The Binomial Model
Exercises

Exercise 5.1. The market model is

ω	$(S_0^1; S_0^2)$	$(S_1^1; S_1^2)$	$(S_2^1; S_2^2)$	C
ω_1	$\left(\frac{1}{1{,}21}; 2\right)$	$\left(\frac{1}{1{,}1}; 2\right)$	$(1; 2)$	0
ω_2	$\left(\frac{1}{1{,}21}; 2\right)$	$\left(\frac{1}{1{,}1}; 2\right)$	$(1; 3)$	0
ω_3	$\left(\frac{1}{1{,}21}; 2\right)$	$\left(\frac{1}{1{,}1}; 4\right)$	$(1; 3)$	0
ω_4	$\left(\frac{1}{1{,}21}; 2\right)$	$\left(\frac{1}{1{,}1}; 4\right)$	$(1; 4)$	1
ω_5	$\left(\frac{1}{1{,}21}; 2\right)$	$\left(\frac{1}{1{,}1}; 4\right)$	$(1; 6)$	3

$$\begin{aligned} \text{where } S_t^1(\omega) &= \text{price of a share of the riskless asset at time } t, \\ \text{and } S_t^2(\omega) &= \text{price of a share of the risky asset at time } t. \end{aligned}$$

In general, the "true" probability measure on $(\Omega, \mathcal{F})$ is not known. However, each market participant has its own opinion about $\mathbb{P}$. On2 of them believe that it is very likely that at time $t = 2$, the price of the risky asset will be below 3 dollars. That is why he offers on the market a call option with strike price $K = 3$, that is, the contingent claim payoff is $C(\omega) = \max\{S_2^2(\omega) - 3; 0\}$.

a) What is the probability space ?
b) Show that this market model does not admit arbitrage opportunity.
c) Find a self-financing trading strategy ϕ that allows the option seller to hedge.
d) What is $V_0(\phi)$?
e) Find all martingale measures.
f) For each martingale measure, compute the expected discounted option payoff. What do you notice?

Exercise 5.2. Three assets are modeled using the stochastic process

Ω	$t = 0$	$t = 1$
ω_1	$(1, 1, 1)$	$(1 + r, x_1, y_1)$
ω_2	$(1, 1, 1)$	$(1 + r, x_1, y_2)$
ω_3	$(1, 1, 1)$	$(1 + r, x_2, y_1)$
ω_4	$(1, 1, 1)$	$(1 + r, x_2, y_2)$

Without loss of generality, we may assume that $x_1 < x_2$, $y_1 < y_2$ and $x_1 < y_1$.
a) What are the conditions so that the market model do not admit arbitrage opportunities?
b) Is the market model complete? Justify.

Exercise 5.3. A two-period market model is formed with a risky asset $\{S(t) : t = 0, 1, 2\}$ paying dividends $\{D(t) : t = 0, 1, 2\}$ and a bank account $\{B(t) : t = 0, 1, 2\}$ such that $B(t) = (1 + r)^t$. Without loss of generality, we may assume that $s_{11} + d_{11} < s_{12} + d_{12}$, $s_{21} + d_{21} < s_{22} + d_{22}$ and $s_{23} + d_{23} < s_{24} + d_{24}$.

$t = 0$		$t = 1$		$t = 2$	ω	$\mathbb{Q}$
				s_{24}, d_{24} $(1 + r)^2$	ω_4	q_4
			$\nearrow$			
		s_{12}, d_{12} $1 + r$	$\longrightarrow$	s_{23}, d_{23} $(1 + r)^2$	ω_3	q_3
	$\nearrow$					
$s_0, 0$ 1						
	$\searrow$					
		s_{11}, d_{11} $1 + r$	$\longrightarrow$	s_{22}, d_{22} $(1 + r)^2$	ω_2	q_2
			$\searrow$			
				s_{21}, d_{21} $(1 + r)^2$	ω_1	q_1

a) Why should we revise the self-financing condition in that case? What should it be?
b) Construct the risk neutral measure.
c) What are the conditions for no arbitrage?
d) What is the time $t = 0$ price of the contingent claim paying at time $t = 1$ the amount f_{11} if ω_1 or ω_2 happen and f_{12} if ω_3 or ω_4 occur ?

Solutions

1 Exercise 5.1

a)

$$\begin{aligned}
\Omega &= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \\
\mathcal{F} &= \text{ all possible subsets of } \Omega \\
\mathcal{F}_0 &= \{\varnothing, \Omega\} \\
\mathcal{F}_1 &= \sigma\left\{\{\omega_1, \omega\}, \{\omega_3, \omega_4, \omega_5\}\right\} \\
\mathcal{F}_2 &= \mathcal{F}
\end{aligned}$$

b) it can be seen as several single period binomial trees. It is also possible to go back to the definition, just like in the Appendix 1.

For the period going from $t = 0$ to $t = 1$, it is a one-period binomial tree. The prevailing interest rate is provided by the riskless asset:

$$r = \frac{\frac{1}{1.1} - \frac{1}{1.21}}{\frac{1}{1.21}} = 0.1.$$

Since $2 < 2 \times (1 + 0.1) = 2.2 < 4$, there is no arbitrage opportunity for that period.
For the time period going from $t = 1$ to $t = 2$, we have to consider two cases : $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4, \omega_5\}$. On $\{\omega_1, \omega_2\}$, it is a one-period binomial model. The interest rate comes from the riskless asset :

$$r = \frac{1 - \frac{1}{1.1}}{\frac{1}{1.1}} = 0.1.$$

Since $2 < 2 \times (1 + 0.1) = 2.2 < 3$, there is no arbitrage. On $\{\omega_3, \omega_4, \omega_5\}$, it is a one-period trinomial model. The interest rate is

$$r = \frac{1 - \frac{1}{1.1}}{\frac{1}{1.1}} = 0.1.$$

There will be arbitrage opportunities if there is a portfolio $\phi = (\phi_1, \phi_2)$ such that

$$\begin{aligned}
&(A1) \quad \forall \omega \in \{\omega_3, \omega_4, \omega_5\}, \; V_\phi(0, \omega) = 0 \\
&(A2) \quad \forall \omega \in \{\omega_3, \omega_4, \omega_5\}, \; V_\phi(1, \omega) \geq 0 \\
&(A3) \quad \exists \omega \in \{\omega_3, \omega_4, \omega_5\}, \; V_\phi(1, \omega) > 0.
\end{aligned}$$

The condition $(A1)$ implies that

$$0 = V_\phi(0,\omega) = \frac{1}{1.1}\phi_1 + 4\phi_2 \Leftrightarrow \phi_1 = -4.4\phi_2.$$

For any portfolio $\phi = (-4.4\phi_2, \phi_2)$,

$$\begin{aligned} V_\phi(1,\omega_3) &= -4.4\phi_2 + 3\phi_2 = -1.4\phi_2, \\ V_\phi(1,\omega_4) &= -4.4\phi_2 + 4\phi_2 = -0.4\phi_2, \\ V_\phi(1,\omega_5) &= -4.4\phi_2 + 6\phi_2 = 1.6\phi_2. \end{aligned}$$

Consequently, if $\phi_2 < 0$, then $V_\phi(1,\omega_5) < 0$ which is in conflict with $(A2)$. If $\phi_2 > 0$, then the Condition $(A2)$ is not respected since $V_\phi(1,\omega_3) < 0$ and $V_\phi(1,\omega_4) < 0$. If $\phi_2 = 0$, the condition $(A3)$ is not respected since $V_\phi(1,\omega_4) = V_\phi(1,\omega_5) = V_\phi(1,\omega_6) = 0$. Therefore, there is no arbitrage opportunity.

c) The answer is

	$(\phi_1^1; \phi_1^2)$	$(\phi_2^1; \phi_2^2)$
$\{\omega_1, \omega_2\}$	$\left(-\frac{14}{10}; \frac{7}{11}\right)$	$(0; 0)$
$\{\omega_3, \omega_4, \omega_5\}$	$\left(-\frac{14}{10}; \frac{7}{11}\right)$	$(-3; 1)$

Indeed, we are looking for a self-financing strategy ϕ such that $V_2(\phi) = C$. Since ϕ_2 is $\mathcal{F}_1-$measurable,

$$\phi_2(\omega_1) = \phi_2(\omega_2) \text{ and } \phi_2(\omega_3) = \phi_2(\omega_4) = \phi_2(\omega_5).$$

Because $V_2(\phi) = C$, we have that

$$(i) \qquad 0 = C(\omega_1) = V_2(\phi, \omega_1) = \phi_2^1(\omega_1) + 2\phi_2^2(\omega_1)$$

$$(ii) \qquad 0 = C(\omega_2) = V_2(\phi, \omega_2) = \phi_2^1(\omega_2) + 3\phi_2^2(\omega_2) = \phi_2^1(\omega_1) + 3\phi_2^2(\omega_1)$$

$$(iii) \qquad 0 = C(\omega_3) = V_2(\phi, \omega_3) = \phi_2^1(\omega_3) + 3\phi_2^2(\omega_3)$$

$$(iv) \qquad 1 = C(\omega_4) = V_2(\phi, \omega_4) = \phi_2^1(\omega_4) + 4\phi_2^2(\omega_4) = \phi_2^1(\omega_3) + 4\phi_2^2(\omega_3)$$

$$(v) \qquad 3 = C(\omega_5) = V_2(\phi, \omega_5) = \phi_2^1(\omega_5) + 6\phi_2^2(\omega_5) = \phi_2^1(\omega_3) + 6\phi_2^2(\omega_3).$$

The Equations (i) and (ii) imply that $\phi_2^1(\omega_1) = 0$ and $\phi_2^2(\omega_1) = 0$. The Equations (iii), (iv) and (v) imply that $\phi_2^1(\omega_3) = -3$ and $\phi_2^2(\omega_3) = 1$. Since $\phi_1 = (\phi_1^1, \phi_1^2)$ is $\mathcal{F}_0-$measurable, the portfolio ϕ_1 is constant, that is,

$$\forall \omega \in \Omega, \ \phi_1(\omega) = (\phi_1^1, \phi_1^2).$$

Since the strategy ϕ is self-financing, it must satisfies $\forall \omega \in \Omega$,

$$\begin{aligned} \phi_2^1(\omega) S_1^1(\omega) + \phi_2^2(\omega) S_1^2(\omega) &= \phi_1^1(\omega) S_1^1(\omega) + \phi_1^2(\omega) S_1^2(\omega) \\ &= \phi_1^1 S_1^1(\omega) + \phi_1^2 S_1^2(\omega). \end{aligned}$$

Hence,

$$\begin{aligned} \forall \omega &\in \{\omega_1, \omega_2\}, \\ \frac{1}{1,1}\phi_1^1 + 2\phi_1^2 &= \frac{1}{1,1}\phi_2^1(\omega) + 2\phi_2^2(\omega) = 0 \Leftrightarrow \phi_1^1 = -2,2\phi_1^2 \\ \\ \forall \omega &\in \{\omega_3, \omega_4, \omega_5\}, \\ \frac{1}{1,1}\phi_1^1 + 4\phi_1^2 &= \frac{1}{1,1}\phi_2^1(\omega) + 4\phi_2^2(\omega) = \frac{-3}{1,1} + 4 = \frac{1,4}{1,1} \Leftrightarrow \phi_1^1 = 1,4 - 4,4\phi_1^2. \end{aligned}$$

Consequently,

$$\begin{aligned} -2,2\phi_1^2 &= 1,4 - 4,4\phi_1^2 \Leftrightarrow \phi_1^2 = \frac{1,4}{2,2} = \frac{7}{11} \\ \phi_1^1 &= -2,2\phi_1^2 = -2,2\frac{1,4}{2,2} = -1,4. \end{aligned}$$

Therefore $\phi_1^1 = -1,4$ and $\phi_1^2 = (1,4)/(2,2) = 7/11 \cong 0,\ 63636$.

d)

$$V_0(\phi) = \phi_1^1 S_0^1 + \phi_1^2 S_0^2 = \frac{-1,4}{1,21} + \frac{7}{11} 2 = \frac{14}{121} \cong 0,1157$$

e) Depending on you free variable, the answer is

ω	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$
ω_1	$0,72$	$0,72$	$0,72$
ω_2	$0,18$	$0,18$	$0,18$
ω_3	$0 < \mathbb{Q}\{\omega_3\} < \frac{4}{75};$	$\mathbb{Q}\{\omega_3\} = \frac{4}{75} - \frac{2}{3}\mathbb{Q}\{\omega_4\}$	$\mathbb{Q}\{\omega_3\} = 2\mathbb{Q}\{\omega_5\} - 0,04$
ω_4	$\mathbb{Q}\{\omega_4\} = 0,08 - \frac{3}{2}\mathbb{Q}\{\omega_3\}$	$0 < \mathbb{Q}\{\omega_4\} < \frac{8}{100}$	$\mathbb{Q}\{\omega_4\} = 0,14 - 3\mathbb{Q}\{\omega_5\}$
ω_5	$\mathbb{Q}\{\omega_5\} = 0,02 + \frac{1}{2}\mathbb{Q}\{\omega_3\}$	$\mathbb{Q}\{\omega_5\} = \frac{14}{300} - \frac{1}{3}\mathbb{Q}\{\omega_4\}$	$0,02 < \mathbb{Q}\{\omega_5\} < \frac{14}{300}$

Indeed, we must verify that $\mathrm{E}^{\mathbb{Q}}\left[\frac{S_i^2}{S_i^1} | \mathcal{F}_0\right](\omega) = \frac{S_{i-1}^2(\omega)}{S_{i-1}^1(\omega)}$ for $i = 1,2$. Hence, the first constraint is

$$\forall \omega \in \Omega,\ \mathrm{E}^{\mathbb{Q}}\left[\frac{S_1^2}{S_1^1} | \mathcal{F}_0\right](\omega) = \frac{S_0^2(\omega)}{S_0^1(\omega)}$$

$$\Leftrightarrow \quad 2,2\mathbb{Q}\{\omega_1, \omega_2\} + 4,4\mathbb{Q}\{\omega_3, \omega_4, \omega_5\} = 2,42$$

$$\Leftrightarrow \quad 2,2\mathbb{Q}\{\omega_1, \omega_2\} + 4,4(1 - \mathbb{Q}\{\omega_1, \omega_2\}) = 2,42$$

$$\Leftrightarrow \quad \mathbb{Q}\{\omega_1, \omega_2\} = 0,9 \text{ and } \mathbb{Q}\{\omega_3, \omega_4, \omega_5\} = 1 - \mathbb{Q}\{\omega_1, \omega_2\} = 0,1.$$

The second constraint is

$$
\begin{aligned}
& \forall\omega \in \{\omega_1,\omega_2\}, \; \mathrm{E}^{\mathbb{Q}}\left[\frac{S_2^2}{S_2^1}|\mathcal{F}_1\right](\omega) = \frac{S_1^2(\omega)}{S_1^1(\omega)} \\
\Leftrightarrow \;& 2\frac{\mathbb{Q}\{\omega_1\}}{\mathbb{Q}\{\omega_1,\omega_2\}} + 3\frac{\mathbb{Q}\{\omega_2\}}{\mathbb{Q}\{\omega_1,\omega_2\}} = 2,2 \\
\Leftrightarrow \;& 2\frac{\mathbb{Q}\{\omega_1\}}{\mathbb{Q}\{\omega_1,\omega_2\}} + 3\left(1-\frac{\mathbb{Q}\{\omega_1\}}{\mathbb{Q}\{\omega_1,\omega_2\}}\right) = 2,2 \\
\Leftrightarrow \;& \mathbb{Q}\{\omega_1\} = 0,8\mathbb{Q}\{\omega_1,\omega_2\} = 0,8\cdot 0,9 = 0,72 \\
& \text{et } \mathbb{Q}\{\omega_2\} = \mathbb{Q}\{\omega_1,\omega_2\} - \mathbb{Q}\{\omega_1\} = 0,9-0,72 = 0,18.
\end{aligned}
$$

Finally, the third constraint is

$$
\begin{aligned}
\forall\omega \;\; \in \;\; & \{\omega_3,\omega_4,\omega_5\}, \; \mathrm{E}^{\mathbb{Q}}\left[\frac{S_2^2}{S_2^1}|\mathcal{F}_1\right](\omega) = \frac{S_1^2(\omega)}{S_1^1(\omega)} \\
\Leftrightarrow \;& 3\frac{\mathbb{Q}\{\omega_3\}}{\mathbb{Q}\{\omega_3,\omega_4,\omega_5\}} + 4\frac{\mathbb{Q}\{\omega_4\}}{\mathbb{Q}\{\omega_3,\omega_4,\omega_5\}} + 6\frac{\mathbb{Q}\{\omega_5\}}{\mathbb{Q}\{\omega_3,\omega_4,\omega_5\}} = 4,4 \\
\Leftrightarrow \;& \begin{array}{l} 3\frac{\mathbb{Q}\{\omega_3\}}{\mathbb{Q}\{\omega_3,\omega_4,\omega_5\}} + 4\frac{\mathbb{Q}\{\omega_4\}}{\mathbb{Q}\{\omega_3,\omega_4,\omega_5\}} \\ +6\left(1-\frac{\mathbb{Q}\{\omega_3\}}{\mathbb{Q}\{\omega_3,\omega_4,\omega_5\}}-\frac{\mathbb{Q}\{\omega_4\}}{\mathbb{Q}\{\omega_3,\omega_4,\omega_5\}}\right) \end{array} = 4,4 \\
\Leftrightarrow \;& 3\mathbb{Q}\{\omega_3\} + 2\mathbb{Q}\{\omega_4\} = 1,6\mathbb{Q}\{\omega_3,\omega_4,\omega_5\} = 1,6\cdot 0,1 = 0,16 \\
\Leftrightarrow \;& \begin{array}{l} 0 < \mathbb{Q}\{\omega_3\} < \frac{4}{75};\; \mathbb{Q}\{\omega_4\} = 0,08 - \frac{3}{2}\mathbb{Q}\{\omega_3\} \\ \text{and } \mathbb{Q}\{\omega_5\} = 0,02 + \frac{1}{2}\mathbb{Q}\{\omega_3\} \end{array} \\
\Leftrightarrow \;& \begin{array}{l} \mathbb{Q}\{\omega_3\} = \frac{4}{75} - \frac{2}{3}\mathbb{Q}\{\omega_4\};\; 0 < \mathbb{Q}\{\omega_4\} < \frac{8}{100} \\ \text{and } \mathbb{Q}\{\omega_5\} = \frac{14}{300} - \frac{1}{3}\mathbb{Q}\{\omega_4\} \end{array} \\
\Leftrightarrow \;& \begin{array}{l} \mathbb{Q}\{\omega_3\} = 2\mathbb{Q}\{\omega_5\} - 0,04;\; \mathbb{Q}\{\omega_4\} = 0,14 - 3\mathbb{Q}\{\omega_5\} \\ \text{and } 0,02 < \mathbb{Q}\{\omega_5\} < \frac{14}{300}. \end{array}
\end{aligned}
$$

f)

$$
\begin{aligned}
\mathrm{E}^{\mathbb{Q}}\left(\frac{C}{1,21}\right) &= \frac{1}{1,21}\mathbb{Q}\{\omega_4\} + \frac{3}{1,21}\mathbb{Q}\{\omega_5\} \\
&= \frac{0,14 - 3\mathbb{Q}\{\omega_5\} + 3\mathbb{Q}\{\omega_5\}}{1,21} \\
&= \frac{14}{121} \cong 0,1157.
\end{aligned}
$$

The option price does not depend on the risk neutral measure and corresponds to the initial value of the replicating investment strategy (see Question 4).

Appendix 1

A strategy ϕ is an arbitrage opportunity if

$$(i) \qquad V_0(\phi) = 0$$

$$(ii) \quad \exists t,\ V_t(\phi) \geq 0 \text{ and } P(V_t(\phi) > 0) > 0.$$

Therefore, the model do not admit arbitrage if all self-financing trading strategies ϕ satisfying $V_0(\phi) = 0$ are such that

$$\forall t,\ \exists \omega \in \Omega \text{ such that } V_t(\phi, \omega) < 0 \text{ or } \mathbb{P}(V_t(\phi) > 0) = 0$$

which is equivalent to

$$\forall t,\ \exists \omega \in \Omega \text{ such that } V_t(\phi, \omega) < 0 \text{ or } \forall \omega \in \Omega,\ V_t(\phi, \omega) = 0.$$

Let ϕ, be a self-financing trading strategy with $V_0(\phi) = 0$. Since $\phi(1) = (\phi_1^1, \phi_1^2)$ is $\mathcal{F}_0-$measurable, the portfolio $\phi(1)$ is constant, that is

$$\forall \omega \in \Omega,\ \phi_1(\omega) = (\phi_1^1, \phi_1^2).$$

Hence,

$$0 = V_0(\phi) = \frac{1}{1,21}\phi_1^1 + 2\phi_1^2 \Leftrightarrow \phi_1^1 = -2,42\phi_1^2.$$

At $t = 1$, the portfolio market value is

$$\begin{aligned} \forall \omega &\in \{\omega_1, \omega_2\}, \\ V_1(\phi, \omega) &= \frac{1}{1,1}\phi_1^1 + 2\phi_1^2 = -\frac{2,42}{1,1}\phi_1^2 + 2\phi_1^2 = -0,2\phi_1^2 \\ \\ \forall \omega &\in \{\omega_3, \omega_4, \omega_5\}, \\ V_1(\phi, \omega) &= \frac{1}{1,1}\phi_1^1 + 4\phi_1^2 = -\frac{2,42}{1,1}\phi_1^2 + 4\phi_1^2 = 1,8\phi_1^2. \end{aligned}$$

Therefore, if $\phi_1^2 \neq 0$ then $\exists \omega \in \Omega$ such that $V_1(\phi, \omega) < 0$ (assuming that $\mathbb{P}(\{\omega_1, \omega_2\}) > 0$ and $\mathbb{P}(\{\omega_3, \omega_4, \omega_5\}) > 0$) which means that there is no arbitrage. What about $\phi_1^2 = 0$? The trading strategy can be revised. Recall that ϕ_2 is $\mathcal{F}_1-$measurable, which implies that

$$\phi_2(\omega_1) = \phi_2(\omega_2) \text{ and } \phi_2(\omega_3) = \phi_2(\omega_4) = \phi_2(\omega_5).$$

Since ϕ must be self-financing, the new portfolio ϕ_2 must satisfy

$$\phi_2 S_1 = \phi_1 S_1 = V_1(\phi) = 0.$$

Therefore,

$$\begin{aligned} \forall\omega &\in \{\omega_1, \omega_2\}, \\ \frac{1}{1,1}\phi_2^1(\omega) + 2\phi_2^2(\omega) &= 0 \Rightarrow \phi_2^1(\omega) = -2,2\phi_2^2(\omega) = -2,2\phi_2^2(\omega_1) \end{aligned}$$

$$\begin{aligned} \forall\omega &\in \{\omega_3, \omega_4, \omega_5\}, \\ \frac{1}{1,1}\phi_2^1(\omega) + 4\phi_2^2(\omega) &= 0 \Rightarrow \phi_2^1(\omega) = -4,4\phi_2^2(\omega) = -4,4\phi_2^2(\omega_3). \end{aligned}$$

The market value of $\phi(2)$ at $t = 2$ is

$$\begin{aligned} V_2(\phi, \omega_1) &= \phi_2^1(\omega_1) + 2\phi_2^2(\omega_1) \\ &= -2,2\phi_2^2(\omega_1) + 2\phi_2^2(\omega_1) = -0,2\phi_2^2(\omega_1) \end{aligned}$$

$$\begin{aligned} V_2(\phi, \omega_2) &= \phi_2^1(\omega_2) + 3\phi_2^2(\omega_2) \\ &= \phi_2^1(\omega_1) + 3\phi_2^2(\omega_1) \\ &= -2,2\phi_2^2(\omega_1) + 3\phi_2^2(\omega_1) = 0,8\phi_2^2(\omega_1) \end{aligned}$$

$$\begin{aligned} V_2(\phi, \omega_3) &= \phi_2^1(\omega_3) + 3\phi_2^2(\omega_3) \\ &= -4,4\phi_2^2(\omega_3) + 3\phi_2^2(\omega_3) = -1,4\phi_2^2(\omega_3) \end{aligned}$$

$$\begin{aligned} V_2(\phi, \omega_4) &= \phi_2^1(\omega_4) + 4\phi_2^2(\omega_4) \\ &= \phi_2^1(\omega_3) + 4\phi_2^2(\omega_3) \\ &= -4,4\phi_2^2(\omega_3) + 4\phi_2^2(\omega_3) = -0,4\phi_2^2(\omega_3) \end{aligned}$$

$$\begin{aligned} V_2(\phi, \omega_4) &= \phi_2^1(\omega_4) + 6\phi_2^2(\omega_4) \\ &= \phi_2^1(\omega_3) + 6\phi_2^2(\omega_3) \\ &= -4,4\phi_2^2(\omega_3) + 6\phi_2^2(\omega_3) = 1,6\phi_2^2(\omega_3) \end{aligned}$$

Therefore, is $\phi_2^2(\omega_1) \neq 0$ then $\exists\omega \in \Omega$ such that $V_2(\phi, \omega) < 0$ (assuming that $\mathbb{P}(\omega_1) > 0$ and $\mathbb{P}(\omega_2) > 0$), which means that there is no arbitrage. If $\phi_2^2(\omega_1) = 0$ but $\phi_2^2(\omega_3) \neq 0$ then $\exists\omega \in \Omega$ such that $V_2(\phi, \omega) < 0$ (assuming that $\mathbb{P}(\omega_3) > 0$ or $\mathbb{P}(\omega_4) > 0$ and $\mathbb{P}(\omega_5) > 0$), which also means that there is no arbitrage. Finally, if $\phi_2^2(\omega_1) = 0$ and $\phi_2^2(\omega_3) = 0$, then $\forall\omega \in \Omega$, $V_2(\phi, \omega) = 0$. Hence, once again, no arbitrage. ■