Hedging
Exercises

Exercise 13.1. The price evolution of two assets are modelled with Geometrical Brownian motions :

$$\begin{aligned} dS_1(t) &= \mu_1 S_1(t)\ dt + \sigma_1 S_1(t)\ dW_1(t), \\ dS_2(t) &= \mu_2 S_2(t)\ dt + \sigma_2 \rho S_2(t)\ dW_1(t) + \sigma_2 \sqrt{1-\rho^2} S_2(t)\ dW_2(t) \end{aligned}$$

where W_1 and W_2 are two independent $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ −Brownian motions. We assume that the risk free rate r is constant. $\{\mathcal{F}_t : t \geq 0\}$ is the filtration generated by the two Brownian motion (with the usual regularity conditions.

The goal is to hedge the contingent claim C.

a) What is the risk neutral market model.

b) Explain why C is attainable.

c) Assume that the time t value of the contingent claim C is a function of the underlying asset prices and time, that is, $f(t, S_1(t), S_2(t))$. Express the discounted contingent claim value as a diffusion process. More precisely,we search for the coefficient a, b and c such that

$$\begin{aligned} \beta_t f(t, S_1(t), S_2(t)) &= \beta_0 f(0, S_1(0), S_2(0)) \\ &\quad + \int_0^t a(s)\ ds + \int_0^t b(s)\ d\widetilde{W}_1(s) + \int_0^t c(s)\ d\widetilde{W}_2(s) \end{aligned}$$

where $\widetilde{W}_1$, $\widetilde{W}_2$ are two independent $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{Q})$ −Brownian motions. If you have to add some hypothesis, mention it.

d) Use the Martingale Representation Theorem to justify the existence of predictable processes ϕ and ψ such that

$$\begin{aligned} \beta_t f(t, S_1(t), S_2(t)) &= f(0, S_1(0), S_2(0)) \\ &\quad + \int_0^t \phi_u d\beta_u S_1(u) + \int_0^t \psi_u d\beta_u S_2(u). \end{aligned}$$

Why are you allowed to use this theorem?

e) Use the results obtained in c) and d) to get ϕ_t and ψ_t in therms of partial derivatives of f and the market model components. More precisely, show that

$$\phi_t = \frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t)) \text{ and } \psi_t = \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t))$$

f) Construct the hedging strategy for C, that is, the self-financing investment strategy with a time T value of C.

g) If the contingent claim $C = S_1(T) S_2(T)$, then what is the hedging strategy?

Exercise 13.2. Let

$$\begin{aligned} X_t &= \text{price of the underlying asset at time } t \\ \text{and } B_t &= \text{bank account value at time } t, \\ B_0 &= 1. \end{aligned}$$

The model is :

$$\begin{aligned} dX_t &= \kappa(\theta - X_t)\ dt + \sigma\ dW_t \\ dB_t &= r_t B_t\ dt \end{aligned}$$

where the instantaneous risk free rate $\{r_t : t \geq 0\}$ is a time dependent deterministic variable.

a) What is the risk-neutral market model? Justify each step.

b) Find the replicating strategy with a time t value of $f(X_t, B_t, t)$.

Exercise 13.3. The risky asset price dynamics satisfy

$$dS(t) = r(t) S(t)\ dt + \sigma(t) S(t)\ dW(t)$$

where $\{W(t) : t \geq 0\}$ is a $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{Q})-$Brownian motion, $\mathbb{Q}$ is a martingale measure and $\{r(t) : t \geq 0\}$ and $\{\sigma(t) : t \geq 0\}$ are $\{\mathcal{F}_t : t \geq 0\}-$predictable processes (careful ! S is not a Geometrical Brownian motion). The time t value $B(t)$ of the bank account satisfy

$$\begin{aligned} B(t) &= \exp\left(\int_0^t r(s)\ ds\right), \\ B(0) &= 1. \end{aligned}$$

a) Find an expression for the time t value $P(t,T)$ of a zero-coupon bond paying one dollar at time T in terms of the risk free rate $\{r(t) : t \geq 0\}$. (Easy question that do not require computation).

Forward contract

A *forward* contract set at time t and with a maturity of T determines the price at which the underlying contract will be traded at maturity. The convention says that the delivery price is the one that make nil the initial value of the forward contract. The delivery price or $T - forward$ price, determine at time t, is denoted $F(t)$. It is a $\mathcal{F}_t-$measurable random variable, and correspond to the price at which the underlying will be trade at time T by the two parties of the contract.

b) Determine $F(t)$.

c) Determine the value $V(t)$ at time t of the forward contract established at time 0.

d) Assuming that the interest rate r is a deterministic function of time, determine the hedging strategy of the forward contract set at time 0.

Change of numeraire

Any asset with positive price may be used as numeraire. The bank account can serve as numeraire. In this case, the risky asset worth $\frac{S(t)}{B(t)}$ shares of the bank account.

The probability measure $\mathbb{Q}_N$ is said to be *risk neutral for the numeraire* N if the price of any asset, divided by the numeraire value, is a $\mathbb{Q}_N-$martingale. The risk neutral measure $\mathbb{Q}$ is a risk neutral measure for the bank account B.

Let

$$\widehat{\mathbb{Q}}(A) = \frac{1}{N(0)} \mathrm{E}^{\mathbb{Q}} \left[\frac{N(T)}{B(T)} \mathbb{I}_A \right] \text{ for each event } A \in \mathcal{F}_T$$

where $\mathbb{I}_A$ is the indicator function that worth 1 if $\omega \in A$ and zero otherwise.

e) Show that $\widehat{\mathbb{Q}}$ is a probability measure.

f) Knowing that for all $\mathcal{F}_T-$measurable random variable X ,

$$\mathrm{E}^{\widehat{\mathbb{Q}}} \left[X \middle| \mathcal{F}_t \right] = \frac{N(0) B(t)}{N(t)} \mathrm{E}^{\mathbb{Q}} \left[X \frac{N(T)}{N(0) B(T)} \middle| \mathcal{F}_t \right],$$

show that $\widehat{\mathbb{Q}}$ is risk neutral for the numeraire N.

g) Consider the case where the numeraire is the zero-coupon bond of maturity T. Determine the value of the risky asset S in terms of this numeraire and justify intuitively why the risk neutral measure $\mathbb{Q}_{P(\bullet,T)}$ for the numeraire $P(\bullet, T)$ is called the *measure* $T-$*forward*.

Solutions

1 Exercise 13.1

a) Under the risk neutral measure

$$\begin{aligned} dS_1(t) &= rS_1(t)\ dt + \sigma_1 S_1(t)\ d\widetilde{W}_1(t), \\ dS_2(t) &= rS_2(t)\ dt + \sigma_2 \rho S_2(t)\ d\widetilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2} S_2(t)\ d\widetilde{W}_2(t). \end{aligned}$$

b) The martingale measure is unique, son the market model is complete. All contingent claims are attainable.

c) If the function f is twicely continuously differentiable, then

$$
\begin{aligned}
& df\left(t, S_1(t), S_2(t)\right) \\
= \; & \frac{\partial f}{\partial t}\left(t, S_1(t), S_2(t)\right)\; dt \\
& +\frac{\partial f}{\partial s_1}\left(t, S_1(t), S_2(t)\right)\; dS_1(t)+\frac{\partial f}{\partial s_2}\left(t, S_1(t), S_2(t)\right)\; dS_2(t) \\
& +\frac{1}{2}\frac{\partial^2 f}{\partial s_1^2}\left(t, S_1(t), S_2(t)\right)\; d\left\langle S_1\right\rangle(t)+\frac{1}{2}\frac{\partial^2 f}{\partial s_2^2}\left(t, S_1(t), S_2(t)\right)\; d\left\langle S_2\right\rangle(t) \\
& +\frac{\partial^2 f}{\partial s_1 \partial s_2}\left(t, S_1(t), S_2(t)\right)\; d\left\langle S_1, S_2\right\rangle(t) \\
= \; & \frac{\partial f}{\partial t}\left(t, S_1(t), S_2(t)\right)\; dt \\
& +\frac{\partial f}{\partial s_1}\left(t, S_1(t), S_2(t)\right) r S_1(t)\; dt+\frac{\partial f}{\partial s_1}\left(t, S_1(t), S_2(t)\right) \sigma_1 S_1(t)\; d\widetilde{W}_1(t) \\
& +\frac{\partial f}{\partial s_2}\left(t, S_1(t), S_2(t)\right)\; r S_2(t)\; dt+\frac{\partial f}{\partial s_2}\left(t, S_1(t), S_2(t)\right)\; \sigma_2 \rho S_2(t)\; d\widetilde{W}_1(t) \\
& +\frac{\partial f}{\partial s_2}\left(t, S_1(t), S_2(t)\right)\; \sigma_2 \sqrt{1-\rho^2} S_2(t)\; d\widetilde{W}_2(t) \\
& +\frac{1}{2}\frac{\partial^2 f}{\partial s_1^2}\left(t, S_1(t), S_2(t)\right)\; \sigma_1^2 S_1^2(t)\, dt+\frac{1}{2}\frac{\partial^2 f}{\partial s_2^2}\left(t, S_1(t), S_2(t)\right)\; \sigma_2^2 S_2^2(t)\, dt \\
& +\frac{\partial^2 f}{\partial s_1 \partial s_2}\left(t, S_1(t), S_2(t)\right)\; \rho \sigma_1 \sigma_2 S_1(t) S_2(t)\, dt \\
= \; & \left[\begin{array}{c}
\frac{\partial f}{\partial t}\left(t, S_1(t), S_2(t)\right) \\
+\frac{\partial f}{\partial s_1}\left(t, S_1(t), S_2(t)\right) r S_1(t)+\frac{\partial f}{\partial s_2}\left(t, S_1(t), S_2(t)\right)\; r S_2(t) \\
+\frac{1}{2}\frac{\partial^2 f}{\partial s_1^2}\left(t, S_1(t), S_2(t)\right)\; \sigma_1^2 S_1^2(t)+\frac{1}{2}\frac{\partial^2 f}{\partial s_2^2}\left(t, S_1(t), S_2(t)\right)\; \sigma_2^2 S_2^2(t) \\
+\frac{\partial^2 f}{\partial s_1 \partial s_2}\left(t, S_1(t), S_2(t)\right)\; \rho \sigma_1 \sigma_2 S_1(t) S_2(t)
\end{array}\right] dt \\
& +\left[\frac{\partial f}{\partial s_1}\left(t, S_1(t), S_2(t)\right) \sigma_1 S_1(t)+\frac{\partial f}{\partial s_2}\left(t, S_1(t), S_2(t)\right)\; \sigma_2 \rho S_2(t)\right]\; d\widetilde{W}_1(t) \\
& +\frac{\partial f}{\partial s_2}\left(t, S_1(t), S_2(t)\right)\; \sigma_2 \sqrt{1-\rho^2} S_2(t)\; d\widetilde{W}_2(t).
\end{aligned}
$$

The multiplication rule leads to

$$
\begin{aligned}
d\beta_t f\left(t, S_1(t), S_2(t)\right) &= \beta_t df\left(t, S_1(t), S_2(t)\right)+f\left(t, S_1(t), S_2(t)\right) d\beta_t \\
&= \beta_t df\left(t, S_1(t), S_2(t)\right)-r f\left(t, S_1(t), S_2(t)\right) \beta_t dt
\end{aligned}
$$

$$
\begin{aligned}
a(t) &= \beta_t \left[\begin{array}{c} \frac{\partial f}{\partial t}(t, S_1(t), S_2(t)) \\ +\frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t)) \, r S_1(t) + \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t)) \;\; r S_2(t) \\ +\frac{1}{2}\frac{\partial^2 f}{\partial s_1^2}(t, S_1(t), S_2(t)) \;\; \sigma_1^2 S_1^2(t) + \frac{1}{2}\frac{\partial^2 f}{\partial s_2^2}(t, S_1(t), S_2(t)) \;\; \sigma_2^2 S_2^2(t) \\ +\frac{\partial^2 f}{\partial s_1 \partial s_2}(t, S_1(t), S_2(t)) \;\; \rho\sigma_1\sigma_2 S_1(t) S_2(t) \\ -r f(t, S_1(t), S_2(t)) \end{array} \right] \\
b(t) &= \beta_t \left[\frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t)) \, \sigma_1 S_1(t) + \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t)) \;\; \sigma_2 \rho S_2(t) \right] \\
c(t) &= \beta_t \left[\frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t)) \;\; \sigma_2 \sqrt{1-\rho^2} S_2(t) \right].
\end{aligned}
$$

d) The discounted value of the contingent claim is a $\mathbb{Q}-$martingale. Moreover, the discounted value of any primary asset are also $\mathbb{Q}-$martingales with positive diffusion coefficients, that is,

$$
\begin{aligned}
\beta_t S_1(t) &= S_1(0) + \int_0^t \underbrace{\sigma_1 \beta_u S_1(u)}_{>0} d\widetilde{W}_1(u) \\
\beta_t S_2(t) &= S_2(0) + \int_0^t \underbrace{\sigma_2 \rho \beta_u S_2(u)}_{>0} d\widetilde{W}_1(u) + \int_0^t \underbrace{\sigma_2 \sqrt{1-\rho^2} \beta_u S_2(u)}_{>0} d\widetilde{W}_2(u).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{Q}} \left[\int_0^T \sigma_1^2 \beta_u^2 S_1^2(u) \, du \right] \\
= \; & \sigma_1^2 \int_0^T \mathrm{E}^{\mathbb{Q}} \left[\beta_u^2 S_1^2(u) \right] du \\
= \; & \sigma_1^2 \int_0^T S_1^2(0) \, \mathrm{E}^{\mathbb{Q}} \left[\left(\exp(-rT) \exp\left(\left(r - \frac{\sigma_1^2}{2} \right) u + \sigma_1 W_u \right) \right)^2 \right] du \\
= \; & \sigma_1^2 \int_0^T S_1^2(0) \, \mathrm{E}^{\mathbb{Q}} \left[\exp\left(-\sigma_1^2 u + 2\sigma_1 W_u \right) \right] du \\
= \; & \sigma_1^2 S_1^2(0) \int_0^T \exp\left(-\sigma_1^2 u + 2\sigma_1^2 u \right) du \\
= \; & \sigma_1^2 S_1^2(0) \int_0^T \exp\left(\sigma_1^2 u \right) du = S_1^2(0) \left(e^{\sigma_1^2 T} - 1 \right) < \infty.
\end{aligned}
$$

Similarly,

$$\begin{aligned}\mathrm{E}^{\mathbb{Q}}\left[\int_0^T \sigma_2^2\rho^2\beta_u^2 S_2^2(u)\,du\right] &< \infty \\ \text{and } \mathrm{E}^{Q}\left[\int_0^T \sigma_2^2\left(1-\rho^2\right)\beta_u^2 S_2^2(u)\,du\right] &< \infty.\end{aligned}$$

Consequently, the Martingale Representation Theorem implies the existence of predictable processes ϕ and ψ such that

$$\begin{aligned}& \beta_t f(t, S_1(t), S_2(t)) \\ = \;& \beta_0 f(0, S_1(0), S_2(0)) \\ & + \int_0^t \phi_u d\beta_u S_1(u) + \int_0^t \psi_u d\beta_u S_2(u) \\ = \;& f(0, S_1(0), S_2(0)) \\ & + \int_0^t \phi_u \sigma_1 \beta_u S_1(u)\, d\widetilde{W}_1(u) \\ & + \int_0^t \psi_u \sigma_2 \rho \beta_u S_2(u)\, d\widetilde{W}_1(u) + \int_0^t \psi_u \sigma_2 \sqrt{1-\rho^2}\beta_u S_2(u)\, d\widetilde{W}_2(u) \\ = \;& f(0, S_1(0), S_2(0)) \\ & + \int_0^t \left[\phi_u \sigma_1 \beta_u S_1(u) + \psi_u \sigma_2 \rho \beta_u S_2(u)\right] d\widetilde{W}_1(u) \\ & + \int_0^t \psi_u \sigma_2 \sqrt{1-\rho^2}\beta_u S_2(u)\, d\widetilde{W}_2(u).\end{aligned}$$

e) Since the equations in c) and in d) are both diffusion processes for the discounted contingent claim value, $\beta_t f(t, S_1(t), S_2(t))$, then their diffusion coefficients must be equal. Therefore, the coefficients of $\widetilde{W}_2$ have to be equal :

$$\beta_t\left[\frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t))\;\sigma_2\sqrt{1-\rho^2}S_2(t)\right] = \psi_t\sigma_2\sqrt{1-\rho^2}\beta_t S_2(t)$$

which is verified if and only if

$$\psi_t = \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t)).$$

The coefficients of $\widetilde{W}_1$ have to be equal:

$$\begin{aligned}& \beta_t\left[\frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t))\,\sigma_1 S_1(t) + \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t))\;\sigma_2\rho S_2(t)\right] \\ = \;& \left[\phi_t\sigma_1\beta_t S_1(t) + \psi_t\sigma_2\rho\beta_t S_2(t)\right].\end{aligned}$$

This equality is satisfied if and only if

$$\begin{aligned}
\phi_t &= \frac{\frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t))\,\sigma_1 S_1(t) + \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t))\ \ \sigma_2 \rho S_2(t) - \psi_t \sigma_2 \rho S_2(t)}{\sigma_1 S_1(t)} \\
&= \frac{\frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t))\,\sigma_1 S_1(t) + \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t))\ \ \sigma_2 \rho S_2(t) - \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t))\,\sigma_2 \rho S_2(t)}{\sigma_1 S_1(t)} \\
&= \frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t))
\end{aligned}$$

f) Let

$$\begin{aligned}
\phi_t &= \text{the number of share of asset 1} \\
\psi_t &= \text{the number of share of asset 2} \\
\text{and } \theta_t &= \text{the number of share of the riskless asset.}
\end{aligned}$$

In our case, the riskless asset is $B(t) = \exp(rt)$. We have shown that

$$\begin{aligned}
\phi_t &= \frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t)) \\
\text{and } \psi_t &= \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t)).
\end{aligned}$$

It suffices to set

$$\begin{aligned}
\theta_t &= \beta_t f(t, S_1(t), S_2(t)) - \phi_t \beta_t S_1(t) - \psi_t \beta_t S_2(t) \\
&= \beta_t \left[f(t, S_1(t), S_2(t)) - S_1(t) \frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t)) - S_2(t) \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t)) \right].
\end{aligned}$$

g) Since $C = S_1(T) S_2(T)$, then its time t value is

$$
\begin{aligned}
& f(t, S_1(t), S_2(t)) \\
= & \frac{1}{\beta_t} \mathrm{E}^Q \left[\beta_T S_1(T) S_2(T) | \mathcal{F}_t \right] \\
= & e^{rt} \mathrm{E}^Q \left[\begin{array}{c} e^{-rT} S_1(t) \exp\left[\left(r - \frac{\sigma_1^2}{2} \right)(T-t) + \sigma_1 (W_1(T) - W_1(t)) \right] \\ S_2(t) \exp\left[\begin{array}{c} \left(r - \frac{\sigma_2^2}{2} \right)(T-t) + \sigma_2 \rho (W_1(T) - W_1(t)) \\ + \sigma_2 \sqrt{1-\rho^2} (W_2(T) - W_2(t)) \end{array} \right] \end{array} | \mathcal{F}_t \right] \\
= & \exp[-r(T-t)] S_1(t) S_2(t) \mathrm{E}^Q \left[\begin{array}{c} \exp\left[\left(2r - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} \right)(T-t) \right] \\ \exp\left[(\sigma_1 + \sigma_2 \rho)(W_1(T) - W_1(t)) \right] \\ \exp\left[\sigma_2 \sqrt{1-\rho^2} (W_2(T) - W_2(t)) \right] \end{array} | \mathcal{F}_t \right] \\
= & \exp\left[\left(r - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} \right)(T-t) \right] S_1(t) S_2(t) \\
& \times \mathrm{E}^Q \left[\exp\left[(\sigma_1 + \sigma_2 \rho)(W_1(T) - W_1(t)) \right] \right] \mathrm{E}^Q \left[\exp\left[\sigma_2 \sqrt{1-\rho^2} (W_2(T) - W_2(t)) \right] \right] \\
= & \exp\left[\left(r - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} \right)(T-t) \right] S_1(t) S_2(t) \\
& \times \exp\left[\frac{(\sigma_1 + \sigma_2 \rho)^2 (T-t)}{2} \right] \exp\left[\frac{\sigma_2^2 (1-\rho^2)}{2} (T-t) \right] \\
= & S_1(t) S_2(t) \exp\left[(r + \sigma_1 \sigma_2 \rho)(T-t) \right].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\phi_t &= \frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t)) = S_2(t) \exp\left[(r + \sigma_1 \sigma_2 \rho)(T-t) \right] \\
\psi_t &= \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t)) = S_1(t) \exp\left[(r + \sigma_1 \sigma_2 \rho)(T-t) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_t &= \beta_t f(t, S_1(t), S_2(t)) - \phi_t \beta_t S_1(t) - \psi_t \beta_t S_2(t) \\
&= \exp(-rt) S_1(t) S_2(t) \exp\left[(r + \sigma_1 \sigma_2 \rho)(T-t) \right] \\
&\quad - S_2(t) \exp\left[(r + \sigma_1 \sigma_2 \rho)(T-t) \right] \exp(-rt) S_1(t) \\
&\quad - S_1(t) \exp\left[(r + \sigma_1 \sigma_2 \rho)(T-t) \right] \exp(-rt) S_2(t) \\
&= -\exp(-rt) S_1(t) S_2(t) \exp\left[(r + \sigma_1 \sigma_2 \rho)(T-t) \right].
\end{aligned}
$$

Verification. At time T, we hold $S_2(T)$ shares of asset 1, $S_1(T)$ shares of asset 2 and we are short of $\exp(-rt)\, S_1(T)\, S_2(T)$ shares of the bank account. The time t value of this investment strategy is

$$\begin{aligned}
V_t &= \phi_t S_1(t) + \psi_t S_2(t) + \theta_t B(t) \\
&= S_2(t) \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] S_1(t) + S_1(t) \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] S_2(t) \\
&\quad - \exp(-rt)\, S_1(t)\, S_2(t) \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] \exp(-rt) \\
&= S_1(t)\, S_2(t) \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right].
\end{aligned}$$

In particular,

$$V_T = S_1(T)\, S_2(T).$$

Other verification. $a(t)$ has to be equal to 0. But

$$\begin{aligned}
&\left[\begin{array}{c}
\frac{\partial f}{\partial t}(t, S_1(t), S_2(t)) \\
+\frac{\partial f}{\partial s_1}(t, S_1(t), S_2(t))\, rS_1(t) + \frac{\partial f}{\partial s_2}(t, S_1(t), S_2(t))\;\; rS_2(t) \\
+\frac{1}{2}\frac{\partial^2 f}{\partial s_1^2}(t, S_1(t), S_2(t))\;\; \sigma_1^2 S_1^2(t) + \frac{1}{2}\frac{\partial^2 f}{\partial s_2^2}(t, S_1(t), S_2(t))\;\; \sigma_2^2 S_2^2(t) \\
+\frac{\partial^2 f}{\partial s_1 \partial s_2}(t, S_1(t), S_2(t))\;\; \rho\sigma_1\sigma_2 S_1(t)\, S_2(t) \\
-rf(t, S_1(t), S_2(t))
\end{array}\right] \\
= &\left[\begin{array}{c}
-(r+\sigma_1\sigma_2\rho)\, s_1 s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] \\
+s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] rs_1 + s_1 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right]\;\; rs_2 \\
0+0 \\
+\exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right]\;\; \rho\sigma_1\sigma_2 s_1 s_2 \\
-rs_1 s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right]
\end{array}\right] \\
= &\left[\begin{array}{c}
-(r+\sigma_1\sigma_2\rho)\, s_1 s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] \\
+rs_1 s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] + rs_1 s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] \\
+\rho\sigma_1\sigma_2 s_1 s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right] \\
-rs_1 s_2 \exp\left[(r+\sigma_1\sigma_2\rho)(T-t)\right]
\end{array}\right] \\
= &\; 0.
\end{aligned}$$

2 Exercise 13.2

a) The multiplication rule implies that

$$\begin{aligned} dB_t^{-1}X_t &= B_t^{-1}dX_t + X_t dB_t^{-1} + d\left\langle X, B^{-1}\right\rangle_t \\ &= B_t^{-1}\left(\kappa\left(\theta - X_t\right)\ dt + \sigma\ dW_t\right) + X_t\left(-r_t B_t^{-1}\ dt\right) \\ &= \left(\kappa\theta B_t^{-1} - \left(r_t+\kappa\right)B_t^{-1}X_t\right)\ dt + \sigma\ dW_t \end{aligned}$$

Let us introduce the γ process : $W_t^* = W_t + \int_0^t \gamma_s ds$ and

$$\begin{aligned} dB_t^{-1}X_t &= \left(\left(\kappa\theta - \sigma\gamma_t\right)B_t^{-1} - \left(r_t+\kappa\right)B_t^{-1}X_t\right)\ dt + \sigma B_t^{-1}\ d\left(W_t + \int_0^t \gamma_s ds\right) \\ &= \left(\left(\kappa\theta - \sigma\gamma_t\right)B_t^{-1} - \left(r_t+\kappa\right)B_t^{-1}X_t\right)\ dt + \sigma\ dW_t^*. \end{aligned}$$

To cancel the drift term,

$$\left(\kappa\theta - \sigma\gamma_t\right)B_t^{-1} - \left(r_t+\kappa\right)B_t^{-1}X_t = 0 \Leftrightarrow \gamma_t = \frac{-\left(r_t+\kappa\right)X_t + \kappa\theta}{\sigma}.$$

Assuming that the Novikov condition[1] $\mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right]$ is satisfied, we can apply Girsanov theorem and state that there exists a measure $\mathbb{Q}$ under which W^* is a Brownian

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[1]

$$\begin{aligned} & \mathrm{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] \\ =\ & \mathrm{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{-\left(r_t+\kappa\right)X_t}{\sigma} + \frac{\kappa\theta}{\sigma}\right)^2 dt\right)\right] \\ =\ & \mathrm{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \left(\left(\frac{\left(r_t+\kappa\right)X_t}{\sigma}\right)^2 - 2\frac{\left(r_t+\kappa\right)X_t}{\sigma}\frac{\kappa\theta}{\sigma} + \left(\frac{\kappa\theta}{\sigma}\right)^2\right) dt\right)\right] \\ =\ & \exp\left(\frac{1}{2}\left(\frac{\kappa\theta}{\sigma}\right)^2 T\right)\mathrm{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \left(\left(\frac{\left(r_t+\kappa\right)X_t}{\sigma}\right)^2 - 2\frac{\left(r_t+\kappa\right)X_t}{\sigma}\frac{\kappa\theta}{\sigma}\right) dt\right)\right] \\ \leq\ & \exp\left(\frac{1}{2}\left(\frac{\kappa\theta}{\sigma}\right)^2 T\right)\sqrt{\mathrm{E}^P\left[\exp\left(\int_0^T \left(\frac{\left(r_t+\kappa\right)X_t}{\sigma}\right)^2 dt\right)\right]}\sqrt{\mathrm{E}^P\left[\exp\left(2\int_0^T \frac{-\left(r_t+\kappa\right)X_t}{\sigma}\frac{\kappa\theta}{\sigma}dt\right)\right]} \\ & \text{(Cauchy-Schwartz inequality)} \\ =\ & \exp\left(\frac{1}{2}\left(\frac{\kappa\theta}{\sigma}\right)^2 T\right)\sqrt{\mathrm{E}^P\left[\exp\left(\int_0^T \left(\frac{\left(r_t+\kappa\right)X_t}{\sigma}\right)^2 dt\right)\right]}\sqrt{\mathrm{E}^P\left[\exp\left(2\int_0^T \frac{-\left(r_t+\kappa\right)X_t}{\sigma}\frac{\kappa\theta}{\sigma}dt\right)\right]} \end{aligned}$$

TO BE COMPLETED

motion. Let find the SDE of X in function of the $\mathbb{Q}-$Brownian motion:

$$\begin{aligned}
dX_t &= \kappa(\theta - X_t)\ dt + \sigma\ dW_t \\
&= \kappa(\theta - X_t)\ dt + \sigma\ d\left(W_t^* - \int_0^t \gamma_s ds\right) \\
&= (\kappa\theta - \sigma\gamma_t - \kappa X_t)\ dt + \sigma\ dW_t^* \\
&= \left(\kappa\theta - \sigma\left(\frac{-(r_t+\kappa)X_t}{\sigma} + \frac{\kappa\theta}{\sigma}\right) - \kappa X_t\right)\ dt + \sigma\ dW_t^* \\
&= r_t X_t\ dt + \sigma\ dW_t^*.
\end{aligned}$$

b) According to Itô's lemma

$$\begin{aligned}
& df(X_t, B_t, t) \\
= & \frac{\partial f}{\partial x}(X_t, B_t, t)\ dX_t + \frac{\partial f}{\partial b}(X_t, B_t, t)\ dB_t + \frac{\partial f}{\partial t}(X_t, B_t, t)\ dt \\
& + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t, B_t, t)\ d\langle X\rangle_t + \frac{1}{2}\frac{\partial^2 f}{\partial b^2}(X_t, B_t, t)\ d\langle B\rangle_t + \frac{\partial^2 f}{\partial x \partial b}(X_t, B_t, t)\ d\langle X, B\rangle_t \\
= & \frac{\partial f}{\partial x}(X_t, B_t, t)\ \ (r_t X_t\ dt + \sigma\ dW_t^*) + \frac{\partial f}{\partial b}(X_t, B_t, t)\ r_t B_t\ dt + \frac{\partial f}{\partial t}(X_t, B_t, t)\ dt \\
& + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t, B_t, t)\ \sigma^2 dt \\
= & \left(r_t X_t \frac{\partial f}{\partial x}(X_t, B_t, t) + r_t B_t \frac{\partial f}{\partial b}(X_t, B_t, t) + \frac{\partial f}{\partial t}(X_t, B_t, t)\ \ + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(X_t, B_t, t)\right)\ dt \\
& + \sigma \frac{\partial f}{\partial x}(X_t, B_t, t)\ dW_t^*
\end{aligned}$$

$$\begin{aligned}
& dB_t^{-1} f(X_t, B_t, t) \\
= & \left(\begin{array}{c} r_t B_t^{-1} X_t \frac{\partial f}{\partial x}(X_t, B_t, t) + r_t \frac{\partial f}{\partial b}(X_t, B_t, t) + B_t^{-1}\frac{\partial f}{\partial t}(X_t, B_t, t) \\ + \frac{\sigma^2}{2} B_t^{-1} \frac{\partial^2 f}{\partial x^2}(X_t, B_t, t) - r_t B_t^{-1} f(X_t, B_t, t) \end{array}\right)\ dt \\
& + \sigma B_t^{-1} \frac{\partial f}{\partial x}(X_t, B_t, t)\ dW_t^*
\end{aligned}$$

According the Martingale Representation Theorem,there is a predictable process ϕ such that

$$\begin{aligned}
dB_t^{-1} f(X_t, B_t, t) &= \phi_t\ dB_t^{-1} X_t \\
&= \phi_t\ \sigma B_t^{-1}\ dW_t^* \\
&= \phi_t \sigma B_t^{-1}\ dW_t^*.
\end{aligned}$$

Therefore

$$\begin{aligned}0 &= r_t B_t^{-1} X_t \frac{\partial f}{\partial x}(X_t, B_t, t) + r_t \frac{\partial f}{\partial b}(X_t, B_t, t) + B_t^{-1} \frac{\partial f}{\partial t}(X_t, B_t, t) \\ &\quad + \frac{\sigma^2}{2} B_t^{-1} \frac{\partial^2 f}{\partial x^2}(X_t, B_t, t) - r_t B_t^{-1} f(X_t, B_t, t),\end{aligned}$$

and

$$\phi_t \sigma B_t^{-1} = \sigma B_t^{-1} \frac{\partial f}{\partial x}(X_t, B_t, t).$$

The last equation implies that the number of shares of the risky asset is

$$\phi_t = \frac{\partial f}{\partial x}(X_t, B_t, t).$$

The number of shares of the riskless asset is,

$$\alpha_t = B_t^{-1} f(X_t, B_t, t) - \frac{\partial f}{\partial x}(X_t, B_t, t) B_t^{-1} X_t.$$

3 Exercise 13.3

3.1 Solution a)

$$\begin{aligned}P(t, T) &= \mathrm{E}^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(T)}\right| \mathcal{F}_t\right] \\ &= \mathrm{E}^{\mathbb{Q}}\left[\left.\exp\left(-\int_t^T r(s)\ ds\right)\right| \mathcal{F}_t\right].\end{aligned}$$

3.2 Solution b)

The delivery price $F(t)$ satisfies

$$\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(T)}(S(T) - F(t))\right| \mathcal{F}_t\right] = 0.$$

But,

$$\begin{aligned}0 &= \mathrm{E}^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(T)}(S(T) - F(t))\right| \mathcal{F}_t\right] \\ &= B(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{S(T)}{B(T)}\right| \mathcal{F}_t\right] - B(t) F(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{B(T)}\right| \mathcal{F}_t\right] \\ &= B(t) \frac{S(t)}{B(t)} - B(t) F(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{B(T)}\right| \mathcal{F}_t\right]\end{aligned}$$

which implies that

$$F(t) = \frac{S(t)}{\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(T)}\right|\mathcal{F}_t\right]} = \frac{S(t)}{P(t,T)}.$$

3.3 Solution c)

$$\begin{aligned}
V(t) &= B(t)\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{S(T)-F(0)}{B(T)}\right|\mathcal{F}_t\right] \\
&= B(t)\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{S(T)-\frac{S(0)}{P(0,T)}}{B(T)}\right|\mathcal{F}_t\right] \\
&= B(t)\left(\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{S(T)}{B(T)}\right|\mathcal{F}_t\right] - \frac{S(0)}{P(0,T)}\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{B(T)}\right|\mathcal{F}_t\right]\right) \\
&= B(t)\left(\frac{S(t)}{B(t)} - \frac{S(0)}{P(0,T)}\frac{P(t,T)}{B(t)}\right) \text{ since } \frac{S}{B} \text{ is a } \mathbb{Q}-\text{martingale} \\
&= S(t) - \frac{S(0)}{P(0,T)}P(t,T).
\end{aligned}$$

Therefore, $V(0) = S(0) - \frac{S(0)}{P(0,T)}P(0,T) = 0$ (the initial contract value is nil) and $V(T) = S(T) - \frac{S(0)}{P(0,T)} = S(T) - F(0)$ (At maturity, the contract value corresponds to its cash flow).

3.4 Solution d)

Assuming deterministic interest rate, the time t value of the forward contract is

$$\begin{aligned}
V(t) &= S(t) - \frac{S(0)}{P(0,T)}P(t,T) \\
&= S(t) - S(0)\exp\left(\int_0^t r(s)\ ds\right) \\
&= S(t) - S(0)B(t).
\end{aligned}$$

It is a twicely continuously differentiable function of S and the bank account B. To replicates this contract, it suffices to hold one share of the risky asset and be short of $S(0)$ shares of the bank account.

3.5 Solution e)

First, note that $\widehat{\mathbb{Q}}(A) \geq 0$ since $N(T) > 0$, $B(T) > 0$, $N(0) > 0$ and $\mathbb{I}_A \geq 0$. Second,

$$\widehat{\mathbb{Q}}(\Omega) = \frac{1}{N(0)} \mathrm{E}^{\mathbb{Q}} \left[\frac{N(T)}{B(T)} \right] = \frac{1}{N(0)} \frac{N(0)}{B(0)} = 1$$

where the second equality is justify by the fact that $\mathbb{Q}$ is risk neutral for the bank account, which implies that N/B is a $\mathbb{Q}-$martingale. Finally, if $A_1, A_2, ...$ are disjoint events

$$\widehat{\mathbb{Q}}\left(\bigcup_{i=1}^{\infty} A_i\right) = \frac{1}{N(0)} \mathrm{E}^{\mathbb{Q}} \left[\frac{N(T)}{B(T)} \mathbb{I}_{\bigcup_{i=1}^{\infty} A_i} \right] = \sum_{i=1}^{\infty} \frac{1}{N(0)} \mathrm{E}^{\mathbb{Q}} \left[\frac{N(T)}{B(T)} \mathbb{I}_{A_i} \right] = \sum_{i=1}^{\infty} \widehat{\mathbb{Q}}(A_i).$$

3.6 Solution f)

Let $\{Y(t) : t \geq 0\}$ be the value of an asset of the model. We have that Y/B is a $\mathbb{Q}-$martingale. We want to show that Y/N is a $\widehat{\mathbb{Q}}-$martingale. First, since Y and N are $\{\mathcal{F}_t : t \geq 0\}$ $-$adapted, Y/N is. Second, we need to check the integrability condition : $\mathrm{E}^{\widehat{\mathbb{Q}}} \left[\left| \frac{Y(t)}{N(t)} \right| \right] < \infty$... Finally, for all $0 \leq s \leq t$, we have

$$\begin{aligned}
& \mathrm{E}^{\widehat{\mathbb{Q}}} \left[\left. \frac{Y(t)}{N(t)} \right| \mathcal{F}_s \right] \\
= & \frac{N(0) B(s)}{N(s)} \mathrm{E}^{\mathbb{Q}} \left[\left. \frac{Y(t)}{N(t)} \frac{N(T)}{N(0) B(T)} \right| \mathcal{F}_s \right] \\
= & \frac{B(s)}{N(s)} \mathrm{E}^{\mathbb{Q}} \left[\left. \frac{Y(t)}{N(t)} \mathrm{E}^{\mathbb{Q}} \left[\left. \frac{N(T)}{B(T)} \right| \mathcal{F}_t \right] \right| \mathcal{F}_s \right] \text{ (law of iterated conditional expectation)} \\
= & \frac{B(s)}{N(s)} \mathrm{E}^{\mathbb{Q}} \left[\left. \frac{Y(t)}{N(t)} \frac{N(t)}{B(t)} \right| \mathcal{F}_s \right] \text{ since } \frac{N}{B} \text{ is a } \mathbb{Q} - \text{martingale} \\
= & \frac{B(s)}{N(s)} \mathrm{E}^{\mathbb{Q}} \left[\left. \frac{Y(t)}{B(t)} \right| \mathcal{F}_s \right] \\
= & \frac{B(s)}{N(s)} \frac{Y(s)}{B(s)} \text{ since } \frac{Y}{B} \text{ is a } \mathbb{Q} - \text{martingale} \\
= & \frac{Y(s)}{N(s)}. \ \square
\end{aligned}$$

3.7 Solution g)

In this case, the risky asset worth$\frac{S(t)}{P(t,T)}$ shares of the zero-coupon bond for all $0 \leq t \leq T$, that is, the risky asset worth $F(t)$shares of the zero-coupon bond where $F(t)$ is the delivery price.

$$\mathrm{E}^{\mathbb{Q}_{P(\bullet,T)}}\left[\left.\frac{S(t)}{P(t,T)}\right|\mathcal{F}_s\right]=\frac{S(s)}{P(s,T)}=F(s).$$

4 Exercise 13.4

The model consist of a bank account $B(t)$, a stochastic interest rate r, a risky asset $S(t)$ with a stochastic volatility $\sigma(t)$:

$$\begin{aligned} dS(t) &= r(t)\,S(t)\,dt + \sigma(t)\,S(t)\,dW^S(t), \\ dB(t) &= r(t)\,B(t)\,dt, \\ dr(t) &= \mu_r(t)\,dt + \sigma_r(t)\,dW^r(t), \\ d\sigma(t) &= \mu_\sigma(t)\,dt + \sigma_\sigma(t)\,dW^\sigma(t). \end{aligned}$$

The $\mathbb{Q}-$Brownian motions W^S, W^r and W^σ are independent. We assume that the drift and diffusion coefficients are such that the system of SDE admits a unique solution. Intuitively, this market model is incomplete because there is less tradable assets than there are sources of randomness. To complete the market, we add a contingent claim of maturity T^* whose time t value is $C_t = f(t, S_t, B_t, \sigma_t)$. Itô's lemma imply that

$$\begin{aligned} dC &= df \\ &= f_t dt + f_S dS + f_B dB + f_\sigma d\sigma \\ &\quad + \frac{1}{2} f_{SS} d\langle S\rangle + \frac{1}{2} f_{BB} d\langle B\rangle + \frac{1}{2} f_{\sigma\sigma} d\langle \sigma\rangle + f_{SB} d\langle S, B\rangle + f_{S\sigma} d\langle S, \sigma\rangle + f_{B\sigma} d\langle B, \sigma\rangle \\ &= \left(f_t + f_S rS + f_B rB + f_\sigma \mu_\sigma + \frac{1}{2} f_{SS}\sigma^2 S^2 + \frac{1}{2} f_{\sigma\sigma}\sigma_\sigma^2\right) dt + f_S \sigma S dW^S + f_\sigma \sigma_\sigma(t)\, dW^\sigma. \end{aligned}$$

the discounted value of C satisfies

$$\begin{aligned} & dB^{-1}C \\ =\ & B^{-1}dC + C dB^{-1} \\ =\ & B^{-1}\left(f_t + f_S rS + f_B rB + f_\sigma \mu_\sigma + \frac{1}{2} f_{SS}\sigma^2 S^2 + \frac{1}{2} f_{\sigma\sigma}\sigma_\sigma^2 - rf\right) dt \\ & + f_S \sigma B^{-1} S dW^S + f_\sigma B^{-1} \sigma_\sigma(t)\, dW^\sigma. \end{aligned}$$

For $B^{-1}C$ a $\mathbb{Q}-$martingale, the drift term as to be nil, that is,

$$f_t + f_S rS + f_B rB + f_\sigma \mu_\sigma + \frac{1}{2} f_{SS}\sigma^2 S^2 + \frac{1}{2} f_{\sigma\sigma}\sigma_\sigma^2 - rf = 0.$$

Therefore,

$$dC = rf dt + f_S \sigma S dW^S + f_\sigma \sigma_\sigma dW^\sigma.$$

We consider a second contingent claim $T < T^*$ whose time t value is $g(t, S_t, B_t, C_t)$. Itô's lemma imply that

$$\begin{aligned} dg &= g_t dt + g_S dS + g_B dB + g_C dC \\ &\quad + \frac{1}{2} g_{SS} d\langle S\rangle + \frac{1}{2} g_{BB} d\langle B\rangle + \frac{1}{2} g_{CC} d\langle C\rangle + f_{SB} d\langle S,B\rangle + f_{SC} d\langle S,C\rangle + f_{BC} d\langle B,C\rangle \\ &= \begin{pmatrix} g_t + g_S rS + g_B rB + g_C rf \\ +\frac{1}{2} g_{SS}\sigma^2 S^2 + \frac{1}{2} g_{CC}\left(f_S^2\sigma^2 S^{2S} + f_\sigma^2\sigma_\sigma^2\right) + f_{SC} f_S \sigma^2 S^2 \end{pmatrix} dt \\ &\quad + (g_S + g_C f_S)\,\sigma S dW^S + g_C f_\sigma \sigma_\sigma dW^\sigma. \end{aligned}$$

The discounted value satisfy

$$\begin{aligned} & dB^{-1} g \\ = & \begin{pmatrix} g_t + g_S rS + g_B rB + g_C rf \\ +\frac{1}{2} g_{SS}\sigma^2 S^2 + \frac{1}{2} g_{CC}\left(f_S^2\sigma^2 S^{2S} + f_\sigma^2\sigma_\sigma^2\right) + f_{SC} f_S \sigma^2 S^2 - rg \end{pmatrix} dt \\ & + (g_S + g_C f_S)\,\sigma B^{-1} S dW^S + g_C B^{-1} f_\sigma \sigma_\sigma dW^\sigma \end{aligned}$$

On the other hand, the Martingale Representation Theorem implies the existence of a predictable processes ϕ and ψ such that

$$\begin{aligned} dB^{-1} g &= \phi dB^{-1} S + \psi dB^{-1} f \\ &= (\phi + \psi f_S)\,\sigma B^{-1} S dW^S + \psi f_\sigma B^{-1} \sigma_\sigma dW^\sigma. \end{aligned}$$

We conclude that

$$\begin{aligned} \psi f_\sigma B^{-1}\sigma_\sigma &= g_C B^{-1} f_\sigma \sigma_\sigma \\ (\phi + \psi f_S)\,\sigma B^{-1} S &= (g_S + g_C f_S)\,\sigma B^{-1} S \\ 0 &= g_t + g_S rS + g_B rB + g_C rf \\ &\quad + \frac{1}{2} g_{SS}\sigma^2 S^2 + \frac{1}{2} g_{CC}\left(f_S^2\sigma^2 S^{2S} + f_\sigma^2\sigma_\sigma^2\right) + f_{SC} f_S \sigma^2 S^2 - rg \end{aligned}$$

which is equivalent

$$\begin{aligned} \psi &= g_C, \\ \phi &= g_S + g_C f_S - \psi f_S = g_S \end{aligned}$$

and

$$0 = g_t + g_S rS + g_B rB + g_C rf + \frac{1}{2} g_{SS}\sigma^2 S^2 + \frac{1}{2} g_{CC}\left(f_S^2\sigma^2 S^{2S} + f_\sigma^2\sigma_\sigma^2\right) + f_{SC} f_S \sigma^2 S^2 - rg.$$

We note that once the market have been completed, the delta-hedging is applicable.