Stochastic integral

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Stochastic Calculus I

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Introduction

The Ito integral

- The theories of stochastic integral and stochastic differential equations have initially been developed by Kiyosi Ito around 1940 (one of the first important papers was published in 1942).

Riemann integral I

- Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function. Typically, when we speak of the integral of a function f, we refer to the Riemann integral, $\int_a^b f(t)\, dt$, which calculates the area under the curve $t \longrightarrow f(t)$ between the bounds a and b.

Riemann integral II

- First, we must realize that not all functions are Riemann integrable.
- Let's explain a little further : in order to construct the Riemann integral of the function f on the interval $[a, b]$, we divide such an interval into n subintervals of equal length $\left(\frac{b-a}{n}\right)$ and, for each of them, we determine the greatest and the smallest value taken by the function f.

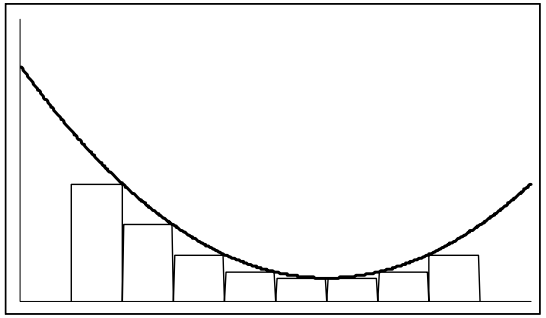
Riemann integral III

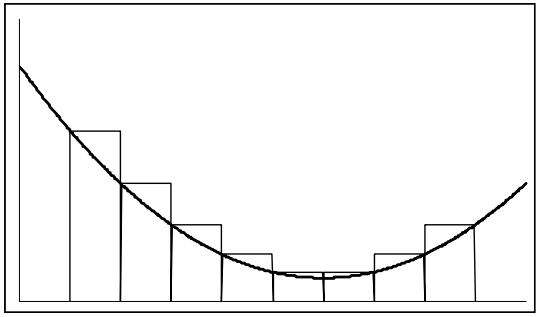
- Thus, for every $k \in \{1, ..., n\}$, the smallest value taken by the function f on the interval $\left[a + (k-1)\frac{b-a}{n}, a + k\frac{b-a}{n}\right]$ is

$$\underline{f}_k^{(n)} \equiv \inf\left\{ f(t) \,\middle|\, a + (k-1)\frac{b-a}{n} \leq t \leq a + k\frac{b-a}{n} \right\}$$

and the greatest is

$$\overline{f}_k^{(n)} \equiv \sup\left\{ f(t) \,\middle|\, a + (k-1)\frac{b-a}{n} \leq t \leq a + k\frac{b-a}{n} \right\}.$$





Riemann integral IV

- The area under the curve $t \longrightarrow f(t)$ between the bounds $a + (k-1)\frac{b-a}{n}$ and $a + k\frac{b-a}{n}$ is therefore greater than the area $\frac{b-a}{n}\underline{f}_k^{(n)}$ of the rectangle of base $\frac{b-a}{n}$ and height $\underline{f}_k^{(n)}$, and smaller than the area $\frac{b-a}{n}\overline{f}_k^{(n)}$ of the rectangle with the same base but with height $\overline{f}_k^{(n)}$.
- As a consequence, if we define

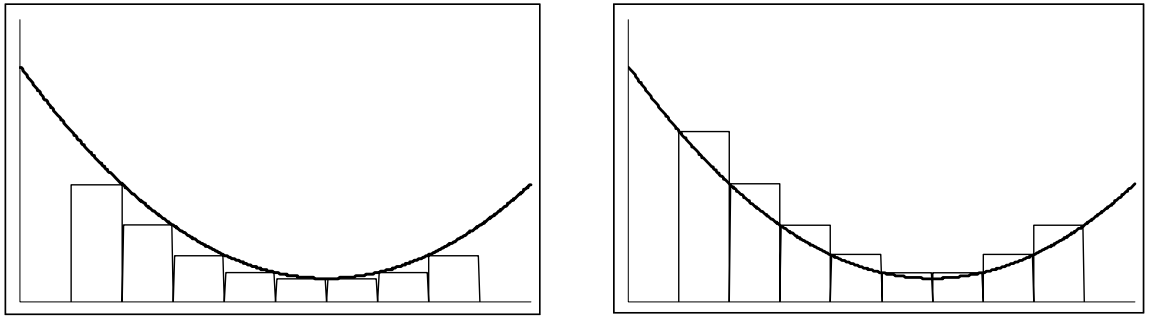
$$\underline{I}_n(f) \equiv \sum_{k=1}^{n} \frac{b-a}{n}\underline{f}_k^{(n)} \text{ and } \overline{I}_n(f) \equiv \sum_{k=1}^{n} \frac{b-a}{n}\overline{f}_k^{(n)}$$

then

$$\underline{I}_n(f) \leq \int_a^b f(t)\,dt \leq \overline{I}_n(f).$$

Riemann integral V

Illustrations of $\underline{I}_n(f)$ and $\overline{I}_n(f)$



Riemann integral VI

- Riemann integrable functions are those for which the limits $\lim_{n\to\infty} \underline{I}_n(f)$ and $\lim_{n\to\infty} \overline{I}_n(f)$ exist and are equal.
- We then define the Riemann integral of the function f as

$$\int_a^b f(t)\,dt \equiv \lim_{n\to\infty} \underline{I}_n(f) = \lim_{n\to\infty} \overline{I}_n(f).$$

- It is possible to show that the integral $\int_a^b f(t)\,dt$ exists for any function f continuous on the interval $[a, b]$. Many other functions are Riemann integrable.

Riemann integral VII

- However, there exist functions that are not Riemann integrable.
- For example, let's consider the indicator function $\mathbb{I}_{\mathbb{Q}} : [0,1] \to \{0,1\}$ which takes the value of 1 if the argument is rational and 0 otherwise.
- Then, for all natural number n, $\underline{I}_n = 0$ and $\overline{I}_n = 1$, which implies that the Riemann integral is not defined for such a function.

Riemann integral VIII

- We will examine a very simple case : the Riemann integral for boxcar functions, i.e. the functions $f : [0; \infty) \to \mathbb{R}$ which admit the representation $f(t) = c\mathbb{I}_{(a,b]}(t)$, $a < b$, $c \in \mathbb{R}$ where $\mathbb{I}_{(a,b]}$ represents the indicator function

$$\mathbb{I}_{(a,b]}(t) = \begin{cases} 1 & \text{if } t \in (a, b] \\ 0 & \text{if } t \notin (a, b] \end{cases} .$$

Riemann integral IX

If $f(t) = c\mathbb{I}_{(a,b]}(t)$ then the Riemann integral of f is

$$\begin{aligned}\int_0^t f(s)\,ds &= \int_0^t c\mathbb{I}_{(a,b]}(s)\,ds \\ &= \begin{cases} 0 & \text{if } t \leq a \\ c(t-a) & \text{if } a < t \leq b \\ c(b-a) & \text{if } t > b. \end{cases}\end{aligned}$$

Ito integral I

- Let $\{W_t : t \geq 0\}$ be a standard Brownian motion built on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.
- Technical condition. Since we will work with almost sure equalities [1], we require that the set of events which have zero probability to occur are included in the sigma-algebra $\mathcal{F}_0$, i.e. that the set

$$\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\} \subset \mathcal{F}_0.$$

That way, if X is $\mathcal{F}_t-$measurable and $Y = X$ $\mathbb{P}-$almost surely, then we know that Y is $\mathcal{F}_t-$measurable.

Ito integral II

- Let $\{X_t : t \geq 0\}$ be a predictable stochastic process. It is tempting to define the stochastic integral of X with respect to W, pathwise, by using a generalization of the Stieltjes integral:

$$\forall \omega \in \Omega, \quad \int X(\omega)\, dW(\omega).$$

- That would be possible if the paths of Brownian motion W had bounded variation, i.e. if they could be expressed as the difference of two non-decreasing functions.
- But, since Brownian motion paths do not have bounded variation, it is not possible to use such an approach.
- We must define the stochastic integral with respect to the Brownian motion as a whole, i.e. as a stochastic process in itself, and not pathwise.

[1] $X = Y$ $P-$almost surely if the set of ω for which X is different from Y has a probability of zero, i.e.

$$P\{\omega \in \Omega : X(\omega) \neq Y(\omega)\} = 0.$$

Basic process I

Ito integral

Definition

We call X a **basic stochastic process** if X admits the following representation:

$$X_t(\omega) = C(\omega)\,\mathbb{I}_{(a,b]}(t)$$

where $a < b \in \mathbb{R}$ and C is a random variable, $\mathcal{F}_a-$measurable and square-integrable, i.e. $E^{\mathbb{P}}\left[C^2\right] < \infty$.

Note that such a process is adapted to the filtration $\mathbb{F}$. Indeed,

$$X_t = \begin{cases} 0 & \text{if } 0 \leq t \leq a & \text{which is } \mathcal{F}_0-\text{measurable therefore } \mathcal{F}_t-\text{measurable} \\ C & \text{if } a < t \leq b & \text{which is } \mathcal{F}_a-\text{measurable therefore } \mathcal{F}_t-\text{measurable} \\ 0 & \text{if } b < t & \text{which is } \mathcal{F}_0-\text{measurable therefore } \mathcal{F}_t-\text{measurable.} \end{cases}$$

Basic process II

Ito integral

- In fact, X is more than $\mathbb{F}-$adapted, it is $\mathbb{F}-$predictable, but the notion of *predictable process* is more delicate to define when we work with continuous-time processes.
- Note, nevertheless, that adapted processes with continuous path are predictable.
- Intuitively, if X_t represents the number of security shares held at time t, then $X_t(\omega) = C(\omega)\mathbb{I}_{(a,b]}(t)$ means that immediately after prices are announced at time a and based on the information available at time a (C being $\mathcal{F}_a-$measurable) we buy $C(\omega)$ security shares which we hold until time b. At that time, we sell them all.

Basic process III

Ito integral

Theorem

$\mathbb{F}-$adapted processes, the paths of which are left-continuous, are $\mathbb{F}-$predictable processes. In particular, $\mathbb{F}-$adapted processes with continuous paths are $\mathbb{F}-$predictible. (cf. Revuz and Yor)

Basic process IV

Ito integral

Definition

The **stochastic integral** of X with respect to the Brownian motion is defined as

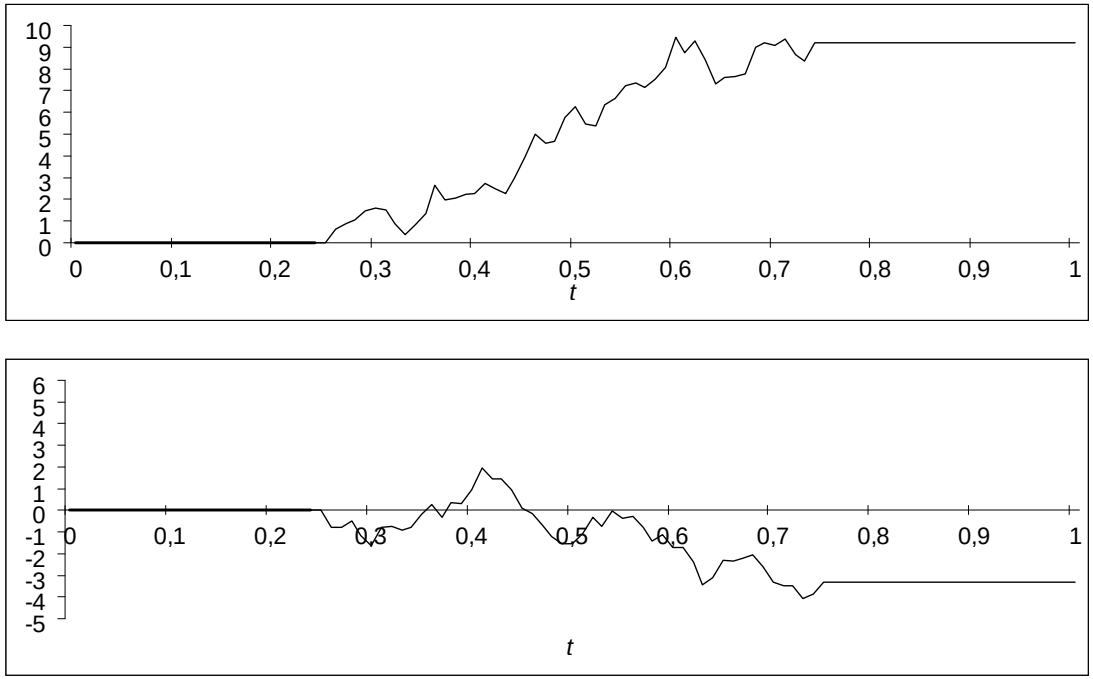
$$\begin{aligned} & \left(\int_0^t X_s dW_s\right)(\omega) \\ = \quad & C(\omega)\left(W_{t\wedge b}(\omega) - W_{t\wedge a}(\omega)\right) \\ = \quad & \begin{cases} 0 & \text{if } 0 \leq t \leq a \\ C(\omega)\left(W_t(\omega) - W_a(\omega)\right) & \text{if } a < t \leq b \\ C(\omega)\left(W_b(\omega) - W_a(\omega)\right) & \text{if } b < t. \end{cases} \end{aligned}$$

- Note that, for all t, the integral $\int_0^t X_s \, dW_s$ is a random variable.
- Moreover, $\left\{\int_0^t X_s \, dW_s : t \geq 0\right\}$ is a stochastic process.

Basic process V

Ito integral

Paths of a basic stochastic process, and its stochastic integral with respect to the Brownian motion



Integrale_stochastique.xls

Basic process VI

Ito integral

- Note that

$$\int_0^t X_s dW_s = \int_0^\infty X_s \mathbb{I}_{(0,t]}(s)\, dW_s.$$

- **Exercise**. Verify it!

Basic process VII

Ito integral

- **Interpretation.** If, for example, the Brownian motion W represents the variation of the share present value relative to its initial value ($W_t = S_t - S_0$, where S_t is the share present value at time t), then $\int_0^t X_s dW_s$ is the present value of the profit earned by the investor.
  - This example is somewhat lame: since the Brownian motion is not lower bounded, the example implies that it is possible for the share price to take negative values. Oops!
- Note on the graphs that, over the time period when the investor holds a constant number of security shares, the present value of his or her profit fluctuates. That is only caused by share price fluctuations.

Basic process VIII

Ito integral

Theorem

Lemma 1. *If X is a basic stochastic process, then*

$$\left\{ \int_0^t X_s dW_s : t \geq 0 \right\}$$

is a $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ −martingale.

Proof of Lemma 1. Recall that, if M is a martingale and τ a stopping time, then the stopped stochastic process $M^\tau = \{M_{t\wedge\tau} : t \geq 0\}$ is also a martingale.
Since the Brownian motion is a martingale, and τ_a and τ_b where $\forall \omega \in \Omega$, $\tau_a(\omega) = a$ and $\tau_b(\omega) = b$ are stopping times, then the stochastic processes $W^{\tau_a} = \{W_{t\wedge a} : t \geq 0\}$ and $W^{\tau_b} = \{W_{t\wedge b} : t \geq 0\}$ are both martingales.

Basic process IX

Ito integral

- Now, let's verify condition $(M1)$.

If $t \leq a$, then

$$\begin{aligned} \mathrm{E}^{\mathbb{P}}\left[\left|\int_0^t X_u dW_u\right|\right] &= \mathrm{E}^{\mathbb{P}}\left[\left|C\left(W_{t\wedge b} - W_{t\wedge a}\right)\right|\right] \\ &= \mathrm{E}^{\mathbb{P}}\left[\left|C\left(W_t - W_t\right)\right|\right] \\ &= 0. \end{aligned}$$

Basic process X

Ito integral

By contrast, if $t > a$, then

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{P}}\left[\left|\int_0^t X_u dW_u\right|\right] \\
= \; & \mathrm{E}^{\mathbb{P}}\left[\left|C\left(W_{t\wedge b} - W_a\right)\right|\right] = \mathrm{E}^{\mathbb{P}}\left[|C|\,\left|W_{t\wedge b} - W_a\right|\right] \\
\leq \; & \sqrt{\mathrm{E}^{\mathbb{P}}\left[C^2\left(W_{t\wedge b} - W_a\right)^2\right]} \text{(see next page)} \\
= \; & \sqrt{\mathrm{E}^{\mathbb{P}}\left[C^2\mathrm{E}^{\mathbb{P}}\left[\left(W_{t\wedge b} - W_a\right)^2 \middle| \mathcal{F}_a\right]\right]} \; C \text{ being } \mathcal{F}_a - \text{mble.} \\
= \; & \sqrt{\mathrm{E}^{\mathbb{P}}\left[C^2\mathrm{E}^{\mathbb{P}}\left[\left(W_{t\wedge b} - W_a\right)^2\right]\right]} \\
& W_{t\wedge b} - W_a \text{ being independent from } \mathcal{F}_a. \\
= \; & \sqrt{\left(t\wedge b - a\right)\mathrm{E}^{\mathbb{P}}\left[C^2\right]} < \infty
\end{aligned}
$$

Basic process XI

Ito integral

Justification for the inequality: for any random variable Y such that $\mathrm{E}\left[Y^2\right] < \infty$, we have

$$0 \leq Var\left[Y\right] = \mathrm{E}\left[Y^2\right] - \left(\mathrm{E}\left[Y\right]\right)^2$$

which implies that

$$\mathrm{E}\left[Y\right] \leq \sqrt{\mathrm{E}\left[Y^2\right]}.$$

Basic process XII

Ito integral

- With respect to condition $(M2)$,

$$
\begin{aligned}
&\left(\int_0^t X_s dW_s\right) \\
= &\begin{cases} 0 & \text{if } 0 \leq t \leq a & \mathcal{F}_0 - \text{measurable therefore } \mathcal{F}_t - \text{measurable} \\ C\left(W_t - W_a\right) & \text{if } a < t \leq b & \mathcal{F}_t - \text{measurable,} \\ C\left(W_b - W_a\right) & \text{if } b < t. & \mathcal{F}_b - \text{measurable therefore } \mathcal{F}_t - \text{measurable} \end{cases}
\end{aligned}
$$

Basic process XIII

Ito integral

- Condition $(M3)$ is also verified since $\forall s, t \in \mathbb{R}$, $0 \leq s < t$,

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^t X_u dW_u \,|\mathcal{F}_s\right] = \mathrm{E}^{\mathbb{P}}\left[C\left(W_{t\wedge b} - W_{t\wedge a}\right)|\mathcal{F}_s\right].$$

Basic process XIV

But, if $s \leq a$, then $\mathcal{F}_s \subseteq \mathcal{F}_a$. Since C is $\mathcal{F}_a-$ measurable, then

$$
\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[\int_0^t X_u dW_u \,|\mathcal{F}_s\right] &= \mathrm{E}^{\mathbb{P}}\left[\mathrm{E}^{\mathbb{P}}\left[C\left(W_{t\wedge b} - W_{t\wedge a}\right)|\mathcal{F}_a\right]|\mathcal{F}_s\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[C\mathrm{E}^{\mathbb{P}}\left[\left(W_{t\wedge b} - W_{t\wedge a}\right)|\mathcal{F}_a\right]|\mathcal{F}_s\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[C\underbrace{\left(W_{a\wedge b} - W_{a\wedge a}\right)}_{=W_a - W_a = 0}|\mathcal{F}_s\right] \\
&\quad \text{since } W^{\tau_a} \text{ and } W^{\tau_b} \text{ are martingales,} \\
&= 0 \\
&= \int_0^s X_u dW_u.
\end{aligned}
$$

Basic process XV

Ito integral

If $s > a$, then C is $\mathcal{F}_s-$ measurable and, as a consequence,

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[\int_0^t X_u dW_u \,|\mathcal{F}_s\right] &= \mathrm{E}^{\mathbb{P}}\left[C\left(W_{t\wedge b} - W_{t\wedge a}\right)|\mathcal{F}_s\right] \\
&= C\mathrm{E}^{\mathbb{P}}\left[\left(W_{t\wedge b} - W_{t\wedge a}\right)|\mathcal{F}_s\right] \\
&= C\left(W_{s\wedge b} - W_{s\wedge a}\right) \\
&= \int_0^s X_u dW_u.
\end{aligned}$$

The process $\int X_s dW_s$ is indeed a martingale. ■

Moments I

Basic process

- We will need a few results later on, that we shall now start to develop.
- Since the Ito integral for basic processes is a martingale, then

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^t X_s dW_s\right] = \mathrm{E}^{\mathbb{P}}\left[\int_0^0 X_s dW_s\right] = 0.$$

- The next result will allow us to calculate variances and covariances.

Moments II

Basic process

Theorem

Lemma 2. *If X and Y are basic processes, then for all $t \geq 0$,*

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^t X_s Y_s ds\right] = \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t X_s dW_s\right)\left(\int_0^t Y_s dW_s\right)\right].$$

Moments III

Basic process

Proof of Lemma 2. It will be sufficient to show that

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right] = \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^\infty X_s dW_s\right)\left(\int_0^\infty Y_s dW_s\right)\right] \quad (1)$$

since

$$X = C\mathbb{I}_{(a,b]} \text{ and } Y = \widetilde{C}\mathbb{I}_{\left(\widetilde{a},\widetilde{b}\right]}$$

being basic processes,

$$\begin{aligned} X\mathbb{I}_{(0,t]} &= C\mathbb{I}_{(a\wedge t, b\wedge t]}, \\ Y\mathbb{I}_{(0,t]} &= \widetilde{C}\mathbb{I}_{\left(\widetilde{a}\wedge t, \widetilde{b}\wedge t\right]} \\ \text{et } XY\mathbb{I}_{(0,t]} &= C\widetilde{C}\mathbb{I}_{\left((a\vee\widetilde{a})\wedge t, \left(b\wedge\widetilde{b}\right)\wedge t\right]} \end{aligned}$$

are also basic processes.

Moments IV

Basic process

Thus, if equation (1) is verified, we can use it to establish the third equality in the following calculation:

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{P}}\left[\int_0^t X_s Y_s ds\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s \mathbb{I}_{(0,t]}(s)\, ds\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty \left(X_s \mathbb{I}_{(0,t]}(s)\right)\left(Y_s \mathbb{I}_{(0,t]}(s)\right) ds\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^\infty X_s \mathbb{I}_{(0,t]}(s)\, dW_s\right)\left(\int_0^\infty Y_s \mathbb{I}_{(0,t]}(s)\, dW_s\right)\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t X_s dW_s\right)\left(\int_0^t Y_s dW_s\right)\right].
\end{aligned}
$$

Moments V

Basic process

So, let's establish Equation (1)

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right] = \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^\infty X_s dW_s\right)\left(\int_0^\infty Y_s dW_s\right)\right].$$

Three cases must be treated separately:

$$\begin{aligned}
&(i) \quad (a,b] \cap \left(\widetilde{a},\widetilde{b}\right] = \varnothing, \\
&(ii) \quad (a,b] = \left(\widetilde{a},\widetilde{b}\right], \\
&(iii) \quad (a,b] \neq \left(\widetilde{a},\widetilde{b}\right] \text{ and } (a,b] \cap \left(\widetilde{a},\widetilde{b}\right] \neq \varnothing.
\end{aligned}$$

Moments VI

Basic process

- **Proof of** (i).

Since $(a, b] \cap \left(\widetilde{a}, \widetilde{b}\right] = \varnothing$, we can assume, without loss of generality, that $a < b \leq \widetilde{a} < \widetilde{b}$. Since

$$X_s Y_s = C\widetilde{C}\mathbb{I}_{(a,b]}(s)\,\mathbb{I}_{\left(\widetilde{a},\widetilde{b}\right]}(s) = 0,$$

then $E^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right] = 0$.

Moments VII

Basic process

Now, since C, $\widetilde{C}$ and $(W_b - W_a)$ are $\mathcal{F}_{\widetilde{a}}-$measurable,

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^\infty X_s dW_s\right)\left(\int_0^\infty Y_s dW_s\right)\right] \\
= \; & \mathrm{E}^{\mathbb{P}}\left[C\left(W_b - W_a\right)\ \widetilde{C}\left(W_{\widetilde{b}} - W_{\widetilde{a}}\right)\right] \\
= \; & \mathrm{E}^{\mathbb{P}}\left[\mathrm{E}^{\mathbb{P}}\left[C\left(W_b - W_a\right)\ \widetilde{C}\left(W_{\widetilde{b}} - W_{\widetilde{a}}\right)|\mathcal{F}_{\widetilde{a}}\right]\right] \\
= \; & \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\left(W_b - W_a\right)\ \underbrace{\mathrm{E}^{\mathbb{P}}\left[W_{\widetilde{b}} - W_{\widetilde{a}}|\mathcal{F}_{\widetilde{a}}\right]}_{=W_{\widetilde{a}}-W_{\widetilde{a}}=0}\right] \\
= \; & 0 = \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right]
\end{aligned}
$$

thus establishing Equation (1) in this particular case. ■

Moments VIII

Basic process

- **Proof of** (ii). Let's now assume that $(a, b] = \left(\widetilde{a}, \widetilde{b}\right]$.

Then,

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right] &= \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty C\widetilde{C}\mathbb{I}_{(a,b]} dt\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\int_0^\infty \mathbb{I}_{(a,b]} dt\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\,(b-a)\right] \\
&= (b-a)\,\mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\right].
\end{aligned}$$

Moments IX

Basic process

On the other hand,

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^\infty X_s dW_s\right)\left(\int_0^\infty Y_s dW_s\right)\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[C\left(W_b - W_a\right)\ \widetilde{C}\left(W_b - W_a\right)\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\mathrm{E}^{\mathbb{P}}\left[\left(W_b - W_a\right)^2 \left|\mathcal{F}_a\right.\right]\right] \text{ since } C \text{ and } \widetilde{C} \text{ are } \mathcal{F}_a - \text{mbles} \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\mathrm{E}^{\mathbb{P}}\left[\left(W_b - W_a\right)^2\right]\right] \\
& \text{since } W_b - W_a \text{ is independent of } \mathcal{F}_a \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\left(b-a\right)\right] \text{ since } W_b - W_a \text{ is } \mathcal{N}\left(0, b-a\right) \\
= \quad & \left(b-a\right)\mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\right] = \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right]
\end{aligned}
$$

thus establishing Equation (1) for this other particular case.■

Moments X

Basic process

- **Proof of** (iii). We have that $(a,b] \neq \left(\widetilde{a},\widetilde{b}\right]$ and $(a,b] \cap \left(\widetilde{a},\widetilde{b}\right] \neq \varnothing$.

$a^* = a \vee \widetilde{a}$ and $b^* = b \wedge \widetilde{b}$. On the one hand,

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right] &= \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty C\mathbb{I}_{(a,b]}\widetilde{C}\mathbb{I}_{\left(\widetilde{a},\widetilde{b}\right]}dt\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty C\widetilde{C}\mathbb{I}_{(a^*,b^*]}dt\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\int_0^\infty \mathbb{I}_{(a^*,b^*]}dt\right] \\
&= \mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\left(b^*-a^*\right)\right] \\
&= \left(b^*-a^*\right)\mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\right].
\end{aligned}$$

Moments XI

Basic process

In what follows, in case there were intervals of the form $(\alpha, \beta]$ where $\alpha \geq \beta$, then we define $(\alpha, \beta] = \varnothing$. On the other hand, since

$$\begin{aligned}
& \int_0^\infty X_s dW_s \\
= \quad & C\left(W_b - W_a\right) \\
= \quad & C\left(W_b - W_{b^*} + W_{b^*} - W_{a^*} + W_{a^*} - W_a\right) \\
= \quad & C\left(W_b - W_{b^*}\right) + C\left(W_{b^*} - W_{a^*}\right) + C\left(W_{a^*} - W_a\right) \\
= \quad & \int_0^\infty C\mathbb{I}_{(b^*,b]} dW_s + \int_0^\infty C\mathbb{I}_{(a^*,b^*]} dW_s + \int_0^\infty C\mathbb{I}_{(a,a^*]} dW_s, \qquad (2)
\end{aligned}$$

Moments XII

Basic process

$$
\begin{aligned}
& \int_0^\infty Y_s dW_s \\
= \quad & \widetilde{C}\left(W_{\widetilde{b}} - W_{\widetilde{a}}\right) \\
= \quad & \widetilde{C}\left(W_{\widetilde{b}} - W_{b^*} + W_{b^*} - W_{a^*} + W_{a^*} - W_{\widetilde{a}}\right) \\
= \quad & \widetilde{C}\left(W_{\widetilde{b}} - W_{b^*}\right) + \widetilde{C}\left(W_{b^*} - W_{a^*}\right) + \widetilde{C}\left(W_{a^*} - W_{\widetilde{a}}\right) \\
= \quad & \int_0^\infty \widetilde{C}\mathbb{I}_{\left(b^*,\widetilde{b}\right]} dW_s + \int_0^\infty \widetilde{C}\mathbb{I}_{\left(a^*,b^*\right]} dW_s + \int_0^\infty \widetilde{C}\mathbb{I}_{\left(\widetilde{a},a^*\right]} dW_s \qquad (3)
\end{aligned}
$$

Moments XIII

Basic process

and

$$
\begin{aligned}
(a, a^*] \cap (\widetilde{a}, a^*] &= \varnothing, \ (a, a^*] \cap (a^*, b^*] = \varnothing, \ (a, a^*] \cap \left(b^*, \widetilde{b}\right] = \varnothing, \\
(a^*, b^*] \cap (\widetilde{a}, a^*] &= \varnothing, \ (a^*, b^*] \cap \left(b^*, \widetilde{b}\right] = \varnothing, \\
(b^*, b] \cap (\widetilde{a}, a^*] &= \varnothing, \ (b^*, b] \cap (a^*, b^*] = \varnothing, \ (b^*, b] \cap \left(b^*, \widetilde{b}\right] = \varnothing,
\end{aligned}
$$

we can use the results obtained in points (i) and (ii) in order to complete the proof: using the expressions from lines (2) and

Moments XIV

Basic process

(3),

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^\infty X_s dW_s\right)\left(\int_0^\infty Y_s dW_s\right)\right] \\
= \; & \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty \left(C\mathbb{I}_{(a^*,b^*]} dW_s\right)\left(\int_0^\infty \widetilde{C}\mathbb{I}_{(a^*,b^*]} dW_s\right)\right] \\
= \; & (b^* - a^*)\,\mathrm{E}^{\mathbb{P}}\left[C\widetilde{C}\right] \\
= \; & \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_s Y_s ds\right]. \blacksquare
\end{aligned}
$$

The proof of Lemma 2 is now complete. ■

Simple process I

Ito integral

Definition

We call X a **simple stochastic process** if X is a finite sum of basic processes :

$$X_t(\omega) = \sum_{i=1}^{n} C_i(\omega) \mathbb{I}_{(a_i, b_i]}(t).$$

- Since such a process is a sum of $\mathbb{F}-$predictable processes, then it is itself predictable with respect to the filtration $\mathbb{F}$.

Simple process II

Ito integral

Definition

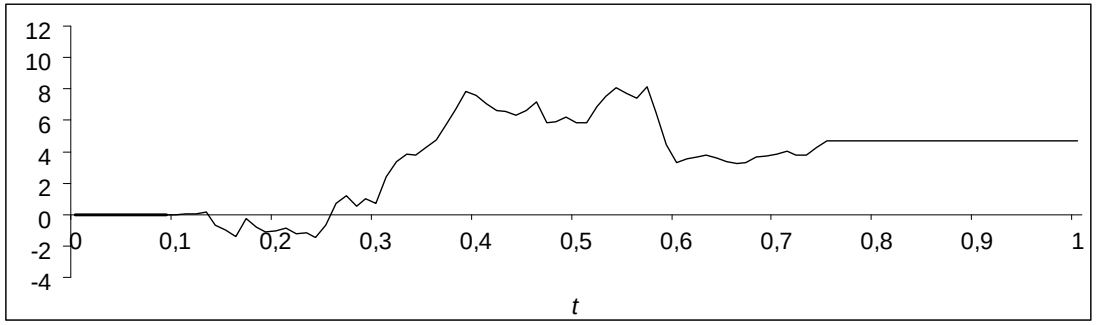
The **stochastic integral** of X with respect to the Brownian motion is defined as the sum of the stochastic integrals of the basic processes which constitute X :

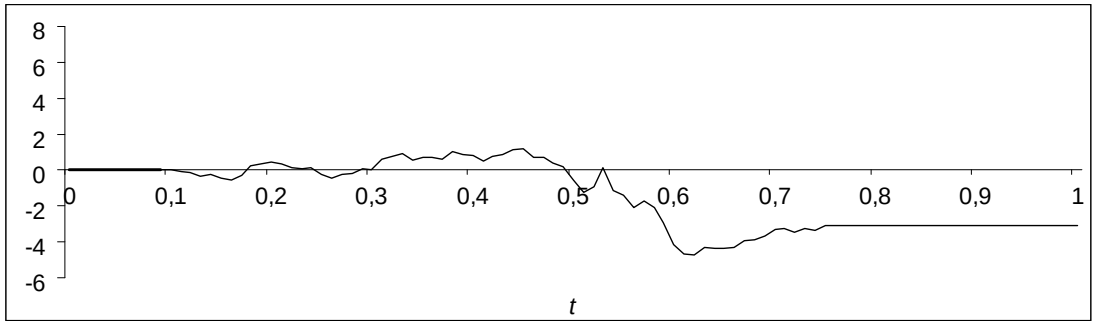
$$\int_0^t X_s dW_s = \sum_{i=1}^{n} \int_0^t C_i \mathbb{I}_{(a_i, b_i]}(s)\, dW_s.$$

Simple process III

Ito integral

Paths of a simple stochastic process and its stochastic integral with respect to the Brownian motion





The simple process is of the form $C_1 \mathbb{I}_{\left(\frac{1}{10},\frac{6}{10}\right]} + C_2 \mathbb{I}_{\left(\frac{1}{2},\frac{3}{4}\right]}$

Simple process IV

Ito integral

Integrale_stochastique.xls

Simple process V

Ito integral

- First, we must ensure that such a definition contains no inconsistencies, i.e. if the simple stochastic process X admits at least two representations in terms of basic stochastic processes, say $\sum_{i=1}^{n} C_i \mathbb{I}_{(a_i,b_i]}$ and $\sum_{j=1}^{m} \widetilde{C}_j \mathbb{I}_{\left(\widetilde{a}_j,\widetilde{b}_j\right]}$, then the integral is defined in a unique manner:

$$\sum_{i=1}^{n} \int_0^t C_i \mathbb{I}_{(a_i,b_i]}(s)\, dW_s = \sum_{j=1}^{m} \int_0^t \widetilde{C}_j \mathbb{I}_{\left(\widetilde{a}_j,\widetilde{b}_j\right]}(s)\, dW_s.$$

- **Exercise**. Prove that the definition of the stochastic integral for simple processes does not depend on the representation chosen.

Simple process VI

Theorem

Lemma 3. *If X is a simple stochastic process, then $\left\{\int_0^t X_s dW_s : t \geq 0\right\}$ is a $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})-$martingale.*

Proof of Lemma 3. The stochastic integral

$$\int_0^t X_s dW_s = \sum_{i=1}^n \int_0^t C_i \mathbb{I}_{(a_i, b_i]}(s)\, dW_s$$

of the simple process $X = \sum_{i=1}^n C_i \mathbb{I}_{(a_i, b_i]}$ is the sum of the stochastic integrals of the basic processes it is made up of. Since a finite sum of martingales is also a martingale, the results comes from the fact that stochastic integrals of basic processes are martingales (Lemma 1). ■

Simple process VII

Ito integral

The next result is rather technical and the proof can be found in the Appendix. It will useful to us later on.

Theorem

Lemma 4. *If X is a simple process, then for all $t \geq 0$,*

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^t X_s^2 ds\right] = \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t X_s dW_s\right)^2\right].$$

Simple process VIII

That lemma is, besides, very useful to calculate the variance of a stochastic integral. Indeed,

$$
\begin{aligned}
& \mathrm{Var}^{\mathbb{P}}\left[\int_0^t X_s dW_s\right] \\
= \; & \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t X_s dW_s\right)^2\right] - \left(\underbrace{\mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t X_s dW_s\right)\right]}_{=0}\right)^2 \\
= \; & \mathrm{E}^{\mathbb{P}}\left[\int_0^t X_s^2 ds\right] = \int_0^t \mathrm{E}^{\mathbb{P}}\left[X_s^2\right] ds.
\end{aligned}
$$

It is possible to extend this calculation to other processes X and toestablish in a similar manner a method to calculate the covariance between two stochastic integrals (see the Appendix).

Predictable processes I

Ito integral

- We would like to extend the class of processes for which the stochastic integral with respect to the Brownian motion can be defined. We will choose the class of $\mathbb{F}-$predictable processes for which there exists a sequence of simple processes approaching them.

Theorem

$\mathbb{F}-$adapted processes, the paths of which are left-continuous are $\mathbb{F}-$predictable processes. In particular, $\mathbb{F}-$adapted processes with continuous paths are $\mathbb{F}-$predictible.

- Let X be an $\mathbb{F}-$predictable process for which there exists a sequence $\left\{X^{(n)} : n \in \mathbb{N}\right\}$ of simple processes converging to X when n increases to infinity.

Predictable processes II

Ito integral

- You can't speak of convergence without speaking of distance. Indeed, how does one measure whether a sequence of stochastic processes "approaches" some other process? What is the distance between two stochastic processes? To answer such a question, we must define the space of stochastic processes on which we work as well as the norm we put on that space.
  - Recall that $\|\bullet\|_{\mathcal{A}} : \mathcal{A} \to [0, \infty)$ is a norm on the space $\mathcal{A}$ if

$$\begin{array}{ll} (i) & \|X\|_{\mathcal{A}} = 0 \Leftrightarrow X = 0; \\ (ii) & \forall X \in \mathcal{A} \text{ and } \forall a \in \mathbb{R},\ \|aX\|_{\mathcal{A}} = |a|\,\|X\|_{\mathcal{A}}; \\ (iii) & \forall X, Y \in \mathcal{A},\ \|X+Y\|_{\mathcal{A}} \leq \|X\|_{\mathcal{A}} + \|Y\|_{\mathcal{A}} \\ & \text{(triangle inequality).} \end{array}$$

Predictable processes III

Ito integral

- Let

$$\mathcal{A} = \left\{ X \,\middle|\, \begin{array}{l} X \text{ is a } \mathbb{F} - \text{predictable process} \\ \text{such that } \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_t^2 dt\right] < \infty \end{array} \right\}.$$

- **Exercise, optional and rather difficult!** The function $\|\bullet\|_{\mathcal{A}} : \mathcal{A} \to [0, \infty)$ defined as

$$\|X\|_{\mathcal{A}} = \sqrt{\mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_t^2 dt\right]}$$

is a norm on the space $\mathcal{A}$.

Predictable processes IV

Ito integral

- It is said that the sequence $\left\{X^{(n)} : n \in \mathbb{N}\right\} \subseteq \mathcal{A}$ converges to $X \in \mathcal{A}$ when n increases to infinity if and only if

$$\begin{aligned}\lim_{n\longrightarrow\infty}\left\|X^{(n)}-X\right\|_{\mathcal{A}} &= \lim_{n\longrightarrow\infty}\sqrt{\mathrm{E}^{\mathbb{P}}\left[\int_0^\infty\left(X_t^{(n)}-X_t\right)^2dt\right]}\\ &= 0.\end{aligned}$$

Predictable processes V

Ito integral

- It is tempting to define the stochastic integral of X with respect to the Brownian motion as the limit of the stochastic integrals of the simple processes, i.e.

$$\int_0^t X_s dW_s = \lim_{n \to \infty} \int_0^t X_s^{(n)} dW_s.$$

Predictable processes VI

Ito integral

- The latter equation raises two issues, one of which being whether such a limit exists.
- For **example**, let's assume we work on the space of strictly positive numbers $A \equiv \{x \in \mathbb{R} \,|\, x > 0\}$ and we study the sequence $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. That sequence does not converge in the space A since $\lim_{n\to\infty} \frac{1}{n} = 0 \notin A$. Of course, that sequence converges in space $\mathbb{R}$.
  - The question is: does $\lim_{n\to\infty} \int_0^t X_s^{(n)} dW_s$ exist in the space in which we work?
  - The second issue is that this equation implies that, the greater n , the closer $\int_0^t X_s^{(n)} dW_s$ is to $\int_0^t X_s dW_s$. But, what is the distance between two stochastic integrals?

Predictable processes VII

Ito integral

- Recall that, in Lemma 3, we have established that, for all n, the integral $\int X_s^{(n)} dW_s$ of a simple process is a martingale.
- Moreover, it is possible to establish that

$$\mathrm{E}\left[\left(\int X_s^{(n)} dW_s\right)^2\right] < \infty.$$

Predictable processes VIII

- As a consequence, the sequence used to approach $\int X_s \, dW_s$ is contained in the space

$$\mathcal{M} = \left\{ M \,\middle|\, \begin{array}{l} M \text{ is a martingale} \\ \text{such that } \sqrt{\sup_{t \geq 0} \mathrm{E}^{\mathbb{P}} \left[M_t^2 \right]} < \infty \end{array} \right\}.$$

- **Exercise, optional and laborious!** The function $\|\bullet\|_{\mathcal{M}} : \mathcal{M} \to [0, \infty)$ defined as

$$\|M\|_{\mathcal{M}} = \sqrt{\sup_{t \geq 0} \mathrm{E}^{\mathbb{P}} \left[M_t^2 \right]}$$

is a norm on $\mathcal{M}$.

Predictable processes IX

Ito integral

- Thus, we say that the sequence $\left\{M^{(n)} : n \in \mathbb{N}\right\}$ of martingales belonging to the normed space $(\mathcal{M}, \|\bullet\|_{\mathcal{M}})$ converges to M if and only if $M \in \mathcal{M}$ and

$$\begin{aligned}\lim_{n\to\infty}\left\|M^{(n)} - M\right\|_{\mathcal{M}} &= \lim_{n\to\infty}\sqrt{\sup_{t\geq 0}\mathrm{E}^{\mathbb{P}}\left[\left(M_t^{(n)} - M_t\right)^2\right]} \\ &= 0.\end{aligned}$$

Predictable processes X

Ito integral

- The idea behind the construction of $\int_0^t X_s \, dW_s$ is that, on the one hand, we have the space $(\mathcal{A}, \|\bullet\|_{\mathcal{A}})$ of the processes for which we construct the stochastic integral and, on the other hand, we have the space $(\mathcal{M}, \|\bullet\|_{\mathcal{M}})$ containing the stochastic integrals of the processes in the first space.
- But, we have chosen the norms in such a way that, if the process X is a simple process, then $\|X\|_{\mathcal{A}} = \left\|\int X_s \, dW_s\right\|_{\mathcal{M}}$ (Ref.: Lemma 5 in the Appendix).

Predictable processes XI

- This implies that, if the sequence of simple processes $\left\{X^{(n)} : n \in \mathbb{N}\right\}$ is a Cauchy sequence in the first space, i.e.

$$\left\|X^{(n)} - X^{(m)}\right\|_{\mathcal{A}} \xrightarrow{m,n\to\infty} 0,$$

then the sequence $\left\{\int_0^t X_s^{(n)} dW_s : n \in \mathbb{N}\right\}$ of stochastic integrals is a Cauchy sequence in the second space:

$$\left\|\int_0^t X_s^{(n)} dW_s - \int X_s^{(m)} dW_s\right\|_{\mathcal{M}} \xrightarrow{m,n\to\infty} 0.$$

- The sequence $\left\{M^{(n)} : n \in \mathbb{N}\right\}$ is a Cauchy sequence on the normed space $(\mathcal{M}, \|\bullet\|_{\mathcal{M}})$ if

$$\lim_{n,m\to\infty} \left\|M^{(n)} - M^{(m)}\right\|_{\mathcal{M}} = 0.$$

Predictable processes XII

Ito integral

- If $\left\{M^{(n)} : n \in \mathbb{N}\right\}$ is a Cauchy sequence, we cannot, in general, state that such a sequence converges, since we are not certain that the limit-point belongs to the set $\mathcal{M}$.

Predictable processes XIII

Ito integral

- The only thing left to verify is that the limit-point of the sequence of stochastic integrals $\left\{ \int_0^t X_s^{(n)} dW_s : n \in \mathbb{N} \right\}$ is indeed an element of $\mathcal{M}$ (ref.: Lemma 6 in the Appendix).
- We will then be able to define the stochastic integral of the predictable process $X = \lim_{n\to\infty} X^{(n)}$ as $\lim_{n\to\infty} \int_0^t X_s^{(n)} dW_s$.

In summary

Predictible process

- We have defined the stochastic integral with respect to the $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ −Brownian motion for all X $\mathbb{F}$−predictable processes satisfying the condition

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^\infty X_t^2 dt\right] < \infty.$$

- For each of these processes, we have established that the family of stochastic integrals $\left\{\int_0^t X_s dW_s : t \geq 0\right\}$ is a $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ −martingale.

Generalization I

Predictible process

- It is possible to extend the class of processes for which the stochastic integral can be defined.
- It involves local martingales as well as semi-martingales.
- We refer to the book by Richard Durrett those who would like to know more.
- Let's however mention that it is possible, for such processes, that the family of stochastic integrals $\left\{ \int_0^t X_s dW_s : t \geq 0 \right\}$ is no longer a martingale.

Generalization II

Predictible process

Here is a result allowing us to verify whether the stochastic integral with which we work is a martingale :

Theorem

Lemma. *If X is a $F-$predictible process such that $E^{\mathbb{P}}\left[\int_0^t X_s^2 ds\right] < \infty$ then $\left\{\int_0^s X_s dW_s : 0 \leq s \leq t\right\}$ is a $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})-$martingale (ref. Revuz and Yor, Corollary 1.25, page 124).*

Generalization III

Predictible process

- It is possible to define the stochastic integral with respect to other stochastic processes than the Brownian motion by using the same construction as the one we have just provided.

References I

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- KARATZAS, Ioannis and SHREVE, Steven E. (1988). *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.
- KARLIN, Samuel and TAYLOR, Howard M. (1975). *A First Course in Stochastic Processes,* second edition, Academic Press, New York.
- LAMBERTON, Damien and LAPEYRE, Bernard (1991). *Introduction au calcul stochastique appliqué à la finance*, Éllipses, Paris.
- REVUZ, Daniel and YOR, Marc (1991). *Continuous Martingale and Brownian Motion*, Springer-Verlag, New York.

Lemma 4 I

Here is a second result similar to Lemma 2, but for simple processes.

Theorem

Lemma 4. *If X is a simple process, then for all $t \geq 0$,*

$$\mathrm{E}^{\mathbb{P}}\left[\int_0^t X_s^2 \, ds\right] = \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t X_s \, dW_s\right)^2\right].$$

Lemma 4 II

Proof of Lemma 4. Let $X = \sum_{i=1}^{n} C_i \mathbb{I}_{(a_i,b_i]}$ be a simple process.

$$
\begin{aligned}
& \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t X_s dW_s\right)^2\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[\left(\sum_{i=1}^{n}\int_0^t C_i \mathbb{I}_{(a_i,b_i]}(s)\, dW_s\right)^2\right] \\
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}^{\mathbb{P}}\left(\int_0^t C_i \mathbb{I}_{(a_i,b_i]}(s)\, dW_s\right)\left(\int_0^t C_j \mathbb{I}_{(a_j,b_j]}(s)\, dW_s\right) \\
= \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}^{\mathbb{P}}\left[\int_0^t C_i \mathbb{I}_{(a_i,b_i]}(s)\ \ C_j \mathbb{I}_{(a_j,b_j]}(s)\ \ ds\right] \text{ by Lemma 2.} \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[\int_0^t \sum_{i=1}^{n}\sum_{j=1}^{n} C_i \mathbb{I}_{(a_i,b_i]}(s)\ \ C_j \mathbb{I}_{(a_j,b_j]}(s)\ \ ds\right] \\
= \quad & \mathrm{E}^{\mathbb{P}}\left[\int_0^t \left(\sum_{i=1}^{n} C_i \mathbb{I}_{(a_i,b_i]}(s)\right)^2 ds\right] = \mathrm{E}^{\mathbb{P}}\left[\int_0^t X_s^2\ ds\right]. \ \blacksquare
\end{aligned}
$$

Lemma 5 I

Theorem

Lemma 5. *For any simple process Y, $\|Y\|_{\mathcal{A}} = \left\|\int Y_s dW_s\right\|_{\mathcal{M}}$.*

Proof of Lemma 5. Note that $\forall \omega \in \Omega$, the function $t \to \int_0^t Y_s^2(\omega)\, ds$ as well as the function $t \to E^{\mathbb{P}}\left[\int_0^t Y_s^2 ds\right]$

Lemma 5 II

are non-decreasing. As a consequence,

$$
\begin{aligned}
\|Y\|_{\mathcal{A}}^2 &= \mathrm{E}^{\mathbb{P}}\left[\int_0^\infty Y_s^2 ds\right] = \mathrm{E}^{\mathbb{P}}\left[\lim_{t\to\infty}\int_0^t Y_s^2 ds\right] \\
&= \lim_{t\to\infty} \mathrm{E}^{\mathbb{P}}\left[\int_0^t Y_s^2 ds\right] \\
&\quad \text{by monotone convergence theorem.} \\
&= \sup_{t\geq 0} \mathrm{E}^{\mathbb{P}}\left[\int_0^t Y_s^2 ds\right] \\
&= \sup_{t\geq 0} \mathrm{E}^{\mathbb{P}}\left[\left(\int_0^t Y_s dW_s\right)^2\right] \quad \text{by Lemma 4.} \\
&= \left\|\int Y_s dW_s\right\|_{\mathcal{M}}^2 \quad \text{by the very definition of } \|\bullet\|_{\mathcal{M}}. \ \blacksquare
\end{aligned}
$$

Lemma 5 III

From the latter result, it follows that the sequence $\left\{\int X_s^{(n)} dW_s : n \in \mathbb{N}\right\}$ of stochastic integrals of simple processes with respect to the Brownian motion is a Cauchy sequence since

$$\begin{aligned} & \lim_{n,m\to\infty} \left\| \int X_s^{(n)} dW_s - \int X_s^{(m)} dW_s \right\|_{\mathcal{M}} \\ = & \lim_{n,m\to\infty} \left\| X^{(n)} - X^{(m)} \right\|_{\mathcal{A}} \\ = & 0 \end{aligned}$$

where the first equality is a direct consequence from Lemma 5, $X^{(n)}$ and $X^{(m)}$ being simple stochastic processes, and the second equality comes from the fact that the sequence $\left\{X^{(n)} : n \in \mathbb{N}\right\}$ is a Cauchy sequence since it converges to X.

Lemma 6 I

Theorem

Lemma 6. *The normed space* $(\mathcal{M}, \|\bullet\|_{\mathcal{M}})$ *is complete*[a] *(Durrett, 1996, Theorem 4.6, page 57).*

[a] A normed space is said to be complete if any Cauchy sequence converges, i.e. the limit-point of any Cauchy sequence belongs to the space.

Since $\left\{\int X_s^{(n)} dW_s : n \in \mathbb{N}\right\}$ is a Cauchy sequence on the complete space $(\mathcal{M}, \|\bullet\|_{\mathcal{M}})$ then such a sequence converges, thus establishing the existence of what we have called $\int X_s dW_s$, and its limit-point belongs to $(\mathcal{M}, \|\bullet\|_{\mathcal{M}})$, which implies that the stochastic integral of X with respect to W, $\int X_s dW_s$, is a martingale, even for some adapted processes which are not simple processes.

Comments I

- It is possible to prove that the definition of the stochastic integral, for the predictable stochastic processes $X \in \mathcal{A}$ which own a sequence of simple processes approaching them, does not depend on the sequence chosen, i.e. if $\left\{X^{(n)} : n \in \mathbb{N}\right\}$ and $\left\{\widetilde{X}^{(n)} : n \in \mathbb{N}\right\}$ are such two sequences of simple processes, that $\lim_{n\to\infty} \left\|X^{(n)} - X\right\|_{\mathcal{A}} = 0$ and $\lim_{n\to\infty} \left\|\widetilde{X}^{(n)} - X\right\|_{\mathcal{A}} = 0$, then
$$\lim_{n\to\infty} \int X_s^{(n)} dW_s = \lim_{n\to\infty} \int \widetilde{X}^{(n)} dW_s.$$

Comments II

- Last comment: for any predictable stochastic process $X \in (\mathcal{A}, \|\bullet\|_{\mathcal{A}})$, there exists such a sequence $\left\{X^{(n)} : n \in \mathbb{N}\right\}$ of simple processes that

$$\lim_{n\to\infty} \left\| X^{(n)} - X \right\|_{\mathcal{A}} = 0.$$

(Durrett, 1996, Lemma 4.5, page 57)