Stochastic differential equation (SDE) and Ito's lemma

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Stochastic Calculus I

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Ordinary differential eq. I

- When we model some situations, we don't know a priori which function to use, since we only know the local behavior of our system.
- **For example**, assume that $f(t)$ represents a commodity price at time t. We write

$$f(t+\Delta t) - f(t) = \mu \, \Delta t \, f(t) \text{ where } \mu \geq 0$$

in order to mean that the variation $f(t+\Delta t) - f(t)$ of the commodity price over a time period is is proportional to the length Δt of the time period considered as well as the commodity price $f(t)$ at the start of the period, i.e. $\mu \, \Delta t \, f(t)$, μ being a constant.

Ordinary differential eq. II

- By dividing both sides of the equality by Δt, we obtain

$$\frac{f(t+\Delta t)-f(t)}{\Delta t}=\mu\ f(t).$$

- Let's now take the limit when Δt tends to zero:

$$\begin{aligned}\frac{d}{dt}f(t) &= \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\mu\ f(t)\\ &= \mu\ f(t).\end{aligned}$$

Ordinary differential eq. III

- Recall that we consider the equation

$$\frac{d}{dt} f(t) = \mu \, f(t).$$

- The notation commonly used for differential equations allows us to rewrite the equation above as:

$$d \, f(t) = \mu \, f(t) \, dt. \tag{1}$$

- Note that, technically speaking, the object $d \, f(t)$ is not well-defined. The latter equation is only a notation to express that "the derivative of the function is proportional to the function itself", i.e. $\frac{d}{dt} f(t) = \mu \, f(t)$.
- The unknown, in that equation, is the function f. We are looking for the functions that satisfy that equality.

Ordinary differential eq. IV

- Recall that we are studying the equation

$$d\ f(t) = \mu\ f(t)\ \ dt. \tag{1}$$

- It is possible to show that the function defined for all $t \in \mathbb{R}$ as

$$f(t) = ce^{\mu t}, \text{ where } c \text{ is any constant,} \tag{2}$$

  satisfies equation (1).
- Indeed, in such a case,

$$\frac{d}{dt} f(t) = \frac{d}{dt} ce^{\mu t} = \mu ce^{\mu t} = \mu f(t).$$

Ordinary differential eq. V

- The initial condition helps determine the constant c. We know commodity price f_0 today. As a consequence,

$$f_0 = f(0) = ce^{\mu \times 0} = c,$$

which yields that the commodity price at time t is

$$f(t) = f_0 e^{\mu t}.$$

- In this example, knowing the infinitesimal behavior of the commodity price $(d\ f(t) = \mu\ f(t)\ dt)$ and the initial price f_0 is sufficient to determine accurately the price at any time.

Ordinary differential eq. VI

- Recall that we are considering the equation

$$d\ f\left(t\right) = \mu\ f\left(t\right)\ \ dt. \tag{1}$$

- equation (1) is an example of ordinary differential equation and it behaves very charmingly since there exists at least one function f which satisfies equation (1) and, in addition, it is possible to show that this function is necessarily of the form described in (2) :

$$f\left(t\right) = f_0 e^{\mu t}.$$

Ordinary differential eq. VII

- There exist ordinary differential equations much less nice. **For example**,

$$d\ f(t) = \frac{f(t)}{t^2} dt. \tag{3}$$

- The solution to that equation has form

$$f(t) = ce^{-\frac{1}{t}} \text{ where } c \text{ is a constant.}$$

- We must now specify c by using the initial condition.
- But $f(0) = 0$ whatever c, which implies that (3) possesses an infinity of solutions when $f(0) = 0$ and possesses none when $f(0) \neq 0$.

Introduction I

Stochastic differential equations

- Now assume that the stochastic process $S = \{S_t : t \geq 0\}$ represents the evolution of a risky asset price.
- We don't know, in general, the law that governs such a process, but we may have an idea of its local behavior.

Introduction II

Stochastic differential equations

- **For example**, over a short time interval of length Δt, it is possible that such a price tends to vary proportionally to the period length and the asset price at the beginning of the period. We write, to begin with,

$$S_{t+\Delta t} - S_t = \mu \; S_t \; \Delta t.$$

If, in general, prices increase, then μ is a positive constant and if the prices tend to decrease, then μ is negative.

Introduction III

Stochastic differential equations

- $S_{t+\Delta t} - S_t = \mu \; S_t \; \Delta t$
- There is however a problem with the latter equation: we are not certain that the price varies proportionally to the period length and the asset price, we only claim that it has a *tendency* to do so.
- Therefore, an unpredictable error needs to be incorporated into our equation.
- We can however control the *magnitude* of such a random error.

Introduction IV

Stochastic differential equations

- **For example**, we can assume that it depends on the asset price at the beginning of the period.
  - Indeed, we observe that, the higher the price, the more the risky asset price can diverge from the trend.
  - Moreover, the random error must also depend on the length of the time interval considered: the longer the interval, the greater the chance that the price diverges from the trend.
  - That is why we add a stochastic term to our initial equation.

Introduction V

Stochastic differential equations

- The stochastic term to be added to our initial equation leads us to the equation

$$S_{t+\Delta t} - S_t = \mu \; S_t \; \Delta t + \sigma \; S_t \; \sqrt{\Delta t} \; \xi_t \tag{4}$$

  where

  - σ is a positive constant and
  - ξ_t is a random variable with distribution $N(0,1)$ independent from $\{S_u : 0 \leq u \leq t\}$.

- The latter condition is important, since we must not be able to predict the error ξ_t from observing the behavior of the risky asset price prior to date t.

Introduction VI

Stochastic differential equations

- Such an equation is random and must be satisfied by "almost" every ω, which is to say that

$$\Pr\left[\left\{\omega \in \Omega : S_{t+\Delta t}(\omega) - S_t(\omega) = \mu\ S_t(\omega)\ \Delta t + \sigma\ S_t(\omega)\ \sqrt{\Delta t}\xi_t(\omega)\right\}\right]$$

worth one.

Introduction VII

Stochastic differential equations

- With respect to the magnitude of the random error, note that $\sqrt{\Delta t}\ \xi_t$ is $N(0, \Delta t)$-distributed. Moreover,

$$\begin{aligned} \mathrm{E}\left[\sigma S_t \sqrt{\Delta t}\xi_t \ \mid \sigma\left\{S_u : u \in \{0, \Delta t, ..., t\}\right\}\right] &= \sigma S_t \sqrt{\Delta t}\mathrm{E}\left[\xi_t\right] \\ &= 0, \end{aligned}$$

$$\begin{aligned} \mathrm{E}\left[\left(\sigma S_t \sqrt{\Delta t}\xi_t\right)^2 \ \mid \sigma\left\{S_u : u \in \{0, \Delta t, ..., t\}\right\}\right] &= \sigma^2 S_t^2 \Delta t \mathrm{E}\left[\xi_t^2\right] \\ &= \sigma^2 S_t^2 \Delta t, \end{aligned}$$

  which implies that the conditional standard deviation of the error term is $\sigma S_t \sqrt{\Delta t}$.
- Thus the longer the time interval Δt or the higher the security price S_t, the greater the standard deviation of the random error.

Introduction VIII

Stochastic differential equations

- This implies that the values that the random error may take are more dispersed around its expectation (which is zero).

Introduction IX

Stochastic differential equations

- Recall that

$$S_{t+\Delta t} - S_t = \mu\ S_t\ \Delta t + \sigma\ S_t\ \sqrt{\Delta t}\ \xi_t. \tag{4}$$

- Let's rewrite equation (4) for the next period:

$$S_{t+2\Delta t} - S_{t+\Delta t} = \mu\ S_{t+\Delta t}\ \Delta t + \sigma\ S_{t+\Delta t}\ \sqrt{\Delta t}\ \xi_{t+\Delta t}.$$

- If we don't want to be able to predict the error $\xi_{t+\Delta t}$, the latter must be independent from $\{S_u : u \in \{0, \Delta t, ..., t + \Delta t\}\}$.
- For that reason, we introduce the Brownian motion since it is a Gaussian process, the increments of which are mutually independent:

$$S_{t+\Delta t} - S_t = \mu\ S_t\ \Delta t + \sigma\ S_t\ \left(W_{t+\Delta t} - W_t\right). \tag{5}$$

- Note that the law of $W_{t+\Delta t} - W_t$ is the same as the law of $\sqrt{\Delta t}\xi_t$: both are $N\left(0, \Delta t\right)$-distributed.

Introduction X

Stochastic differential equations

- Let $W = \{W_t : t \geq 0\}$ be a Brownian motion built on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that the filtration $\mathbb{F}$ is the one generated by the Brownian motion, plus it includes all zero-probability events, i.e. for all $t \geq 0$,

$$\mathcal{F}_t = \sigma\left(\mathcal{N} \text{ and } W_s : 0 \leq s \leq t\right).$$

Introduction XI

Stochastic differential equations

- The risky asset price today $(t = 0)$ is known with certainty. S_0 is thus $(\varnothing, \Omega)$ −measurable and therefore $\mathcal{F}_0$−measurable. Let's go back to equation (5) and let's set $t = 0$.

$$S_{\Delta t} = \underbrace{\underbrace{S_0}_{\mathcal{F}_0-\text{measurable}} + \underbrace{\mu\, S_0\, \Delta t}_{\mathcal{F}_0-\text{measurable}} + \underbrace{\sigma\, S_0}_{\mathcal{F}_0-\text{measurable}} \underbrace{(W_{\Delta t} - W_0)}_{\substack{\mathcal{F}_{\Delta t}-\text{measurable} \\ \text{indpendent from } \mathcal{F}_0}}}_{\mathcal{F}_{\Delta t}-\text{measurable}}$$

- We observe that $S_{\Delta t}$ is $\mathcal{F}_{\Delta t}$−measurable.

Introduction XII

Stochastic differential equations

- We can show by induction that $S_{n\ \Delta t}$ is $\mathcal{F}_{n\ \Delta t}-$measurable, for any positive integer n.
- Indeed, let's assume there exists $k \in \{0, 1, 2, ...\}$ such that $S_{k\ \Delta t}$ is $\mathcal{F}_{k\ \Delta t}-$measurable. Then

$$S_{(k+1)\ \Delta t} = \underbrace{\underbrace{S_{k\ \Delta t}}_{\mathcal{F}_{k\ \Delta t}-\text{measurable}} + \underbrace{\mu\ S_{k\ \Delta t}\ \Delta t}_{\mathcal{F}_{k\ \Delta t}-\text{measurable}} + \underbrace{\sigma\ S_{k\ \Delta t}}_{\mathcal{F}_{k\ \Delta t}-\text{measurable}} \underbrace{\left(W_{(k+1)\ \Delta t} - W_{k\ \Delta t}\right)}_{\substack{\mathcal{F}_{(k+1)\ \Delta t}-\text{measurable} \\ \text{independent from } \mathcal{F}_{k\ \Delta t}}}}_{\mathcal{F}_{(k+1)\ \Delta t}-\text{measurable}}$$

which implies that $S_{(k+1)\ \Delta t}$ is $\mathcal{F}_{(k+1)\ \Delta t}-$measurable.

- Our process S lives on the same filtered probability space as the Brownian motion W that we have used to build that process. ■

Introduction XIII

Stochastic differential equations

- Let's go back to equation (5)

$$S_{t+\Delta t} - S_t = \mu \; S_t \; \Delta t + \sigma \; S_t \; \left(W_{t+\Delta t} - W_t\right).$$

- When time intervals of length Δt become of infinitesimal length, we obtain an equation of the type

$$dS_t = \mu \; S_t \; dt + \sigma \; S_t \; dW_t.$$

Introduction XIV

Stochastic differential equations

- The latter equation is an example of stochastic differential equation and it should raise a few questions:
  1. the term $\sigma\ S_t\ dW_t$ is not well defined, particularly if we recall that the Brownian motion paths are nowhere differentiable!
  2. does a solution to that equation exist? and
  3. if that solution exists, is it unique and how can it be found?
- Note that the solution to a stochastic differential equation is not, as is the case for ordinary differential equations, a function, but rather a stochastic process.

Introduction XV

Stochastic differential equations

- In order to answer those questions, let's consider a stochastic differential equation in a more general form

$$dX(t) = \underbrace{b(X(t), t)}_{\text{drift coefficient}} dt + \underbrace{a(X(t), t)}_{\text{diffusion coefficient}} dW(t). \quad (6)$$

where the functions $a : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $b : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ are measurable functions.

Introduction XVI

Stochastic differential equations

- What do we mean by $a(X(t), t)\, dW(t)$? We haven't defined that term. Actually, Equation (6) is the differential form of the integral equation:

$$X(t) = X(0) + \int_0^t b(X(u), u)\ du + \int_0^t a(X(u), u)\ dW(u)$$

and we now know the meaning of the term

$$\int_0^t a(X(u), u)\, dW(u).$$

Introduction XVII

Stochastic differential equations

- There doesn't always exist a solution to that equation and we will see a few results which give the conditions that b and a functions must meet for a solution to exist.
- To answer questions (2) and (3), we need a few additional tools.

Fundamental theorem of calculus I

- Itô's lemma is the stochastic equivalent of the fundamental theorem of calculus. It will allow us to determine the stochastic differential equation satisfied by some given stochastic processes.

Theorem

The **fundamental theorem of calculus** *stipulates that, if* $\frac{df}{dx} : \mathbb{R} \to \mathbb{R}$ *represents the derivative of the function* $f : \mathbb{R} \to \mathbb{R}$, *then*

$$f(b) - f(a) = \int_a^b \frac{df}{dx}(x) \; dx.$$

Richard R. Goldberg, Theorem 7.8A, page 205.

Fundamental theorem of calculus II

- **Example**. if $f(x) = x^2$, $a = 0$ and $b = t$, then

$$t^2 = \int_0^t 2x \, dx.$$

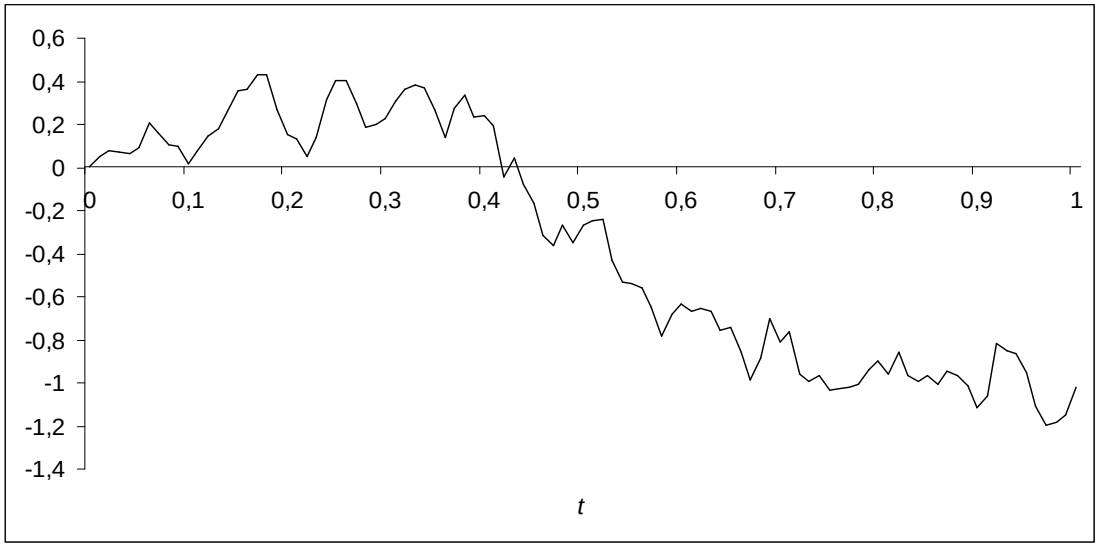
- Is such a rule still valid in the context of stochastic calculus? Is

$$W_t^2 = \int_0^t 2W_s \, dW_s \ ? \tag{7}$$

- We have observed, when constructing the stochastic integral, that the paths of the process

Fundamental theorem of calculus III

$\left\{ \int_0^t W_s \, dW_s : 0 \leq t \leq T \right\}$ could be negative at some times



Fundamental theorem of calculus IV

- Recall: is

$$W_t^2 = \int_0^t 2W_s \, dW_s \ ? \tag{7}$$

- But the left side of equality (7) is necessarily non-negative, while the right side can take negative values. There is a contradiction and we conclude that equation (7) is false.

First version I

Itô's lemma

- There exists, in the framework of stochastic calculus, an equivalent of the fundamental theorem of calculus that was established by K. Itô. Note that a term is added.

First version II

Itô's lemma

Theorem

Itô's lemma (first version). *Let W be a Brownian motion built on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function, the first two derivatives of which exist and are continuous. Then $\forall 0 \leq t \leq T$,*

$$f(W_t) - f(W_0) \overset{\mathbb{P}-a.s.}{=} \int_0^t \frac{df}{dw}(W_s)\ dW_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dw^2}(W_s)\ ds. \tag{8}$$

In its differential form, equation (8) is written

$$df(W_t) = \frac{df}{dw}(W_t)\ dW_t + \frac{1}{2}\frac{d^2 f}{dw^2}(W_t)\ dt.$$

First version III

Itô's lemma

For example, if $f(x) = x^2$, then

$$f(W_t) = W_t^2 \text{ and } f(W_0) = W_0^2 = 0$$

and

$$\frac{df}{dw}(W_s) = 2W_s \text{ and } \frac{d^2 f}{dw^2}(W_s) = 2.$$

By substituting into Itô's equation, we obtain

$$W_t^2 \overset{\mathbb{P}-\text{a.s.}}{=} 2\int_0^t W_s \, dW_s + \int_0^t 1 ds = 2\int_0^t W_s \, dW_s + t$$

which implies

$$\int_0^t W_s \, dW_s = \frac{W_t^2 - t}{2}.$$

Idea of the proof I

Itô's lemma

Let's assume a division of time $0 = t_0 < t_1 < ... < t_n = t$.
One can assume that $t_i - t_{i-1} = t/n$.
By expanding f as a Taylor series about point x_0, one finds

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{1}{2} f''(\xi)(x - x_0)^2$$

where $\min(x_0, x) \leq \xi \leq \max(x_0, x)$.
By applying that result to random points
($x = W_{t_i}$ and $x_0 = W_{t_{i-1}}$),

$$f(W_{t_i}) - f(W_{t_{i-1}}) = f'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) + \frac{1}{2} f''(\xi_i)(W_{t_i} - W_{t_{i-1}})^2$$

where $\min(W_{t_{i-1}}, W_{t_i}) \leq \xi_i \leq \max(W_{t_{i-1}}, W_{t_i})$.

Idea of the proof II

Itô's lemma

$$
\begin{aligned}
& f(W_t) - f(W_0) \\
= & \sum_{i=1}^{n} \left(f(W_{t_i}) - f(W_{t_{i-1}}) \right) \\
= & \sum_{i=1}^{n} f'(W_{t_{i-1}}) (W_{t_i} - W_{t_{i-1}}) + \frac{1}{2} \sum_{i=1}^{n} f''(\xi_i) (W_{t_i} - W_{t_{i-1}})^2
\end{aligned}
$$

By taking the limit when $n \to \infty$ on both sides,

$$
\begin{aligned}
f(W_t) - f(W_0) = & \lim_{n\to\infty} \sum_{i=1}^{n} f'(W_{t_{i-1}}) (W_{t_i} - W_{t_{i-1}}) \\
& + \frac{1}{2} \lim_{n\to\infty} \sum_{i=1}^{n} f''(\xi_i) (W_{t_i} - W_{t_{i-1}})^2 .
\end{aligned}
$$

Idea of the proof III

Itô's lemma

The objective is to show that

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} f'\left(W_{t_{i-1}}\right)\left(W_{t_i} - W_{t_{i-1}}\right) &= \int_0^t f'\left(W_s\right) dW_s \\ \text{et} \quad \lim_{n\to\infty} \sum_{i=1}^{n} f''\left(\xi_i\right)\left(W_{t_i} - W_{t_{i-1}}\right)^2 &= \int_0^t f''\left(W_s\right) ds. \end{aligned}$$

Idea of the proof IV

Itô's lemma

First,

$$\int_0^t f'(W_s)\,dW_s = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f'(W_s)\,dW_s$$

which implies that

$$\begin{aligned}
& \int_0^t f'(W_s)\,dW_s - \sum_{i=1}^n f'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \\
= & \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f'(W_s)\,dW_s - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f'(W_{t_{i-1}})\,dW_t \\
= & \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(f'(W_s) - f'(W_{t_{i-1}})\right) dW_s \\
& \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Idea of the proof V

Itô's lemma

Second,

$$
\begin{aligned}
& \int_0^t f''(W_s)\,ds - \sum_{i=1}^n f''(\xi_i)\,(W_{t_i} - W_{t_{i-1}})^2 \\
= & \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f''(W_s)\,ds - \sum_{i=1}^n f''(\xi_i)\,(W_{t_i} - W_{t_{i-1}})^2 \\
= & \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(f''(W_s) - f''(\xi_i)\right) ds \\
& + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f''(\xi_i)\,ds - \sum_{i=1}^n f''(\xi_i)\,(W_{t_i} - W_{t_{i-1}})^2 \\
= & \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(f''(W_s) - f''(\xi_i)\right) ds \\
& - \sum_{i=1}^n f''(\xi_i)\left((W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right)
\end{aligned}
$$

Idea of the proof VI

Itô's lemma

- The first term $\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(f''(W_s) - f''(\xi_i)\right) ds$ converges to zero since $\min\left(W_{t_{i-1}}, W_{t_i}\right) \leq \xi_i \leq \max\left(W_{t_{i-1}}, W_{t_i}\right)$ implies that $\xi_i - W_s \longrightarrow 0$. f'' being continuous, we deduce that $f''(\xi_i) - f''(W_s) \longrightarrow 0$.
- The second term $\sum_{i=1}^{n} f''(\xi_i) \left(\left(W_{t_i} - W_{t_{i-1}}\right)^2 - \left(t_i - t_{i-1}\right)\right)$ tends also to 0 since

$$\begin{aligned} \mathrm{E}\left[\left(W_{t_i} - W_{t_{i-1}}\right)^2 - \left(t_i - t_{i-1}\right)\right] &= 0 \\ \mathrm{Var}\left[\left(W_{t_i} - W_{t_{i-1}}\right)^2 - \left(t_i - t_{i-1}\right)\right] &= \left(t_i - t_{i-1}\right) \longrightarrow 0. \end{aligned}$$

- Such a proof is only roughly sketched since we have omitted to specify some technical details. (ref. R. Durrett)

Second version I

Itô's lemma

- The proof given by K. Itô applies to much more complex situations that the one described above. Here is a first generalization:

Second version II

Theorem

Itô's lemma (second version). *Let W be a Brownian motion built on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a function, the first and second partial derivatives of which exist and are continuous. Then $\forall 0 \leq t \leq T$,*

$$\begin{aligned} & f(W_t, t) - f(W_0, 0) \\ = & \int_0^t \left(\frac{\partial f}{\partial t}(W_s, s) + \frac{1}{2} \frac{\partial^2 f}{\partial w^2}(W_s, s) \right) ds \\ & + \int_0^t \frac{\partial f}{\partial w}(W_s, s) \, dW_s \end{aligned}$$

In its differential form, we have

$$df(W_t, t) = \left(\frac{\partial f}{\partial t}(W_t, t) + \frac{1}{2} \frac{\partial^2 f}{\partial w^2}(W_t, t) \right) dt + \frac{\partial f}{\partial w}(W_t, t) \, dW_t.$$

Example I

The Black-Scholes model

- The stochastic process $S = \{S_t : 0 \leq t \leq T\}$ represents the price evolution of a risky asset where

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right],$$

  μ and σ being constants, and W represents a standard Brownian motion.
- Since W_t is $N(0, t)$-distributed,

$$\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \text{ follows a distribution } N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2\right.$$

  and S_t is lognormally distributed.

Example II

The Black-Scholes model

An intermission: the moments of a lognormally-distributed random variable

If Z represents a standard normal random variable and if a and b are constants then

$$\mathrm{E}\left[\exp\left[b + aZ\right]\right] = \exp\left[b + \frac{a^2}{2}\right].$$

So, if $S(0)$ is independent from the Brownian motion,

$$\begin{aligned}
\mathrm{E}\left[S(t)\right] &= \mathrm{E}\left[S(0)\right]\mathrm{E}\left[\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]\right] \\
&= \mathrm{E}\left[S(0)\right]\exp\left[\mu t\right]
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[S^2(t)\right] &= \mathrm{E}\left[S^2(0)\right]\mathrm{E}\left[\exp\left[2\left(\mu - \frac{\sigma^2}{2}\right)t + 2\sigma W_t\right]\right] \\
&= \mathrm{E}\left[S^2(0)\right]\exp\left[2\mu t + \sigma^2 t\right]
\end{aligned}$$

$$\mathrm{Var}\left[S(t)\right] = \mathrm{E}\left[S^2(0)\right]\exp\left[2\mu t + \sigma^2 t\right] - \mathrm{E}^2\left[S(0)\right]\exp\left[2\mu t\right].$$

Example III

The Black-Scholes model

- Using Itô's lemma, it is possible to show that the stochastic process S defined as

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]$$

is a solution to the integral equation

$$S_t - S_0 = \mu \int_0^t S_u \, du + \sigma \int_0^t S_u \, dW_u.$$

Example IV

The Black-Scholes model

- Recall that

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right].$$

- Indeed, if $f(w,t) = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma w\right]$, then
  - $f(W_t, t) = S_t$
  - $f(W_0, 0) = S_0$
  - $\frac{\partial f}{\partial t}(w,t) = \left(\mu - \frac{\sigma^2}{2}\right) f(w,t)$
  - $\frac{\partial f}{\partial w}(w,t) = \sigma f(w,t)$
  - $\frac{\partial^2 f}{\partial w^2}(w,t) = \sigma^2 f(w,t)$.

Example V

The Black-Scholes model

Thus,

$$
\begin{aligned}
& S_t - S_0 \\
= \quad & f(W_t, t) - f(W_0, 0) \\
= \quad & \int_0^t \frac{\partial f}{\partial w}(W_u, u)\, dW_u + \int_0^t \left(\frac{\partial f}{\partial t}(W_u, u) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(W_u, u) \right) du \\
= \quad & \int_0^t \sigma f(W_u, u)\, dW_u + \int_0^t \left(\left(\mu - \frac{\sigma^2}{2} \right) f(W_u, u) + \frac{1}{2}\sigma^2 f(W_u, u) \right) du \\
= \quad & \int_0^t \sigma f(W_u, u)\, dW_u + \int_0^t \mu f(W_u, u)\, du \\
= \quad & \int_0^t \sigma S_u dW_u + \int_0^t \mu S_u du.
\end{aligned}
$$

Example VI

The Black-Scholes model

We have just proved that the stochastic process $S = \{S_t : 0 \leq t \leq T\}$ where

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]$$

is a solution to the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Itô process I

Definition

An **Itô process** is defined to be a process $X = \{X_t : 0 \leq t \leq T\}$ taking its values in $\mathbb{R}$ such that:

$$X_t \equiv X_0 + \int_0^t K_s \; ds + \int_0^t H_s \; dW_s \tag{9}$$

where $K = \{K_t : 0 \leq t \leq T\}$ and $H = \{H_t : 0 \leq t \leq T\}$ are processes adapted to the filtration $\{\mathcal{F}_t\}$, satisfying $\mathbb{P}\left[\int_0^T |K_s| \; ds < \infty\right] = 1$ and $\mathbb{P}\left[\int_0^T |H_s|^2 \; ds < \infty\right] = 1$.

Written in its differential form, the Itô process becomes

$$dX_t = K_t \; dt + H_t \; dW_t.$$

Damien Lamberton and Bernard Lapeyre, page 53.

Itô process II

- **Remark**. First, note that, if

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^T |H_s|^2 \; ds\right] < \infty, \tag{10}$$

then the process $M = \{M_t : 0 \leq t \leq T\}$ where

$$M_t = \int_0^t H_s \; dW_s$$

is a $(\{\mathcal{F}_t\}, \mathbb{P})$ −martingale.

- **Remark**. As a consequence, if the drift coefficient is zero, i.e. $K_t = 0$ for all $0 \leq t \leq T$, then the Itô process X where

$$X_t \equiv X_0 + \int_0^t H_s \; dW_s$$

is a $(\{\mathcal{F}_t\}, \mathbb{P})$ −martingale if and only if the condition (10) is satisfied.

Daniel Revuz and Marc Yor, page 129, proposition 1.23.

Quadratic variation I

- **Recall that**

$$X_t \equiv X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s \tag{11}$$

Definition

Let X be the Itô process defined by equation (11), then the **quadratic variation** process of X,

$$\langle X \rangle = \{ \langle X \rangle_t : 0 \leq t \leq T \},$$

is defined to be

$$\langle X \rangle_t = \int_0^t H_s^2 \, ds.$$

Quadratic variation II

- **Example**. The Brownian motion is an Itô process since

$$W_t = \int_0^t 1 dW_s.$$

As a consequence,

$$\langle W \rangle_t = \int_0^t 1^2 ds = t.$$

Third version I

Itô's lemma

Theorem

Itô's lemma (third version). *Let X be an Itô process and let $f : \mathbb{R} \to \mathbb{R}$ be a function, the first two derivatives of which exist and are continuous. Then $\forall 0 \leq t \leq T$,*

$$
\begin{aligned}
& f(X_t) - f(X_0) \qquad (12) \\
& \overset{\mathbb{P}-a.s.}{=} \int_0^t \frac{df}{dx}(X_s)\ dX_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dx^2}(X_s)\ d\langle X\rangle_s \\
& \overset{\mathbb{P}-a.s.}{=} \int_0^t \frac{df}{dx}(X_s)\, H_s\ dW_s \\
& + \int_0^t \left(\frac{df}{dx}(X_s)\ K_s + \frac{1}{2}\frac{d^2 f}{dx^2}(X_s)\ H_s^2\right)\ ds,
\end{aligned}
$$

Third version II

Itô's lemma

In its differential form, we find

$$df\left(X_t\right) = \frac{df}{dx}\left(X_t\right) H_t \; dW_t + \left(\frac{df}{dx}\left(X_t\right) \; K_t + \frac{1}{2}\frac{d^2 f}{dx^2}\left(X_t\right) \; H_t^2\right) \; dt.$$

Third version III

Itô's lemma

Justification attempt!

Itô's lemma for Itô processes is

$$f(X_t) - f(X_0) \overset{\mathbb{P}-\text{a.s.}}{=} \int_0^t \frac{df}{dx}(X_s)\ dX_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dx^2}(X_s)\ d\langle X\rangle_s .$$

We recognize equation (8) as a particular case of the equality above since $\langle W\rangle_t = t$.

Third version IV

Itô's lemma

But

$$
\begin{aligned}
& f(X_t) - f(X_0) \\
= & \int_0^t \frac{df}{dx}(X_s)\ dX_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dx^2}(X_s)\ d\langle X\rangle_s \\
= & \int_0^t \frac{df}{dx}(X_s)\ (K_s\ ds + H_s\ dW_s) + \frac{1}{2}\int_0^t \frac{d^2 f}{dx^2}(X_s)\ \left(H_s^2 ds\right) \\
= & \int_0^t \frac{df}{dx}(X_s)\ K_s\ ds + \int_0^t \frac{df}{dx}(X_s)\ H_s\ dW_s + \frac{1}{2}\int_0^t \frac{d^2 f}{dx^2}(X_s)\ H_s^2\ ds \\
= & \int_0^t \frac{df}{dx}(X_s)\ H_s\ dW_s + \int_0^t \left(\frac{df}{dx}(X_s)\ K_s + \frac{1}{2}\frac{d^2 f}{dx^2}(X_s)\ H_s^2\right)\ ds. \blacksquare
\end{aligned}
$$

Third version V

- The preceding calculation is valid provided the following exercise has been verified:
- **Exercise**. If

$$Y_t = \int_0^t K_s \, ds + \int_0^t H_s \, dW_s$$

where W is a $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ −Brownian motion, then

$$\int_0^t X_s \, dY_s = \int_0^t X_s K_s \, ds + \int_0^t X_s H_s \, dW_s$$

and

$$\int_0^t X_s \, d\langle Y \rangle_s = \int_0^t X_s H_s^2 \, ds$$

where X is a predictable process.

Example I

Return

- Let's assume that the evolution $S = \{S_t : 0 \leq t \leq T\}$ of a risky asset satisfies the stochastic differential equation

$$dS_t = \mu S_t \; dt + \sigma S_t \; dW_t.$$

- The return R_t on such a risky asset, when accumulation is continuous, is defined as

$$R_t = \ln \frac{S_t}{S_0}.$$

- Using Itô's lemma, we can show that R satisfies the stochastic differential equation

$$dR_t = \left(\mu - \frac{\sigma^2}{2}\right) \; dt + \sigma dW_t. \tag{13}$$

Example II

Return

- **Intermission: Fubini's theorem.** What follows is a particular case of Fubini's theorem.

Theorem

If $Y = \{Y_t : 0 \leq t \leq T\}$ is an adapted stochastic process such that, for all $0 \leq t \leq T$, $Y_t \geq 0$ then

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^t Y_s \, ds\right] = \int_0^t \mathrm{E}^{\mathbb{P}}\left[Y_s\right] \, ds,$$

i.e. one can exchange integral and expectation.

Donald L.Cohn, page 159, proposition 5.2.1.

Example III

Return

- Recall that

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t. \tag{14}$$

First, let's verify that S truly is an Itô process such that $H_t = \sigma S_t$ and $K_t = \mu S_t$.

- Indeed, since we have already shown that

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right] \quad (\mathcal{F}_t - \text{measurable})$$

is a solution to the equation (14), the processes $K = \{K_t = \mu S_t : 0 \leq t \leq T\}$ and $H = \{H_t = \sigma S_t : 0 \leq t \leq T\}$ are $\{\mathcal{F}_t\}$ −adapted.

Example IV

Return

- Moreover, since S_t is a non-negative random variable, we can use Fubini's theorem and exchangethe $\mathrm{E}^{\mathbb{P}}$ and $\int$ operators, which yields

$$
\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[\int_0^T |K_u|\, du\right] &= |\mu|\, \mathrm{E}^{\mathbb{P}}\left[\int_0^T S_u du\right] \\
&= |\mu| \int_0^T \mathrm{E}^{\mathbb{P}}\left[S_u\right] du \\
&= |\mu| \int_0^T \mathrm{E}^{\mathbb{P}}\left[S_0\right] \exp\left[\mu u\right] du \\
&= |\mu|\, \mathrm{E}^{\mathbb{P}}\left[S_0\right] \int_0^T \exp\left[\mu u\right] du \\
&= |\mu|\, \mathrm{E}^{\mathbb{P}}\left[S_0\right] \frac{e^{T\mu} - 1}{\mu} < \infty
\end{aligned}
$$

Example V

Return

$$
\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[\int_0^T H_u^2 du\right] &= \sigma^2 \mathrm{E}^{\mathbb{P}}\left[\int_0^T S_u^2 du\right] \\
&= \sigma^2 \int_0^T \mathrm{E}^{\mathbb{P}}\left[S_u^2\right] du \\
&= \sigma^2 \int_0^T \mathrm{E}^{\mathbb{P}}\left[S_0^2\right] \exp\left[2\mu u + \sigma^2 u\right] du \\
&= \sigma^2 \mathrm{E}^{\mathbb{P}}\left[S_0^2\right] \int_0^T \exp\left[2\mu u + \sigma^2 u\right] du \\
&= \sigma^2 \mathrm{E}^{\mathbb{P}}\left[S_0^2\right] \frac{e^{T\left(2\mu+\sigma^2\right)} - 1}{2\mu + \sigma^2} \\
&< \infty.
\end{aligned}
$$

Example VI

Return

- Hence, we can deduce that

$$\int_0^T |K_s| \; ds < \infty \text{ and } \int_0^T H_s^2 \; ds < \infty \; \mathbb{P} - \text{a.s.}$$

Example VII

Return

- Now, let's apply Itô's lemma. If $f(x) = \ln \frac{x}{x_0}$ then

$$\frac{df}{dx}(x) = \frac{1}{x} \text{ and } \frac{d^2 f}{dx^2}(x) = -\frac{1}{x^2}.$$

Thus,

$$\begin{aligned}
& dR_t \\
= \; & df(S_t) \\
= \; & \frac{df}{dx}(S_t)\ H_t\ dW_t + \left(\frac{df}{dx}(S_t)\ K_t + \frac{1}{2}\frac{d^2 f}{dx^2}(S_t)\ H_t^2\right)\ dt \\
= \; & \frac{1}{S_t}\sigma S_t\ dW_t + \left(\frac{1}{S_t}\mu S_t - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2\right)\ dt \\
= \; & \sigma\ dW_t + \left(\mu - \frac{\sigma^2}{2}\right)\ dt.
\end{aligned}$$

Example VIII

Return

- But the integral form of equation (13)

$$dR_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t$$

is

$$\begin{aligned} R_t &= R_0 + \sigma \int_0^t dW_s + \left(\mu - \frac{\sigma^2}{2}\right) \int_0^t ds \\ &= \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t \text{ car } R_0 = \ln \frac{S_0}{S_0} = \ln 1 = 0 \\ &\sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) t; \sigma^2 t\right). \end{aligned}$$

In the Black-Scholes world, returns are Gaussian.

Fourth version I

Itô's lemma

Theorem

Itô's lemma (fourth version). *Let X be an Itô process, i.e. $dX_t = K_t\ dt + H_t\ dW_t$, and let $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a function, the first and second partial derivatives of which exist and are continuous. Then $\forall 0 \leq t \leq T$,*

$$\begin{aligned} & f(X_t, t) - f(X_0, 0) \\ & \overset{\mathbb{P}-a.s.}{=} \int_0^t \frac{\partial f}{\partial x}(X_s, s)\ dX_s + \int_0^t \frac{\partial f}{\partial t}(X_s, s)\ ds \\ & + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(X_s, s)\ d\langle X \rangle_s \end{aligned}$$

Fourth version II

Itô's lemma

We can also write

$$
\begin{aligned}
& f\left(X_t, t\right) - f\left(X_0, 0\right) \\
& \overset{\mathbb{P}-\text{a.s.}}{=} \int_0^t \frac{\partial f}{\partial x}\left(X_s, s\right) \; H_s \; dW_s \\
& + \int_0^t \left(\frac{\partial f}{\partial x}\left(X_s, s\right) \; K_s + \frac{\partial f}{\partial t}\left(X_s, s\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(X_s, s\right) \; H_s^2 \right) \; ds.
\end{aligned}
$$

Fourth version III

Itô's lemma

In the differential form, we obtain

$$
\begin{aligned}
& df\left(X_t, t\right) \\
= \quad & \frac{\partial f}{\partial X}\left(X_t, t\right) \; H_t \; dW_t \\
& + \left(\frac{\partial f}{\partial X}\left(X_t, t\right) \; K_t + \frac{\partial f}{\partial t}\left(X_t, t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\left(X_t, t\right) \; H_t^2\right) \; dt.
\end{aligned}
$$

Relation with Taylor series I

- The Taylor series expansion of the continuous function $f : \mathbb{R} \to \mathbb{R}$ (the first derivatives of which exist and are continuous) about point x_0 is

$$f(x) \cong f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2}$$

where $\cong$ means that it is an approximation.

- If the continuous function $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, then Itô's lemma is stated as follows:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X \rangle_s$$

where X is an Itô process.

Relation with Taylor series II

- If $f'(X_s)$ and $f''(X_s)$ could be respectively replaced with $f'(X_0)$ and $f''(X_0)$ (that would be a good approximation if t is small), then we could write

$$
\begin{aligned}
& f(X_t) \\
= \; & f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s \\
\cong \; & f(X_0) + \int_0^t f'(X_0)\, dX_s + \frac{1}{2}\int_0^t f''(X_0)\, d\langle X\rangle_s \\
= \; & f(X_0) + f'(X_0)\int_0^t dX_s + \frac{1}{2} f''(X_0)\int_0^t d\langle X\rangle_s \\
= \; & f(X_0) + f'(X_0)(X_t - X_0) + \frac{1}{2} f''(X_0)\left(\langle X\rangle_t - \langle X\rangle_0\right).
\end{aligned}
$$

which looks more like the Taylor series expansion. But the latter approximation is not Itô's lemma.

Solution to an SDE I

Itô's lemma

- The solution to a stochastic differential equation (SDE) consists in finding the stochastic process(es) satisfying thegiven equation.
- But Itô's lemma allows to do the opposite, i.e. it helps find the SDE satisfied by a given stochastic process. If it turns out that the stochastic process chosen satisfies the given SDE, then we will have found a solution to that equation. In that sense, Itô's lemma provides us with solutions to SDEs.
- However, it is not a systematic way to obtain solutions. It requires educated guesses!

Quadratic covariation

Definition

Let X and Y be two Itô process such that

$$dX_t = K_t\ dt + H_t\ dW_t \text{ and } dY_t = \widetilde{K}_t\ dt + \widetilde{H}_t\ dW_t$$

then the **quadratic covariation** process $\langle X, Y \rangle$ is defined for all $t \in [0, T]$ as

$$\langle X, Y \rangle_t = \int_0^t H_s \widetilde{H}_s ds.$$

Multiplication rule I

- The multiplication rule is useful when we want to study, for example, the behavior of the present value of an asset while knowing the processes for both the asset price evolution and the discount factor.

Multiplication rule II

Theorem

The multiplication rule. *Let X and Y be two Itô processes such that*

$$dX_t = K_t\ dt + H_t\ dW_t \text{ and } dY_t = \widetilde{K}_t\ dt + \widetilde{H}_t\ dW_t.$$

The multiplication rule is

$$\begin{aligned} & dX_t Y_t \\ = \ & X_t\ dY_t + Y_t\ dX_t + d\left\langle X, Y\right\rangle_t \\ = \ & X_t\left(\widetilde{K}_t\ dt + \widetilde{H}_t\ dW_t\right) + Y_t\left(K_t\ dt + H_t\ dW_t\right) + H_t\widetilde{H}_t dt \\ = \ & \left(X_t\widetilde{K}_t + Y_t K_t + H_t\widetilde{H}_t\right) dt + \left(X_t\widetilde{H}_t + Y_t H_t\right) dW_t. \end{aligned}$$

Example I

Present value of an asset

- Let's assume that the stochastic process X, satisfying the equation
$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t,$$
is the price evolution of a risky asset.
- The process $\{\beta_t = e^{-rt} : t \geq 0\}$ is the discount factor. Note that
$$\frac{d}{dt}\beta_t = -re^{-rt} = -r\beta_t,$$
i.e. the process β is an Itô process satisfying the equation differential
$$d\beta_t = -r\beta_t \, dt.$$

Example II

Present value of an asset

- The process $Y = \beta X$ represents the evolution of the present value of the asset. The multiplication rule yields that

$$\begin{aligned} dY_t &= d\beta_t X_t \\ &= \beta_t \, dX_t + X_t \, d\beta_t + d\langle \beta, X \rangle_t \\ &= \beta_t \, (\mu X_t \, dt + \sigma X_t \, dW_t) + X_t \, (-r\beta_t dt) \\ &= (\mu - r)\, \beta_t X_t \, dt + \sigma \beta_t X_t \, dW_t \\ &= (\mu - r)\, Y_t \, dt + \sigma Y_t dW_t. \end{aligned}$$

- In its integral form, the latter equation is written

$$Y_t = Y_0 + (\mu - r) \int_0^t Y_s \, ds + \sigma \int_0^t Y_s \, dW_s. \qquad (15)$$

Introduction

Multidimensional Itô's lemma

The phenomenon we would like to model may contain several sources of uncertainty. That is why we consider Itô-type processes which involve several Brownian motions.

Itô process I

Multidimensional Itô's lemma

- Let $W^{(1)}, ..., W^{(n)}$ be independent Brownian motions built on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})$.
- Let $K^{(i)}$, $i \in \{1, 2, ..., m\}$ and $H^{(i,k)}$, where $i \in \{1, 2, ..., m\}$ and $k \in \{1, 2, ..., n\}$, be $\{\mathcal{F}_t\}$ −adapted processes.
  - $\mathbb{P}\left[\sum_{i=1}^{m} \int_0^T \left|K_s^{(i)}\right| ds < \infty\right] = 1$,
  - $\mathbb{P}\left[\int_0^T H_s^{(i,k)} H_s^{(j,k^*)} ds < \infty\right] = 1$, $i, j \in \{1, 2, ..., m\}$ and $k, k^* \in \{1, 2, ..., n\}$.

Itô process II

Multidimensional Itô's lemma

Definition

For all $i \in \{1, 2, ..., m\}$, we define the process $X^{(i)}$ as

$$X_t^{(i)} = X_0^{(i)} + \int_0^t K_s^{(i)} ds + \sum_{k=1}^{n} \int_0^t H_s^{(i,k)} dW_s^{(k)} \tag{16}$$

where $X_0^{(1)}, ..., X_0^{(m)}$ are $\mathcal{F}_0-$measurable random variables.

Quadratic covariation I

Multidimensional Itô's lemma

Definition

The **quadratic covariation** process $\left\langle X^{(i)}, X^{(j)} \right\rangle = \left\{ \left\langle X^{(i)}, X^{(j)} \right\rangle_t : 0 \leq t \leq T \right\}$ defined as

$$\left\langle X^{(i)}, X^{(j)} \right\rangle_t = \sum_{k=1}^{n} \int_0^t H_s^{(i,k)} H_s^{(j,k)} ds.$$

Note that

$$\left\langle X^{(i)}, X^{(j)} \right\rangle = \left\langle X^{(j)}, X^{(i)} \right\rangle$$

and

$$\left\langle X^{(i)}, X^{(i)} \right\rangle = \left\langle X^{(i)} \right\rangle.$$

Quadratic covariation II

Multidimensional Itô's lemma

- **Example.** We want to calculate the quadratic covariation of two correlated Brownian motions.
- Let's set B_1 and B_2, two Brownian motions such that $\mathrm{Corr}[B_1(t), B_2(t)] = \rho$ for all $t > 0$.
- it is possible to write $B_1(t) = W_1(t)$ and $B_2(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t)$ where W_1 and W_2 are independent Brownian motions.
- Since $dB_1(t) = 1dW_1(t) + 0dW_2(t)$ and $dB_2(t) = \rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)$.
- Then $\langle B_1, B_2 \rangle_t = \int_0^t 1\rho ds + \int_0^t 0\sqrt{1-\rho^2} ds = \rho t$

Itô's lemma I

Multidimensional Itô's lemma

- *Let* $W^{(1)}, ..., W^{(n)}$ *be independent Brownian motions built on the filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})$ *in the Itô processes defined in equation (16).*

Itô's lemma II

Multidimensional Itô's lemma

Theorem

Let $f : \mathbb{R}^m \times [0, \infty) \to \mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial s}, \frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_i \partial s}$, $i, j \in \{1, ..., m\}$ exist and are continuous. Then $\forall t \geq 0$,

$$
\begin{aligned}
& f\left(X_t^{(1)}, ..., X_t^{(m)}, t\right) - f\left(X_0^{(1)}, ..., X_0^{(m)}, 0\right) \\
& \overset{\mathbb{P}-a.s.}{=} \sum_{i=1}^{m} \int_0^t \frac{\partial f}{\partial x_i}\left(X_s^{(1)}, ..., X_s^{(m)}, s\right) dX_s^{(i)} \\
& + \int_0^t \frac{\partial f}{\partial t}\left(X_s^{(1)}, ..., X_s^{(m)}, s\right) ds \\
& + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}\left(X_s^{(1)}, ..., X_s^{(m)}, s\right) d\left\langle X^{(i)}, X^{(j)} \right\rangle_s .
\end{aligned}
$$

Example I

Exchange rate

- Assume that the stochastic process X, satisfying equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

represents the price evolution of a stock in British pounds.

- Note that $d\langle X\rangle_t = \sigma^2 X_t^2 dt$.
- Assume also that the Canadian dollar-value evolution of the British Pound is modeled by a process C where

$$dC_t = \alpha C_t dt + \beta C_t d\widetilde{W}_t + \gamma C_t dW_t.$$

- If the Brownian motions W and $\widetilde{W}$ are built on the same filtered probability space and are independent, then

$$\begin{aligned} d\langle C\rangle_t &= \left(\beta^2 C_t^2 + \gamma^2 C_t^2\right) dt = \left(\beta^2 + \gamma^2\right) C_t^2 dt \\ d\langle X, C\rangle_t &= \left(\sigma X_t \gamma C_t + 0\beta C_t\right) dt = \sigma\gamma X_t C_t dt. \end{aligned}$$

Example II

Exchange rate

- The price evolution Y of the stock in Canadian dollars is such that, at any time,

$$Y_t = X_t C_t.$$

Example III

Exchange rate

- Let a function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, c) = xc.$$

  - Note that $Y_t = f(X_t, C_t)$, $\forall t \geq 0$,
  - that the first derivatives are $\frac{\partial f}{\partial x}(x, c) = c$ and $\frac{\partial f}{\partial c}(x, c) = x$
  - and that the second derivatives are

$$\frac{\partial^2 f}{\partial x^2}(x, c) = \frac{\partial^2 f}{\partial c^2}(x, c) = 0 \text{ and } \frac{\partial^2 f}{\partial x \partial c}(x, c) = 1.$$

Example IV

Exchange rate

Itô's lemma yields

$$
\begin{aligned}
& dY_t = df\left(X_t, C_t\right) \\
= \; & \frac{\partial f}{\partial x}\left(X_t, C_t\right) dX_t + \frac{\partial f}{\partial c}\left(X_t, C_t\right) dC_t \\
& + \frac{1}{2}\left(\begin{array}{c} \frac{\partial^2 f}{\partial x^2}\left(X_t, C_t\right) d\left\langle X\right\rangle_t + \frac{\partial^2 f}{\partial x \partial c}\left(X_t, C_t\right) d\left\langle X, C\right\rangle_t \\ + \frac{\partial^2 f}{\partial c \partial x}\left(C_t, X_t\right) d\left\langle C, X\right\rangle_t + \frac{\partial^2 f}{\partial c^2}\left(X_t, C_t\right) d\left\langle C\right\rangle_t \end{array}\right) \\
= \; & C_t dX_t + X_t dC_t + d\left\langle X, C\right\rangle_t \\
= \; & C_t\left(\mu X_t dt + \sigma X_t dW_t\right) + X_t\left(\alpha C_t dt + \beta C_t d\widetilde{W}_t + \gamma C_t dW_t\right) \\
& + \sigma\gamma X_t C_t dt \\
= \; & \left(\mu X_t C_t + \alpha X_t C_t + \sigma\gamma X_t C_t\right) dt \\
& + \left(\sigma X_t C_t + \gamma C_t X_t\right) dW_t + \beta X_t C_t d\widetilde{W}_t \\
= \; & \left(\mu + \alpha + \sigma\gamma\right) Y_t dt + \left(\sigma + \gamma\right) Y_t dW_t + \beta Y_t d\widetilde{W}_t.
\end{aligned}
$$

What is a solution? I

- There exist two types of solutions to a stochastic differential equation: strong solutions and weak solutions.
- Consider the stochastic differential equation

$$dX_t = \mu\left(X_t\right)\ dt + \sigma\left(X_t\right)\ dW_t$$

where $\{W_t : t \geq 0\}$ is a $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ −standard Brownian motion.

Definition

A strong solution consists of finding a stochastic process X existing on the same filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ as the Brownian motion and satisfying the equation

$$X_t = X_0 + \int_0^t \mu\left(X_s\right)\ ds + \int_0^t \sigma\left(X_s\right)\ dW_s.$$

What is a solution? II

- The strong solution is said to be unique if, when X and $\widetilde{X}$ represent two strong solutions to the same stochastic differential equation, we have

$$\mathbb{P}\left[X_t = \widetilde{X}_t,\ t \geq 0\right] = 1.$$

What is a solution? III

Definition

We have a weak solution if we can construct:

1) filtered probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \left\{\widetilde{\mathcal{F}}_t : t \geq 0\right\}, \widetilde{\mathbb{P}}\right)$,

2) a $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \left\{\widetilde{\mathcal{F}}_t : t \geq 0\right\}, \widetilde{\mathbb{P}}\right)$ −standard Brownian motion $\left\{\widetilde{W}_t : t \geq 0\right\}$ and

3) a $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \left\{\widetilde{\mathcal{F}}_t : t \geq 0\right\}, \widetilde{\mathbb{P}}\right)$ −stochastic process $\left\{\widetilde{X}_t : t \geq 0\right\}$ such that

$$\widetilde{X}_t = \widetilde{X}_0 + \int_0^t \mu\left(\widetilde{X}_s\right)\, ds + \int_0^t \sigma\left(\widetilde{X}_s\right)\, d\widetilde{W}_s$$

- It is possible to have several strong solutions to a same equation.

What is a solution? IV

- If the strong solution is unique, then there will also be a unique weak solution. By contrast, if the weak solution is unique, there may be several strong solutions.

Gaussian models I

Definition

Some stochastic processes that give rise to **Gaussian models** are of the form

$$dX_t = \left(\mu_1(t) X_t + \mu_2(t)\right) \; dt + \sigma(t) \; dW_t \tag{17}$$

where $\mu_1(\bullet)$, $\mu_2(\bullet)$ and $\sigma(\bullet)$ deterministic functions of time and $\{W_t : 0 \leq t \leq T\}$ is a $(\{\mathcal{F}_t\}, \mathbb{P})$ −standard Brownian motion.

Gaussian models II

Theorem

If $\mathbb{P}\left[\int_0^T \frac{\sigma^2(s)}{\Phi^2(s)}\, ds < \infty\right] = 1$, *then the strong solution to the stochastic differential equation (17) is*

$$X_t = \Phi(t)\left[X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)}\, ds + \int_0^t \frac{\sigma(s)}{\Phi(s)}\, dW_s\right]$$

where the function $\Phi(\bullet)$ *is the solution to the ordinary differential equation*

$$d\Phi(t) = \mu_1(t)\,\Phi(t)\; dt, \qquad \Phi(0) = 1.$$

Bisière, page 117.

Gaussian models III

- **Proof**. Let's set

$$Y_t \equiv \int_0^t \frac{\sigma(s)}{\Phi(s)}\, dW_s,\ 0 \leq t \leq T. \tag{18}$$

- The process Y is an Itô process if

$$\mathbb{P}\left[\int_0^T \frac{\sigma^2(s)}{\Phi^2(s)}\, ds < \infty\right] = 1. \tag{19}$$

- Note that $dY_t = \frac{\sigma(t)}{\Phi(t)}\, dW_t$ and $d\langle Y\rangle_t = \frac{\sigma^2(t)}{\Phi^2(t)}\, dt$.

Gaussian models IV

- Let's set $g(t, y) = \Phi(t)\left[X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)}\, ds + y\right]$.
- Then $g(t, Y_t) = X_t$ and $g(0, Y_0) = \underbrace{\Phi(0)}_{=1} X_0 = X_0$.

Gaussian models V

- Recall that

$$g(t,y) = \Phi(t)\left[X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)}\,ds + y\right].$$

- Moreover,

$$\begin{aligned}
&\frac{\partial g}{\partial t}(t,y) \\
= \quad & \underbrace{\frac{\partial \Phi}{\partial t}(t)}_{=\mu_1(t)\Phi(t)} \underbrace{\left[X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)}\,ds + y\right]}_{=\frac{g(t,y)}{\Phi(t)}} + \Phi(t)\frac{\mu_2(t)}{\Phi(t)} \\
= \quad & \mu_1(t)\,g(t,y) + \mu_2(t),
\end{aligned}$$

$\frac{\partial g}{\partial y}(t,y) = \Phi(t)$, and $\frac{\partial^2 g}{\partial y^2}(t,y) = 0.$

Gaussian models VI

Using Itô's lemma, we obtain

$$
\begin{aligned}
& dX_t \\
= \; & dg\left(t, Y_t\right) \\
= \; & \frac{\partial g}{\partial t}\left(t, Y_t\right)\ dt + \frac{\partial g}{\partial y}\left(t, Y_t\right)\ dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}\left(t, Y_t\right)\ d\left\langle Y\right\rangle_t \\
= \; & \left(\mu_1\left(t\right) g\left(t, Y_t\right) + \mu_2\left(t\right)\right)\ dt + \Phi\left(t\right) \underbrace{dY_t}_{=\frac{\sigma(t)}{\Phi(t)}\ dW_t} \\
= \; & \left(\mu_1\left(t\right) g\left(t, Y_t\right) + \mu_2\left(t\right)\right)\ dt + \sigma\left(t\right)\ dW_t. \ \blacksquare
\end{aligned}
$$

Gaussian models VII

- **Why Gaussian?**
- It is possible to show that the joint distribution of $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ is the multivariate normal law.

Ornstein-Uhlenbeck Process I

Gaussian Models

- **Vasicek (1977)**
- The spot interest rate satisfies

$$dr_t = \kappa\left(\theta - r_t\right)\ dt + \sigma\ dW_t. \tag{20}$$

- Referring to the general form,

$$dX_t = \left(\mu_1\left(t\right)X_t + \mu_2\left(t\right)\right)\ dt + \sigma\left(t\right)\ dW_t,$$

  we conclude that $\mu_1\left(t\right) = -\kappa,\ \mu_2\left(t\right) = \kappa\theta$ et $\sigma\left(t\right) = \sigma$.
- Bisière, pages 115-117.

Ornstein-Uhlenbeck Process II

Gaussian Models

- Since the ordinary differential equation

$$d\Phi(t) = \mu_1(t)\Phi(t)\ dt, \qquad \Phi(0) = 1$$

becomes

$$d\Phi(t) = -\kappa\Phi(t)\ dt, \qquad \Phi(0) = 1,$$

the function $\Phi(\bullet)$ is

$$\Phi(t) = \exp(-\kappa t),\ 0 \leq t \leq T.$$

Ornstein-Uhlenbeck Process III

Gaussian Models

- The process Y where

$$Y_t = \int_0^t \frac{\sigma(s)}{\Phi(s)}\, dW_s \text{ becomes } Y_t = \sigma \int_0^t \exp(\kappa s)\, dW_s$$

Which is an Itô's process because

$$\begin{aligned}
\mathbb{P}\left[\int_0^T \frac{\sigma^2(s)}{\Phi^2(s)}\, ds < \infty\right] &= \mathbb{P}\left[\sigma^2 \int_0^T \exp(2\kappa s)\, ds < \infty\right] \\
&= \mathbb{P}\left[\frac{\sigma^2}{2\kappa}\left(e^{2T\kappa} - 1\right) < \infty\right] \\
&= 1.
\end{aligned}$$

Ornstein-Uhlenbeck Process IV

Gaussian Models

The general solution

$$X_t = \Phi(t)\left[X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)}\,ds + \int_0^t \frac{\sigma(s)}{\Phi(s)}\,dW_s\right]$$

is

$$\begin{aligned} r_t &= \exp(-\kappa t)\left[r_0 + \underbrace{\int_0^t \kappa\theta\exp(\kappa s)\,ds}_{=\theta(\exp(\kappa t)-1)} + \int_0^t \sigma\exp(\kappa s)\,dW_s\right] \\ &= r_0\exp(-\kappa t) + \theta(1-\exp(-\kappa t)) + \int_0^t \sigma\exp(-\kappa(t-s))\,dW_s \\ &= \theta + (r_0-\theta)\exp(-\kappa t) + \int_0^t \sigma\exp(-\kappa(t-s))\,dW_s. \end{aligned}$$

Affine Models I

- A model is said to belong to the exponential affine class if there is determinisic functions $a(t, T)$ and $b(t, T)$ for which the time t value of a zero-coupon bond paying 1 dollar at maturity T is

$$P(t, T) = \exp\left(a(t, T)\, r(t) + b(t, T)\right).$$

- In that case, the riskless interest rate as the form

$$dr_t = \left(\mu_1(t)\, r_t + \mu_2(t)\right)\, dt + \sqrt{\sigma_1(t)\, r_t + \sigma_2(t)}\, dW_t$$

  where $\mu_1(\bullet)$, $\mu_2(\bullet)$, $\sigma_1(\bullet)$ and $\sigma_2(\bullet)$ are deterministic function of time.

- Bisière, pages 116-117 et 128-129.

Affine Models II

- The Gaussian models belong the the exponential-affine classe of model..
- **Square-root Processus (Feller's branching diffusion)** Cox-Ingersoll-Ross (1985), generalised by Hull-White (1990)

$$dr_t = \kappa\left(\theta - r_t\right)\ dt + \sigma\sqrt{r_t}\ dW_t.$$

Bisière, pages 116-117.

Lipschitz Coefficients

Theorem

Consider $dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$. *If* $b(\bullet, \bullet)$ *and* $\sigma(\bullet, \bullet)$ *are continuous functions such that there is a constant* $K < \infty$, *for which*

$$\begin{aligned} |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| &\leq K|x-y| \\ |b(t,x)| + |\sigma(t,x)| &\leq K(1+|x|) \\ \mathrm{E}\left[X_0^2\right] &< \infty \end{aligned}$$

then, for all $T \geq 0$, *the equation (EDS) admits a unique solution in the interval* $[0, T]$. *Moreover, this solution verifies*

$$\mathrm{E}\left[\sup_{0 \leq t \leq T} |X_t|^2\right] < \infty.$$

There are sufficent but not nessary conditions. (Lamberton et Lapeyre, p. 59-60.)

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