)\}$, $x_i = X(\omega_i) \geq 0$. As a consequence,

$$\begin{aligned}
\pi(X) &= \frac{L(X)}{L(S_T^0)} \text{ from the very definition of } \pi. \\
&= \frac{L\left(\sum_{i=1}^{Card(\Omega)} x_i \mathbb{I}_{\omega_i}\right)}{L(S_T^0)} \\
&= \underbrace{\frac{1}{L(S_T^0)}}_{>0} \sum_{i=1}^{Card(\Omega)} \underbrace{x_i}_{\geq 0} \underbrace{L(\mathbb{I}_{\omega_i})}_{>0} \geq 0.
\end{aligned}$$

Point (v) IV

Proof of Theorem 2.7

- Secondly, $\pi(X) = 0 \Leftrightarrow X = 0$.

Indeed,

$$
\begin{aligned}
0 = \pi(X) &= \underbrace{\frac{1}{L(S_T^0)}}_{>0} \sum_{i=1}^{Card(\Omega)} \underbrace{x_i}_{\geq 0} \underbrace{L(\mathbb{I}_{\omega_i})}_{>0} \\
\Leftrightarrow \quad & \forall i \in \{1, ..., Card(\Omega)\}, x_i = 0 \\
\Leftrightarrow \quad & X = 0.
\end{aligned}
$$

Point (v) V

Proof of Theorem 2.7

- Thirdly, since L is a linear form, we have that $\forall a, b \geq 0$ and $\forall X_1, X_2 \in \mathbf{X}$,

$$\begin{aligned}
\pi\left(aX_1 + bX_2\right) &= \frac{L\left(aX_1 + bX_2\right)}{L\left(S_T^0\right)} \\
&= \frac{aL\left(X_1\right) + bL\left(X_2\right)}{L\left(S_T^0\right)} \\
&= a\pi\left(X_1\right) + b\pi\left(X_2\right). \blacksquare
\end{aligned}$$

Point (v) VI

Proof of Theorem 2.7

- We now need to ensure that our price system is consistent with the market model, i.e. for all admissible strategy $\overrightarrow{\phi} \in \Phi$, we must establish the equality

$$\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right).$$

Point (v) VII

Proof of Theorem 2.7

Let $\overrightarrow{\phi} \in \Phi$ be any admissible strategy. We build a new trading strategy $\overrightarrow{\psi}$ from the $\overrightarrow{\phi}$:

$$\psi_t^k = \begin{cases} \phi_t^0 - V_0\left(\overrightarrow{\phi}\right) & \text{if } k = 0 \\ \phi_t^k & \text{if } k \in \{1, ...K\} \end{cases}$$

i.e. at time $t = 0$, we borrow an amount of $V_0\left(\overrightarrow{\phi}\right)$ by selling short $V_0\left(\overrightarrow{\phi}\right)$ shares of the riskless security in order to acquire the portfolio $\overrightarrow{\phi}_0$.

Point (v) VIII

Proof of Theorem 2.7

Since $\overrightarrow{\phi}$ is self-financing, then $\overrightarrow{\psi}$ is also self-financing. Indeed, $\forall t \in \{1, ..., T-1\}$

$$\begin{aligned}
\overrightarrow{\psi}_t \overrightarrow{S}_t &= \left(\phi_t^0 - V_0\left(\overrightarrow{\phi}\right)\right) S_t^0 + \sum_{k=1}^{K} \phi_t^k S_t^k \\
&= \sum_{k=0}^{K} \phi_t^k S_t^k - V_0\left(\overrightarrow{\phi}\right) S_t^0 \\
&= \sum_{k=0}^{K} \phi_{t+1}^k S_t^k - V_0\left(\overrightarrow{\phi}\right) S_t^0 \\
&= \left(\phi_{t+1}^0 - V_0\left(\overrightarrow{\phi}\right)\right) S_t^0 + \sum_{k=1}^{K} \phi_{t+1}^k S_t^k \\
&= \overrightarrow{\psi}_{t+1} \overrightarrow{S}_t.
\end{aligned}$$

Point (v) IX

Proof of Theorem 2.7

However, it is possible that $\overrightarrow{\psi}$ is not admissible. We now want to show that π is consistent with the market model.
Since

$$\begin{aligned} V_0\left(\overrightarrow{\psi}\right) &= \overrightarrow{\psi}_1 \overrightarrow{S}_0 \\ &= \left(\phi_1^0 - V_0\left(\overrightarrow{\phi}\right)\right) S_0^0 + \sum_{k=1}^{K} \phi_1^k S_0^k \\ &= \sum_{k=0}^{K} \phi_1^k S_0^k - V_0\left(\overrightarrow{\phi}\right) S_0^0 \\ &= V_0\left(\overrightarrow{\phi}\right) - V_0\left(\overrightarrow{\phi}\right) S_0^0 = 0, \end{aligned}$$

then $V_T\left(\overrightarrow{\psi}\right) \in \mathbf{X}_0$.

Point (v) X

Proof of Theorem 2.7

As a consequence, since $L(X) = 0 \ \forall X \in \mathbf{X}_0$,

$$\pi\left(V_T\left(\overrightarrow{\psi}\right)\right) = \frac{L\left(V_T\left(\overrightarrow{\psi}\right)\right)}{L\left(S_T^0\right)} = \frac{0}{L\left(S_T^0\right)} = 0.$$

Point (v) XI

Proof of Theorem 2.7

On the other hand, since

$$\begin{aligned} V_T\left(\overrightarrow{\psi}\right) &= \overrightarrow{\psi}_T \overrightarrow{S}_T \\ &= \left(\phi_T^0 - V_0\left(\overrightarrow{\phi}\right)\right) S_T^0 + \sum_{k=1}^{K} \phi_T^k S_T^k \\ &= \sum_{k=0}^{K} \phi_T^k S_T^k - V_0\left(\overrightarrow{\phi}\right) S_T^0 \\ &= V_T\left(\overrightarrow{\phi}\right) - V_0\left(\overrightarrow{\phi}\right) S_T^0 \end{aligned}$$

Point (v) XII

Proof of Theorem 2.7

then

$$
\begin{aligned}
0 &= \pi\left(V_T\left(\overrightarrow{\psi}\right)\right) \\
&= \pi\left(V_T\left(\overrightarrow{\phi}\right) - V_0\left(\overrightarrow{\phi}\right) S_T^0\right) \\
&= \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) - V_0\left(\overrightarrow{\phi}\right) \underbrace{\pi\left(S_T^0\right)}_{=\frac{L(S_T^0)}{L(S_T^0)}=1} \\
&= \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) - V_0\left(\overrightarrow{\phi}\right)
\end{aligned}
$$

hence

$$
\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right).
$$

We have just finished building a price system consistent with the market model. ■

Point (i)

Proof of Theorem 2.7

- Let $\mathbf{Y}$ be the set of all random variables built on the probability space $(\Omega, \mathcal{F}, P)$. We want to prove that $\mathbf{Y}$ is a vector space.
- $\mathbf{Y}$ *is a vector space*

In order to show that $\mathbf{Y}$ is a vector space, we must verify that

$$\forall a, b \in \mathbb{R} \text{ and } \forall Y_1, Y_2 \in \mathbf{Y},\ aY_1 + bY_2 \in \mathbf{Y}.$$

This is indeed the case, since all finite linear combination of $(\Omega, \mathcal{F})$ −random variables is a $(\Omega, \mathcal{F})$ −random variable. We must also verify the other conditions stated in the appendix.■

Point (ii) I

Proof of Theorem 2.7

- $\mathbf{X}_0$ *is a vector subspace of* $\mathbf{Y}$ *where*

$$\mathbf{X}_0 = \left\{ X \in \mathbf{Y} \;\middle|\; \begin{array}{c} \exists \overrightarrow{\phi} \text{ self-financing satisfying} \\ V_T\left(\overrightarrow{\phi}\right) = X \text{ and } V_0\left(\overrightarrow{\phi}\right) = 0 \end{array} \right\}$$

- We must show that

$$\forall a, b \in \mathbb{R} \text{ and } \forall X_1, X_2 \in \mathbf{X}_0,\ aX_1 + bX_2 \in \mathbf{X}_0.$$

Point (ii) II

Proof of Theorem 2.7

Let X_1 and X_2, be two elements of $\mathbf{X}_0$ arbitrarily chosen. Since $X_1 \in \mathbf{X}_0$, then there exists a self-financing strategy $\overrightarrow{\phi}$ satisfying $V_T\left(\overrightarrow{\phi}\right) = X_1$ and $V_0\left(\overrightarrow{\phi}\right) = 0$. Similarly, we know there exists a self-financing strategy $\overrightarrow{\varphi}$ such that $V_T\left(\overrightarrow{\varphi}\right) = X_2$ and $V_0\left(\overrightarrow{\varphi}\right) = 0$.

Let a and b, be any two real numbers. Since $a\overrightarrow{\phi} + b\overrightarrow{\varphi}$ is also a self-financing strategy satisfying

$$V_T\left(a\overrightarrow{\phi} + b\overrightarrow{\varphi}\right) = aV_T\left(\overrightarrow{\phi}\right) + bV_T\left(\overrightarrow{\varphi}\right) = aX_1 + bX_2$$

and

$$V_0\left(a\overrightarrow{\phi} + b\overrightarrow{\varphi}\right) = aV_0\left(\overrightarrow{\phi}\right) + bV_0\left(\overrightarrow{\varphi}\right) = 0,$$

then $aX_1 + bX_2$ belongs to the set $\mathbf{X}_0$. Again we must verify the other conditions stated in the appendix.■

Point (iii)

Proof of Theorem 2.7

- $\mathbf{X} = \mathbf{X}^+ \cup \{0\}$ *is a cone de* $\mathbf{Y}$.

Proof. Let any $X \in \mathbf{X}$. Since, for any real number $c \geq 0$, cX is a non-negative random variable, then cX is a contingent claim, which implies that $cX \in \mathbf{X}$. ■

- $\mathbf{X}^+$ *is convex.*

Proof. Let $0 < \alpha < 1$ and $X_1, X_2 \in \mathbf{X}$.

$$\forall \omega \in \Omega,\ \alpha X_1(\omega) + (1-\alpha) X_2(\omega) \geq 0,$$

hence $\alpha X_1 + (1-\alpha) X_2 \in \mathbf{X}$.

- The proof of Theorem 2.7 is complete. ■

Statement

Corollary p 228

Corollary

Corollary on page 228. *If the market model contains no arbitrage, then there is a single price associated with any attainable contingent claim X and it satisfies $\pi = \mathrm{E}^Q\left[\beta_T X\right]$ for each martingale measure $Q \in P$.*

Interpretation I

Corollary p 228

- Proposal 2.6 associates a price system to each of the martingale measures.
- But it is also possible that there exists an infinity of such measures. So, it is also possible that there is an infinity of price systems.
- The corollary states that, if X is any attainable contingent claim, and π and $\widetilde{\pi}$ are two different prices systems, then $\pi(X) = \widetilde{\pi}(X)$, i.e. the price of the attainable contingent claim is the same, whatever price system is used to price it.

Proof I

Corollary p 228

- Let X be any attainable contingent claim.
- Since it is attainable, there exists at least one admissible strategy $\overrightarrow{\varphi}$ such that $V_T\left(\overrightarrow{\varphi}\right) = X$.
- Let $\overrightarrow{\varphi}$ and $\overrightarrow{\phi} \in \Phi$ be two admissible strategies such that

$$V_T\left(\overrightarrow{\varphi}\right) = X \text{ and } V_T\left(\overrightarrow{\phi}\right) = X. \tag{9}$$

- Note that it is possible that $\overrightarrow{\varphi} = \overrightarrow{\phi}$, in particular if there is only a single admissible strategy allowing to attain X.

Proof II

Corollary p 228

- Since our market model contains no arbitrage opportunities (by hypothesis), then there exists at least one price system consistent with the market model (by Theorem 2.7 followed by Proposal 2.6). If there only exists one such system, then there will only be a single price associated with the attainable contingent claim X.

Proof III

Corollary p 228

- We want to verify that, if there exist several price systems consistent with the market model, then there will always only be a single price associated with X, i.e. if π and $\widetilde{\pi}$ are two different price systems, then

$$\pi(X) = \widetilde{\pi}(X). \tag{10}$$

- From Definition (2.5a), page 226, of price systems, we have that $X = 0 \Leftrightarrow \pi(X) = 0 = \widetilde{\pi}(X)$. As a consequence, the equation (10) is trivially satisfied when $X = 0$. So let's assume there exists an $\omega^* \in \Omega$ for which $X(\omega^*) > 0$.

Proof IV

Corollary p 228

- Let π, $\widetilde{\pi} \in \Pi$, $\pi \neq \widetilde{\pi}$ be two price systems consistent with the market model. Since they are consistent, we have $\forall \overrightarrow{\phi} \in \Phi$,

$$\pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right) \text{ and } \widetilde{\pi}\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right). \quad (11)$$

- In particular, because of equalities (9), we can write

$$\begin{aligned} \pi(X) &= \pi\left(V_T\left(\overrightarrow{\varphi}\right)\right) = V_0\left(\overrightarrow{\varphi}\right) = \overrightarrow{\varphi}_1 \overrightarrow{S}_0 \quad (12) \\ \widetilde{\pi}(X) &= \widetilde{\pi}\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right) = \overrightarrow{\phi}_1 \overrightarrow{S}_0 \end{aligned}$$

- We want to show that $\pi(X) = \widetilde{\pi}(X)$. Let's assume the contrary and let's show that, in such a case, the market contains arbitrage opportunities, which contradicts the premise of the corollary.

Proof V

Corollary p 228

- So, let's assume that $\pi(X) \neq \widetilde{\pi}(X)$ (since $\pi(X)$ and $\widetilde{\pi}(X)$ are real numbers, we can, without loss of generality, assume that $\pi(X) < \widetilde{\pi}(X)$).
- We are going to build ourselves a strategy that generates arbitrage opportunities.
- To begin with, note that, due to the inequality $\pi(X) < \widetilde{\pi}(X)$ the two strategies chosen on line (9) are different.

Indeed, from the equations on line (12),

$$0 < \widetilde{\pi}(X) - \pi(X) = \overrightarrow{\phi}_1 \overrightarrow{S}_0 - \overrightarrow{\varphi}_1 \overrightarrow{S}_0 = \left(\overrightarrow{\phi}_1 - \overrightarrow{\varphi}_1\right) \overrightarrow{S}_0$$

hence $\overrightarrow{\phi}_1 - \overrightarrow{\varphi}_1 \neq \overrightarrow{0}$.

Proof VI

Corollary p 228

- Let's build a new strategy $\overrightarrow{\psi}$ consisting of a fortunate mix of the first two:

$$\overrightarrow{\psi} = \overrightarrow{\varphi} - \frac{\pi(X)}{\widetilde{\pi}(X)} \overrightarrow{\phi}.$$

Recall that $X \neq 0$ implies, from Property (2.5a), p. 226, that $\widetilde{\pi}(X) > 0$.

Proof VII

Corollary p 228

- We must verify that $\overrightarrow{\psi}$ is self-financing, i.e.

$$\forall t \in \{0, ..., T-1\}, \ \overrightarrow{\psi}_t \overrightarrow{S}_t = \overrightarrow{\psi}_{t+1} \overrightarrow{S}_t$$

then we will show that $\overrightarrow{\psi}$ allows us to create an arbitrage opportunity.

Proof VIII

Corollary p 228

Since strategies $\overrightarrow{\varphi}$ and $\overrightarrow{\phi}$ are self-financing, then $\forall t \in \{0, ..., T-1\}$

$$\begin{aligned}
\overrightarrow{\psi}_t \overrightarrow{S}_t &= \left(\overrightarrow{\varphi}_t - \frac{\pi(X)}{\widetilde{\pi}(X)} \overrightarrow{\phi}_t\right) \overrightarrow{S}_t \\
&= \overrightarrow{\varphi}_t \overrightarrow{S}_t - \frac{\pi(X)}{\widetilde{\pi}(X)} \overrightarrow{\phi}_t \overrightarrow{S}_t \\
&= \overrightarrow{\varphi}_{t+1} \overrightarrow{S}_t - \frac{\pi(X)}{\widetilde{\pi}(X)} \overrightarrow{\phi}_{t+1} \overrightarrow{S}_t \\
&= \left(\overrightarrow{\varphi}_{t+1} - \frac{\pi(X)}{\widetilde{\pi}(X)} \overrightarrow{\phi}_{t+1}\right) \overrightarrow{S}_t \\
&= \overrightarrow{\psi}_{t+1} \overrightarrow{S}_t
\end{aligned}$$

thus proving that $\overrightarrow{\psi}$ is also self-financing.

Proof IX

Corollary p 228

In order to show that $\overrightarrow{\psi}$ allows us to create an arbitrage opportunity, we must verify that:

(i) $V_0\left(\overrightarrow{\psi}\right) = 0$,

(ii) $V_T\left(\overrightarrow{\psi}\right) \geq 0$,

(iii) $\mathrm{E}^P\left[V_T\left(\overrightarrow{\psi}\right)\right] > 0$.

Indeed, using the lemma on page 228, we can conclude from (i), (ii) and (iii) that there exists an arbitrage opportunity. Note that we don't need to verify that strategy $\overrightarrow{\psi}$ is admissible.

Proof - Point (i)

Corollary p 228

Using the relations established on line number (12),

$$
\begin{aligned}
V_0\left(\overrightarrow{\psi}\right) &= \overrightarrow{\psi}_1 \overrightarrow{S}_0 \\
&= \left(\overrightarrow{\varphi}_1 - \frac{\pi(X)}{\widetilde{\pi}(X)} \overrightarrow{\phi}_1\right) \overrightarrow{S}_0 \\
&= \overrightarrow{\varphi}_1 \overrightarrow{S}_0 - \frac{\pi(X)}{\widetilde{\pi}(X)} \overrightarrow{\phi}_1 \overrightarrow{S}_0 \\
&= \pi(X) - \frac{\pi(X)}{\widetilde{\pi}(X)} \widetilde{\pi}(X) \\
&= 0. \blacksquare
\end{aligned}
$$

Proof - Point (ii)

Corollary p 228

From the choice of strategies $\overrightarrow{\varphi}$ and $\overrightarrow{\phi}$ (ref.: line (9)),

$$
\begin{aligned}
V_T\left(\overrightarrow{\psi}\right) &= \overrightarrow{\psi}_T \overrightarrow{S}_T \\
&= \left(\overrightarrow{\varphi}_T - \frac{\pi(X)}{\widetilde{\pi}(X)}\overrightarrow{\phi}_T\right)\overrightarrow{S}_T \\
&= \overrightarrow{\varphi}_T \overrightarrow{S}_T - \frac{\pi(X)}{\widetilde{\pi}(X)}\overrightarrow{\phi}_T \overrightarrow{S}_T \\
&= V_T\left(\overrightarrow{\varphi}\right) - \frac{\pi(X)}{\widetilde{\pi}(X)} V_T\left(\overrightarrow{\phi}\right) \\
&= X - \frac{\pi(X)}{\widetilde{\pi}(X)} X \\
&= \underbrace{\left(1 - \frac{\pi(X)}{\widetilde{\pi}(X)}\right)}_{>0 \text{ since } \pi(X)<\widetilde{\pi}(X)} X \geq 0. \blacksquare
\end{aligned}
$$

Proof - Point (iii) I

Corollary p 228

Since X is such that $X(\omega^*) > 0$ for some $\omega^* \in \Omega$,

$$\begin{aligned}
\mathrm{E}^P\left[V_T\left(\overrightarrow{\psi}\right)\right] &= \mathrm{E}^P\left[\left(1-\frac{\pi(X)}{\widetilde{\pi}(X)}\right)X\right] \\
&= \sum_{\omega\in\Omega}\underbrace{\left(1-\frac{\pi(X)}{\widetilde{\pi}(X)}\right)}_{>0}\underbrace{X(\omega)}_{\geq 0}\underbrace{P(\omega)}_{>0} \\
&\geq \left(1-\frac{\pi(X)}{\widetilde{\pi}(X)}\right)X(\omega^*)P(\omega^*) \\
&> 0 \blacksquare
\end{aligned}$$

The proof of the corollary is thus completed.

Statement

Proposal 2.8

Theorem

Proposal 2.8. *If* $\overrightarrow{\phi}$ *is an admissible strategy, then the process* $\beta V\left(\overrightarrow{\phi}\right)$, *representing its discounted market value, is a* $Q-$*martingale for each measure* $Q \in P$.

Proof I

Proposal 2.8

- Let $\overrightarrow{\phi} \in \Phi$ be any admissible strategy and $Q \in \mathbb{P}$, a martingale measure arbitrarily chosen.
- In order to show that the adapted process $\beta V\left(\overrightarrow{\phi}\right)$ is a $Q-$martingale, it is sufficient to verify that $\forall t \in \{1, ..., T\}$,

$$\mathrm{E}^{Q}\left[\beta_t V_t\left(\overrightarrow{\phi}\right) - \beta_{t-1} V_{t-1}\left(\overrightarrow{\phi}\right) | \mathcal{F}_{t-1}\right] = 0.$$

Proof II

Proposal 2.8

Since

$$\beta_t V_t\left(\overrightarrow{\phi}\right) = \begin{cases} \beta_0 \overrightarrow{\phi}_1 \overrightarrow{S}_0 & \text{if } t = 0 \\ \beta_t \overrightarrow{\phi}_t \overrightarrow{S}_t & \text{if } t \in \{1, ..., T\} \end{cases}$$

then, by setting $\overrightarrow{\phi}_0 = \overrightarrow{\phi}_1$ we have $\forall t \in \{0, ..., T\}$, $\beta_t V_t\left(\overrightarrow{\phi}\right) = \beta_t \overrightarrow{\phi}_t \overrightarrow{S}_t$

Proof III

Proposal 2.8

As a consequence, $\forall t \in \{1, ..., T\}$,

$$\begin{aligned}
& \mathrm{E}^Q\left[\beta_t V_t\left(\overrightarrow{\phi}\right) - \beta_{t-1} V_{t-1}\left(\overrightarrow{\phi}\right) | \mathcal{F}_{t-1}\right] \\
= \; & \mathrm{E}^Q\left[\beta_t \overrightarrow{\phi}_t \overrightarrow{S}_t - \beta_{t-1} \overrightarrow{\phi}_{t-1} \overrightarrow{S}_{t-1} | \mathcal{F}_{t-1}\right] \\
= \; & \mathrm{E}^Q\left[\beta_t \overrightarrow{\phi}_t \overrightarrow{S}_t - \beta_{t-1} \overrightarrow{\phi}_t \overrightarrow{S}_{t-1} | \mathcal{F}_{t-1}\right] \\
& \text{car } \overrightarrow{\phi} \text{ is self-financing} \\
= \; & \overrightarrow{\phi}_t \quad \underbrace{\mathrm{E}^Q\left[\beta_t \overrightarrow{S}_t - \beta_{t-1} \overrightarrow{S}_{t-1} | \mathcal{F}_{t-1}\right]}_{= \overrightarrow{0}} \\
& \qquad \text{since the components of } \beta \overrightarrow{S} \text{ are } Q\text{-martingales} \\
& \text{since } \overrightarrow{\phi} \text{ is predictable} \\
= \; & 0. \blacksquare
\end{aligned}$$

Statement

Proposal 2.9

Theorem

Proposal 2.9. *If $X \in \mathbf{X}$ is an attainable contingent claim, then*

$$\beta_t V_t\left(\overrightarrow{\phi}\right) = \mathrm{E}^Q\left[\beta_T X \,|\mathcal{F}_t\right]$$

for any trading strategy $\overrightarrow{\phi}$ which generates X and for each measure $Q \in P$.

Interpretation I

Proposal 2.9

- If the contingent claim X can be replicated with strategy $\overrightarrow{\phi}$, $V_T\left(\overrightarrow{\phi}\right) = X$, then the market value of such a contingent claim must be, at any time, equal to the market value of the strategy, otherwise, there is a way to create an arbitrage opportunity.

Indeed, if at time t, the market value of the contingent claim X, let's denote it X_t, is less than $V_t\left(\overrightarrow{\phi}\right)$, then let's sell short portfolio $\overrightarrow{\phi}_t$, and, with part of the proceeds from such a sale, let's purchase for an amount X_t the contingent claim X. With the other part of the proceeds, $V_t\left(\overrightarrow{\phi}\right) - X_t$, let's purchase $\left(V_t\left(\overrightarrow{\phi}\right) - X_t\right)/S_t^0$ shares of the riskless security. Our initial investment is nil. On the time horizon $t = T$, we use the contingent claim to pay back the portfolio that had been sold short, since, at that time, their respective market values are equal. We now only have to sell the shares of the riskless security to make a profit of $\frac{V_t\left(\overrightarrow{\phi}\right) - X_t}{S_t^0} S_T^0$.

Interpretation II

Proposal 2.9

- Such a proposal allows us to determine, using the equivalent martingale measure, the market value of the contingent claim X at any time since

$$\begin{aligned} & \text{the market value of } X \text{ at time } t \\ = \quad & V_t\left(\overrightarrow{\phi}\right) \\ = \quad & \frac{1}{\beta_t}\beta_t V_t\left(\overrightarrow{\phi}\right) \\ = \quad & \frac{1}{\beta_t}\mathrm{E}^Q\left[\beta_T X \,|\mathcal{F}_t\right]. \end{aligned}$$

Proof

Proposal 2.9

$$\begin{aligned}
\beta_t V_t\left(\overrightarrow{\phi}\right) &= \mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\phi}\right)|\mathcal{F}_t\right] \\
&\quad \text{from Proposal 2.8, page 230} \\
&= \mathrm{E}^Q\left[\beta_T X\,|\mathcal{F}_t\right] \\
&\quad \text{since } \overrightarrow{\phi} \text{ generates } X \text{ therefore } V_T\left(\overrightarrow{\phi}\right) = X. \blacksquare
\end{aligned}$$

Alternative approach

Pricing a European-style contingent claim

- What follows cannot be found as such in the article under study. However, such an alternative approach allows us to answer some questions not addressed in the article.
- The corollary on page 228 shows how to obtain prices for attainable contingent claims. What about the other contingent claims, those which are not attainable?
- The price of a European-style contingent claim may be studied from two different perspectives: from the contingent claim seller's and from the contingent claim buyer's.

Seller's perspective

Pricing a European-style contingent claim

The seller's first goal is to ensure that, if he invests the proceeds x from selling the contingent claim X adequately, then, at time T when the buyer exercises his claim, he is able to meet his obligation, i.e. pay the amount X. The minimum price acceptable to the seller of the contingent claim X is therefore

$$x_{\sup} = \inf \left\{ x \geq 0 \,\middle|\, \begin{array}{c} \exists \overrightarrow{\varphi} \in \Phi \text{ such that} \\ V_0\left(\overrightarrow{\varphi}\right) = x \text{ and } V_T\left(\overrightarrow{\varphi}\right) \geq X \end{array} \right\}.$$

Buyer's perspective

Pricing a European-style contingent claim

If the buyer borrows an amount x at time $t = 0$ in order to purchase the contingent claim X, then, at time T when he exercises his claim, he wishes to be able to pay back his debt, i.e. there must exist a self-financing trading strategy ϕ such that $V_0\left(\overrightarrow{\phi}\right) = -x$ and $V_T\left(\overrightarrow{\phi}\right) + X \geq 0$. Thus, the maximum price which the buyer of the contingent claim X is willing to pay is

$$x_{\inf} = \sup\left\{ x \geq 0 \,\middle|\, \begin{array}{c} \exists\, \overrightarrow{\phi} \in \Phi \text{ such that} \\ V_0\left(\overrightarrow{\phi}\right) = -x \text{ and } V_T\left(\overrightarrow{\phi}\right) + X \geq 0 \end{array} \right\}.$$

Theorem

Pricing a European-style contingent claim

Theorem

If the market model contains no arbitrage opportunities, then, for any martingale measure $Q \in \mathbb{P}$,

$$x_{\inf} \leq \mathrm{E}^{Q}\left[\beta_T X\right] \leq x_{\sup}.$$

Interpretation

Pricing a European-style contingent claim

- Since the minimum price acceptable to the seller of the contingent claim X, x_{sup}, is always greater than or equal to the maximum price that the buyer of the contingent claim X is willing to pay, x_{inf}, we must have that

$$x_{\text{inf}} = \mathrm{E}^{Q}\left[\beta_T X\right] = x_{\text{sup}}$$

  for the transaction to be concluded without any of the two parties taking on a risk.
- But, if there exist several martingale measures, say Q and Q^*, it is possible that, for some non-attainable contingent claims, we have

$$\mathrm{E}^{Q}\left[\beta_T X\right] < \mathrm{E}^{Q^*}\left[\beta_T X\right].$$

  In such a case, the two parties cannot agree upon a price, since

$$x_{\text{inf}} \leq \mathrm{E}^{Q}\left[\beta_T X\right] < \mathrm{E}^{Q^*}\left[\beta_T X\right] \leq x_{\text{sup}}.$$

Proof - second inequality I

Pricing a European-style contingent claim

- **To be shown.** $\mathrm{E}^Q\left[\beta_T X\right] \leq x_{\sup}$.

Let's choose arbitrarily

$$x_0 \in \left\{ x \geq 0 \,\middle|\, \begin{array}{c} \exists \overrightarrow{\varphi} \in \Phi \text{ such that} \\ V_0\left(\overrightarrow{\varphi}\right) = x \text{ and } V_T\left(\overrightarrow{\varphi}\right) \geq X \end{array} \right\}.$$

Due to such a choice, there exists a strategy $\overrightarrow{\varphi} \in \Phi$ such that $V_0\left(\overrightarrow{\varphi}\right) = x_0$ and $V_T\left(\overrightarrow{\varphi}\right) \geq X$. Thus, based on that last inequality, we state that

$$\mathrm{E}^Q\left[\beta_T X\right] \leq \mathrm{E}^Q\left[\beta_T V_T\left(\overrightarrow{\varphi}\right)\right]. \tag{13}$$

Proof - second inequality II

Pricing a European-style contingent claim

Since the market contains no arbitrage opportunities, the stochastic process $\beta V\left(\overrightarrow{\varphi}\right)$ is a $(Q, \mathbb{F})$ −martingale (Harrison and Pliska, 1981, Proposal 2.8, p. 230). As a consequence,

$$\mathrm{E}^{Q}\left[\beta_{T} V_{T}\left(\overrightarrow{\varphi}\right)\right]=\mathrm{E}^{Q}\left[\beta_{0} V_{0}\left(\overrightarrow{\varphi}\right)\right]=\beta_{0} x_{0}=x_{0}.$$

By substitution into inequality (13), we obtain

$$\mathrm{E}^{Q}\left[\beta_{T} X\right] \leq x_{0}.$$

Since the choice of x_0 was arbitrary, then

$$\begin{aligned} & \mathrm{E}^{Q}\left[\beta_{T} X\right] \\ \leq & \inf\left\{x_{0} \geq 0 \,\middle|\, \begin{array}{c} \exists \overrightarrow{\varphi} \in \Phi \text{ such that } V_{0}\left(\overrightarrow{\varphi}\right)=x_{0} \\ \text{and } V_{T}\left(\overrightarrow{\varphi}\right) \geq X \end{array}\right\} \\ = & x_{\sup}. \end{aligned}$$

Proof - first inequality I

Pricing a European-style contingent claim

- **To be shown.** $x_{\inf} \leq \mathrm{E}^{Q}\left[\beta_T X\right]$

Let's choose arbitrarily

$$x_0 \in \left\{ x \geq 0 \;\middle|\; \begin{array}{c} \exists \overrightarrow{\phi} \in \Phi \text{ such that} \\ V_0\left(\overrightarrow{\phi}\right) = -x \text{ and } V_T\left(\overrightarrow{\phi}\right) + X \geq 0 \end{array} \right\}$$

thus establishing that there exists an admissible strategy $\overrightarrow{\phi} \in \Phi$ satisfying $V_0\left(\overrightarrow{\phi}\right) = -x_0$ and $V_T\left(\overrightarrow{\phi}\right) + X \geq 0$. Thus, given that $\beta_T > 0$,

$$\begin{aligned} 0 &\leq \mathrm{E}^{Q}\left[\beta_T\left(V_T\left(\overrightarrow{\phi}\right) + X_T\right)\right] \\ &= \mathrm{E}^{Q}\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] + \mathrm{E}^{Q}\left[\beta_T X\right]. \end{aligned}$$

Proof - first inequality II

Pricing a European-style contingent claim

On the other hand, since $\beta V\left(\overrightarrow{\phi}\right)$ is a $(Q, \mathbb{F})$ −martingale,

$$\mathrm{E}^{Q}\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] = \mathrm{E}^{Q}\left[\beta_0 V_0\left(\overrightarrow{\phi}\right)\right] = -\beta_0 x_0 = -x_0.$$

As a consequence,

$$0 \leq \mathrm{E}^{Q}\left[\beta_T V_T\left(\overrightarrow{\phi}\right)\right] + \mathrm{E}^{Q}\left[\beta_T X\right] = -x_0 + \mathrm{E}^{Q}\left[\beta_T X\right]$$

hence

$$\mathrm{E}^{Q}\left[\beta_T X\right] \geq x_0.$$

Since the choice of x_0 was arbitrary,

$$\begin{aligned} & \mathrm{E}^{Q}\left[\beta_T X\right] \\ \geq \quad & \sup\left\{ x_0 \geq 0 \;\middle|\; \begin{array}{c} \exists\, \overrightarrow{\phi} \in \Phi \text{ such that} \\ V_0\left(\overrightarrow{\phi}\right) = -x_0 \text{ and } V_T\left(\overrightarrow{\phi}\right) + X \geq 0 \end{array} \right\} \\ = \quad & x_{\inf}. \blacksquare \end{aligned}$$

Risk-neutral measures I

- Why are martingale measures also called risk-neutral measures?
- To answer that question, we assume that the evolution of the price of the riskless security is modelled by a predictable process. That is not unreasonable since, in the case of a bond, we know at least one period ahead what the price will be. Indeed, since interest rates are given at the time of purchase, we can predict the value of the bond.

Risk-neutral measures II

- For any of the K risky securities, the expected return, under a martingale measure Q, is equal to the average return of the riskless security. Indeed,

$$
\begin{aligned}
& \mathrm{E}^Q\left[\frac{S_t^k - S_{t-1}^k}{S_{t-1}^k}\right] = \mathrm{E}^Q\left[\frac{\beta_t S_t^k - \beta_t S_{t-1}^k}{\beta_t S_{t-1}^k}\right] \qquad (14) \\
= \quad & \mathrm{E}^Q\left[\mathrm{E}^Q\left[\frac{\beta_t S_t^k - \beta_t S_{t-1}^k}{\beta_t S_{t-1}^k}\,|\mathcal{F}_{t-1}\right]\right] = \mathrm{E}^Q\left[\frac{\mathrm{E}^Q\left[\beta_t S_t^k - \beta_t S_{t-1}^k\,|\mathcal{F}_{t-1}\right]}{\beta_t S_{t-1}^k}\right] \\
= \quad & \mathrm{E}^Q\left[\frac{\mathrm{E}^Q\left[\beta_t S_t^k\,|\mathcal{F}_{t-1}\right] - \mathrm{E}^Q\left[\beta_t S_{t-1}^k\,|\mathcal{F}_{t-1}\right]}{\beta_t S_{t-1}^k}\right] \\
= \quad & \mathrm{E}^Q\left[\frac{\beta_{t-1} S_{t-1}^k - \beta_t S_{t-1}^k}{\beta_t S_{t-1}^k}\right] \text{ since } \beta S^k \text{ is a } Q-\text{martingale.} \\
= \quad & \mathrm{E}^Q\left[\frac{\beta_{t-1} - \beta_t}{\beta_t}\right] = \mathrm{E}^Q\left[\frac{\frac{1}{S_{t-1}^0} - \frac{1}{S_t^0}}{\frac{1}{S_t^0}}\right] \text{ since the discount factor } \beta_t = \frac{1}{S_t^0}. \\
= \quad & \mathrm{E}^Q\left[\frac{S_t^0 - S_{t-1}^0}{S_{t-1}^0}\right].
\end{aligned}
$$

Complete market

Definition

A market is said to be complete if it contains no arbitrage opportunity and if all contingent claims are attainable.

Theorem

A market is complete if and only if there exists a single martingale measure.

Proof - Part 1 I

Complete market

- Since the market is complete, it contains no arbitrage opportunity. From Theorem 2.7, we conclude that there exists at least one martingale measure.
- We must now prove that it is unique.
- Let's assume that several such measures exist, and let's choose two of them, Q and $\widehat{Q}$, arbitrarily. In order to establish uniqueness, it will be enough to show that

$$\forall \omega \in \Omega,\ Q(\omega) = \widehat{Q}(\omega).$$

Proof - Part 1 II

Complete market

Let $\omega^* \in \Omega$ be any element of the sample space. Let's define the contingent claim X as follows:

$$X = \frac{1}{\beta_T} \mathbb{I}_{\omega^*}$$

while π_Q and $\pi_{\widehat{Q}}$ represent price systems associated with Q and $\widehat{Q}$ respectively.
Thus,

$$\begin{aligned}
\pi_Q(X) &= \mathrm{E}^Q[\beta_T X] \text{ (from Proposal 2.6)} \\
&= \mathrm{E}^Q\left[\beta_T \frac{1}{\beta_T} \mathbb{I}_{\omega^*}\right] \text{ (by definition of } X\text{)} \\
&= \mathrm{E}^Q[\mathbb{I}_{\omega^*}] \\
&= Q[\omega^*]
\end{aligned}$$

Proof - Part 1 III

Complete market

and, using the same approach, $\pi_{\widehat{Q}}(X) = \widehat{Q}[\omega^*]$
Since the market is complete, the contingent claim X is attainable, and the corollary on page 228 then implies that

$$\pi_Q(X) = \pi_{\widehat{Q}}(X) \tag{15}$$

hence

$$Q[\omega^*] = \pi_Q(X) = \pi_{\widehat{Q}}(X) = \widehat{Q}[\omega^*]. \blacksquare$$

Proof - Part 2 I

Complete market

- We want to show that, in a market without arbitrage, if there exists no more than a single martingale measure, then all contingent claims are attainable.
- We are going to prove the contrapositive, i.e. if the market contains no arbitrage opportunity and it is non- complete, then there exist at least two martingale measures.

Proof - Part 2 II

Complete market

- Since the market contains no arbitrage opportunity, then, from Theorem 2.7, we know that there exists at least one measure martingale, which we denote Q.
- Proposal 2.6 shows that the price system associated with Q can be defined, for all contingent claim X, as follows:

$$\begin{aligned} \pi_Q[X] &= \mathrm{E}^Q[\beta_T X] \\ &= \sum_{\omega \in \Omega} \beta_T(\omega) X(\omega) Q(\omega). \end{aligned}$$

Proof - Part 2 III

Complete market

- Moreover, since the market is non-complete, then there exists at least one non-attainable contingent claim, which we will call Y.
- That contingent claim Y does not belong to the vector subspace generated by the attainable random variables since there exists no admissible portfolio allowing to replicate the cash flow generated by Y
  - (recall that all finite linear combination of attainable random variables is itself attainable using a portfolio made up of that same linear combination of portfolios allowing to attain each of the random variables individually).

Proof - Part 2 IV

Complete market

- We can therefore build (see appendix) a random variable Z such that $|Z| < 1$, $(1+Z)$ is a non-null contingent claim and, for any attainable contingent claim X,

$$\begin{aligned}\pi\left(X\left(1+Z\right)\right) &= \mathrm{E}^{Q}\left[\beta_T X\left(1+Z\right)\right] \\ &= \mathrm{E}^{Q}\left[\beta_T X\right] + \underbrace{\mathrm{E}^{Q}\left[\beta_T X Z\right]}_{=0} = \pi\left(X\right).\end{aligned}$$

- Let's define the function $\widehat{\pi} : \mathbf{X} \longrightarrow [0, \infty)$ as follows:

$$\widehat{\pi}\left[X\right] = \sum_{\omega \in \Omega} \beta_T\left(\omega\right) X\left(\omega\right)\left(1+Z\left(\omega\right)\right) Q\left(\omega\right) = \pi\left(X\left(1+Z\right)\right).$$

- $\widehat{\pi}$ is a consistent price system since it satisfies the three criteria in the definition.

Proof - Part 2 V

Complete market

1. First

$$\widehat{\pi}[X] = 0 \Leftrightarrow \sum_{\omega\in\Omega} \underbrace{\beta_T(\omega)}_{>0} X(\omega) \underbrace{(1+Z(\omega))}_{>0} \underbrace{Q(\omega)}_{>0} = 0. \Leftrightarrow \pi(X) = 0$$

2. Second, for all non-negative constants a and b, and for all contingent claims X_1 and X_2,

$$\begin{aligned} & \widehat{\pi}[aX_1 + bX_2] \\ = & \sum_{\omega\in\Omega} \beta_T(\omega)(aX_1(\omega) + bX_2(\omega))(1+Z(\omega))Q(\omega) \\ = & a\widehat{\pi}(X_1) + b\widehat{\pi}(X_2). \end{aligned}$$

3. Third,

$$\widehat{\pi}\left(V_T\left(\overrightarrow{\phi}\right)\right) = \pi\left(V_T\left(\overrightarrow{\phi}\right)\right) = V_0\left(\overrightarrow{\phi}\right)$$

since $V_T\left(\overrightarrow{\phi}\right)$ is an attainable contingent claim and π is consistent.

Proof - Part 2 VI

Complete market

- But $\widehat{\pi}$ is different from π as a price system since

$$\begin{aligned} & \widehat{\pi}\left[\mathbb{I}_{Z>0}\right] - \pi\left(\mathbb{I}_{Z>0}\right) \\ = \quad & \mathrm{E}^{Q}\left[\beta_T \mathbb{I}_{Z>0}\left(1+Z\right)\right] - \mathrm{E}^{Q}\left[\beta_T \mathbb{I}_{Z>0}\right] \\ = \quad & \mathrm{E}^{Q}\left[\beta_T \mathbb{I}_{Z>0} Z\right] > 0. \end{aligned}$$

- Since there exists more than one price system, then, from Proposal 2.6, we can build more than one martingale measure. ■

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Linear algebra review I

Appendix

Definition

A *vector space* on the body $\mathbb{R}$ of real numbers is a set V such that, for all $u, v, w \in V$ and for all $a, b \in \mathbb{R}$,

1. $u + v \in V$ (closed under addition)
2. $au \in V$ (closed under scalar multiplication)
3. $u + v = v + u$ (commutative)
4. $u + (v + w) = (u + v) + w$ (associative)
5. V has a remarkable element, denoted $\mathbf{0}$, such that $u + \mathbf{0} = \mathbf{0} + u = u$ (neutral element for the addition)
6. $u + (-1)u = \mathbf{0}$ ($(-1)u$, denoted $-u$, is the opposite of u)
7. $1u = u$ (unitary)
8. $a(u + v) = au + av$ (distributive)
9. $(a + b)u = au + bu$ (distributive)
10. $(ab)u = a(bu)$ (distributive)

Linear algebra review II

Appendix

The elements of a vector space are called *vectors*. A *vector subspace* of V is a subset U of V which is itself a vector space (Leroux, p. 165-166).

Linear algebra review III

Definition

Let V be a vector space on the body $\mathbb{R}$ of the real numbers. A **scalar product** on V is a function, denoted

$$\langle \bullet, \bullet \rangle : V \times V \to \mathbb{R}$$

which is symmetric, bilinear, and positive definite, i.e. such that, for all $u, u^{'}, v, v^{'} \in V$ and $c \in \mathbb{R}$, we have:

1. $\langle u, v \rangle = \langle v, u \rangle$ (symmetric)
2. $\langle cu, v \rangle = c \langle u, v \rangle = \langle u, cv \rangle$ (bilinear)
3. $\left\langle u + u^{'}, v \right\rangle = \langle u, v \rangle + \left\langle u^{'}, v \right\rangle$ (bilinear)
4. $\left\langle u, v + v^{'} \right\rangle = \langle u, v \rangle + \left\langle u, v^{'} \right\rangle$ (bilinear)
5. $\langle u, u \rangle \geq 0$ (positive)
6. $\langle u, u \rangle = 0 \Leftrightarrow u = \mathbf{0}$ (positive)

(Leroux, p. 328).

Linear algebra review IV

Appendix

Definition

We say that a vector space on the body $\mathbb{R}$ of real numbers, equipped with a scalar product, is a **Euclidean space**, (Leroux, p. 329).

Let's now assume that our sample space Ω only contains n elements.
Let X be a random variable built on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. X may be viewed as a point in $\mathbb{R}^n$ since

$$(X(\omega_1), ..., X(\omega_n)) \in \mathbb{R}^n$$

Let $\mathcal{V}$ be the set of random variables built on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Linear algebra review V

Appendix

Theorem

The set $\mathcal{V}$ of random variables built on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a vector space.

Proof of the theorem. $\mathcal{V}$ is closed under the addition since the sum of two random variables is a random variable. $\mathcal{V}$ also closed under scalar multiplication since, if U is a random variable and a is a real number, then aU is a random variable. It is also easy to verify that the other eight conditions in the definition of a vector space are met. For example, $\mathbf{0}$ is the random variable that always takes the value 0, whatever the state of the world ω_i may be. ■

Linear algebra review VI

Appendix

Theorem

The expectation is a scalar product on $\mathcal{V}$.

Proof of the theorem. Let $U, U', V, V' \in \mathcal{V}$ be random variables and $c \in \mathbb{R}$.

$$\mathrm{E}^P[UV] = \mathrm{E}^P[VU] \qquad \text{(symmetric)}$$

$$\mathrm{E}^P[cUV] = c\mathrm{E}^P[UV] = \mathrm{E}^P[UcV] \qquad \text{(bilinear)}$$

$$\begin{aligned} \mathrm{E}^P\left[\left(U+U'\right)V\right] &= \mathrm{E}^P[UV] + \mathrm{E}^P\left[U'V\right] \\ \mathrm{E}^P\left[U\left(V+V'\right)\right] &= \mathrm{E}^P[UV] + \mathrm{E}^P\left[UV'\right] \end{aligned} \qquad \text{(bilinear)}$$

$$\begin{aligned} \mathrm{E}^P[UU] = \mathrm{E}^P\left[U^2\right] &\geq 0 \\ \mathrm{E}^P[UU] = \mathrm{E}^P\left[U^2\right] = 0 &\Leftrightarrow U = \mathbf{0} \end{aligned} \qquad \text{(positive)}$$

Linear algebra review VII

Appendix

Corollary

Corollary. *The set V of random variables built on the filtered probability space$(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the expectation $\mathrm{E}^P[\bullet]$ is a Euclidean space when Ω only has a finite number of elements.*

Linear algebra review VIII

Appendix

Definition

Let V be a Euclidean space and let u and v be two elements of V. The distance $d(u, v)$ between vectors u and v is defined as follows:

$$d(u, v) = \sqrt{\langle u - v, u - v \rangle}$$

(Leroux, p. 329).

In the example under consideration, the distance between two random variables is

$$d(U, V) = \sqrt{\langle U - V, U - V \rangle} = \sqrt{\mathrm{E}^P\left[(U - V)^2\right]}.$$

Linear algebra review - Complete market I

Appendix

Theorem

Proposal. *Let V be a Euclidean space, let v be a vector of V and U be a vector subspace of V. There exists an element u^* of U such that $v - u^*$ is orthogonal to U i.e.*

$$\forall u \in U, \ \langle v - u^*, u \rangle = 0.$$

Linear algebra review - Complete market II

Appendix

Proof of the proposal. The vector u^* in question is the one that minimizes the distance between vector v and the subspace U, i.e.

$$d\left(u^*, v\right) = \min_{u \in U} d(u, v).$$

Note that it remains to prove that such a u^* exists. Such a proof is provided by Theorem 9.5, p.334 in Leroux.
Let u be any vector of U. If $u = \mathbf{0}$ then we obviously have

$$\langle v - u^*, u \rangle = \langle v - u^*, \mathbf{0} \rangle = 0.$$

Therefore we will assume that $u \neq \mathbf{0}$. For any real number c, $cu + u^* \in U$ so that

$$d\left(u^*, v\right) = \min_{u \in U} d(u, v) \leq d(cu + u^*, v).$$

Linear algebra review - Complete market III

Appendix

By raising to the square, we get

$$
\begin{aligned}
& \langle u^* - v, u^* - v \rangle \\
= \; & \left(d\left(u^*, v\right)\right)^2 \\
\leq \; & \left(d\left(cu + u^*, v\right)\right)^2 \\
= \; & \langle cu + u^* - v, cu + u^* - v \rangle \\
= \; & \langle cu, cu + u^* - v \rangle + \langle u^* - v, cu + u^* - v \rangle \\
= \; & \langle cu, cu \rangle + \langle cu, u^* - v \rangle + \langle u^* - v, cu \rangle + \langle u^* - v, u^* - v \rangle \\
= \; & c^2 \langle u, u \rangle + 2c \langle u^* - v, u \rangle + \langle u^* - v, u^* - v \rangle
\end{aligned}
$$

so

$$0 \leq c^2 \langle u, u \rangle + 2c \langle u^* - v, u \rangle .$$

Linear algebra review - Complete market IV

Appendix

Since $u \neq \mathbf{0}$, $\langle u, u \rangle > 0$. So, let's set

$$c = -\frac{\langle u^* - v, u \rangle}{\langle u, u \rangle}.$$

By substituting into the inequality, we get

$$\begin{aligned} 0 &\leq \left(\frac{-\langle u^* - v, u \rangle}{\langle u, u \rangle} \right)^2 \langle u, u \rangle + 2 \frac{-\langle u^* - v, u \rangle}{\langle u, u \rangle} \langle u^* - v, u \rangle \\ &= -\frac{\langle u^* - v, u \rangle^2}{\langle u, u \rangle} \leq 0. \end{aligned}$$

From there, it follows that

$$-\frac{\langle u^* - v, u \rangle^2}{\langle u, u \rangle} = 0,$$

Linear algebra review - Complete market V

Appendix

hence $\langle u^* - v, u \rangle = 0$. But, since u is arbitrarily chosen in the subspace U, we have that $\langle u^* - v, u \rangle = \langle v - u^*, u \rangle = 0$ for all $u \in U$. ■

Linear algebra review - Complete market VI

Appendix

Now let's use that result for our case. Let $\mathcal{U}$ be a vector subspace of $\mathcal{V}$ and V be a random variable of $\mathcal{V}$. There exists a random variable U^* in $\mathcal{U}$ such that

$$\mathrm{E}^P\left[\left(V - U^*\right) U\right] = 0 \text{ for all } U \in \mathcal{U}.$$

In particular, if $V \notin \mathcal{U}$, then the random variable $V - U^* \neq \mathbf{0}$.