# Risk Neutral Modelling Exercises

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**Exercise 12.1**. Assume that the price evolution of a given asset satisfies

$$dX_t = \mu_t X_t \ dt + \sigma X_t \ dW_t$$

where $\mu_t = \mu (1 + \sin (t))$ and $W = \{W_t : t \geq 0\}$ is a $(\Omega, \mathcal{F}, \mathbb{P})$ −Brownian motion.

**a)** Show that the SDE admits a unique strong solution.

**b)** The riskless asset as a time $t$ instantaneous return at time of $r_t = r (1 + \sin (t))$. Find the SDE of the risky asset price in the risk neutral framework. What allows you to pretend that the risk neutral measure exists?

**c)** What is this risk neutral measure? Express it in function of $\mathbb{P}$.

**Exercise 12.2**. The instantaneous interest rate $\{r_t : t \geq 0\}$ of the riskless asset satisfies the ordinary differential equation

$$d \ r_t = c (\rho - r_t) \ dt,$$

which implies that

$$r_t = \rho + (r_0 - \rho) e^{-ct}.$$

The riskless asset price evolution $\{S_t^* : t \geq 0\}$ satisfies

$$S_t^* = \exp \left[ \int_0^t r_u \ du \right].$$

**a)** Show that the price evolution satisfies

$$dS_t^* = r_t S_t^* \ dt.$$

**b)** The risky asset price evolution $\{S_t : t \geq 0\}$ is characterized by

$$dS_t = \mu S_t \; dt + \sigma S_t \; dW_t$$

where $\{W_t : t \geq 0\}$ is a $(\Omega, \mathcal{F}, \mathbb{P})$ −Brownian motion.
Show how to change the probability measure such that we get the risky asset price dynamics under the risk neutral measure.

**Exercise 12.3.** The prices of the assets are characterize with :

$$\begin{aligned} dS_1(t) &= \mu_1 S_1(t) \; dt + \sigma_1 S_1(t) \; dW_1(t), \\ dS_2(t) &= \mu_2 S_2(t) \; dt + \sigma_2 \rho S_2(t) \; dW_1(t) + \sigma_2 \sqrt{1-\rho^2} S_2(t) \; dW_2(t) \end{aligned}$$

where $W_1$ and $W_2$ are independent $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ −Brownian motions. The risk free interest rate $r$ is assumed to be constant. $\{\mathcal{F}_t : t \geq 0\}$ is the filtration generated by the Brownian motions, with the usual regularity conditions.
Find the risk neutral market model.

**Exercise 12.4.** The value in American dollar of the yen satisfies

$$dU(t) = \mu_U U(t) \; dt + \sigma_U U(t) \; dW_U(t).$$

The value in Canadian dollars of the yen evolves like

$$dC(t) = \mu_C C(t) \; dt + \sigma_C C(t) \; dW_C(t).$$

The value in Canadian dollars of an American dollar is

$$dV(t) = \mu_V V(t) \; dt + \sigma_V V(t) \; dW_V(t).$$

$W_U$, $W_C$ and $W_V$ are $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, P)$ −Brownian motions such that for all $t > 0$

$$\begin{aligned} \text{Corr}^{\mathbb{P}}[W_U(t), W_C(t)] &= \rho_{UC}, \\ \text{Corr}^{\mathbb{P}}[W_U(t), W_V(t)] &= \rho_{UV} \\ \text{and Corr}^{\mathbb{P}}[W_V(t), W_C(t)] &= \rho_{VC}. \end{aligned}$$

Moreover,

$$\begin{aligned} dA(t) &= r_U A(t) \; dt; \; A(0) = 1 \\ dB(t) &= r_C B(t) \; dt, \; B(0) = 1 \\ dD(t) &= r_J D(t) \; dt, \; D(0) = 1 \end{aligned}$$

represent the evolutions of the riskless asset in the United-States, Canada and Japan, respectively. More precisely, $A(t)$ is in American dollars, $B(t)$ is expressed in Canadian dollars and $D(t)$ is in yens.

**Questions**.
a) What are the conditions so that there is no arbitrage opportunity?
b) Is the market model complete?
c) What are the evolutions of the exchange rates in the risk neutral framework? Interpret your results

**Exercise 12.5**. Assume that $W$, $\widehat{W}$ and $\widetilde{W}$ are independent $(\Omega, \mathcal{F}, \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})$ $-$Brownian motions. Let

$$\widetilde{B}_t \equiv \theta W_t + \sqrt{1-\theta^2}\widetilde{W}_t, \ 0 \leq t \leq T$$

and

$$\widehat{B}_t \equiv \kappa W_t + \sqrt{1-\kappa^2}\widehat{W}_t, \ 0 \leq t \leq T.$$

a) Show that $\left\{\widehat{B}_t : 0 \leq t \leq T\right\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ $-$Brownian motion.
b) Compute $\mathrm{Corr}^{\mathbb{P}}\left[\widehat{B}_t, \widetilde{B}_t\right]$ for all $0 < t \leq T$.

The price evolution of two risky assets is

$$\begin{aligned} dX_t &= \mu_X X_t \ dt + \sigma_X X_t \ d\widetilde{B}_t \\ &= \mu_X X_t \ dt + \sigma_X \theta X_t \ dW_t + \sigma_X \sqrt{1-\theta^2} X_t \ d\widetilde{W}_t \end{aligned}$$

and

$$\begin{aligned} dY_t &= \mu_Y Y_t \ dt + \sigma_Y Y_t \ d\widehat{B}_t \\ &= \mu_Y Y_t \ dt + \sigma_Y \kappa Y_t \ dW_t + \sigma_Y \sqrt{1-\kappa^2} Y_t \ d\widehat{W}_t. \end{aligned}$$

We can interpret $W$ as the common chock, whereas $\widetilde{W}$ and $\widehat{W}$ are the idiosyncratic chocks of $X$ and $Y$ respectively.

c) Determine the model in the risk neutral environment , assuming that the instantaneous riskless interest rate is $r$.

d) Is this market model complete? Justify.

e) Can you price the contract that promises the difference $C = X_T - Y_T$ between to asset? Justify all steps. Interpret your results.

# Solutions

## 1 Exercise 12.1

a) We need to verify that.

$$
\begin{array}{ll}
(i) & |b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq K \ |x-y|, \ \forall t \geq 0 \\
(ii) & |b(x,t)| + |\sigma(x,t)| \leq K \ (1+|x|), \ \forall t \geq 0 \\
(iii) & \mathrm{E}\left[X_0^2\right] < \infty.
\end{array}
$$

Let $K = 2|\mu| + |\sigma|$.

$$
\begin{aligned}
& |b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \\
= \ & |\mu(1+\sin(t))x - \mu(1+\sin(t))y| + |\sigma x - \sigma y| \\
= \ & |\mu|(1+\sin(t))|x-y| + |\sigma||x-y| \\
\leq \ & 2|\mu||x-y| + |\sigma||x-y| = (2|\mu| + |\sigma|)|x-y| \leq K|x-y|
\end{aligned}
$$

$$
\begin{aligned}
& |b(x,t)| + |\sigma(x,t)| \\
= \ & |\mu(1+\sin(t))x| + |\sigma x| \\
\leq \ & 2|\mu||x| + \sigma|x| \leq K|x|.
\end{aligned}
$$

If $\mathrm{E}\left[X_0^2\right] < \infty$, then the three conditions are satisfied.

b)

$$
\begin{aligned}
dX_t &= \mu_t X_t \ dt + \sigma X_t \ dW_t \\
&= r_t X_t \ dt + (\mu_t - r_t) X_t \ dt + \sigma X_t \ dW_t \\
&= r_t X_t \ dt + \sigma X_t \ d\left(W_t + \int_0^t \frac{\mu_s - r_s}{\sigma} \ ds\right).
\end{aligned}
$$

Let

$$
\gamma_s = \frac{\mu_s - r_s}{\sigma}, \qquad s \geq 0
$$

and note that the function $s \to \gamma_s$ is continuous, which implies that the process $\{\gamma_s : s \geq 0\}$

is predictable. Since

$$\begin{aligned}
\int_0^T \gamma_s^2 \, ds &= \int_0^T \left(\frac{\mu_s - r_s}{\sigma}\right)^2 \, ds \\
&= \int_0^T \left(\frac{\mu\,(1+\sin(s)) - r\,(1+\sin(s))}{\sigma}\right)^2 \, ds \\
&= \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (4 - 4\cos T - \cos T \sin T + 3T) < \infty,
\end{aligned}$$

then $\mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty$. We can apply the Cameron-Martin-Girsanov theorem :there is a martingale measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $\widetilde{W} = \left\{\widetilde{W}_t : t \in [0, T]\right\}$ define as

$$\widetilde{W}_t = W_t + \int_0^t \gamma_s ds, \qquad t \geq 0.$$

is a $(\Omega, \mathcal{F}, \mathbb{Q})$ −Brownian motion. Therefore,

$$dX_t = r_t X_t \; dt + \sigma X_t \; d\widetilde{W}_t.$$

c) According to the Cameron-Martin-Girsanov theorem, the Radon-Nikodym derivative is

$$\begin{aligned}
\frac{d\mathbb{Q}}{d\mathbb{P}} &= \exp\left[-\int_0^T \gamma_t \; dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right] \\
&= \exp\left[\begin{array}{c} -\int_0^T (1+\sin t)\, \frac{\mu-r}{\sigma} \; dW_t \\ -\frac{1}{2}\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 (4 - 4\cos T - \cos T \sin T + 3T) \end{array}\right] \\
&= \exp\left[\begin{array}{c} -\frac{\mu-r}{\sigma} \; W_T - \frac{\mu-r}{\sigma}\int_0^T \sin t \; dW_t \\ -\left(\frac{\mu-r}{\sigma}\right)^2 \left(1 - \cos T - \frac{1}{4}\cos T \sin T + \frac{3}{4}T\right) \end{array}\right].
\end{aligned}$$

Consequently,

$$\mathbb{Q}(A) = \mathrm{E}^{\mathbb{P}}\left[\exp\left[\begin{array}{c} -\frac{\mu-r}{\sigma} \; W_T - \frac{\mu-r}{\sigma}\int_0^T \sin t \; dW_t \\ -\left(\frac{\mu-r}{\sigma}\right)^2 \left(1 - \cos T - \frac{1}{4}\cos T \sin T + \frac{3}{4}T\right) \end{array}\right] \delta_A\right].$$

## 2 Exercise 12.2

a) Let $Y_t = \int_0^t r_u \, du$. $Y$ is an Itô process ($dY_t = K_t \, dt + H_t \, dW_t$)for which $K_t = r_t$ and $H_t = 0$ since

$$\begin{aligned}
\int_0^T |K_s| \, ds &= \int_0^T |r_s| \, ds = \int_0^T \left|\rho + (r_0 - \rho) e^{-cs}\right| \, ds \\
&\leq \int_0^T \left(\rho + (r_0 + \rho) e^{-cs}\right) \, ds = \rho T + (r_0 + \rho) \frac{1 - e^{-cT}}{c} < \infty.
\end{aligned}$$

Note that

$$dY_t = r_t \, dt \text{ and } \langle Y \rangle_t = \int_0^T H_s^2 \, ds = 0.$$

Let $f(t, y) = e^y$. We have $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = f$, $f(t, Y_t) = S_t^*$ and $f(0, Y_0) = 1 = S_0^*$. From Itô's lemma,

$$\begin{aligned}
dS_t^* &= df(t, Y_t) \\
&= \frac{\partial f}{\partial t}(t, Y_t) \, dt + \frac{\partial f}{\partial y}(t, Y_t) \, dY_t + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(t, Y_t) \, d\langle Y \rangle_t \\
&= f(t, Y_t) \, dY_t \\
&= S_t^* r_t \, dt
\end{aligned}$$

b)

$$\begin{aligned}
dS_t &= \mu S_t \, dt + \sigma S_t \, dW_t \\
&= r_t S_t \, dt + \sigma S_t \frac{\mu - r_t}{\sigma} \, dt + \sigma S_t \, dW_t \\
&= r_t S_t \, dt + \sigma S_t \, d\left(W_t + \int_0^t \frac{\mu - r_u}{\sigma} du\right)
\end{aligned}$$

Let $\gamma_t = \frac{\mu - r_t}{\sigma}$. We need to verify that

$$\mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty$$

is verify to use Girsanov theorem.

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] &= \mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu - r_t}{\sigma}\right)^2 dt\right)\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu - (\rho + (r_0 - \rho)\, e^{-ct})}{\sigma}\right)^2 dt\right)\right] \\
&= \exp\left(\frac{1}{2}\int_0^T \left(\frac{\mu - \rho - (r_0 - \rho)\, e^{-ct}}{\sigma}\right)^2 dt\right) \\
&\leq \exp\left(\int_0^T \left(C_1 + C_2 e^{-ct} + C_3 e^{-2ct}\right) dt\right) \\
&< \infty
\end{aligned}$$

# 3 Exercise 12.4

## 3.1 The main ideas

To show that there is no arbitrage opportunity, we need to find a risk neutral measure under which the discounted price process of tradable assets are martingale.

What are these **tradable assets**? In the perspective of a local investor, these are the three riskless assets, all expressed in Canadian dollars. The value in Canadian dollars of the American riskless asset $A^*(t) = V(t)\, A(t)$ satisfies

$$\begin{aligned}
dA^*(t) &= V(t)\;\, dA(t) + A(t)\;\, dV(t) + d\langle A, V\rangle(t) \\
&= (r_U + \mu_V)\, A^*(t)\;\, dt + \sigma_V A^*(t)\;\, dW_V(t)\,.
\end{aligned}$$

Similarly, the value in Canadian dollars of the Japanese riskless asset $D^*(t) = C(t)\, D(t)$ is characterize with

$$\begin{aligned}
dD^*(t) &= C(t)\;\, dD(t) + D(t)\;\, dC(t) + d\langle D, C\rangle(t) \\
&= (r_J + \mu_C)\, D^*(t)\;\, dt + \sigma_C D^*(t)\;\, dW_C(t)\,.
\end{aligned}$$

But the value in Canadian dollars of the American riskless asset is also $A^{**}(t) = \frac{C(t)}{U(t)} A(t)$. Indeed, $\frac{1}{U(t)} A(t)$ is the value in yen of the American bank account and $C(t)\,\frac{1}{U(t)} A(t)$ is the value in Canadian dollars of $\frac{1}{U(t)}A(t)$ which is expressed in yens. Using Itô's lemma,

$$\begin{aligned} dA^{**}(t) &= -\frac{C(t)}{U^2(t)}A(t)\ dU(t) + \frac{1}{U(t)}A(t)\ dC(t) + \frac{C(t)}{U(t)}\ dA(t) \\ &\quad +\frac{1}{2}\frac{2C(t)}{U^3(t)}A(t)\ d\langle U\rangle(t) - \frac{1}{U^2(t)}A(t)\ d\langle C,U\rangle(t) \\ &= \left(r_U + \mu_C - \mu_U + \sigma_U^2 - \sigma_C\sigma_U\rho_{CU}\right)A^{**}(t)\ dt \\ &\quad +\sigma_C A^{**}(t)\ dW_C(t) - \sigma_U A^{**}(t)\ dW_U(t). \end{aligned}$$

Similarly, the value in Canadian dollars of the Japanese bank account satisfies $D^{**}(t) = V(t)U(t)D(t)$. Itô's lemma gives

$$\begin{aligned} dD^{**}(t) &= (r_J + \mu_U + \mu_V + \sigma_U\sigma_V\rho_{UV})D^{**}(t)\ dt \\ &\quad +\sigma_U D^{**}(t)\ dW_U(t) + \sigma_V D^{**}(t)\ dW_V(t). \end{aligned}$$

Finally, $\frac{1}{C(t)}B(t)$ is the value in yen of the Canadian bank account; $\frac{U(t)}{C(t)}B(t)$ is expressed in American dollars and $B^*(t) = \frac{V(t)U(t)}{C(t)}B(t)$ is reverted back to Canadian dollars. Itô's lemma gives

$$\begin{aligned} dB^*(t) &= -\frac{V(t)U(t)}{C^2(t)}B(t)\ dC(t) + \frac{U(t)}{C(t)}B(t)\ dV(t) + \frac{V(t)}{C(t)}B(t)\ dU(t) + \frac{V(t)U(t)}{C(t)}\ dB(t) \\ &\quad +\frac{1}{2}\frac{2V(t)U(t)}{C^3(t)}B(t)\ d\langle C\rangle(t) \\ &\quad -\frac{V(t)}{C^2(t)}B(t)\ d\langle C,U\rangle(t) - \frac{U(t)}{C^2(t)}B(t)\ d\langle C,V\rangle(t) + \frac{1}{C(t)}B(t)\ d\langle U,V\rangle(t) \\ &= \left(-\mu_C + \mu_V + \mu_U + r_C + \sigma_C^2 - \sigma_C\sigma_U\rho_{CU} - \sigma_C\sigma_V\rho_{CV} + \sigma_U\sigma_V\rho_{UV}\right)B^*(t)\ dt \\ &\quad -\sigma_C B^*(t)\ dW_C(t) + \sigma_U B^*(t)\ dW_U(t) + \sigma_V B^*(t)\ dW_V(t). \end{aligned}$$

Since the value in Canadian dollars of the American bank account has two different expressions, , $A^*(t)$ and $A^{**}(t)$, these two processes must be the same

$$\begin{aligned} dA^*(t) &= (r_U + \mu_V)A^*(t)\ dt + \sigma_V A^*(t)\ dW_V(t) \\ dA^{**}(t) &= \left(r_U + \mu_C - \mu_U + \sigma_U^2 - \sigma_C\sigma_U\rho_{CU}\right)A^{**}(t)\ dt \\ &\quad +\sigma_C A^{**}(t)\ dW_C(t) - \sigma_U A^{**}(t)\ dW_U(t). \end{aligned}$$

Therefore,

$$r_U + \mu_V = r_U + \mu_C - \mu_U + \sigma_U^2 - \sigma_C\sigma_U\rho_{CU} \quad (1)$$

$$\sigma_V W_V(t) = \sigma_C W_C(t) - \sigma_U W_U(t). \quad (2)$$

Similarly, the value in Canadian dollars of the Japanese bank account have two expressions:

$$\begin{aligned} dD^{*}(t) &= (r_J + \mu_C) D^{*}(t) \ dt + \sigma_C D^{*}(t) \ dW_C(t) \\ dD^{**}(t) &= (r_J + \mu_U + \mu_V + \sigma_U \sigma_V \rho_{UV}) D^{**}(t) \ dt \\ &\quad + \sigma_U D^{**}(t) \ dW_U(t) + \sigma_V D^{**}(t) \ dW_V(t) \end{aligned}$$

Therefore

$$\begin{aligned} r_J + \mu_C &= r_J + \mu_U + \mu_V + \sigma_U \sigma_V \rho_{UV} \\ \sigma_C W_C(t) &= \sigma_U W_U(t) + \sigma_V W_V(t). \end{aligned}$$

These four restrictions may be rewritten as

$$\begin{aligned} \mu_V &= \mu_C - \mu_U + \sigma_U^2 - \sigma_C \sigma_U \rho_{CU}, \\ \mu_V &= \mu_C - \mu_U - \sigma_U \sigma_V \rho_{UV}, \\ \sigma_V W_V(t) &= \sigma_C W_C(t) - \sigma_U W_U(t). \end{aligned}$$

From the two first ones, we conclude that

$$\sigma_U = \sigma_C \rho_{CU} - \sigma_V \rho_{UV}.$$

From the last one,

$$\begin{aligned} \rho_{UV} &= \text{Corr}\left[W_U(t), W_V(t)\right] \\ &= \text{Corr}\left[W_U(t), \frac{\sigma_C W_C(t) - \sigma_U W_U(t)}{\sigma_V}\right] \\ &= \frac{\sigma_C}{\sigma_V} \text{Corr}\left[W_U(t), W_C(t)\right] - \frac{\sigma_U}{\sigma_V} \text{Corr}\left[W_U(t), W_U(t)\right] \\ &= \frac{\sigma_C \rho_{CU} - \sigma_U}{\sigma_V} \\ &= \frac{\sigma_C \rho_{CU} - (\sigma_C \rho_{CU} - \sigma_V \rho_{UV})}{\sigma_V} \\ &= \rho_{UV} \end{aligned}$$

which do not brings new information. Therefore

$$\begin{aligned} \sigma_U &= \sigma_C \rho_{CU} - \sigma_V \rho_{UV} \\ \mu_U &= \mu_C - \mu_V - \sigma_U \sigma_V \rho_{UV}. \end{aligned}$$

The second step consist in finding the risk neutral measures. The Choleski decomposition allows to express the dependent Brownian motions as a linear combination of independent Brownian motions:

$$\begin{aligned} W_V(t) &= a_{11}B_1(t) + a_{12}B_2(t) + a_{13}B_3(t) \\ W_U(t) &= a_{21}B_1(t) + a_{22}B_2(t) + a_{23}B_3(t) \\ W_C(t) &= a_{31}B_1(t) + a_{32}B_2(t) + a_{33}B_3(t). \end{aligned}$$

Let

$$\widetilde{B}_i(t) = B_i(t) + \int_0^t \alpha_i(s)\,ds,\ \ i = 1, 2, 3$$

and

$$\begin{aligned} \widetilde{W}_V(t) &= a_{11}\widetilde{B}_1(t) + a_{12}\widetilde{B}_2(t) + a_{13}\widetilde{B}_3(t) = W_V(t) + \int_0^t \gamma_V(s)\,ds \\ \widetilde{W}_U(t) &= a_{21}\widetilde{B}_1(t) + a_{22}\widetilde{B}_2(t) + a_{23}\widetilde{B}_3(t) = W_U(t) + \int_0^t \gamma_U(s)\,ds \\ \widetilde{W}_C(t) &= a_{31}\widetilde{B}_1(t) + a_{32}\widetilde{B}_2(t) + a_{33}\widetilde{B}_3(t) = W_C(t) + \int_0^t \gamma_C(s)\,ds \end{aligned}$$

where

$$\begin{aligned} \gamma_V(t) &= a_{11}\alpha_1(t) + a_{12}\alpha_2(t) + a_{13}\alpha_3(t) \\ \gamma_U(t) &= a_{21}\alpha_1(t) + a_{22}\alpha_2(t) + a_{23}\alpha_3(t) \\ \gamma_C(t) &= a_{31}\alpha_1(t) + a_{32}\alpha_2(t) + a_{33}\alpha_3(t). \end{aligned}$$

Therefore

$$\begin{aligned} dA^*(t) &= (r_U + \mu_V - \sigma_V\gamma_V)\,A^*(t)\ \ dt + \sigma_V A^*(t)\ \ d\widetilde{W}_V(t) \\ dA^{**}(t) &= \left(r_U + \mu_C - \mu_U + \sigma_U^2 - \sigma_C\sigma_U\rho_{CU} - \sigma_C\gamma_C + \sigma_U\gamma_U\right) A^{**}(t)\ \ dt \\ &\quad +\sigma_C A^{**}(t)\ \ d\widetilde{W}_C(t) - \sigma_U A^{**}(t)\ \ d\widetilde{W}_U(t) \\ dD^*(t) &= (r_J + \mu_C - \sigma_C\gamma_C)\,D^*(t)\ \ dt + \sigma_C D^*(t)\ \ d\widetilde{W}_C(t) \\ dD^{**}(t) &= (r_J + \mu_U + \mu_V + \sigma_U\sigma_V\rho_{UV} - \sigma_U\gamma_U - \sigma_V\gamma_V)\,D^{**}(t)\ \ dt \\ &\quad +\sigma_U D^{**}(t)\ \ d\widetilde{W}_U(t) + \sigma_V D^{**}(t)\ \ d\widetilde{W}_V(t) \\ dB^*(t) &= \begin{pmatrix} -\mu_C + \mu_V + \mu_U + r_C + \sigma_C^2 - \sigma_C\sigma_U\rho_{CU} - \sigma_C\sigma_V\rho_{CV} + \sigma_U\sigma_V\rho_{UV} \\ +\sigma_C\gamma_C - \sigma_U\gamma_U - \sigma_V\gamma_V \end{pmatrix} B^*(t)\ \ dt \\ &\quad -\sigma_C B^*(t)\ \ d\widetilde{W}_C(t) + \sigma_U B^*(t)\ \ d\widetilde{W}_U(t) + \sigma_V B^*(t)\ \ d\widetilde{W}_V(t). \end{aligned}$$

To risk neutralize the system, the drift parameter must be the Canadian risk free rate:

$$
\begin{array}{rcl}
r_U+\mu_V-\sigma_V\gamma_V &=& r_C\\
r_U+\mu_C-\mu_U+\sigma_U^2-\sigma_C\sigma_U\rho_{CU}-\sigma_C\gamma_C+\sigma_U\gamma_U &=& r_C\\
r_J+\mu_C-\sigma_C\gamma_C &=& r_C\\
(r_J+\mu_U+\mu_V+\sigma_U\sigma_V\rho_{UV}-\sigma_U\gamma_U-\sigma_V\gamma_V) &=& r_C\\
\left(\begin{array}{c} -\mu_C+\mu_V+\mu_U+r_C+\sigma_C^2-\sigma_C\sigma_U\rho_{CU}-\sigma_C\sigma_V\rho_{CV}+\sigma_U\sigma_V\rho_{UV}\\ +\sigma_C\gamma_C-\sigma_U\gamma_U-\sigma_V\gamma_V\end{array}\right) &=& r_C.
\end{array}
$$

On matrix form, the first three expressions become

$$
\begin{bmatrix} -\sigma_V & 0 & 0\\ 0 & \sigma_U & -\sigma_C\\ 0 & 0 & -\sigma_C \end{bmatrix}\begin{bmatrix}\gamma_V\\ \gamma_U\\ \gamma_C\end{bmatrix}+\begin{bmatrix} r_U+\mu_V-r_C\\ r_U+\mu_C-\mu_U+\sigma_U^2-\sigma_C\sigma_U\rho_{CU}-r_C\\ r_J+\mu_C-r_C\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.
$$

The solution is

$$
\begin{bmatrix}\gamma_V\\ \gamma_U\\ \gamma_C\end{bmatrix}=\begin{bmatrix}\frac{1}{\sigma_V}(r_U+\mu_V-r_C)\\ \frac{1}{\sigma_U}(r_J-r_U+\mu_U-\sigma_U^2+\sigma_C\sigma_U\rho_{CU})\\ \frac{1}{\sigma_C}(r_J+\mu_C-r_C)\end{bmatrix}.
$$

The price of risk associated to the $W$s, $(\gamma_V,\gamma_U,\gamma_C)$, have to exist. For that, $\sigma_V$, $\sigma_U$ and $\sigma_C$ have to be strictly positive. Since $\gamma_V,\gamma_U,\gamma_C$ are constants, $\alpha_1,\alpha_2,\alpha_3$ are also constants. The Novikov condition is satisfied. We can apply Girsanov theorem to find $\mathbb{Q}$ under which $\widetilde{B}_1$, $\widetilde{B}_2$, $\widetilde{B}_3$ are independent $\mathbb{Q}-$Brownian motions. Consequently $\widetilde{W_V},\widetilde{W_U},\widetilde{W_C}$ are correlated $\mathbb{Q}-$Brownian motions having the same correlation structure as the $\mathbb{P}-$Brownian motions $W_V$, $W_U,W_C$. Since the price of risk $\gamma_V,\gamma_U,\gamma_C$ are uniquely determined, the risk neutral measure is unique and the market model is complete.

To rule out arbitrage opportunities, the prices of risk $\gamma_V,\gamma_U,\gamma_C$ are substituted in the fourth and fifth equations. Replacing $\gamma_V,\gamma_U$ and $\gamma_C$ in the fourth equation leads to

$$\sigma_V\rho_{UV}+\sigma_U-\sigma_C\rho_{CU}=0.$$

Similarly, working with the fifth equation,

$$
\begin{array}{rcl}
0 &=& \sigma_C^2-2\sigma_C\sigma_U\rho_{CU}-\sigma_C\sigma_V\rho_{CV}+\sigma_U\sigma_V\rho_{UV}+\sigma_U^2\\
&=& \sigma_C(\sigma_C-\sigma_U\rho_{CU}-\sigma_V\rho_{CV})+\sigma_U\underbrace{(\sigma_V\rho_{UV}+\sigma_U-\sigma_C\rho_{CU})}_{=0}.
\end{array}
$$

Therefore

$$\sigma_C-\sigma_U\rho_{CU}-\sigma_V\rho_{CV}=0. \tag{3}$$

For the martingale measure to exist, we need that the yen\American dollar exchange rate volatility satisfies

$$\sigma_U = \sigma_C \rho_{CU} - \sigma_V \rho_{UV} \text{ and } \sigma_U = \frac{\sigma_C - \sigma_V \rho_{CV}}{\rho_{CU}}.$$

The SDE under $\mathbb{Q}$ are

$$\begin{aligned} dU(t) &= \left(r_U - r_J + \sigma_U^2 - \sigma_C \sigma_U \rho_{CU}\right) U(t) \ dt + \sigma_U U(t) \ d\widetilde{W}_U(t) \\ &= \left(r_U - r_J - \sigma_U \sigma_V \rho_{UV}\right) U(t) \ dt + \sigma_U U(t) \ d\widetilde{W}_U(t), \\ dC(t) &= \left(r_C - r_J\right) C(t) \ dt + \sigma_C C(t) \ d\widetilde{W}_C(t), \\ dV(t) &= \left(r_C - r_U\right) V(t) \ dt + \sigma_V V(t) \ d\widetilde{W}_V(t). \end{aligned}$$

# 4 Exercise 12.5

## 4.1 Question a)

Recall that

$$\widehat{B}_t \equiv \kappa W_t + \sqrt{1-\kappa^2}\widehat{W}_t.$$

First

$$\widehat{B}_0 \equiv \kappa W_0 + \sqrt{1-\kappa^2}\widehat{W}_0 = 0.$$

Second, for all $t_0 < t_1 < ... < t_n$, the increments

$$\widehat{B}_{t_1} - \widehat{B}_{t_0}, \ \widehat{B}_{t_2} - \widehat{B}_{t_1}, ..., \widehat{B}_{t_n} - \widehat{B}_{t_{n-1}}$$

are independent since for $0 \le r < s \le t < u$,

$$\begin{aligned} & \operatorname{Cov}\left[\widehat{B}_s - \widehat{B}_r, \widehat{B}_u - \widehat{B}_t\right] \\ = \ & \operatorname{Cov}\left[\kappa\left(W_s - W_r\right) + \sqrt{1-\kappa^2}\left(\widehat{W}_s - \widehat{W}_r\right), \kappa\left(W_u - W_t\right) + \sqrt{1-\kappa^2}\left(\widehat{W}_u - \widehat{W}_t\right)\right] \\ = \ & \operatorname{Cov}\left[\kappa\left(W_s - W_r\right), \kappa\left(W_u - W_t\right)\right] \\ & + \operatorname{Cov}\left[\kappa\left(W_s - W_r\right), \sqrt{1-\kappa^2}\left(\widehat{W}_u - \widehat{W}_t\right)\right] \\ & + \operatorname{Cov}\left[\sqrt{1-\kappa^2}\left(\widehat{W}_s - \widehat{W}_r\right), \kappa\left(W_u - W_t\right)\right] \\ & + \operatorname{Cov}\left[\sqrt{1-\kappa^2}\left(\widehat{W}_s - \widehat{W}_r\right), \sqrt{1-\kappa^2}\left(\widehat{W}_u - \widehat{W}_t\right)\right] \\ = \ & 0 \end{aligned}$$

The independence property will be established once we will have prove that $\widehat{B}_t - \widehat{B}_s$ is Gaussian.

Third, because a linear combination of a multivariate Gaussian random variables is Gaussian, $\widehat{B}_t$ is Gaussian. Moreover,

$$\mathrm{E}^{\mathbb{P}}\left[\widehat{B}_t\right] = \kappa \mathrm{E}^{\mathbb{P}}\left[W_t\right] + \sqrt{1-\kappa^2}\mathrm{E}^{\mathbb{P}}\left[\widehat{W}_t\right] = 0$$

and

$$\begin{aligned}
\mathrm{Var}^{\mathbb{P}}\left[\widehat{B}_t\right] &= \kappa^2 \mathrm{Var}^{\mathbb{P}}\left[W_t\right] + \left(1-\kappa^2\right)\mathrm{Var}^{\mathbb{P}}\left[\widehat{W}_t\right] \\
&= \kappa^2 t + \left(1-\kappa^2\right) t \\
&= t.
\end{aligned}$$

Finally, the path of $W$ and $\widehat{W}$ being continuous, the ones of $\widehat{B}$ are also continuous.

#### 4.1.1 Question b)

$$\begin{aligned}
& \mathrm{Corr}^{\mathbb{P}}\left[\widehat{B}_t, \widetilde{B}_t\right] \\
= & \frac{\mathrm{Cov}^{\mathbb{P}}\left[\widehat{B}_t, \widetilde{B}_t\right]}{\sqrt{\mathrm{Var}^{\mathbb{P}}\left[\widehat{B}_t\right]}\sqrt{\mathrm{Var}^{\mathbb{P}}\left[\widetilde{B}_t\right]}} \\
= & \frac{\mathrm{Cov}^{\mathbb{P}}\left[\kappa W_t + \sqrt{1-\kappa^2}\widehat{W}_t, \theta W_t + \sqrt{1-\theta^2}\widetilde{W}_t\right]}{\sqrt{t}\sqrt{t}} \\
= & \frac{1}{t}\left(\begin{array}{c} \kappa\theta \mathrm{Cov}^{\mathbb{P}}\left[W_t, W_t\right] + \kappa\sqrt{1-\theta^2}\mathrm{Cov}^{\mathbb{P}}\left[W_t, \widetilde{W}_t\right] \\ +\sqrt{1-\kappa^2}\theta \mathrm{Cov}^{\mathbb{P}}\left[\widehat{W}_t, W_t\right] + \sqrt{1-\kappa^2}\sqrt{1-\theta^2}\mathrm{Cov}^{\mathbb{P}}\left[\widehat{W}_t, \widetilde{W}_t\right] \end{array}\right) \\
= & \frac{\kappa\theta \mathrm{Cov}^{\mathbb{P}}\left[W_t, W_t\right]}{t} \\
= & \kappa\theta.
\end{aligned}$$

#### 4.1.2 Question c)

We have

$$\begin{aligned}
dX_t &= \mu_X X_t\, dt + \sigma_X \theta X_t\, dW_t + \sigma_X \sqrt{1-\theta^2} X_t\, d\widetilde{W}_t \\
\text{and } dY_t &= \mu_Y Y_t\, dt + \sigma_Y \kappa Y_t\, dW_t + \sigma_Y \sqrt{1-\kappa^2} Y_t\, d\widehat{W}_t.
\end{aligned}$$

Let

$$\beta_t \equiv \exp(-rt), \ 0 \leq t \leq T.$$

$\beta_t$ is the discounted factor at $t$. This process is deterministic and satisfies

$$d\beta_t = -r\beta_t \ dt.$$

The evolution of the discounted price processes satisfies

$$\begin{aligned} d\beta_t X_t &= \left(\mu_X - r - \sigma_X \theta \gamma_t - \sigma_X \sqrt{1-\theta^2}\widetilde{\gamma}_t\right) \ \beta_t X_t \ dt \\ &\quad + \sigma_X \theta \ \beta_t X_t \ d\left(W_t + \int_0^t \gamma_s ds\right) + \sigma_X \sqrt{1-\theta^2} \ \beta_t X_t \ d\left(\widetilde{W}_t + \int_0^t \widetilde{\gamma}_s ds\right) \\ \text{and } d\beta_t Y_t &= \left(\mu_Y - r - \sigma_Y \kappa \gamma_t - \sigma_Y \sqrt{1-\kappa^2}\widehat{\gamma}_t\right) \ \beta_t Y_t \ dt \\ &\quad + \sigma_Y \kappa \ \beta_t Y_t \ d\left(W_t + \int_0^t \gamma_s ds\right) + \sigma_Y \sqrt{1-\kappa^2} \ \beta_t Y_t \ d\left(\widehat{W}_t + \int_0^t \widehat{\gamma}_s ds\right). \end{aligned}$$

We find $\gamma, \widetilde{\gamma}$ and $\widehat{\gamma}$ such that the drift terms are nil:

$$\begin{aligned} \mu_X - r - \sigma_X \theta \gamma_t - \sigma_X \sqrt{1-\theta^2}\widetilde{\gamma}_t &= 0 \\ \mu_Y - r - \sigma_Y \kappa \gamma_t - \sigma_Y \sqrt{1-\kappa^2}\widehat{\gamma}_t &= 0 \end{aligned}$$

We need to solve

$$\begin{pmatrix} \sigma_X \theta & \sigma_X \sqrt{1-\theta^2} & 0 \\ \sigma_Y \kappa & 0 & \sigma_Y \sqrt{1-\kappa^2} \end{pmatrix} \begin{pmatrix} \gamma_t \\ \widetilde{\gamma}_t \\ \widehat{\gamma}_t \end{pmatrix} - \begin{pmatrix} \mu_X - r \\ \mu_Y - r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solution exists but it is not unique. Need to satisfy the Novikov condition to apply Girsanov theorem. For $\widehat{\gamma}_t \in \mathbb{R}$,

$$\begin{aligned} d\beta_t X_t &= \sigma_X \theta \ \beta_t X_t \ dW_t^* + \sigma_X \sqrt{1-\theta^2} \ \beta_t X_t \ d\widetilde{W}_t^* \\ \text{and } d\beta_t Y_t &= \sigma_Y \kappa \ \beta_t Y_t \ dW_t^* + \sigma_Y \sqrt{1-\kappa^2} \ \beta_t Y_t \ d\widehat{W}_t^* \end{aligned}$$

where $W^*$, $\widetilde{W}^*$ and $\widehat{W}^*$are $\left(\{\mathcal{F}_t\}, \mathbb{Q}^{\widehat{\gamma}}\right)$ −Brownian motions. We have

$$\begin{aligned} dX_t &= rX_t \ dt + \sigma_X \theta X_t \ dW_t^* + \sigma_X \sqrt{1-\theta^2} X_t \ d\widetilde{W}_t^* \\ \text{and } dY_t &= rY_t \ dt + \sigma_Y \kappa Y_t \ dW_t + \sigma_Y \sqrt{1-\kappa^2} Y_t \ d\widehat{W}_t. \end{aligned}$$

### 4.1.3 Question d)

Infinitely many martingale measures, the market is incomplete.

### 4.1.4 Question e)

The contract is accessible since it suffices to hold 1 share of the first asset and $-1$ share of the second one.

$$\begin{aligned}
\mathrm{E}^{\mathbb{Q}}\left[\exp(-rT)C\right] &= \mathrm{E}^{\mathbb{Q}}\left[\exp(-rT)\left(X_T - Y_T\right)\right] \\
&= \mathrm{E}^{\mathbb{Q}}\left[B_T^{-1}X_T\right] - \mathrm{E}^{\mathbb{Q}}\left[B_T^{-1}Y_T\right] \\
&= \mathrm{E}^{\mathbb{Q}}\left[B_0^{-1}X_0\right] - \mathrm{E}^{\mathbb{Q}}\left[B_0^{-1}Y_0\right] \\
&\quad \text{martingale property} \\
&= X_0 - Y_0.
\end{aligned}$$