Stochastic integral

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Stochastic Calculus I

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The theories of stochastic integral and stochastic differential equations have initially been developed by Kiyosi Ito around 1940 (one of the first important papers was published in 1942).
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Typically, when we speak of the integral of a function $f$, we refer to the Riemann integral, $\int_{a}^{b} f(t) \, dt$, which calculates the area under the curve $t \to f(t)$ between the bounds $a$ and $b$. 
First, we must realize that not all functions are Riemann integrable.

Let’s explain a little further: in order to construct the Riemann integral of the function $f$ on the interval $[a, b]$, we divide such an interval into $n$ subintervals of equal length $\left(\frac{b-a}{n}\right)$ and, for each of them, we determine the greatest and the smallest value taken by the function $f$. 
Thus, for every \( k \in \{1, \ldots, n\} \), the smallest value taken by the function \( f \) on the interval \([a + (k - 1) \frac{b-a}{n}, a + k \frac{b-a}{n}]\) is

\[
\underline{f}_k^{(n)} \equiv \inf \left\{ f(t) \mid a + (k - 1) \frac{b-a}{n} \leq t \leq a + k \frac{b-a}{n} \right\}
\]

and the greatest is

\[
\bar{f}_k^{(n)} \equiv \sup \left\{ f(t) \mid a + (k - 1) \frac{b-a}{n} \leq t \leq a + k \frac{b-a}{n} \right\}.
\]
Riemann integral IV

- The area under the curve $t \to f(t)$ between the bounds $a + (k - 1) \frac{b-a}{n}$ and $a + k \frac{b-a}{n}$ is therefore greater than the area $\frac{b-a}{n} f_k^{(n)}$ of the rectangle of base $\frac{b-a}{n}$ and height $f_k^{(n)}$, and smaller than the area $\frac{b-a}{n} \overline{f}_k^{(n)}$ of the rectangle with the same base but with height $\overline{f}_k^{(n)}$.

- As a consequence, if we define

$$ l_n(f) \equiv \sum_{k=1}^{n} \frac{b-a}{n} f_k^{(n)} \quad \text{and} \quad \overline{l}_n(f) \equiv \sum_{k=1}^{n} \frac{b-a}{n} \overline{f}_k^{(n)} $$

then

$$ l_n(f) \leq \int_a^b f(t) \, dt \leq \overline{l}_n(f). $$
Riemann integral V

Illustrations of $I_n(f)$ and $\bar{I}_n(f)$
Riemann integrable functions are those for which the limits \( \lim_{n \to \infty} L_n(f) \) and \( \lim_{n \to \infty} \bar{I}_n(f) \) exist and are equal.

We then define the Riemann integral of the function \( f \) as

\[
\int_a^b f(t) \, dt \equiv \lim_{n \to \infty} L_n(f) = \lim_{n \to \infty} \bar{I}_n(f).
\]

It is possible to show that the integral \( \int_a^b f(t) \, dt \) exists for any function \( f \) continuous on the interval \([a, b]\). Many other functions are Riemann integrable.
However, there exist functions that are not Riemann integrable.

For example, let’s consider the indicator function \( \mathbb{1}_\mathbb{Q} : [0, 1] \rightarrow \{0, 1\} \) which takes the value of 1 if the argument is rational and 0 otherwise.

Then, for all natural number \( n \), \( \underline{I}_n = 0 \) and \( \bar{I}_n = 1 \), which implies that the Riemann integral is not defined for such a function.
We will examine a very simple case: the Riemann integral for boxcar functions, i.e. the functions $f : [0; \infty) \to \mathbb{R}$ which admit the representation $f(t) = c I_{(a,b)}(t), \ a < b, \ c \in \mathbb{R}$ where $I_{(a,b)}$ represents the indicator function

$$I_{(a,b)}(t) = \begin{cases} 1 & \text{if } t \in (a, b] \\ 0 & \text{if } t \notin (a, b] \end{cases}.$$
If \( f(t) = c \mathbb{I}_{(a,b]}(t) \) then the Riemann integral of \( f \) is

\[
\int_0^t f(s) \, ds = \int_0^t c \mathbb{I}_{(a,b]}(s) \, ds
\]

\[
= \begin{cases} 
0 & \text{if } t \leq a \\
(c(t - a)) & \text{if } a < t \leq b \\
c(b - a) & \text{if } t > b.
\end{cases}
\]
Let \( \{ W_t : t \geq 0 \} \) be a standard Brownian motion built on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), \( \mathbb{F} = \{ \mathcal{F}_t : t \geq 0 \} \).

**Technical condition.** Since we will work with almost sure equalities \(^1\), we require that the set of events which have zero probability to occur are included in the sigma-algebra \( \mathcal{F}_0 \), i.e. that the set

\[
\mathcal{N} = \{ A \in \mathcal{F} : \mathbb{P}(A) = 0 \} \subset \mathcal{F}_0.
\]

That way, if \( X \) is \( \mathcal{F}_t \)-measurable and \( Y = X \mathbb{P} \)-almost surely, then we know that \( Y \) is \( \mathcal{F}_t \)-measurable.
Ito integral II

- Let \( \{ X_t : t \geq 0 \} \) be a predictable stochastic process. It is tempting to define the stochastic integral of \( X \) with respect to \( W \), pathwise, by using a generalization of the Stieltjes integral:

\[
\forall \omega \in \Omega, \quad \int X(\omega) \, dW(\omega).
\]

- That would be possible if the paths of Brownian motion \( W \) had bounded variation, i.e. if they could be expressed as the difference of two non-decreasing functions.

- But, since Brownian motion paths do not have bounded variation, it is not possible to use such an approach.

- We must define the stochastic integral with respect to the Brownian motion as a whole, i.e. as a stochastic process in itself, and not pathwise.

\[ X = Y \text{ } P\text{-almost surely if the set of } \omega \text{ for which } X \text{ is different from } Y \text{ has a probability of zero, i.e.} \]

\[ P \{ \omega \in \Omega : X(\omega) \neq Y(\omega) \} = 0. \]
Definition

We call $X$ a **basic stochastic process** if $X$ admits the following representation:

$$X_t (\omega) = C (\omega) \mathbb{I}_{(a,b]} (t)$$

where $a < b \in \mathbb{R}$ and $C$ is a random variable, $\mathcal{F}_a$—measurable and square-integrable, i.e. $E[|C|^2] < \infty$.

Note that such a process is adapted to the filtration $\mathbb{F}$. Indeed,

$$X_t = \begin{cases} 
0 & \text{if } 0 \leq t \leq a \\
C & \text{if } a < t \leq b \\
0 & \text{if } b < t
\end{cases}$$

which is $\mathcal{F}_0$—measurable therefore $\mathcal{F}_t$—measurable

which is $\mathcal{F}_a$—measurable therefore $\mathcal{F}_t$—measurable

which is $\mathcal{F}_0$—measurable therefore $\mathcal{F}_t$—measurable.
Basic process II

In fact, \( X \) is more than \( \mathbb{F} \)-adapted, it is \( \mathbb{F} \)-predictable, but the notion of *predictable process* is more delicate to define when we work with continuous-time processes.

Note, nevertheless, that adapted processes with continuous path are predictable.

Intuitively, if \( X_t \) represents the number of security shares held at time \( t \), then \( X_t (\omega) = C (\omega) \mathbb{1}_{(a,b]} (t) \) means that immediately after prices are announced at time \( a \) and based on the information available at time \( a \) (\( C \) being \( \mathcal{F}_a \)-measurable) we buy \( C (\omega) \) security shares which we hold until time \( b \). At that time, we sell them all.
Theorem

$\mathcal{F}$—adapted processes, the paths of which are left-continuous, are $\mathcal{F}$—predictable processes. In particular, $\mathcal{F}$—adapted processes with continuous paths are $\mathcal{F}$—predictable. (cf. Revuz and Yor)
Basic process IV

Ito integral

Definition

The **stochastic integral** of $X$ with respect to the Brownian motion is defined as

$$
\left( \int_0^t X_s \, dW_s \right)(\omega)
= C(\omega) \left( W_{t \wedge b}(\omega) - W_{t \wedge a}(\omega) \right)
= \begin{cases} 
0 & \text{if } 0 \leq t \leq a \\
C(\omega) \left( W_t(\omega) - W_a(\omega) \right) & \text{if } a < t \leq b \\
C(\omega) \left( W_b(\omega) - W_a(\omega) \right) & \text{if } b < t.
\end{cases}
$$

- Note that, for all $t$, the integral $\int_0^t X_s \, dW_s$ is a random variable.
- Moreover, $\left\{ \int_0^t X_s \, dW_s : t \geq 0 \right\}$ is a stochastic process.
Basic process V
Ito integral

Paths of a basic stochastic process, and its stochastic integral with respect to the Brownian motion

Integrale_stochastique.xls
Basic process VI

Ito integral

Note that

\[ \int_0^t X_s \, dW_s = \int_0^\infty X_s I_{(0,t]}(s) \, dW_s. \]

Exercise. Verify it!
**Interpretation.** If, for example, the Brownian motion $W$ represents the variation of the share present value relative to its initial value ($W_t = S_t - S_0$, where $S_t$ is the share present value at time $t$), then $\int_0^t X_s dW_s$ is the present value of the profit earned by the investor.

This example is somewhat lame: since the Brownian motion is not lower bounded, the example implies that it is possible for the share price to take negative values. Oops!

Note on the graphs that, over the time period when the investor holds a constant number of security shares, the present value of his or her profit fluctuates. That is only caused by share price fluctuations.
Basic process VIII

Ito integral

Theorem

Lemma 1. If $X$ is a basic stochastic process, then

$$\{ \int_0^t X_s dW_s : t \geq 0 \}$$

is a $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$-martingale.

Proof of Lemma 1. Recall that, if $M$ is a martingale and $\tau$ a stopping time, then the stopped stochastic process $M^\tau = \{ M_{t\wedge \tau} : t \geq 0 \}$ is also a martingale.

Since the Brownian motion is a martingale, and $\tau_a$ and $\tau_b$ where $\forall \omega \in \Omega$, $\tau_a (\omega) = a$ and $\tau_b (\omega) = b$ are stopping times, then the stochastic processes $W^{\tau_a} = \{ W_{t\wedge a} : t \geq 0 \}$ and $W^{\tau_b} = \{ W_{t\wedge b} : t \geq 0 \}$ are both martingales.
Now, let's verify condition $(M1)$.

If $t \leq a$, then

$$
\mathbb{E}^P \left[ \int_0^t X_u dW_u \right] = \mathbb{E}^P \left[ C(W_{t\wedge b} - W_{t\wedge a}) \right] = \mathbb{E}^P \left[ |C(W_t - W_t)| \right] = 0.
$$
By contrast, if \( t > a \), then

\[
\mathbb{E}^P \left[ \left| \int_0^t X_u dW_u \right| \right] \\
= \mathbb{E}^P \left[ |C (W_{t\wedge b} - W_a)| \right] = \mathbb{E}^P \left[ |C| |W_{t\wedge b} - W_a| \right] \\
\leq \sqrt{\mathbb{E}^P \left[ C^2 (W_{t\wedge b} - W_a)^2 \right]} (\text{see next page}) \\
= \sqrt{\mathbb{E}^P \left[ C^2 \mathbb{E}^P \left[ (W_{t\wedge b} - W_a)^2 \mid \mathcal{F}_a \right] \right]} \quad C \text{ being } \mathcal{F}_a - \text{mble.} \\
= \sqrt{\mathbb{E}^P \left[ C^2 \mathbb{E}^P \left[ (W_{t\wedge b} - W_a)^2 \right] \right]} \\
W_{t\wedge b} - W_a \text{ being independent from } \mathcal{F}_a. \\
= \sqrt{(t \wedge b - a) \mathbb{E}^P \left[ C^2 \right]} < \infty
\]
Justification for the inequality: for any random variable $Y$ such that $E[Y^2] < \infty$, we have

$$0 \leq \text{Var}[Y] = E[Y^2] - (E[Y])^2$$

which implies that

$$E[Y] \leq \sqrt{E[Y^2]}.$$
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With respect to condition $(M2)$,

\[
\left( \int_0^t X_s \, dW_s \right) = \begin{cases} 
0 & \text{if } 0 \leq t \leq a \\
C \left( W_t - W_a \right) & \text{if } a < t \leq b \\
C \left( W_b - W_a \right) & \text{if } b < t.
\end{cases}
\]

\( \mathcal{F}_0 \) — measurable therefore \( \mathcal{F}_t \) — measurable,
\( \mathcal{F}_t \) — measurable,
\( \mathcal{F}_b \) — measurable therefore \( \mathcal{F}_t \) — measurable.
Condition (M3) is also verified since $\forall s, t \in \mathbb{R}, 0 \leq s < t$,

$$
\mathbb{E}^\mathbb{P} \left[ \int_0^t X_u dW_u \mid \mathcal{F}_s \right] = \mathbb{E}^\mathbb{P} \left[ C \left( W_{t \wedge b} - W_{t \wedge a} \right) \mid \mathcal{F}_s \right].
$$
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Ito integral

But, if \( s \leq a \), then \( \mathcal{F}_s \subseteq \mathcal{F}_a \). Since \( C \) is \( \mathcal{F}_a \)-measurable, then

\[
\mathbb{E}^\mathbb{P} \left[ \int_0^t X_u dW_u \mid \mathcal{F}_s \right] = \mathbb{E}^\mathbb{P} \left[ \mathbb{E}^\mathbb{P} \left[ C \left( W_{t \wedge b} - W_{t \wedge a} \right) \mid \mathcal{F}_a \right] \mid \mathcal{F}_s \right]
\]

\[
= \mathbb{E}^\mathbb{P} \left[ C \mathbb{E}^\mathbb{P} \left[ (W_{t \wedge b} - W_{t \wedge a}) \mid \mathcal{F}_a \right] \mid \mathcal{F}_s \right]
\]

\[
= \mathbb{E}^\mathbb{P} \left[ C \left( W_{a \wedge b} - W_{a \wedge a} \right) \mid \mathcal{F}_s \right]
\]

\[
= \mathbb{E}^\mathbb{P} \left[ C \left( W_{a \wedge b} - W_{a \wedge a} \right) \mid \mathcal{F}_s \right] = 0
\]

since \( W^{\tau_a} \) and \( W^{\tau_b} \) are martingales,

\[
= \int_0^s X_u dW_u.
\]
If $s > a$, then $C$ is $\mathcal{F}_s$ – measurable and, as a consequence,

\[
\begin{align*}
\mathbb{E}^\mathbb{P} \left[ \int_0^t X_u \, dW_u \mid \mathcal{F}_s \right] &= \mathbb{E}^\mathbb{P} [C (W_{t \wedge b} - W_{t \wedge a}) \mid \mathcal{F}_s] \\
&= CE^\mathbb{P} [(W_{t \wedge b} - W_{t \wedge a}) \mid \mathcal{F}_s] \\
&= C (W_{s \wedge b} - W_{s \wedge a}) \\
&= \int_0^s X_u \, dW_u.
\end{align*}
\]

The process $\int X_s \, dW_s$ is indeed a martingale. ■
Moments 1

Basic process

- We will need a few results later on, that we shall now start to develop.
- Since the Ito integral for basic processes is a martingale, then

\[ E_P \left[ \int_0^t X_s dW_s \right] = E_P \left[ \int_0^0 X_s dW_s \right] = 0. \]

- The next result will allow us to calculate variances and covariances.
Theorem

**Lemma 2.** If $X$ and $Y$ are basic processes, then for all $t \geq 0$,

$$
\mathbb{E}_{\mathbb{P}} \left[ \int_0^t X_s Y_s \, ds \right] = \mathbb{E}_{\mathbb{P}} \left[ \left( \int_0^t X_s \, dW_s \right) \left( \int_0^t Y_s \, dW_s \right) \right].
$$
Proof of Lemma 2. It will be sufficient to show that

\[ \mathbb{E}^P \left[ \int_0^\infty X_s Y_s \, ds \right] = \mathbb{E}^P \left[ \left( \int_0^\infty X_s \, dW_s \right) \left( \int_0^\infty Y_s \, dW_s \right) \right] \] (1)

since

\[ X = C \mathcal{I}_{(a,b]} \] and \[ Y = \tilde{C} \mathcal{I}_{(\tilde{a},\tilde{b}]} \]

being basic processes,

\[ X \mathcal{I}_{(0,t]} = C \mathcal{I}_{(a \wedge t, b \wedge t]}, \]

\[ Y \mathcal{I}_{(0,t]} = \tilde{C} \mathcal{I}_{(\tilde{a} \wedge t, \tilde{b} \wedge t]} \]

et \[ X Y \mathcal{I}_{(0,t]} = C \tilde{C} \mathcal{I}_{(a \vee \tilde{a}) \wedge t, (b \wedge \tilde{b}) \wedge t} \]

are also basic processes.
Thus, if equation (1) is verified, we can use it to establish the third equality in the following calculation:

\[
\begin{align*}
\mathbb{E}^P \left[ \int_0^t X_s Y_s \, ds \right] &= \mathbb{E}^P \left[ \int_0^\infty X_s Y_s I_{(0,t]}(s) \, ds \right] \\
&= \mathbb{E}^P \left[ \int_0^\infty \left( X_s I_{(0,t]}(s) \right) \left( Y_s I_{(0,t]}(s) \right) \, ds \right] \\
&= \mathbb{E}^P \left[ \left( \int_0^\infty X_s I_{(0,t]}(s) \, dW_s \right) \left( \int_0^\infty Y_s I_{(0,t]}(s) \, dW_s \right) \right] \\
&= \mathbb{E}^P \left[ \left( \int_0^t X_s \, dW_s \right) \left( \int_0^t Y_s \, dW_s \right) \right]
\end{align*}
\]
So, let’s establish Equation (1)

$$\mathbb{E}^\mathbb{P} \left[ \int_0^\infty X_s Y_s ds \right] = \mathbb{E}^\mathbb{P} \left[ \left( \int_0^\infty X_s dW_s \right) \left( \int_0^\infty Y_s dW_s \right) \right].$$

Three cases must be treated separately:

(i) $$(a, b] \cap \left( \tilde{a}, \tilde{b} \right] = \emptyset,$$

(ii) $$a, b] = \left( \tilde{a}, \tilde{b} \right],$$

(iii) $$(a, b] \neq \left( \tilde{a}, \tilde{b} \right]$$ and $$(a, b] \cap \left( \tilde{a}, \tilde{b} \right] \neq \emptyset.$$
Proof of (i).

Since \((a, b] \cap (\tilde{a}, \tilde{b}] = \emptyset\), we can assume, without loss of generality, that \(a < b \leq \tilde{a} < \tilde{b}\). Since

\[
X_s Y_s = C \tilde{C} \mathbb{I}_{(a,b]}(s) \mathbb{I}_{(\tilde{a},\tilde{b}]}(s) = 0,
\]

then \(E^{\mathbb{P}} \left[ \int_0^\infty X_s Y_s ds \right] = 0\).
Now, since $C$, $\tilde{C}$ and $(W_b - W_a)$ are $\mathcal{F}_{\tilde{a}}$-measurable,

\[
\begin{align*}
E^P \left[ \left( \int_0^\infty X_s dW_s \right) \left( \int_0^\infty Y_s dW_s \right) \right] \\
= E^P \left[ C (W_b - W_a) \tilde{C} (W_{\tilde{b}} - W_{\tilde{a}}) \right] \\
= E^P \left[ E^P \left[ C (W_b - W_a) \tilde{C} (W_{\tilde{b}} - W_{\tilde{a}}) \mid \mathcal{F}_{\tilde{a}} \right] \right] \\
= E^P \left[ C \tilde{C} (W_b - W_a) E^P \left[ W_{\tilde{b}} - W_{\tilde{a}} \mid \mathcal{F}_{\tilde{a}} \right] \right] \\
= E^P \left[ \int_0^\infty X_s Y_s ds \right] \\
\text{thus establishing Equation (1) in this particular case.} \quad \blacksquare
\end{align*}
\]
Proof of \((ii)\). Let’s now assume that \((a, b) = (\tilde{a}, \tilde{b})\).

Then,

\[
\begin{align*}
\mathbb{E}^{P} \left[ \int_{0}^{\infty} X_s Y_s ds \right] &= \mathbb{E}^{P} \left[ \int_{0}^{\infty} C \tilde{C} \mathbb{I}_{(a, b]} dt \right] \\
&= \mathbb{E}^{P} \left[ C \tilde{C} \int_{0}^{\infty} \mathbb{I}_{(a, b]} dt \right] \\
&= \mathbb{E}^{P} \left[ C \tilde{C} (b - a) \right] \\
&= (b - a) \mathbb{E}^{P} \left[ C \tilde{C} \right].
\end{align*}
\]
On the other hand,

\[
\begin{align*}
\mathbb{E}^\mathcal{P} & \left[ \left( \int_0^\infty X_s dW_s \right) \left( \int_0^\infty Y_s dW_s \right) \right] \\
& = \mathbb{E}^\mathcal{P} \left[ C (W_b - W_a) \tilde{C} (W_b - W_a) \right] \\
& = \mathbb{E}^\mathcal{P} \left[ C \tilde{C} \mathbb{E}^\mathcal{P} \left[ (W_b - W_a)^2 | \mathcal{F}_a \right] \right] \text{ since } C \text{ and } \tilde{C} \text{ are } \mathcal{F}_a \text{ -mbles} \\
& = \mathbb{E}^\mathcal{P} \left[ C \tilde{C} \mathbb{E}^\mathcal{P} \left[ (W_b - W_a)^2 \right] \right] \text{ since } W_b - W_a \text{ is independent of } \mathcal{F}_a \\
& = \mathbb{E}^\mathcal{P} \left[ C \tilde{C} (b - a) \right] \text{ since } W_b - W_a \text{ is } \mathcal{N} (0, b - a) \\
& = (b - a) \mathbb{E}^\mathcal{P} \left[ C \tilde{C} \right] = \mathbb{E}^\mathcal{P} \left[ \int_0^\infty X_s Y_s ds \right] \\
\end{align*}
\]

thus establishing Equation (1) for this other particular case.
Moments \( X \)

**Basic process**

- **Proof of (iii).** We have that \((a, b) \neq (\tilde{a}, \tilde{b})\) and \((a, b) \cap (\tilde{a}, \tilde{b}) \neq \emptyset\).

\[ a^* = a \lor \tilde{a} \text{ and } b^* = b \land \tilde{b}. \]

On the one hand,

\[
\mathbb{E}^P \left[ \int_0^\infty X_s Y_s \, ds \right] = \mathbb{E}^P \left[ \int_0^\infty C \mathbb{I}_{(a,b]} \tilde{C} \mathbb{I}_{(\tilde{a},\tilde{b})} \, dt \right]
\]

\[
= \mathbb{E}^P \left[ \int_0^\infty C \tilde{C} \mathbb{I}_{(a^*,b^*]} \, dt \right]
\]

\[
= \mathbb{E}^P \left[ C \tilde{C} \int_0^\infty \mathbb{I}_{(a^*,b^*]} \, dt \right]
\]

\[
= \mathbb{E}^P \left[ C \tilde{C} (b^* - a^*) \right]
\]

\[
= (b^* - a^*) \mathbb{E}^P \left[ C \tilde{C} \right].
\]
In what follows, in case there were intervals of the form \((\alpha, \beta]\) where \(\alpha \geq \beta\), then we define \((\alpha, \beta] = \emptyset\). On the other hand, since

\[
\int_0^\infty X_s dW_s = C (W_b - W_a) \\
= C (W_b - W_{b^*} + W_{b^*} - W_{a^*} + W_{a^*} - W_a) \\
= C (W_b - W_{b^*}) + C (W_{b^*} - W_{a^*}) + C (W_{a^*} - W_a) \\
= \int_0^\infty C \Pi_{(b^*, b]} dW_s + \int_0^\infty C \Pi_{(a^*, b^*]} dW_s + \int_0^\infty C \Pi_{(a, a^*]} dW_s,
\]

(2)
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\[
\begin{align*}
\int_0^\infty Y_s dW_s \\
= \tilde{C} (W_b - W_a) \\
= \tilde{C} (W_b - W_{b^*} + W_{b^*} - W_{a^*} + W_{a^*} - W_a) \\
= \tilde{C} (W_b - W_{b^*}) + \tilde{C} (W_{b^*} - W_{a^*}) + \tilde{C} (W_{a^*} - W_a) \\
= \int_0^\infty \tilde{C} \Pi(b^*, b) dW_s + \int_0^\infty \tilde{C} \Pi(a^*, b^*) dW_s + \int_0^\infty \tilde{C} \Pi(\tilde{a}, a^*) dW_s
\end{align*}
\]
and
\[
\begin{align*}
(a, a^\ast] \cap (\tilde{a}, a^\ast] &= \emptyset, \\
(a, a^\ast] \cap (a^\ast, b^\ast] &= \emptyset, \\
(a, a^\ast] \cap (b^\ast, \tilde{b}] &= \emptyset, \\
(a^\ast, b^\ast] \cap (\tilde{a}, a^\ast] &= \emptyset, \\
(a^\ast, b^\ast] \cap (b^\ast, \tilde{b}] &= \emptyset, \\
(b^\ast, b] \cap (\tilde{a}, a^\ast] &= \emptyset, \\
(b^\ast, b] \cap (a^\ast, b^\ast] &= \emptyset, \\
(b^\ast, b] \cap (b^\ast, \tilde{b}] &= \emptyset,
\end{align*}
\]
we can use the results obtained in points (i) and (ii) in order to complete the proof: using the expressions from lines (2) and
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(3),

\[
\mathbb{E}^P \left[ \left( \int_0^\infty X_s dW_s \right) \left( \int_0^\infty Y_s dW_s \right) \right]
= \mathbb{E}^P \left[ \int_0^\infty \left( C \mathbb{I}_{(a^*, b^*]} dW_s \right) \left( \int_0^\infty \tilde{C} \mathbb{I}_{(a^*, b^*]} dW_s \right) \right]
= (b^* - a^*) \mathbb{E}^P \left[ C \tilde{C} \right]
= \mathbb{E}^P \left[ \int_0^\infty X_s Y_s ds \right].
\]

The proof of Lemma 2 is now complete.
We call $X$ a **simple stochastic process** if $X$ is a finite sum of basic processes:

$$X_t(\omega) = \sum_{i=1}^{n} C_i(\omega) \mathbb{I}_{(a_i, b_i]}(t).$$

Since such a process is a sum of $\mathcal{F}$–predictable processes, then it is itself predictable with respect to the filtration $\mathcal{F}$. 
Definition

The **stochastic integral** of $X$ with respect to the Brownian motion is defined as the sum of the stochastic integrals of the basic processes which constitute $X$:

$$
\int_0^t X_s dW_s = \sum_{i=1}^n \int_0^t C_i \mathbb{1}_{(a_i, b_i]}(s) \, dW_s.
$$
Paths of a simple stochastic process and its stochastic integral with respect to the Brownian motion

The simple process is of the form $C_1 I_{\left(\frac{1}{10}, \frac{6}{10}\right]} + C_2 I_{\left(\frac{1}{2}, \frac{3}{4}\right]}$
Simple process IV
Ito integral

Integrale_stochastique.xls
First, we must ensure that such a definition contains no inconsistencies, i.e. if the simple stochastic process $X$ admits at least two representations in terms of basic stochastic processes, say $\sum_{i=1}^{n} C_i \mathbb{1}_{(a_i,b_i]}$ and $\sum_{j=1}^{m} \tilde{C}_j \mathbb{1}_{(\tilde{a}_j,\tilde{b}_j]}$, then the integral is defined in a unique manner:

$$\sum_{i=1}^{n} \int_{0}^{t} C_i \mathbb{1}_{(a_i,b_i]} (s) \, dW_s = \sum_{j=1}^{m} \int_{0}^{t} \tilde{C}_j \mathbb{1}_{(\tilde{a}_j,\tilde{b}_j]} (s) \, dW_s.$$

**Exercise.** Prove that the definition of the stochastic integral for simple processes does not depend on the representation chosen.
Theorem

**Lemma 3.** If $X$ is a simple stochastic process, then

$$\left\{ \int_0^t X_s dW_s : t \geq 0 \right\}$$

is a $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$—martingale.

**Proof of Lemma 3.** The stochastic integral

$$\int_0^t X_s dW_s = \sum_{i=1}^n \int_0^t C_i \mathbb{I}_{(a_i, b_i]} (s) dW_s$$

of the simple process $X = \sum_{i=1}^n C_i \mathbb{I}_{(a_i, b_i]}$ is the sum of the stochastic integrals of the basic processes it is made up of. Since a finite sum of martingales is also a martingale, the results comes from the fact that stochastic integrals of basic processes are martingales (Lemma 1). $\blacksquare$
The next result is rather technical and the proof can be found in the Appendix. It will useful to us later on.

**Theorem**

**Lemma 4.** If $X$ is a simple process, then for all $t \geq 0$,

$$
\mathbb{E}^P \left[ \int_0^t X_s^2 \, ds \right] = \mathbb{E}^P \left[ \left( \int_0^t X_s \, dW_s \right)^2 \right].
$$
That lemma is, besides, very useful to calculate the variance of a stochastic integral. Indeed,

\[
\text{Var}^\text{IP} \left[ \int_0^t X_s dW_s \right] = 
\text{E}^\text{IP} \left[ \left( \int_0^t X_s dW_s \right)^2 \right] - \left( \text{E}^\text{IP} \left[ \int_0^t X_s dW_s \right] \right)^2
= \text{E}^\text{IP} \left[ \int_0^t X_s^2 ds \right] = \int_0^t \text{E}^\text{IP} [X_s^2] ds.
\]

It is possible to extend this calculation to other processes \( X \) and to establish in a similar manner a method to calculate the covariance between two stochastic integrals (see the Appendix).
We would like to extend the class of processes for which the stochastic integral with respect to the Brownian motion can be defined. We will choose the class of $\mathbb{F}$—predictable processes for which there exists a sequence of simple processes approaching them.

**Theorem**

$\mathbb{F}$—adapted processes, the paths of which are left-continuous are $\mathbb{F}$—predictable processes. In particular, $\mathbb{F}$—adapted processes with continuous paths are $\mathbb{F}$—predictable.

Let $X$ be an $\mathbb{F}$—predictable process for which there exists a sequence $\{X^{(n)} : n \in \mathbb{N}\}$ of simple processes converging to $X$ when $n$ increases to infinity.
You can’t speak of convergence without speaking of distance. Indeed, how does one measure whether a sequence of stochastic processes "approaches" some other process? What is the distance between two stochastic processes? To answer such a question, we must define the space of stochastic processes on which we work as well as the norm we put on that space.

Recall that \( \| \bullet \|_A : A \to [0, \infty) \) is a norm on the space \( A \) if

1. \( \| X \|_A = 0 \iff X = 0; \)
2. \( \forall X \in A \text{ and } \forall a \in \mathbb{R}, \| aX \|_A = |a| \| X \|_A; \)
3. \( \forall X, Y \in A, \| X + Y \|_A \leq \| X \|_A + \| Y \|_A \) (triangle inequality).
Predictable processes III
Ito integral

- Let

\[ \mathcal{A} = \left\{ X \mid X \text{ is a } \mathcal{F} - \text{predictable process such that } \mathbb{E}^{\mathbb{P}} \left[ \int_0^\infty X_t^2 \, dt \right] < \infty \right\}. \]

- Exercise, optional and rather difficult! The function \( \| \cdot \|_\mathcal{A} : \mathcal{A} \to [0, \infty) \) defined as

\[ \| X \|_\mathcal{A} = \sqrt{\mathbb{E}^{\mathbb{P}} \left[ \int_0^\infty X_t^2 \, dt \right]} \]

is a norm on the space \( \mathcal{A} \).
It is said that the sequence \( \{X^{(n)} : n \in \mathbb{N}\} \subseteq \mathcal{A} \) converges to \( X \in \mathcal{A} \) when \( n \) increases to infinity if and only if

\[
\lim_{n \to \infty} \left\| X^{(n)} - X \right\|_{\mathcal{A}} = \lim_{n \to \infty} \sqrt{\mathbb{E}^P \left[ \int_0^\infty (X^{(n)}_t - X_t)^2 \ dt \right]} = 0.
\]
It is tempting to define the stochastic integral of $X$ with respect to the Brownian motion as the limit of the stochastic integrals of the simple processes, i.e.

$$\int_0^t X_s \, dW_s = \lim_{n \to \infty} \int_0^t X_s^{(n)} \, dW_s.$$
The latter equation raises two issues, one of which being whether such a limit exists.

For example, let's assume we work on the space of strictly positive numbers $A \equiv \{ x \in \mathbb{R} \mid x > 0 \}$ and we study the sequence $\{ \frac{1}{n} : n \in \mathbb{N} \}$. That sequence does not converge in the space $A$ since $\lim_{n \to \infty} \frac{1}{n} = 0 \notin A$. Of course, that sequence converges in space $\mathbb{R}$.

The question is: does $\lim_{n \to \infty} \int_0^t X_s^{(n)} dW_s$ exist in the space in which we work?

The second issue is that this equation implies that, the greater $n$, the closer $\int_0^t X_s^{(n)} dW_s$ is to $\int_0^t X_s dW_s$. But, what is the distance between two stochastic integrals?
Recall that, in Lemma 3, we have established that, for all \( n \), the integral \( \int X_s^{(n)} \, dW_s \) of a simple process is a martingale.

Moreover, it is possible to establish that

\[
E \left[ \left( \int X_s^{(n)} \, dW_s \right)^2 \right] < \infty.
\]
Predictable processes VIII

Ito integral

- As a consequence, the sequence used to approach \( \int X_s \, dW_s \) is contained in the space

\[
\mathcal{M} = \left\{ M \mid M \text{ is a martingale such that} \sup_{t \geq 0} E^P [M_t^2] < \infty \right\}.
\]

- Exercise, optional and laborious! The function \( \| \bullet \|_\mathcal{M} : \mathcal{M} \to [0, \infty) \) defined as

\[
\|M\|_\mathcal{M} = \sqrt{\sup_{t \geq 0} E^P [M_t^2]}
\]

is a norm on \( \mathcal{M} \).
Thus, we say that the sequence \( \{ M^{(n)} : n \in \mathbb{N} \} \) of martingales belonging to the normed space \((\mathcal{M}, \| \cdot \|_{\mathcal{M}})\) converges to \( M \) if and only if \( M \in \mathcal{M} \) and

\[
\lim_{n \to \infty} \left\| M^{(n)} - M \right\|_{\mathcal{M}} = \lim_{n \to \infty} \sqrt{\sup_{t \geq 0} \mathbb{E}^\mathbb{P} \left[ \left( M^{(n)}_t - M_t \right)^2 \right]} = 0.
\]
The idea behind the construction of $\int_0^t X_s \, dW_s$ is that, on the one hand, we have the space $(\mathcal{A}, \| \cdot \|_\mathcal{A})$ of the processes for which we construct the stochastic integral and, on the other hand, we have the space $(\mathcal{M}, \| \cdot \|_\mathcal{M})$ containing the stochastic integrals of the processes in the first space.

But, we have chosen the norms in such a way that, if the process $X$ is a simple process, then $\| X \|_\mathcal{A} = \| \int X_s \, dW_s \|_\mathcal{M}$ (Ref.: Lemma 5 in the Appendix).
This implies that, if the sequence of simple processes
\( \{X^{(n)} : n \in \mathbb{N}\} \) is a Cauchy sequence in the first space, i.e.
\[
\left\| X^{(n)} - X^{(m)} \right\|_A \xrightarrow{m,n \to \infty} 0,
\]
then the sequence \( \{\int_0^t X_s^{(n)} dW_s : n \in \mathbb{N}\} \) of stochastic integrals is a Cauchy sequence in the second space:
\[
\left\| \int_0^t X_s^{(n)} dW_s - \int_0^t X_s^{(m)} dW_s \right\|_M \xrightarrow{m,n \to \infty} 0.
\]

The sequence \( \{M^{(n)} : n \in \mathbb{N}\} \) is a Cauchy sequence on the normed space \((M, \|\cdot\|_M)\) if
\[
\lim_{n,m \to \infty} \left\| M^{(n)} - M^{(m)} \right\|_M = 0.
\]
If \( \{ M^{(n)} : n \in \mathbb{N} \} \) is a Cauchy sequence, we cannot, in general, state that such a sequence converges, since we are not certain that the limit-point belongs to the set \( M \).
The only thing left to verify is that the limit-point of the sequence of stochastic integrals \( \left\{ \int_0^t X_s^{(n)} \, dW_s : n \in \mathbb{N} \right\} \) is indeed an element of \( \mathcal{M} \) (ref.: Lemma 6 in the Appendix).

We will then be able to define the stochastic integral of the predictable process \( X = \lim_{n \to \infty} X^{(n)} \) as
\[
\lim_{n \to \infty} \int_0^t X_s^{(n)} \, dW_s.
\]
In summary
Predictable process

- We have defined the stochastic integral with respect to the \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) — Brownian motion for all \(X\) \(\mathbb{F}\)—predictable processes satisfying the condition

\[
\mathbb{E}^{\mathbb{P}} \left[ \int_{0}^{\infty} X_t^2 dt \right] < \infty.
\]

- For each of these processes, we have established that the family of stochastic integrals \(\left\{ \int_{0}^{t} X_s dW_s : t \geq 0 \right\}\) is a \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) — martingale.
It is possible to extend the class of processes for which the stochastic integral can be defined.

It involves local martingales as well as semi-martingales.

We refer to the book by Richard Durrett those who would like to know more.

Let’s however mention that it is possible, for such processes, that the family of stochastic integrals
\[ \left\{ \int_0^t X_s dW_s : t \geq 0 \right\} \]
is no longer a martingale.
Generalization II

Predictable process

Here is a result allowing us to verify whether the stochastic integral with which we work is a martingale:

**Theorem**

**Lemma.** If $X$ is a $F-$predictable process such that
\[
E^{\mathbb{P}} \left[ \int_0^t X_s^2 \, ds \right] < \infty \quad \text{then} \quad \left\{ \int_0^s X_s \, dW_s : 0 \leq s \leq t \right\} \quad \text{is a} \\
(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})-\text{martingale} \quad \text{(ref. Revuz and Yor, Corollary 1.25, page 124).}
\]
It is possible to define the stochastic integral with respect to other stochastic processes than the Brownian motion by using the same construction as the one we have just provided.
References I


Here is a second result similar to Lemma 2, but for simple processes.

**Theorem**

**Lemma 4.** If $X$ is a simple process, then for all $t \geq 0$,

$$
\mathbb{E}^\mathbb{P} \left[ \int_0^t X_s^2 \, ds \right] = \mathbb{E}^\mathbb{P} \left[ \left( \int_0^t X_s \, dW_s \right)^2 \right].
$$
Lemma 4 II

Proof of Lemma 4. Let \( X = \sum_{i=1}^{n} C_i \mathbb{I}_{(a_i, b_i]} \) be a simple process.

\[
\begin{align*}
E^P \left[ \left( \int_0^t X_s dW_s \right)^2 \right] &= E^P \left[ \left( \sum_{i=1}^{n} \int_0^t C_i \mathbb{I}_{(a_i, b_i]} (s) dW_s \right)^2 \right] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} E^P \left[ \int_0^t C_i \mathbb{I}_{(a_i, b_i]} (s) dW_s \right] \left( \int_0^t C_j \mathbb{I}_{(a_j, b_j]} (s) dW_s \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} E^P \left[ \int_0^t C_i \mathbb{I}_{(a_i, b_i]} (s) \ C_j \mathbb{I}_{(a_j, b_j]} (s) \ ds \right] \text{ by Lemma 2.} \\
&= E^P \left[ \int_0^t \sum_{i=1}^{n} \sum_{j=1}^{n} C_i \mathbb{I}_{(a_i, b_i]} (s) \ C_j \mathbb{I}_{(a_j, b_j]} (s) \ ds \right] \\
&= E^P \left[ \int_0^t \left( \sum_{i=1}^{n} C_i \mathbb{I}_{(a_i, b_i]} (s) \right)^2 \ ds \right] = E^P \left[ \int_0^t X_s^2 \ ds \right]. \quad \blacksquare
\end{align*}
\]
Lemma 5. For any simple process $Y$, $\| Y \|_\mathcal{A} = \| \int Y_s dW_s \|_\mathcal{M}$.

Proof of Lemma 5. Note that $\forall \omega \in \Omega$, the function $t \rightarrow \int_0^t Y_s^2(\omega) \, ds$ as well as the function $t \rightarrow E^\Pi \left[ \int_0^t Y_s^2 \, ds \right]$
are non-decreasing. As a consequence,

\[ \| Y \|_A^2 = E^P \left[ \int_0^\infty Y_s^2 ds \right] = E^P \left[ \lim_{t \to \infty} \int_0^t Y_s^2 ds \right] \]

\[ = \lim_{t \to \infty} E^P \left[ \int_0^t Y_s^2 ds \right] \]

by monotone convergence theorem.

\[ = \sup_{t \geq 0} E^P \left[ \int_0^t Y_s^2 ds \right] \]

\[ = \sup_{t \geq 0} E^P \left[ \left( \int_0^t Y_s dW_s \right)^2 \right] \]

by Lemma 4.

\[ = \left\| \int Y_s dW_s \right\|_M^2 \]

by the very definition of \( \| \cdot \|_M \).
Lemma 5 III

From the latter result, it follows that the sequence
\[ \left\{ \int X^{(n)}_s \, dW_s : n \in \mathbb{N} \right\} \]
of stochastic integrals of simple processes with respect to the Brownian motion is a Cauchy sequence since

\[
\lim_{n,m \to \infty} \left\| \int X^{(n)}_s \, dW_s - \int X^{(m)}_s \, dW_s \right\|_{\mathcal{M}} = 0
\]

where the first equality is a direct consequence from Lemma 5, \( X^{(n)} \) and \( X^{(m)} \) being simple stochastic processes, and the second equality comes from the fact that the sequence
\[ \left\{ X^{(n)} : n \in \mathbb{N} \right\} \]
is a Cauchy sequence since it converges to \( X \).
Lemma 6. The normed space \((\mathcal{M}, \| \cdot \|_\mathcal{M})\) is complete\(^a\) \((Durrett, 1996, \text{Theorem 4.6, page 57})\).

\(^a\)A normed space is said to be complete if any Cauchy sequence converges, i.e. the limit-point of any Cauchy sequence belongs to the space.

Since \(\left\{ \int X_s^{(n)} \, dW_s : n \in \mathbb{N} \right\}\) is a Cauchy sequence on the complete space \((\mathcal{M}, \| \cdot \|_\mathcal{M})\) then such a sequence converges, thus establishing the existence of what we have called \(\int X_s \, dW_s\), and its limit-point belongs to \((\mathcal{M}, \| \cdot \|_\mathcal{M})\), which implies that the stochastic integral of \(X\) with respect to \(W\), \(\int X_s \, dW_s\), is a martingale, even for some adapted processes which are not simple processes.
It is possible to prove that the definition of the stochastic integral, for the predictable stochastic processes $X \in \mathcal{A}$ which own a sequence of simple processes approaching them, does not depend on the sequence chosen, i.e. if $\left\{X^{(n)} : n \in \mathbb{N}\right\}$ and $\left\{\tilde{X}^{(n)} : n \in \mathbb{N}\right\}$ are such two sequences of simple processes, that
\[
\lim_{n \to \infty} \left\|X^{(n)} - X\right\|_\mathcal{A} = 0 \quad \text{and} \quad \lim_{n \to \infty} \left\|\tilde{X}^{(n)} - X\right\|_\mathcal{A} = 0,
\]
then
\[
\lim_{n \to \infty} \int X^{(n)}_s \, dW_s = \lim_{n \to \infty} \int \tilde{X}^{(n)}_s \, dW_s.
\]
Last comment: for any predictable stochastic process $X \in (\mathcal{A}, \| \cdot \|_{\mathcal{A}})$, there exists such a sequence $
abla X^{(n)} : n \in \mathbb{N}$ of simple processes that

$$\lim_{n \to \infty} \| X^{(n)} - X \|_{\mathcal{A}} = 0.$$  

(Durrett, 1996, Lemma 4.5, page 57)