Stochastic differential equation (SDE) and Ito’s lemma

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Stochastic Calculus I

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When we model some situations, we don’t know a priori which function to use, since we only know the local behavior of our system.

For example, assume that \( f(t) \) represents a commodity price at time \( t \). We write

\[
f(t + \Delta t) - f(t) = \mu \Delta t f(t) \quad \text{where} \quad \mu \geq 0\]

in order to mean that the variation \( f(t + \Delta t) - f(t) \) of the commodity price over a time period is is proportional to the length \( \Delta t \) of the time period considered as well as the commodity price \( f(t) \) at the start of the period, i.e. \( \mu \Delta t f(t) \), \( \mu \) being a constant.
Ordinary differential eq. II

By dividing both sides of the equality by $\Delta t$, we obtain

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = \mu f(t).$$

Let’s now take the limit when $\Delta t$ tends to zero:

$$\frac{d}{dt} f(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \mu f(t) = \mu f(t).$$
Ordinary differential eq. III

- Recall that we consider the equation

\[ \frac{df(t)}{dt} = \mu f(t). \]

- The notation commonly used for differential equations allows us to rewrite the equation above as:

\[ df(t) = \mu f(t) \, dt. \] (1)

- Note that, technically speaking, the object \( df(t) \) is not well-defined. The latter equation is only a notation to express that "the derivative of the function is proportional to the function itself", i.e. \( \frac{df(t)}{dt} = \mu f(t) \).

- The unknown, in that equation, is the function \( f \). We are looking for the functions that satisfy that equality.
Ordinary differential eq. IV

- Recall that we are studying the equation

\[ df(t) = \mu f(t) \, dt. \]  \hspace{1cm} (1)

- It is possible to show that the function defined for all \( t \in \mathbb{R} \) as

\[ f(t) = ce^{\mu t}, \text{ where } c \text{ is any constant}, \]  \hspace{1cm} (2)

satisfies equation (1).

- Indeed, in such a case,

\[
\frac{df(t)}{dt} = \frac{d}{dt} ce^{\mu t} = \mu ce^{\mu t} = \mu f(t).
\]
The initial condition helps determine the constant $c$. We know commodity price $f_0$ today. As a consequence,

$$f_0 = f(0) = ce^{\mu \times 0} = c,$$

which yields that the commodity price at time $t$ is

$$f(t) = f_0 e^{\mu t}.$$

In this example, knowing the infinitesimal behavior of the commodity price ($d f(t) = \mu f(t) \, dt$) and the initial price $f_0$ is sufficient to determine accurately the price at any time.
Recall that we are considering the equation

\[ d f(t) = \mu f(t) \, dt. \]  

(1)

equation (1) is an example of ordinary differential equation and it behaves very charmingly since there exists at least one function \( f \) which satisfies equation (1) and, in addition, it is possible to show that this function is necessarily of the form described in (2):

\[ f(t) = f_0 e^{\mu t}. \]
There exist ordinary differential equations much less nice. For example,

\[ d \, f(t) = \frac{f(t)}{t^2} dt. \]  

(3)

The solution to that equation has form

\[ f(t) = ce^{-\frac{1}{t}} \] where \( c \) is a constant.

We must now specify \( c \) by using the initial condition.

But \( f(0) = 0 \) whatever \( c \), which implies that (3) possesses an infinity of solutions when \( f(0) = 0 \) and possesses none when \( f(0) \neq 0 \).
Now assume that the stochastic process \( S = \{S_t : t \geq 0\} \) represents the evolution of a risky asset price.

We don’t know, in general, the law that governs such a process, but we may have an idea of its local behavior.
Introduction II

Stochastic differential equations

For example, over a short time interval of length $\Delta t$, it is possible that such a price tends to vary proportionally to the period length and the asset price at the beginning of the period. We write, to begin with,

$$ S_{t+\Delta t} - S_t = \mu \ S_t \ \Delta t. $$

If, in general, prices increase, then $\mu$ is a positive constant and if the prices tend to decrease, then $\mu$ is negative.
S_{t+\Delta t} - S_t = \mu S_t \Delta t

There is however a problem with the latter equation: we are not certain that the price varies proportionally to the period length and the asset price, we only claim that it has a tendency to do so.

Therefore, an unpredictable error needs to be incorporated into our equation.

We can however control the magnitude of such a random error.
**Introduction IV**

**Stochastic differential equations**

- **For example**, we can assume that it depends on the asset price at the beginning of the period.
  - Indeed, we observe that, the higher the price, the more the risky asset price can diverge from the trend.
  - Moreover, the random error must also depend on the length of the time interval considered: the longer the interval, the greater the chance that the price diverges from the trend.
  - That is why we add a stochastic term to our initial equation.
The stochastic term to be added to our initial equation leads us to the equation

\[ S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \sqrt{\Delta t} \xi_t \]  

where

- \( \sigma \) is a positive constant and
- \( \xi_t \) is a random variable with distribution \( N(0,1) \) independent from \( \{S_u : 0 \leq u \leq t\} \).

The latter condition is important, since we must not be able to predict the error \( \xi_t \) from observing the behavior of the risky asset price prior to date \( t \).
Such an equation is random and must be satisfied by "almost" every $\omega$, which is to say that

$$\Pr \left\{ \omega \in \Omega : S_{t+\Delta t} (\omega) - S_t (\omega) = \mu S_t (\omega) \Delta t + \sigma S_t (\omega) \sqrt{\Delta t} \xi_t (\omega) \right\}$$

worth one.
With respect to the magnitude of the random error, note that $\sqrt{\Delta t} \xi_t$ is $N(0, \Delta t)$-distributed. Moreover,

$$E \left[ \sigma S_t \sqrt{\Delta t} \xi_t \mid \sigma \{ S_u : u \in \{0, \Delta t, ..., t\} \} \right] = \sigma S_t \Delta t E [\xi_t]$$

$$= 0,$$

$$E \left[ (\sigma S_t \sqrt{\Delta t} \xi_t)^2 \mid \sigma \{ S_u : u \in \{0, \Delta t, ..., t\} \} \right] = \sigma^2 S_t^2 \Delta t E [\xi_t^2]$$

$$= \sigma^2 S_t^2 \Delta t,$$

which implies that the conditional standard deviation of the error term is $\sigma S_t \sqrt{\Delta t}$.

Thus the longer the time interval $\Delta t$ or the higher the security price $S_t$, the greater the standard deviation of the random error.
This implies that the values that the random error may take are more dispersed around its expectation (which is zero).
Recall that
\[ S_{t+\Delta t} - S_t = \mu \, S_t \, \Delta t + \sigma \, S_t \, \sqrt{\Delta t} \, \xi_t. \] (4)

Let’s rewrite equation (4) for the next period:
\[ S_{t+2\Delta t} - S_{t+\Delta t} = \mu \, S_{t+\Delta t} \, \Delta t + \sigma \, S_{t+\Delta t} \, \sqrt{\Delta t} \, \xi_{t+\Delta t}. \]

If we don’t want to be able to predict the error \( \xi_{t+\Delta t} \), the latter must be independent from \( \{S_u : u \in \{0, \Delta t, \ldots, t + \Delta t\}\} \).

For that reason, we introduce the Brownian motion since it is a Gaussian process, the increments of which are mutually independent:
\[ S_{t+\Delta t} - S_t = \mu \, S_t \, \Delta t + \sigma \, S_t \, (W_{t+\Delta t} - W_t). \] (5)

Note that the law of \( W_{t+\Delta t} - W_t \) is the same as the law of \( \sqrt{\Delta t} \xi_t \) : both are \( N(0, \Delta t) \)-distributed.
Let $\mathcal{W} = \{W_t : t \geq 0\}$ be a Brownian motion built on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ such that the filtration $\mathcal{F}$ is the one generated by the Brownian motion, plus it includes all zero-probability events, i.e. for all $t \geq 0$,

$$\mathcal{F}_t = \sigma (\mathcal{N} \text{ and } W_s : 0 \leq s \leq t).$$
The risky asset price today \((t = 0)\) is known with certainty. \(S_0\) is thus \((\emptyset, \Omega)\)–measurable and therefore \(\mathcal{F}_0\)–measurable. Let’s go back to equation (5) and let’s set \(t = 0\).

\[
S_{\Delta t} = \underbrace{S_0}_{\mathcal{F}_0\text{-measurable}} + \underbrace{\mu S_0 \Delta t}_{\mathcal{F}_0\text{-measurable}} + \underbrace{\sigma S_0}_{\mathcal{F}_0\text{-measurable}} + \underbrace{(W_{\Delta t} - W_0)}_{\mathcal{F}_{\Delta t}\text{-measurable independent from } \mathcal{F}_0}
\]

We observe that \(S_{\Delta t}\) is \(\mathcal{F}_{\Delta t}\)–measurable.
Introduction XII

Stochastic differential equations

- We can show by induction that $S_n \Delta t$ is $\mathcal{F}_n \Delta t$–measurable, for any positive integer $n$.

- Indeed, let’s assume there exists $k \in \{0, 1, 2, \ldots\}$ such that $S_k \Delta t$ is $\mathcal{F}_k \Delta t$–measurable. Then

$$S_{(k+1)} \Delta t = S_k \Delta t + \mu S_k \Delta t \Delta t + \sigma S_k \Delta t \left( W_{(k+1)} \Delta t - W_k \Delta t \right)$$

which implies that $S_{(k+1)} \Delta t$ is $\mathcal{F}_{(k+1)} \Delta t$–measurable.

- Our process $S$ lives on the same filtered probability space as the Brownian motion $W$ that we have used to build that process. ■
Introduction XIII
Stochastic differential equations

Let’s go back to equation (5)

\[ S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \left( W_{t+\Delta t} - W_t \right). \]

When time intervals of length \( \Delta t \) become of infinitesimal length, we obtain an equation of the type

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t. \]
The latter equation is an example of stochastic differential equation and it should raise a few questions:

1. the term $\sigma S_t \, dW_t$ is not well defined, particularly if we recall that the Brownian motion paths are nowhere differentiable!
2. does a solution to that equation exist? and
3. if that solution exists, is it unique and how can it be found?

Note that the solution to a stochastic differential equation is not, as is the case for ordinary differential equations, a function, but rather a stochastic process.
In order to answer those questions, let’s consider a stochastic differential equation in a more general form

\[ dX(t) = b(X(t), t)\, dt + a(X(t), t)\, dW(t). \]  

(6)

where the functions \( a : \mathbb{R} \times [0, \infty) \to \mathbb{R} \) and \( b : \mathbb{R} \times [0, \infty) \to \mathbb{R} \) are measurable functions.
What do we mean by $a(X(t), t) \, dW(t)$? We haven’t defined that term. Actually, Equation (6) is the differential form of the integral equation:

$$X(t) = X(0) + \int_0^t b(X(u), u) \, du + \int_0^t a(X(u), u) \, dW(u)$$

and we now know the meaning of the term

$$\int_0^t a(X(u), u) \, dW(u).$$
There doesn’t always exist a solution to that equation and we will see a few results which give the conditions that $b$ and $a$ functions must meet for a solution to exist.

To answer questions (2) and (3), we need a few additional tools.
Fundamental theorem of calculus I

- Itô’s lemma is the stochastic equivalent of the fundamental theorem of calculus. It will allow us to determine the stochastic differential equation satisfied by some given stochastic processes.

**Theorem**

The fundamental theorem of calculus stipulates that, if $\frac{df}{dx} : \mathbb{R} \to \mathbb{R}$ represents the derivative of the function $f : \mathbb{R} \to \mathbb{R}$, then

$$f(b) - f(a) = \int_{a}^{b} \frac{df}{dx}(x) \, dx.$$  

Richard R. Goldberg, Theorem 7.8A, page 205.
Fundamental theorem of calculus II

- **Example.** if $f(x) = x^2$, $a = 0$ and $b = t$, then

  $$t^2 = \int_0^t 2x \, dx.$$  

- Is such a rule still valid in the context of stochastic calculus? Is

  $$W_t^2 = \int_0^t 2W_s \, dW_s ? \quad (7)$$

- We have observed, when constructing the stochastic integral, that the paths of the process
\[ \left\{ \int_0^t W_s \, dW_s : 0 \leq t \leq T \right\} \text{ could be negative at some times} \]
Recall: is

$$W_t^2 = \int_0^t 2W_s \, dW_s \quad ?$$  \hspace{1cm} (7)

But the left side of equality (7) is necessarily non-negative, while the right side can take negative values. There is a contradiction and we conclude that equation (7) is false.
There exists, in the framework of stochastic calculus, an equivalent of the fundamental theorem of calculus that was established by K. Itô. Note that a term is added.
**Theorem**

**Itô’s lemma (first version).** Let $W$ be a Brownian motion built on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function, the first two derivatives of which exist and are continuous. Then $\forall 0 \leq t \leq T,$

$$f(W_t) - f(W_0) \overset{\mathbb{P}-a.s.}{=} \int_0^t \frac{df}{dw}(W_s) \, dW_s + \frac{1}{2} \int_0^t \frac{d^2f}{dw^2}(W_s) \, ds.$$  \hspace{1cm} (8)

In its differential form, equation (8) is written

$$df(W_t) = \frac{df}{dw}(W_t) \, dW_t + \frac{1}{2} \frac{d^2f}{dw^2}(W_t) \, dt.$$
For example, if \( f(x) = x^2 \), then

\[
f(W_t) = W_t^2 \quad \text{and} \quad f(W_0) = W_0^2 = 0
\]

and

\[
\frac{df}{dw}(W_s) = 2W_s \quad \text{and} \quad \frac{d^2f}{dw^2}(W_s) = 2.
\]

By substituting into Itô’s equation, we obtain

\[
W_t^2 \overset{\mathbb{P}}{=} \text{a.s.} \left(2 \int_0^t W_s \, dW_s + \int_0^t 1 \, ds \right) = 2 \int_0^t W_s \, dW_s + t
\]

which implies

\[
\int_0^t W_s \, dW_s = \frac{W_t^2 - t}{2}.
\]
Idea of the proof I
Itô’s lemma

Let’s assume a division of time \( 0 = t_0 < t_1 < \ldots < t_n = t \).
One can assume that \( t_i - t_{i-1} = t/n \).

By expanding \( f \) as a Taylor series about point \( x_0 \), one finds

\[
f (x) - f (x_0) = f' (x_0) (x - x_0) + \frac{1}{2} f'' (\xi) (x - x_0)^2
\]

where \( \min (x_0, x) \leq \xi \leq \max (x_0, x) \).

By applying that result to random points \( (x = W_{t_i} \text{ and } x_0 = W_{t_{i-1}}) \),

\[
f (W_{t_i}) - f (W_{t_{i-1}}) = f' (W_{t_{i-1}}) (W_{t_i} - W_{t_{i-1}}) + \frac{1}{2} f'' (\xi_i) (W_{t_i} - W_{t_{i-1}})^2
\]

where \( \min (W_{t_{i-1}}, W_{t_i}) \leq \xi_i \leq \max (W_{t_{i-1}}, W_{t_i}) \).
Idea of the proof II

Itô’s lemma

\[ f(W_t) - f(W_0) \]

\[ = \sum_{i=1}^{n} (f(W_{t_i}) - f(W_{t_{i-1}})) \]

\[ = \sum_{i=1}^{n} f'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) + \frac{1}{2} \sum_{i=1}^{n} f''(\xi_i)(W_{t_i} - W_{t_{i-1}})^2 \]

By taking the limit when \( n \to \infty \) on both sides,

\[ f(W_t) - f(W_0) = \lim_{n \to \infty} \sum_{i=1}^{n} f'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \]

\[ + \frac{1}{2} \lim_{n \to \infty} \sum_{i=1}^{n} f''(\xi_i)(W_{t_i} - W_{t_{i-1}})^2. \]
The objective is to show that

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f'(W_{t_{i-1}}) (W_{t_i} - W_{t_{i-1}}) = \int_0^t f'(W_s) \, dW_s
\]

and

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f''(\zeta_i) (W_{t_i} - W_{t_{i-1}})^2 = \int_0^t f''(W_s) \, ds.
\]
First,
\[ \int_0^t f'(W_s) \, dW_s = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f'(W_s) \, dW_s \]
which implies that
\[ \int_0^t f'(W_s) \, dW_s - \sum_{i=1}^n f'(W_{t_{i-1}}) (W_{t_i} - W_{t_{i-1}}) \]
\[ = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f'(W_s) \, dW_s - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f'(W_{t_{i-1}}) \, dW_t \]
\[ = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f'(W_s) - f'(W_{t_{i-1}})) \, dW_s \]
\[ \xrightarrow{n \to \infty} 0. \]
Idea of the proof V

Itô’s lemma

Second,

\[
\int_0^t f''(W_s) \, ds - \sum_{i=1}^{n} f''(\xi_i)(W_{t_i} - W_{t_{i-1}})^2
\]

\[
= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f''(W_s) \, ds - \sum_{i=1}^{n} f''(\xi_i)(W_{t_i} - W_{t_{i-1}})^2
\]

\[
= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (f''(W_s) - f''(\xi_i)) \, ds
\]

\[
+ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f''(\xi_i) \, ds - \sum_{i=1}^{n} f''(\xi_i)(W_{t_i} - W_{t_{i-1}})^2
\]

\[
= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (f''(W_s) - f''(\xi_i)) \, ds
\]

\[
- \sum_{i=1}^{n} f''(\xi_i) \left((W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right)
\]
Idea of the proof VI

Itô’s lemma

- The first term \( \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (f''(W_s) - f''(\xi_i)) \, ds \) converges to zero since \( \min(W_{t_{i-1}}, W_{t_i}) \leq \xi_i \leq \max(W_{t_{i-1}}, W_{t_i}) \) implies that \( \xi_i - W_s \to 0 \). \( f'' \) being continuous, we deduce that \( f''(\xi_i) - f''(W_s) \to 0 \).

- The second term
  \[
  \sum_{i=1}^{n} f''(\xi_i) \left( (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1}) \right)
  \]
tends also to 0 since

  \[
  E \left[ (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1}) \right] = 0
  \]

  \[
  \text{Var} \left[ (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1}) \right] = (t_i - t_{i-1}) \to 0.
  \]

- Such a proof is only roughly sketched since we have omitted to specify some technical details. (ref. R. Durrett)
The proof given by K. Itô applies to much more complex situations that the one described above. Here is a first generalization:
Second version II

Itô’s lemma

Theorem

Itô’s lemma (second version). Let \( W \) be a Brownian motion built on the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) and let \( f : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R} \) be a function, the first and second partial derivatives of which exist and are continuous. Then \( \forall 0 \leq t \leq T \),

\[
\begin{align*}
f(W_t, t) - f(W_0, 0) &= \int_0^t \left( \frac{\partial f}{\partial t}(W_s, s) + \frac{1}{2} \frac{\partial^2 f}{\partial w^2}(W_s, s) \right) \, ds \\
&\quad + \int_0^t \frac{\partial f}{\partial w}(W_s, s) \, dW_s
\end{align*}
\]

In its differential form, we have

\[
df(W_t, t) = \left( \frac{\partial f}{\partial t}(W_t, t) + \frac{1}{2} \frac{\partial^2 f}{\partial w^2}(W_t, t) \right) \, dt + \frac{\partial f}{\partial w}(W_t, t) \, dW_t.
\]
The stochastic process $S = \{S_t : 0 \leq t \leq T\}$ represents the price evolution of a risky asset where

$$S_t = S_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right],$$

$\mu$ and $\sigma$ being constants, and $W$ represents a standard Brownian motion.

Since $W_t$ is $N(0, t)$-distributed,

$$\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t$$

follows a distribution $N\left( \left( \mu - \frac{\sigma^2}{2} \right) t, \sigma^2 \right)$

and $S_t$ is lognormally distributed.
Example II
The Black-Scholes model

An intermission: the moments of a lognormally-distributed random variable

If $Z$ represents a standard normal random variable and if $a$ and $b$ are constants then

$$
E[\exp[b + aZ]] = \exp\left[b + \frac{a^2}{2}\right].
$$

So, if $S(0)$ is independent from the Brownian motion,

$$
E[S(t)] = E[S(0)] E\left[\exp\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right]
= E[S(0)] \exp[\mu t]
$$

$$
E\left[S^2(t)\right] = E\left[S^2(0)\right] E\left[\exp\left(2\left(\mu - \frac{\sigma^2}{2}\right) t + 2\sigma W_t\right)\right]
= E\left[S^2(0)\right] \exp\left[2\mu t + \sigma^2 t\right]
$$

$$
\text{Var}[S(t)] = E\left[S^2(0)\right] \exp\left[2\mu t + \sigma^2 t\right] - E^2\left[S(0)\right] \exp\left[2\mu t\right].
$$
Using Itô’s lemma, it is possible to show that the stochastic process $S$ defined as

$$S_t = S_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right]$$

is a solution to the integral equation

$$S_t - S_0 = \mu \int_0^t S_u \, du + \sigma \int_0^t S_u \, dW_u.$$
Example IV
The Black-Scholes model

Recall that

\[ S_t = S_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right]. \]

Indeed, if \( f(w, t) = S_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma w \right] \), then

* \( f(W_t, t) = S_t \)
* \( f(W_0, 0) = S_0 \)
* \( \frac{\partial f}{\partial t}(w, t) = \left( \mu - \frac{\sigma^2}{2} \right) f(w, t) \)
* \( \frac{\partial f}{\partial w}(w, t) = \sigma f(w, t) \)
* \( \frac{\partial^2 f}{\partial w^2}(w, t) = \sigma^2 f(w, t) \).
Thus,

\begin{align*}
S_t - S_0 &= f(W_t, t) - f(W_0, 0) \\
&= \int_0^t \frac{\partial f}{\partial w}(W_u, u) \, dW_u + \int_0^t \left( \frac{\partial f}{\partial t}(W_u, u) + \frac{1}{2} \frac{\partial^2 f}{\partial w^2}(W_u, u) \right) \, du \\
&= \int_0^t \sigma f(W_u, u) \, dW_u + \int_0^t \left( \left( \mu - \frac{\sigma^2}{2} \right) f(W_u, u) + \frac{1}{2} \sigma^2 f(W_u, u) \right) \, du \\
&= \int_0^t \sigma f(W_u, u) \, dW_u + \int_0^t \mu f(W_u, u) \, du \\
&= \int_0^t \sigma S_u \, dW_u + \int_0^t \mu S_u \, du.
\end{align*}
Example VI
The Black-Scholes model

We have just proved that the stochastic process
\[ S = \{ S_t : 0 \leq t \leq T \} \]
where
\[ S_t = S_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right] \]
is a solution to the stochastic differential equation
\[ dS_t = \mu S_t dt + \sigma S_t dW_t. \]
**Definition**

An **Itô process** is defined to be a process \( X = \{X_t : 0 \leq t \leq T\} \) taking its values in \( \mathbb{R} \) such that:

\[
X_t \equiv X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s \quad (9)
\]

where \( K = \{K_t : 0 \leq t \leq T\} \) and \( H = \{H_t : 0 \leq t \leq T\} \) are processes adapted to the filtration \( \{\mathcal{F}_t\} \), satisfying

\[
\mathbb{P} \left[ \int_0^T |K_s| \, ds < \infty \right] = 1 \quad \text{and} \quad \mathbb{P} \left[ \int_0^T |H_s|^2 \, ds < \infty \right] = 1.
\]

Written in its differential form, the Itô process becomes

\[
dX_t = K_t \, dt + H_t \, dW_t.
\]

Damien Lamberton and Bernard Lapeyre, page 53.
**Remark.** First, note that, if

$$
E^P \left[ \int_0^T |H_s|^2 \, ds \right] < \infty,
$$

(10)

then the process $M = \{ M_t : 0 \leq t \leq T \}$ where

$$
M_t = \int_0^t H_s \, dW_s
$$

is a $(\mathcal{F}_t, P)$—martingale.

**Remark.** As a consequence, if the drift coefficient is zero, i.e. $K_t = 0$ for all $0 \leq t \leq T$, then the Itô process $X$ where

$$
X_t \equiv X_0 + \int_0^t H_s \, dW_s
$$

is a $(\mathcal{F}_t, P)$—martingale if and only if the condition (10) is satisfied.

Daniel Revuz and Marc Yor, page 129, proposition 1.23.
Recall that

\[ X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s \quad (11) \]

**Definition**

Let \( X \) be the Itô process defined by equation (11), then the **quadratic variation** process of \( X \),

\[ \langle X \rangle = \{ \langle X \rangle_t : 0 \leq t \leq T \} , \]

is defined to be

\[ \langle X \rangle_t = \int_0^t H_s^2 \, ds. \]
Example. The Brownian motion is an Itô process since

$$W_t = \int_0^t 1 \, dW_s.$$ 

As a consequence,

$$\langle W \rangle_t = \int_0^t 1^2 \, ds = t.$$
Theorem

\textbf{Itô’s lemma (third version).} Let $X$ be an Itô process and let $f : \mathbb{R} \to \mathbb{R}$ be a function, the first two derivatives of which exist and are continuous. Then $\forall 0 \leq t \leq T$, 

\begin{equation}
\begin{aligned}
&f (X_t) - f (X_0) \\
\mathbb{P} \text{-a.s.} &\equiv \int_0^t \frac{df}{dx} (X_s) \, dX_s + \frac{1}{2} \int_0^t \frac{d^2 f}{dx^2} (X_s) \, d\langle X \rangle_s \\
&\quad + \int_0^t \left( \frac{df}{dx} (X_s) \, K_s + \frac{1}{2} \frac{d^2 f}{dx^2} (X_s) \, H_s^2 \right) \, ds,
\end{aligned}
\end{equation}
In its differential form, we find

$$df (X_t) = \frac{df}{dx} (X_t) H_t \ dW_t + \left( \frac{df}{dx} (X_t) K_t + \frac{1}{2} \frac{d^2 f}{dx^2} (X_t) H_t^2 \right) \ dt.$$
Justification attempt!

Itô's lemma for Itô processes is

\[ f(X_t) - f(X_0) \overset{\text{IP-a.s.}}{=} \int_0^t \frac{df}{dx}(X_s) \, dX_s + \frac{1}{2} \int_0^t \frac{d^2f}{dx^2}(X_s) \, d\langle X \rangle_s. \]

We recognize equation (8) as a particular case of the equality above since \( \langle W \rangle_t = t \).
But

\begin{align*}
\int_{0}^{t} & \frac{df}{dx} (X_s) \ dX_s + \frac{1}{2} \int_{0}^{t} \frac{d^2 f}{dx^2} (X_s) \ d \langle X \rangle_s \\
= & \int_{0}^{t} \frac{df}{dx} (X_s) \ (K_s \ ds + H_s \ dW_s) + \frac{1}{2} \int_{0}^{t} \frac{d^2 f}{dx^2} (X_s) \ (H_s^2 \ ds) \\
= & \int_{0}^{t} \frac{df}{dx} (X_s) \ K_s \ ds + \int_{0}^{t} \frac{df}{dx} (X_s) \ H_s \ dW_s + \frac{1}{2} \int_{0}^{t} \frac{d^2 f}{dx^2} (X_s) \ H_s^2 \ ds \\
= & \int_{0}^{t} \frac{df}{dx} (X_s) \ H_s \ dW_s + \int_{0}^{t} \left( \frac{df}{dx} (X_s) \ K_s + \frac{1}{2} \frac{d^2 f}{dx^2} (X_s) \ H_s^2 \right) \ ds.
\end{align*}
Third version V

Itô’s lemma

The preceding calculation is valid provided the following exercise has been verified:

Exercise. If

\[ Y_t = \int_0^t K_s \, ds + \int_0^t H_s \, dW_s \]

where \( W \) is a \((\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})\) — Brownian motion, then

\[ \int_0^t X_s \, dY_s = \int_0^t X_s K_s \, ds + \int_0^t X_s H_s \, dW_s \]

and

\[ \int_0^t X_s \, d\langle Y \rangle_s = \int_0^t X_s H_s^2 \, ds \]

where \( X \) is a predictable process.
Let’s assume that the evolution $S = \{S_t : 0 \leq t \leq T\}$ of a risky asset satisfies the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$  

The return $R_t$ on such a risky asset, when accumulation is continuous, is defined as

$$R_t = \ln \frac{S_t}{S_0}.$$ 

Using Itô’s lemma, we can show that $R$ satisfies the stochastic differential equation

$$dR_t = \left( \mu - \frac{\sigma^2}{2} \right) \, dt + \sigma \, dW_t. \quad (13)$$
Intermission: Fubini’s theorem. What follows is a particular case of Fubini’s theorem.

Theorem

If $Y = \{Y_t : 0 \leq t \leq T\}$ is an adapted stochastic process such that, for all $0 \leq t \leq T$, $Y_t \geq 0$ then

$$E^P \left[ \int_0^t Y_s \, ds \right] = \int_0^t E^P \left[ Y_s \right] \, ds,$$

i.e. one can exchange integral and expectation.

Donald L. Cohn, page 159, proposition 5.2.1.
Example III

Recall that
\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t. \quad (14) \]

First, let's verify that \( S \) truly is an Itô process such that \( H_t = \sigma S_t \) and \( K_t = \mu S_t \).

Indeed, since we have already shown that
\[
S_t = S_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right] \quad (\mathcal{F}_t - \text{measurable})
\]
is a solution to the equation (14), the processes
\[
K = \{ K_t = \mu S_t : 0 \leq t \leq T \} \quad \text{and} \quad H = \{ H_t = \sigma S_t : 0 \leq t \leq T \}
\]
are \( \{ \mathcal{F}_t \} - \text{adapted}. \)
Moreover, since $S_t$ is a non-negative random variable, we can use Fubini’s theorem and exchange the $E^P$ and $\int$ operators, which yields

$$
E^P \left[ \int_0^T |K_u| \, du \right] = |\mu| E^P \left[ \int_0^T S_u \, du \right] 
$$

$$
= |\mu| \int_0^T E^P [S_u] \, du 
$$

$$
= |\mu| \int_0^T E^P [S_0] \exp [\mu u] \, du 
$$

$$
= |\mu| E^P [S_0] \int_0^T \exp [\mu u] \, du 
$$

$$
= |\mu| E^P [S_0] \frac{e^{T \mu} - 1}{\mu} < \infty 
$$
Example V

\[
E^P \left[ \int_0^T H_u^2 \, du \right] = \sigma^2 E^P \left[ \int_0^T S_u^2 \, du \right]
\]

\[
= \sigma^2 \int_0^T E^P \left[ S_u^2 \right] \, du
\]

\[
= \sigma^2 \int_0^T E^P \left[ S_0^2 \right] \exp \left[ 2\mu u + \sigma^2 u \right] \, du
\]

\[
= \sigma^2 E^P \left[ S_0^2 \right] \int_0^T \exp \left[ 2\mu u + \sigma^2 u \right] \, du
\]

\[
= \sigma^2 E^P \left[ S_0^2 \right] \frac{e^{T(2\mu + \sigma^2)} - 1}{2\mu + \sigma^2}
\]

\[
< \infty.
\]
Hence, we can deduce that

\[ \int_0^T |K_s| \, ds < \infty \quad \text{and} \quad \int_0^T H_s^2 \, ds < \infty \, \mathbb{P} - \text{a.s.} \]
Now, let’s apply Itô’s lemma. If \( f(x) = \ln \frac{x}{x_0} \) then
\[
\frac{df}{dx}(x) = \frac{1}{x} \quad \text{and} \quad \frac{d^2f}{dx^2}(x) = -\frac{1}{x^2}.
\]
Thus,
\[
dR_t = df(S_t) = \frac{df}{dx}(S_t) H_t \ dW_t + \left( \frac{df}{dx}(S_t) K_t + \frac{1}{2} \frac{d^2f}{dx^2}(S_t) H_t^2 \right) dt
\]
\[
= \frac{1}{S_t} \sigma S_t \ dW_t + \left( \frac{1}{S_t} \mu S_t - \frac{1}{2} \frac{1}{S_t^2} \sigma^2 S_t^2 \right) dt
\]
\[
= \sigma \ dW_t + \left( \mu - \frac{\sigma^2}{2} \right) dt.
\]
Example VIII

But the integral form of equation (13)

\[ dR_t = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t \]

is

\[
R_t = R_0 + \sigma \int_0^t dW_s + \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t ds \\
= \sigma W_t + \left( \mu - \frac{\sigma^2}{2} \right) t \text{ car } R_0 = \ln \frac{S_0}{S_0} = \ln 1 = 0 \\
\approx N \left( \left( \mu - \frac{\sigma^2}{2} \right) t; \sigma^2 t \right).
\]

In the Black-Scholes world, returns are Gaussian.
Fourth version I

Itô’s lemma

Theorem

**Itô’s lemma (fourth version).** Let $X$ be an Itô process, i.e. $dX_t = K_t \, dt + H_t \, dW_t$, and let $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a function, the first and second partial derivatives of which exist and are continuous. Then $\forall 0 \leq t \leq T$, 

$$f(X_t, t) - f(X_0, 0) \quad \mathbb{P}-a.s. \quad \int_0^t \frac{\partial f}{\partial x} (X_s, s) \, dX_s + \int_0^t \frac{\partial f}{\partial t} (X_s, s) \, ds$$

$$+ \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2} (X_s, s) \, d\langle X \rangle_s$$
Fourth version II

Itô’s lemma

We can also write

$$f(X_t, t) - f(X_0, 0) \equiv_{\text{P-a.s.}} \int_0^t \frac{\partial f}{\partial x}(X_s, s) \, H_s \, dW_s$$

$$+ \int_0^t \left( \frac{\partial f}{\partial x}(X_s, s) \, K_s + \frac{\partial f}{\partial t}(X_s, s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(X_s, s) \, H_s^2 \right) \, ds.$$
In the differential form, we obtain

\[
\frac{df}{dt}(X_t, t) = \frac{\partial f}{\partial X}(X_t, t) H_t \, dW_t \\
+ \left( \frac{\partial f}{\partial X}(X_t, t) K_t + \frac{\partial f}{\partial t}(X_t, t) + \frac{1}{2} \frac{\partial^2 f}{\partial X^2}(X_t, t) H_t^2 \right) dt.
\]
The Taylor series expansion of the continuous function $f : \mathbb{R} \to \mathbb{R}$ (the first derivatives of which exist and are continuous) about point $x_0$ is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2} (x - x_0)^2$$

where $\approx$ means that it is an approximation.

If the continuous function $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, then Itô's lemma is stated as follows:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s) \, d\langle X \rangle_s$$

where $X$ is an Itô process.
Relation with Taylor series II

Itô’s lemma

If \( f' (X_s) \) and \( f'' (X_s) \) could be respectively replaced with \( f' (X_0) \) and \( f'' (X_0) \) (that would be a good approximation if \( t \) is small), then we could write

\[
f (X_t) = f (X_0) + \int_0^t f' (X_s) \, dX_s + \frac{1}{2} \int_0^t f'' (X_s) \, d\langle X \rangle_s
\]

which looks more like the Taylor series expansion. But the latter approximation is not Itô’s lemma.
Solution to an SDE 1

Itô’s lemma

- The solution to a stochastic differential equation (SDE) consists in finding the stochastic process(es) satisfying the given equation.

- But Itô’s lemma allows to do the opposite, i.e. it helps find the SDE satisfied by a given stochastic process. If it turns out that the stochastic process chosen satisfies the given SDE, then we will have found a solution to that equation. In that sense, Itô’s lemma provides us with solutions to SDEs.

- However, it is not a systematic way to obtain solutions. It requires educated guesses!
Quadratic covariation

Definition

Let $X$ and $Y$ be two Itô process such that

$$dX_t = K_t \, dt + H_t \, dW_t \quad \text{and} \quad dY_t = \tilde{K}_t \, dt + \tilde{H}_t \, dW_t$$

then the **quadratic covariation** process $\langle X, Y \rangle$ is defined for all $t \in [0, T]$ as

$$\langle X, Y \rangle_t = \int_0^t H_s \tilde{H}_s \, ds.$$
The multiplication rule is useful when we want to study, for example, the behavior of the present value of an asset while knowing the processes for both the asset price evolution and the discount factor.
Multiplication rule II

**Theorem**

**The multiplication rule.** Let $X$ and $Y$ be two Itô processes such that

$$
\begin{align*}
    dX_t &= K_t \, dt + H_t \, dW_t \\
    dY_t &= \tilde{K}_t \, dt + \tilde{H}_t \, dW_t.
\end{align*}
$$

The multiplication rule is

$$
\begin{align*}
    dX_t \, Y_t &= X_t \, dY_t + Y_t \, dX_t + d\langle X, Y \rangle_t \\
    &= X_t \left( \tilde{K}_t \, dt + \tilde{H}_t \, dW_t \right) + Y_t \left( K_t \, dt + H_t \, dW_t \right) + H_t \tilde{H}_t \, dt \\
    &= \left( X_t \tilde{K}_t + Y_t K_t + H_t \tilde{H}_t \right) \, dt + \left( X_t \tilde{H}_t + Y_t H_t \right) \, dW_t.
\end{align*}
$$
Example I

Present value of an asset

- Let’s assume that the stochastic process \( X \), satisfying the equation
  \[
  dX_t = \mu X_t \, dt + \sigma X_t \, dW_t,
  \]
is the price evolution of a risky asset.

- The process \( \{ \beta_t = e^{-rt} : t \geq 0 \} \) is the discount factor. Note that
  \[
  \frac{d}{dt} \beta_t = -re^{-rt} = -r \beta_t,
  \]
i.e. the process \( \beta \) is an Itô process satisfying the equation differential
  \[
  d\beta_t = -r \beta_t \, dt.
  \]
Example II

Present value of an asset

The process \( Y = \beta X \) represents the evolution of the present value of the asset. The multiplication rule yields that

\[
dY_t = d\beta_t X_t \\
= \beta_t dX_t + X_t d\beta_t + d\langle \beta, X \rangle_t \\
= \beta_t (\mu X_t dt + \sigma X_t dW_t) + X_t (-r \beta_t dt) \\
= (\mu - r) \beta_t X_t dt + \sigma \beta_t X_t dW_t \\
= (\mu - r) Y_t dt + \sigma Y_t dW_t.
\]

In its integral form, the latter equation is written

\[
Y_t = Y_0 + (\mu - r) \int_0^t Y_s ds + \sigma \int_0^t Y_s dW_s. \quad (15)
\]
The phenomenon we would like to model may contain several sources of uncertainty. That is why we consider Itô-type processes which involve several Brownian motions.
Itô process I
Multidimensional Itô’s lemma

- Let \( W^{(1)}, \ldots, W^{(n)} \) be independent Brownian motions built on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})\).
- Let \( K^{(i)}, \ i \in \{1, 2, \ldots, m\} \) and \( H^{(i,k)} \), where \( i \in \{1, 2, \ldots, m\} \) and \( k \in \{1, 2, \ldots, n\} \), be \( \{\mathcal{F}_t\} \)-adapted processes.
  - \( \mathbb{P} \left[ \sum_{i=1}^m \int_0^T K^{(i)}_s \ ds < \infty \right] = 1 \),
  - \( \mathbb{P} \left[ \int_0^T H^{(i,k)}_s H^{(j,k^*)}_s \ ds < \infty \right] = 1, \ i, j \in \{1, 2, \ldots, m\} \) and \( k, k^* \in \{1, 2, \ldots, n\} \).
Itô process II
Multidimensional Itô’s lemma

Definition

For all $i \in \{1, 2, ..., m\}$, we define the process $X^{(i)}$ as

$$X_t^{(i)} = X_0^{(i)} + \int_0^t K_s^{(i)} ds + \sum_{k=1}^n \int_0^t H_s^{(i,k)} dW_s^{(k)}$$  \hspace{1cm} (16)$$

where $X_0^{(1)}, ..., X_0^{(m)}$ are $\mathcal{F}_0$-measurable random variables.
Quadratic covariation I
Multidimensional Itô’s lemma

Definition

The quadratic covariation process
\[
\left\langle X^{(i)}, X^{(j)} \right\rangle = \left\{ \left\langle X^{(i)}, X^{(j)} \right\rangle_t : 0 \leq t \leq T \right\}
\]
defined as

\[
\left\langle X^{(i)}, X^{(j)} \right\rangle_t = \sum_{k=1}^{n} \int_{0}^{t} H_{s}^{(i,k)} H_{s}^{(j,k)} \, ds.
\]

Note that
\[
\left\langle X^{(i)}, X^{(j)} \right\rangle = \left\langle X^{(j)}, X^{(i)} \right\rangle
\]
and
\[
\left\langle X^{(i)}, X^{(i)} \right\rangle = \left\langle X^{(i)} \right\rangle.
\]
Quadratic covariation II
Multidimensional Itô’s lemma

**Example.** We want to calculate the quadratic covariation of two correlated Brownian motions.

Let’s set $B_1$ and $B_2$, two Brownian motions such that $\text{Corr}[B_1(t), B_2(t)] = \rho$ for all $t > 0$.

It is possible to write $B_1(t) = W_1(t)$ and $B_2(t) = \rho W_1(t) + \sqrt{1 - \rho^2} W_2(t)$ where $W_1$ and $W_2$ are independent Brownian motions.

Since $dB_1(t) = 1dW_1(t) + 0dW_2(t)$ and $dB_2(t) = \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t)$.

Then $\langle B_1, B_2 \rangle_t = \int_0^t 1 \rho ds + \int_0^t 0 \sqrt{1 - \rho^2} ds = \rho t$.
Let $W^{(1)}, ..., W^{(n)}$ be independent Brownian motions built on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})$ in the Itô processes defined in equation (16).
Itô's lemma II
Multidimensional Itô's lemma

Theorem

Let $f : \mathbb{R}^m \times [0, \infty) \rightarrow \mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial s}, \frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_i \partial s}, i, j \in \{1, \ldots, m\}$ exist and are continuous. Then $\forall t \geq 0$,

$$f \left( X_t^{(1)}, \ldots, X_t^{(m)}, t \right) - f \left( X_0^{(1)}, \ldots, X_0^{(m)}, 0 \right)$$

$$\mathbb{P}-a.s. \sum_{i=1}^{m} \int_0^t \frac{\partial f}{\partial x_i} \left( X_s^{(1)}, \ldots, X_s^{(m)}, s \right) dX_s^{(i)}$$

$$+ \int_0^t \frac{\partial f}{\partial t} \left( X_s^{(1)}, \ldots, X_s^{(m)}, s \right) ds$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j} \left( X_s^{(1)}, \ldots, X_s^{(m)}, s \right) d \left\langle X^{(i)}, X^{(j)} \right\rangle_s.$$
Example 1
Exchange rate

Assume that the stochastic process $X$, satisfying equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

represents the price evolution of a stock in British pounds.

Note that $d \langle X \rangle_t = \sigma^2 X_t^2 dt$.

Assume also that the Canadian dollar-value evolution of the British Pound is modeled by a process $C$ where

$$dC_t = \alpha C_t dt + \beta C_t d\tilde{W}_t + \gamma C_t dW_t.$$

If the Brownian motions $W$ and $\tilde{W}$ are built on the same filtered probability space and are independent, then

$$d \langle C \rangle_t = (\beta^2 C_t^2 + \gamma^2 C_t^2) dt = (\beta^2 + \gamma^2) C_t^2 dt$$

$$d \langle X, C \rangle_t = (\sigma X_t \gamma C_t + 0 \beta C_t) dt = \sigma \gamma X_t C_t dt.$$
Example II

Exchange rate

The price evolution $Y$ of the stock in Canadian dollars is such that, at any time,

$$Y_t = X_t C_t.$$
Let a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be defined as
\[
f(x, c) = xc.
\]

Note that \( Y_t = f(X_t, C_t), \forall t \geq 0 \),
that the first derivatives are \( \frac{\partial f}{\partial x}(x, c) = c \) and \( \frac{\partial f}{\partial c}(x, c) = x \)
and that the second derivatives are
\[
\frac{\partial^2 f}{\partial x^2}(x, c) = \frac{\partial^2 f}{\partial c^2}(x, c) = 0 \text{ and } \frac{\partial^2 f}{\partial x \partial c}(x, c) = 1.
\]
Example IV
Exchange rate

Itô's lemma yields

\[ dY_t = df (X_t, C_t) \]
\[ = \frac{\partial f}{\partial x} (X_t, C_t) \, dX_t + \frac{\partial f}{\partial c} (X_t, C_t) \, dC_t \]
\[ + \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} (X_t, C_t) \, d\langle X \rangle_t + \frac{\partial^2 f}{\partial x \partial c} (X_t, C_t) \, d\langle X, C \rangle_t \right) \]
\[ + \frac{1}{2} \left( \frac{\partial^2 f}{\partial c^2} (C_t, X_t) \, d\langle C, X \rangle_t + \frac{\partial^2 f}{\partial c \partial x} (X_t, C_t) \, d\langle C \rangle_t \right) \]
\[ = C_t \, dX_t + X_t \, dC_t + d\langle X, C \rangle_t \]
\[ = C_t (\mu X_t \, dt + \sigma X_t \, dW_t) + X_t \left( \alpha C_t \, dt + \beta C_t \, d\widetilde{W}_t + \gamma C_t \, dW_t \right) \]
\[ + \sigma \gamma X_t \, C_t \, dt \]
\[ = (\mu X_t C_t + \alpha X_t C_t + \sigma \gamma X_t C_t) \, dt \]
\[ + (\sigma X_t C_t + \gamma C_t X_t) \, dW_t + \beta X_t C_t \, d\widetilde{W}_t \]
\[ = (\mu + \alpha + \sigma \gamma) \, Y_t \, dt + (\sigma + \gamma) \, Y_t \, dW_t + \beta Y_t \, d\widetilde{W}_t. \]
What is a solution?

- There exist two types of solutions to a stochastic differential equation: strong solutions and weak solutions.
- Consider the stochastic differential equation

\[ dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t \]

where \( \{ W_t : t \geq 0 \} \) is a 
\((\Omega, \mathcal{F}, \{ \mathcal{F}_t : t \geq 0 \}, \mathbb{P})\) — standard Brownian motion.

**Definition**

A strong solution consists of finding a stochastic process \( X \) existing on the same filtered probability space 
\((\Omega, \mathcal{F}, \{ \mathcal{F}_t : t \geq 0 \}, \mathbb{P})\) as the Brownian motion and satisfying the equation

\[ X_t = X_0 + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s. \]
What is a solution? II

The strong solution is said to be unique if, when \( X \) and \( \tilde{X} \) represent two strong solutions to the same stochastic differential equation, we have

\[
\mathbb{P} \left[ X_t = \tilde{X}_t, \ t \geq 0 \right] = 1.
\]
### Definition

We have a weak solution if we can construct:

1) filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t : t \geq 0\}, \tilde{\mathbb{P}})$,

2) a $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t : t \geq 0\}, \tilde{\mathbb{P}})$ — standard Brownian motion $\{\tilde{W}_t : t \geq 0\}$ and

3) a $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t : t \geq 0\}, \tilde{\mathbb{P}})$ — stochastic process $\{\tilde{X}_t : t \geq 0\}$ such that

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \mu(\tilde{X}_s) \, ds + \int_0^t \sigma(\tilde{X}_s) \, d\tilde{W}_s$$

- It is possible to have several strong solutions to a same equation.
What is a solution? IV

- If the strong solution is unique, then there will also be a unique weak solution. By contrast, if the weak solution is unique, there may be several strong solutions.
Definition

Some stochastic processes that give rise to Gaussian models are of the form

\[ dX_t = (\mu_1(t)X_t + \mu_2(t)) \, dt + \sigma(t) \, dW_t \quad (17) \]

where \( \mu_1(\bullet), \mu_2(\bullet) \) and \( \sigma(\bullet) \) deterministic functions of time and \( \{W_t : 0 \leq t \leq T\} \) is a \( (\{\mathcal{F}_t\}, \mathbb{P}) \)–standard Brownian motion.
If \( \mathbb{P} \left[ \int_0^T \frac{\sigma^2(s)}{\Phi^2(s)} \, ds < \infty \right] = 1 \), then the strong solution to the stochastic differential equation (17) is

\[
X_t = \Phi(t) \left[ X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)} \, ds + \int_0^t \frac{\sigma(s)}{\Phi(s)} \, dW_s \right]
\]

where the function \( \Phi(\cdot) \) is the solution to the ordinary differential equation

\[
d\Phi(t) = \mu_1(t) \Phi(t) \, dt, \quad \Phi(0) = 1.
\]

Bisière, page 117.
Gaussian models III

- **Proof.** Let’s set

  \[ Y_t \equiv \int_0^t \frac{\sigma(s)}{\Phi(s)} \, dW_s, \ 0 \leq t \leq T. \]  

  (18)

- The process \( Y \) is an Itô process if

  \[ \mathbb{P} \left[ \int_0^T \frac{\sigma^2(s)}{\Phi^2(s)} \, ds < \infty \right] = 1. \]  

  (19)

- Note that \( dY_t = \frac{\sigma(t)}{\Phi(t)} \, dW_t \) and \( d\langle Y \rangle_t = \frac{\sigma^2(t)}{\Phi^2(t)} \, dt \).
Let’s set $g(t, y) = \Phi(t) \left[ X_0 + \int_0^t \frac{\mu_s(s)}{\Phi(s)} ds + y \right]$.

Then $g(t, Y_t) = X_t$ and $g(0, Y_0) = \Phi(0)X_0 = X_0$. 

$= 1$
Recall that

\[ g(t, y) = \Phi(t) \left[ X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)} \, ds + y \right]. \]

Moreover,

\[ \frac{\partial g}{\partial t}(t, y) = \frac{\partial \Phi}{\partial t}(t) \left[ X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)} \, ds + y \right] + \Phi(t) \frac{\mu_2(t)}{\Phi(t)} \]

\[ = \mu_1(t) \Phi(t) \left[ X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)} \, ds + y \right] + \Phi(t) \frac{\mu_2(t)}{\Phi(t)} \]

\[ = \mu_1(t) g(t, y) + \mu_2(t), \]

and

\[ \frac{\partial g}{\partial y}(t, y) = \Phi(t), \text{ and } \frac{\partial^2 g}{\partial y^2}(t, y) = 0. \]
Using Itô’s lemma, we obtain

\[ dX_t = dg(t, Y_t) \]
\[ = \frac{\partial g}{\partial t}(t, Y_t) \ dt + \frac{\partial g}{\partial y}(t, Y_t) \ dY_t + \frac{1}{2} \frac{\partial^2 g}{\partial y^2}(t, Y_t) \ d\langle Y \rangle_t \]
\[ = (\mu_1(t)g(t, Y_t) + \mu_2(t)) \ dt + \Phi(t) \left( dY_t \right) \]
\[ = \frac{\sigma(t)}{\Phi(t)} \ dW_t. \]
Why Gaussian?

It is possible to show that the joint distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is the multivariate normal law.
Ornstein-Uhlenbeck Process I
Gaussian Models

- **Vasicek (1977)**
  - The spot interest rate satisfies
    \[
    dr_t = \kappa (\theta - r_t) \, dt + \sigma \, dW_t. \tag{20}
    \]
  - Referring to the general form,
    \[
    dX_t = (\mu_1 (t) X_t + \mu_2 (t)) \, dt + \sigma (t) \, dW_t,
    \]
    we conclude that \( \mu_1 (t) = -\kappa \), \( \mu_2 (t) = \kappa \theta \) et \( \sigma (t) = \sigma \).
- Bisière, pages 115-117.
Ornstein-Uhlenbeck Process II
Gaussian Models

Since the ordinary differential equation
\[ d\Phi (t) = \mu_1 (t) \Phi (t) \, dt, \quad \Phi (0) = 1 \]
becomes
\[ d\Phi (t) = -\kappa \Phi (t) \, dt, \quad \Phi (0) = 1, \]
the function \( \Phi (\bullet) \) is
\[ \Phi (t) = \exp (-\kappa t), \; 0 \leq t \leq T. \]
The process \( Y \) where
\[
Y_t = \int_0^t \frac{\sigma(s)}{\Phi(s)} \, dW_s \quad \text{becomes} \quad Y_t = \sigma \int_0^t \exp(\kappa s) \, dW_s
\]

Which is an Itô's process because
\[
\mathbb{P} \left[ \int_0^T \frac{\sigma^2(s)}{\Phi^2(s)} \, ds < \infty \right] = \mathbb{P} \left[ \sigma^2 \int_0^T \exp(2\kappa s) \, ds < \infty \right] = \mathbb{P} \left[ \frac{\sigma^2}{2\kappa} \left( e^{2TK} - 1 \right) < \infty \right] = 1.
\]
Ornstein-Uhlenbeck Process IV
Gaussian Models

The general solution

\[ X_t = \Phi(t) \left[ X_0 + \int_0^t \frac{\mu_2(s)}{\Phi(s)} \, ds + \int_0^t \frac{\sigma(s)}{\Phi(s)} \, dW_s \right] \]

is

\[
\begin{align*}
    r_t & = \exp(-\kappa t) \left[ r_0 + \int_0^t \kappa \theta \exp(\kappa s) \, ds + \int_0^t \sigma \exp(\kappa s) \, dW_s \right] \\
    & = r_0 \exp(-\kappa t) + \theta (1 - \exp(-\kappa t)) + \int_0^t \sigma \exp(-\kappa (t - s)) \, dW_s \\
    & = \theta + (r_0 - \theta) \exp(-\kappa t) + \int_0^t \sigma \exp(-\kappa (t - s)) \, dW_s.
\end{align*}
\]
Affine Models I

- A model is said to belong to the exponential affine class if there is deterministic functions $a(t, T)$ and $b(t, T)$ for which the time $t$ value of a zero-coupon bond paying 1 dollar at maturity $T$ is

$$P(t, T) = \exp(a(t, T) r(t) + b(t, T)).$$

- In that case, the riskless interest rate as the form

$$dr_t = \left(\mu_1(t) r_t + \mu_2(t)\right) dt + \sqrt{\sigma_1(t) r_t + \sigma_2(t)} dW_t$$

where $\mu_1(\bullet)$, $\mu_2(\bullet)$, $\sigma_1(\bullet)$ and $\sigma_2(\bullet)$ are deterministic function of time.

- Bisière, pages 116-117 et 128-129.
The Gaussian models belong the the exponential-affine classe of model..

**Square-root Processus (Feller’s branching diffusion)**
Cox-Ingersoll-Ross (1985), generalised by Hull-White (1990)

\[ dr_t = \kappa (\theta - r_t) \ dt + \sigma \sqrt{r_t} \ dW_t. \]

Bisière, pages 116-117.
Lipschitz Coefficients

**Theorem**

Consider \( dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \). If \( b(\cdot, \cdot) \) and \( \sigma(\cdot, \cdot) \) are continuous functions such that there is a constant \( K < \infty \), for which

\[
|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K |x - y| \\
|b(t, x)| + |\sigma(t, x)| \leq K (1 + |x|) \\
E[X_0^2] < \infty
\]

then, for all \( T \geq 0 \), the equation (EDS) admits a unique solution in the interval \([0, T]\). Moreover, this solution verifies

\[
E \left[ \sup_{0 \leq t \leq T} |X_t|^2 \right] < \infty.
\]

There are sufficient but not necessary conditions. (Lamberton et Lapeyre, p. 59-60.)
Références I

Références II
