Stochastic processes and stopping time

Exercises

Exercise 2.1. Today is Monday and you have one dollar in your piggy bank. Starting tomorrow, every morning until Friday (inclusively), you toss a coin. If the result is tails you withdraw one dollar (if possible) from the piggy bank. If the result is heads you deposit one dollar in it. Model the amount of money in the piggy bank by answering the following questions:

a) What is the sample space?

b) Define a stochastic process describing this situation, and explain its signification. Do not forget to specify the period of time you are using.

c) What \( \sigma \)-algebra should you use to build your measurable space?

d) List the \( \sigma \)-algebras in the filtration generated by the stochastic process for the following days: Monday, Tuesday, Wednesday and Friday?

e) Interpret, as function of the available information, the information structure you built in the previous question for Wednesday.

f) What is the distribution of the amount of money in the piggy bank Friday noon?

g) Show that the first time the piggy bank is empty is a stopping time.

Exercise 2.2. Let \((\Omega, \mathcal{F})\) be a measurable space such that \(\text{Card}(\Omega) < \infty\) and equipped with the filtration \(\mathbb{F} = \{\mathcal{F}_t : t \in \{0, 1, \ldots\}\}\). If the random variables \(\tau_1\) and \(\tau_2\) stopping times with respect to the filtration \(\mathbb{F}\), show that \(\tau_1 \wedge \tau_2 = \max\{\tau_1, \tau_2\}\) is also a stopping time.

Exercise 2.3. You have 20 dollars and decide to play the following game of chance. You toss a coin four times. On the first toss, you win 10 dollars if the result is “tails” and lose 10 dollars if the result is “heads”. On the subsequent tosses, if the result is identical to that of the previous toss, then you don’t win nor lose anything. However, if the result is different from the previous toss, you win 10 dollars in the case this result is “tails” and lose 10 dollars in the case this result is “heads”.

a) Define the random variables and notation needed to model this situation.

b) Give the \( \sigma \)-algebra reflecting the information available after the second toss. Interpret.

c) What is the distribution of the amount of money you own at the end of the game?
Exercice 2.4    Gambler’s ruin.

Two gamblers initially own fortunes of $r$ and $n - r$ dollars respectively ($r$ and $n$ are positive integers such that $r < n$). They decide to gamble until one of them is ruined. Say the second player represents the croupier and the first player always decides of the bets. This first player wins the amount of money he bets with probability $p$ ($0 < p < 1$) or loses it with probability $q = 1 - p$, independently from one round to another. The only amounts he can bet are multiples of one dollar and no loans are admitted.

Let us study two popular strategies for this game:
1) the bold strategy by which the first player always bets the minimum between his fortune and the amount required to win the whole game;
2) the shy strategy by which the first player always bets one dollar.

Let $X_t$ represent the first player’s fortune after the $t$-th round when he follows the shy strategy. Let $Y_t$ represent the fortune of the first player after the $t$-th round when he follows the bold strategy.

In this exercise, we assume each round is settled by the toss of a possibly poorly balanced coin. On any given round, the first player wins if the result of the toss is “tails”. This happens with probability $p$. We consider the results of the first four tosses only. The croupier starts with 4 dollars while the first player starts with 6 dollars.

a) What sample space corresponds to this situation?
b) What $\sigma$-algebra should you use to build your measurable space?
c) List the $\sigma$-algebras in the filtration generated by the process $X = \{X_t : t \in \{0, 1, 2, 3, 4\}\}$ for times 0, 1, 2 et 4.
d) List the $\sigma$-algebras in the filtration generated by the process $Y = \{Y_t : t \in \{0, 1, 2, 3, 4\}\}$ for times 0, 1, 2 et 4.
e) Interpret, as function of the available information, the information structure you built in question c) for time $t = 2$.
f) Give the mass functions of $X_4$ and $Y_4$.

g) Let $\tau$ be a random variable representing the time at which the game stopped, meaning that

$$\tau(\omega) \equiv \min \{t \in \{0, 1, 2, \ldots\} : X_t = 0 \text{ or } X_t = n\}.$$ 

Show that $\tau$ is a stopping time.
Exercice 2.5. Let $\tau_1$ and $\tau_2$ be two $(\Omega, \mathcal{F}, \mathbb{F})$—stopping times with $\Omega$ a sample space containing a finite number of elements, and with $\mathbb{F} = \{\mathcal{F}_t : t \in \{0, 1, 2, \ldots\}\}$, a filtration. Is the sum of those two stopping times also a stopping time? If so, show it. If not, give a counterexample.

Exercice 2.6. You throw two dices. Let $X$ be the result of the first dice, and $Y$ the result of the second dice. You receive a payment of amount $\max(X, Y)$ at time $\tau = \min(X, Y)$. Let the process $\{S_t : t \in \{1, 2, ..., 6\}\}$ represent the payments made to you throughout time.

a) What is the sample space?

b) What is the filtration $\{\mathcal{F}_t : t \in \{1, 2, ..., 6\}\}$ generated by the stochastic process $\{S_t : t \in \{1, 2, ..., 6\}\}$?

c) Is $\tau$ a $\{\mathcal{F}_t : t \in \{1, 2, ..., 6\}\}$—stopping time? Justify your answer.
Solutions

1 Exercise 2.1

a) What is the sample space ?

\[ \Omega = \left\{ \ TTTT, TTTH, TTHT, TTHH, THTT, THTH, THHT, THHH, \\
HTTT, HTHH, HTHT, HTTH, HHTT, HHTH, HHHH \right\} \]

b) Define a stochastic process describing this situation, and explain its signification. Do not forget to specify the period of time you are using.

Let \( t = 0 \) on Monday (today). \( \forall t \in \{0,1,2,3,4\} \)

\[ X_t = \text{money in the piggy bank after the transaction on day } t \text{ happened} \]

c) What \( \sigma \)-algebra should you use to build your measurable space ?

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( X_0 (\omega) )</th>
<th>( X_1 (\omega) )</th>
<th>( X_2 (\omega) )</th>
<th>( X_3 (\omega) )</th>
<th>( X_4 (\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 = TTTT )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_2 = TTTH )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_3 = TTHT )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_4 = TTHH )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \omega_5 = THTT )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_6 = THTH )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_7 = THHT )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_8 = THHH )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( \omega_9 = HTTT )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_{10} = HTTH )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_{11} = HTHT )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_{12} = HTHH )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( \omega_{13} = HHTT )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_{14} = HHTH )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( \omega_{15} = HHHT )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>( \omega_{16} = HHHH )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

The \( \sigma \)-algebra \( F \) must ensure that \( X_0, X_1, X_2, X_3 \) and \( X_4 \) are all random variables. Since all of the 16 trajectories are different from one another, \( F \) is generated by the partition
\{\omega_1, ..., \omega_{16}\}. Hence \[ \mathcal{F} = \text{set of all the events in } \Omega \text{ plus } \emptyset. \]

d) List the \( \sigma \)-algebras in the filtration generated by the stochastic process for the following days: Monday, Tuesday, Wednesday and Friday?

\[ \mathcal{F}_0 = \{\emptyset, \Omega\} \]

\[ \mathcal{F}_1 = \{\emptyset, A, A^c, \Omega\} \]

where \( A = \{TTTT, TTHH, TTHT, THHT, THTT, THTH, THTH, TTHH\} \)

\[ \mathcal{F}_2 = \left\{ \begin{array}{c} \emptyset, B_1, B_2, B_3, B_4, B_1 \cup B_2, B_1 \cup B_3, B_1 \cup B_4, B_2 \cup B_3, B_2 \cup B_4, B_3 \cup B_4, B_1 \cup B_2 \cup B_3, B_1 \cup B_2 \cup B_4, B_1 \cup B_3 \cup B_4, B_2 \cup B_3 \cup B_4, \Omega \end{array} \right\} \]

where \( B_1 = \{TTTT, TTHH, TTHT, TTHH\} \)

\( B_2 = \{THTT, THTH, THHT, TTHH\} \)

\( B_3 = \{HTTT, HTTH, HTHT, HTTH\} \)

\( B_4 = \{HHTT, HHTH, HHHT, HHHH\} \)

\[ \mathcal{F}_4 = \mathcal{F} \]

e) Interpret, as function of the available information, the information structure you built in the previous question for Wednesday.

\[ \mathcal{F}_2 = \left\{ \begin{array}{c} \emptyset, B_1, B_2, B_3, B_4, B_1 \cup B_2, B_1 \cup B_3, B_1 \cup B_4, B_2 \cup B_3, B_2 \cup B_4, B_3 \cup B_4, B_1 \cup B_2 \cup B_3, B_1 \cup B_2 \cup B_4, B_1 \cup B_3 \cup B_4, B_2 \cup B_3 \cup B_4, \Omega \end{array} \right\} \]

where \( B_1 = \{TTTT, TTHH, TTHT, TTHH\} \)

\( B_2 = \{THTT, THTH, THHT, TTHH\} \)

\( B_3 = \{HTTT, HTTH, HTHT, HTTH\} \)

\( B_4 = \{HHTT, HHTH, HHHT, HHHH\} \)
The atoms generating $\mathcal{F}_2$ partition $\Omega$ according to the results of the first two coin tosses (those of Tuesday and Wednesday morning), however they cannot distinguish events involving tosses of the last two mornings (those of Thursday and Friday).

**f) What is the distribution of the amount of money in the piggy bank Friday noon?**

Say the result is “tails” with probability $p$ and “heads” with probability $1 - p$. Note this means the same coin is used all week long.

\[
\begin{array}{cccccc}
\omega & \mathbb{P}(\omega) & X_4(\omega) & \omega & \mathbb{P}(\omega) & X_4(\omega) \\
TTTT & p^4 & 0 & HTTT & p^3(1-p) & 0 \\
TTTH & p^3(1-p) & 1 & HTTH & p^2(1-p)^2 & 1 \\
TTHT & p^3(1-p) & 0 & HHTT & p^2(1-p)^2 & 1 \\
TTHH & p^2(1-p)^2 & 2 & HHTH & p(1-p)^3 & 3 \\
THTT & p^3(1-p) & 0 & HHTT & p^2(1-p)^2 & 1 \\
THTH & p^2(1-p)^2 & 1 & HHHT & p(1-p)^3 & 3 \\
THTTH & p^2(1-p)^2 & 1 & HHHH & (1-p)^4 & 5 \\
\end{array}
\]

The density function is:

\[
x f_X(x) = \mathbb{P}\{\omega \in \Omega : X(\omega) = x\} \quad x \quad \mathbb{P}\{\omega \in \Omega : X(\omega) = x \mid p = \frac{1}{2}\}
\]

\[
\begin{array}{cccccc}
x & x f_X(x) & x \mathbb{P}\{\omega \in \Omega : X(\omega) = x \mid p = \frac{1}{2}\} \\
0 & p^4 + 3p^3(1-p) = -2p^4 + 3p^3 & 0 & \frac{4}{16} = \frac{1}{4} \\
1 & p^3(1-p) + 5p^2(1-p)^2 = +4p^4 - 9p^3 + 5p^2 & 1 & \frac{6}{16} = \frac{3}{8} \\
2 & p^2(1-p)^2 = p^4 - 2p^3 + p^2 & 2 & \frac{1}{16} \\
3 & 4p(1-p)^3 = 4p - 12p^2 + 12p^3 - 4p^4 & 3 & \frac{4}{16} = \frac{1}{4} \\
4 & 0 & 4 & 0 \\
5 & (1-p)^4 = 1 - 4p + 6p^2 - 4p^3 + p^4 & 5 & \frac{1}{16}
\end{array}
\]
The cumulative distribution function is:

\[ F_X(x) = \mathbb{P} \{ \omega \in \Omega : X(\omega) \leq x \} \]

- \( F_X(0) = p^4 + 3p^3 (1 - p) = -2p^4 + 3p^3 \)
- \( F_X(1) = p^4 + 4p^3 (1 - p) + 5p^2 (1 - p)^2 = 2p^4 - 6p^3 + 5p^2 \)
- \( F_X(2) = p^4 + 4p^3 (1 - p) + 6p^2 (1 - p)^2 = 3p^4 - 8p^3 + 6p^2 \)
- \( F_X(3) = p^4 + 4p^3 (1 - p) + 6p^2 (1 - p)^2 + 4p (1 - p)^3 = -p^4 + 4p^3 - 6p^2 + 4p \)
- \( F_X(4) = -p^4 + 4p^3 - 6p^2 + 4p \)
- \( F_X(5) = p^4 + 4p^3 (1 - p) + 6p^2 (1 - p)^2 + 4p (1 - p)^3 + (1 - p)^4 = 1 \)

\textit{g) Show that the first time the piggy bank is empty is a stopping time.}

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( X_0(\omega) )</th>
<th>( X_1(\omega) )</th>
<th>( X_2(\omega) )</th>
<th>( X_3(\omega) )</th>
<th>( X_4(\omega) )</th>
<th>( \tau(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 = TTTT )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_2 = TTTH )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_3 = TTTH )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_4 = TTHH )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_5 = THTT )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_6 = THTH )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_7 = THHT )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_8 = THHH )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( \omega_9 = HTTT )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( \omega_{10} = HTTH )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( \omega_{11} = HTHT )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \omega_{12} = HTHH )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \omega_{13} = HHTT )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \omega_{14} = HHHT )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \omega_{15} = HHHH )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \omega_{16} = HHHH )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>
\[ \{ \omega \in \Omega : \tau(\omega) = 0 \} = \emptyset \in \mathcal{F}_0 \]

\[ \{ \omega \in \Omega : \tau(\omega) = 1 \} = \{ TTTT, TTTH, TTHH, THTT, THTH, THHT, THHH \} \in \mathcal{F}_1 \]

\[ \{ \omega \in \Omega : \tau(\omega) = 2 \} = \emptyset \in \mathcal{F}_2 \]

\[ \{ \omega \in \Omega : \tau(\omega) = 3 \} = \{ HTTT, HTTH \} \in \mathcal{F}_3 \]

\[ \{ \omega \in \Omega : \tau(\omega) = 4 \} = \emptyset \in \mathcal{F}_4. \]

2 Exercise 2.2

Since \( \forall k \in \{ 0, 1, \ldots \} \),

\[ \{ \omega \in \Omega : \tau_1(\omega) \lor \tau_2(\omega) \leq k \} \]

\[ = \{ \omega \in \Omega : \tau_1(\omega) \leq k \text{ et } \tau_2(\omega) \leq k \} \]

\[ = \left\{ \underbrace{\omega \in \Omega : \tau_1(\omega) \leq k}_{\in \mathcal{F}_k} \right\} \cap \left\{ \underbrace{\omega \in \Omega : \tau_2(\omega) \leq k}_{\in \mathcal{F}_k} \right\} \in \mathcal{F}_k, \]
then $\forall t \in \{0, 1, \ldots\}$,

$$\{\omega \in \Omega : \tau_1(\omega) \lor \tau_2(\omega) = t\}$$

$$= \{\omega \in \Omega : t - 1 < \tau_1(\omega) \lor \tau_2(\omega) \leq t\}$$

$$= \{\omega \in \Omega : \tau_1(\omega) \lor \tau_2(\omega) \leq t\} \cap \{\omega \in \Omega : \tau_1(\omega) \lor \tau_2(\omega) > t - 1\}$$

$$= \underbrace{\{\omega \in \Omega : \tau_1(\omega) \lor \tau_2(\omega) \leq t\} \cap \{\omega \in \Omega : \tau_1(\omega) \lor \tau_2(\omega) > t - 1\}}_{\in \mathcal{F}_t}$$

$$\in \mathcal{F}_t. \blacksquare$$

3 Exercise 2.3

a) Define the random variables and notation needed to model this situation.

The sample space contains 16 elements, all listed in the table below. For $i \in \{0, 1, 2, 3, 4\}$, the random variable $X_i$ represents the amount you own after the $i$-th toss.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$X_0$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$\omega$</th>
<th>$X_0$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$tttt$</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>$httt$</td>
<td>20</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>$ttth$</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>20</td>
<td>$htth$</td>
<td>20</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>$thht$</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td>20</td>
<td>30</td>
<td>$htht$</td>
<td>20</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>$tthh$</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>$htth$</td>
<td>20</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$tthh$</td>
<td>20</td>
<td>30</td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>$hthh$</td>
<td>20</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$tthh$</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>30</td>
<td>$hhtt$</td>
<td>20</td>
<td>10</td>
<td>10</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>$thht$</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>$hhth$</td>
<td>20</td>
<td>10</td>
<td>10</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>$thht$</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>$hhht$</td>
<td>20</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>$thht$</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>$hhhh$</td>
<td>20</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

b) Give the $\sigma$-algebra reflecting the information available after the second toss. Interpret.

$$\mathcal{F}_2 = \sigma \left\{ \{tttt, ttth, thht, tthh\}, \{tthh, thht, thht, thhh\}, \{httt, hthh, hhtt, hhth\}, \{hhtt, hhth, hhtt, hhhh\} \right\}.$$  

If we observe the stochastic process $X$ until the second toss, we can establish which results were obtained on the first two tosses, but we are unable to determine the results of the last two tosses.
c) What is the distribution of the amount of money you own at the end of the game?
Let \( q \) be the probability to obtain “tails” on a given toss. Then

\[
\mathbb{P}(X_4 = 30) = q^4 + 2q^3 (1 - q) + q^2 (1 - q)^2 \\
= q^2
\]

\[
\mathbb{P}(X_4 = 20) = 2q^3 (1 - q) + 4q^2 (1 - q)^2 + 2q (1 - q)^3 \\
= -2q^2 + 2q \\
= 2q (1 - q)
\]

\[
\mathbb{P}(X_4 = 10) = q^2 (1 - q)^2 + 2q (1 - q)^3 + (1 - q)^4 \\
= 1 - 2q + q^2 = (1 - q)^2
\]

4 Exercise 2.4

a) What sample space corresponds to this situation?

\[
\Omega = \{TTTT, TTTH, TTHT, TTHH, THTT, THTH, THHT, THHH, HHTT, HHTH, HHTT, HHTH, HHHT, HHHH\}
\]

In the case the bold strategy is used, one might also describe the sample space by

\[
\{T, HHTT, HHTH, HHTT, HH\}
\]

since the game can stop only after a few rounds. It’s a question of interpretation. We may assume that the coin will be tossed four times, even if ruined has occurred and no bet is on the table. This way we can build the two stochastic processes on the same measurable space.
b) What \( \sigma \)-algebra should you use to build your measurable space?

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( X_0 )</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_4 )</th>
<th>( Y_0 )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_3 )</th>
<th>( Y_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 = TTTT )</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_2 = TTTH )</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_3 = TTHT )</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_4 = TTHH )</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_5 = THTT )</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_6 = THTH )</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_7 = THHT )</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_8 = THHH )</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_9 = HHTT )</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>( \omega_{10} = HHTH )</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>( \omega_{11} = HTHT )</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_{12} = HTHH )</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_{13} = HHTT )</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_{14} = HHTH )</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_{15} = HHHT )</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_{16} = HHHH )</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The \( \sigma \)-algebra \( \mathcal{F} \) must ensure that \( X_0, X_1, X_2, X_3 \) and \( X_4 \) as well as \( Y_0, Y_1, Y_2, Y_3 \) and \( Y_4 \) are random variables. Since the 16 trajectories are different from one another, \( \mathcal{F} \) is generated by the partition \( \{ \omega_1, \ldots, \omega_{16} \} \). Hence

\[
\mathcal{F} = \text{set of every event in } \Omega \text{ plus } \varnothing.
\]

Note \( \mathcal{F} \) contains \( 2^{16} = 65,536 \) events.

c) List the \( \sigma \)-algebras in the filtration generated by the process \( X = \{ X_t : t \in \{0, 1, 2, 3, 4\} \} \)
for times 0, 1, 2 and 4.

\[ \mathcal{F}_0 = \{ \emptyset, \Omega \} \]

\[ \mathcal{F}_1 = \{ \emptyset, A, A^c, \Omega \} \]

where \( A = \{ TTTT, TTTT, TTHT, TTHH, THTT, THTH, THHT, THHH \} \)

\[
\mathcal{F}_2 = \left\{ \begin{array}{l}
\emptyset, B_1, B_2, B_3, B_4, B_1 \cup B_2, B_1 \cup B_3, \\
B_1 \cup B_4, B_2 \cup B_3, B_2 \cup B_4, B_3 \cup B_4, \\
B_1 \cup B_2 \cup B_3, B_1 \cup B_2 \cup B_4, \\
B_1 \cup B_3 \cup B_4, B_2 \cup B_3 \cup B_4, \Omega
\end{array} \right\}
\]

where \( B_1 = \{ TTTT, TTTT, TTHT, TTHH \} \)

\( B_2 = \{ THTT, THTH, THHT, THHH \} \)

\( B_3 = \{ HHTT, HHTH, HTHT, HTHH \} \)

\( B_4 = \{ HHTT, HHTH, HHHT, HHHH \} \)

\[ \mathcal{F}_4 = \mathcal{F} \]

d) List the \( \sigma \)-algebras in the filtration generated by the process \( Y = \{ Y_t : t \in \{ 0, 1, 2, 3, 4 \} \} \)
for times 0, 1, 2 and 4.

\[ G_0 = \{ \emptyset, \Omega \} \]

\[ G_1 = \{ \emptyset, A, A^c, \Omega \} \]

where \( A = \{ TTTT, TTTH, TTHH, THTT, THTH, THHT, THHH \} \)

\[ G_2 = \{ \emptyset, B_1, B_3, B_4, B_1 \cup B_3, B_1 \cup B_4, B_3 \cup B_4, B_1 \cup B_3 \cup B_4, \Omega \} \]

where \( B_1 = \{ TTTT, TTTH, TTHH, THTT, THTH, THHT, THHH \} \)

\( B_2 = \{ HTTT, HTTH, HTHT, HTHH \} \)

\( B_3 = \{ HHTT, HHHT, HHHT, HHHH \} \)

\( G_4 = \{ \emptyset, C_1, C_2, C_3, C_4, C_5, \]

\[ C_1 \cup C_2, C_1 \cup C_3, C_1 \cup C_4, C_1 \cup C_5, C_2 \cup C_3, \]

\[ C_2 \cup C_4, C_2 \cup C_5, C_3 \cup C_4, C_3 \cup C_5, C_4 \cup C_5, \]

\[ C_3 \cup C_4 \cup C_5, C_2 \cup C_4 \cup C_5, C_2 \cup C_3 \cup C_5, \]

\[ C_2 \cup C_3 \cup C_4, C_1 \cup C_4 \cup C_5, C_1 \cup C_3 \cup C_5, \]

\[ C_1 \cup C_2 \cup C_3 \cup C_4, C_1 \cup C_2 \cup C_3 \cup C_5, C_1 \cup C_2 \cup C_4 \cup C_5, \]

\[ C_1 \cup C_3 \cup C_4 \cup C_5, C_2 \cup C_3 \cup C_4 \cup C_5, \]

\[ C_1 \cup C_3 \cup C_4 \cup C_5, C_1 \cup C_2 \cup C_3 \cup C_5, \]

\[ C_1 \cup C_2 \cup C_3 \cup C_4, C_1 \cup C_2 \cup C_3 \cup C_5, C_1 \cup C_2 \cup C_4 \cup C_5, \]

\[ C_1 \cup C_3 \cup C_4 \cup C_5, C_2 \cup C_3 \cup C_4 \cup C_5, \]

\[ \Omega \} \]

\( C_1 = \{ TTTT, TTTH, TTHH, THTT, THTH, THHT, THHH \} \)

\( C_2 = \{ HTTT \} \)

\( C_3 = \{ HTTH \} \)

\( C_4 = \{ HTHT, HTTH \} \)

\( C_5 = \{ HHTT, HHHT, HHHT, HHHH \} \)

(e) Interpret, as function of the available information, the information structure you built in question c) for time t=3.

\[ F_2 = \{ \emptyset, B_1, B_2, B_3, B_4, B_1 \cup B_2, B_1 \cup B_3, \]

\[ B_1 \cup B_4, B_2 \cup B_3, B_2 \cup B_4, B_3 \cup B_4, \]

\[ B_1 \cup B_2 \cup B_3, B_1 \cup B_2 \cup B_4, \]

\[ B_1 \cup B_3 \cup B_4, B_2 \cup B_3 \cup B_4, \Omega \} \]

where \( B_1 = \{ TTTT, TTTH, TTHH, TTHH \} \)

\( B_2 = \{ THTT, THHT, THHT, THHH \} \)

\( B_3 = \{ HTTT, HTTH, HTHT, HTHH \} \)

\( B_4 = \{ HHTT, HHHT, HHHT, HHHH \} \)
The atoms generating $F_2$ partition $\Omega$ according to the results of the first two coin tosses, however they cannot distinguish events involving the last two tosses.

f) Give the density functions of $X_4$ and $Y_4$.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$X_0$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$Y_0$</th>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_3$</th>
<th>$Y_4$</th>
<th>$\mathbb{P}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ = TTTT</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>$p^4$</td>
</tr>
<tr>
<td>$\omega_2$ = TTTH</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>$p^3(1 - p)$</td>
</tr>
<tr>
<td>$\omega_3$ = TTHT</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>$p^3(1 - p)$</td>
</tr>
<tr>
<td>$\omega_4$ = TTHH</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>$p^2(1 - p)^2$</td>
</tr>
<tr>
<td>$\omega_5$ = THTT</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>$p^3(1 - p)$</td>
</tr>
<tr>
<td>$\omega_6$ = THHT</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>$p^2(1 - p)^2$</td>
</tr>
<tr>
<td>$\omega_7$ = THHH</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>$p^2(1 - p)^2$</td>
</tr>
<tr>
<td>$\omega_8$ = HHTT</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>$p^3(1 - p)$</td>
</tr>
<tr>
<td>$\omega_9$ = HTTH</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>$p^3(1 - p)$</td>
</tr>
<tr>
<td>$\omega_{10}$ = HTHT</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>6</td>
<td>$p^2(1 - p)^2$</td>
</tr>
<tr>
<td>$\omega_{11}$ = HHTT</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>$p^3(1 - p)^2$</td>
</tr>
<tr>
<td>$\omega_{12}$ = HHHH</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$(1 - p)^4$</td>
</tr>
</tbody>
</table>

$$f_{Y_4}(y) = \begin{cases} 
-p^2 + p^3 - p + 1 & \text{if } y = 0 \\
-p^2 + 2p^3 + p^4 = p^2(1 - p)^2 & \text{if } y = 6 \\
-p^4 + p^3 + p & \text{if } y = 10 \\
0 & \text{otherwise}
\end{cases}$$
\[ f_{X_4}(y) = \begin{cases} 
(1-p)^4 & \text{if } x = 2 \\
4p(1-p)^3 & \text{if } x = 4 \\
6p^2(1-p)^2 & \text{if } x = 6 \\
4p^3(1-p) & \text{if } x = 8 \\
p^4 & \text{if } x = 10 \\
0 & \text{otherwise} \end{cases} \]

\[ g) \quad \text{Let } \tau \text{ be a random variable representing the time at which the game stopped, meaning that } \]

\[ \tau(\omega) \equiv \min\{t \in \{0, 1, 2, \ldots\} : X_t = 0 \text{ or } X_t = n\}. \]

Show that \( \tau \) is a stopping time.

We must show that for every \( t \in \{0, 1, 2, \ldots\} \), the event \( \{\omega \in \Omega : \tau(\omega) = t\} \in \mathcal{F}_t \). But,

\[
\begin{align*}
\{\omega \in \Omega : \tau(\omega) = t\} &= \{\omega \in \Omega : X_0(\omega), X_1(\omega), \ldots, X_{t-1}(\omega) \in \{1, 2, \ldots, n-1\} \text{ and } X_t(\omega) \in \{0, n\}\} \\
&= \bigcap_{k=0}^{t-1} \left\{ \omega \in \Omega : X_k(\omega) \in \{1, 2, \ldots, n-1\} \right\} \cap \left\{ \omega \in \Omega : X_t(\omega) \in \{0, n\} \right\} \\
&= \left( \bigcap_{k=0}^{t-1} \{\omega \in \Omega : X_k(\omega) \in \{1, 2, \ldots, n-1\} \} \right) \cap \{\omega \in \Omega : X_t(\omega) \in \{0, n\}\} \\
&= \mathcal{F}_t
\end{align*}
\]

A simpler solution would be as follows: \( \tau \) is the time of first contact with the set \{0, n\}, therefore by a theorem from the course (see end of chapter 2), it is a stopping time.

5 Exercise 2.5

Yes, the sum of those two stopping times is a stopping time, because \( t \in \{0, 1, 2, \ldots\} \). Indeed, let \( t \in \{0, 1, 2, \ldots\} \) be arbitrary. Then

\[
\begin{align*}
\{\omega \in \Omega : \tau_1 + \tau_2 = t\} &= \bigcup_{k=0}^{t} \left\{ \omega \in \Omega : \tau_1 = k \right\} \cap \left\{ \omega \in \Omega : \tau_2 = t-k \right\} \\
&= \bigcup_{k=0}^{t} \left( \{\omega \in \Omega : \tau_1 = k\} \cap \{\omega \in \Omega : \tau_2 = t-k\} \right) \in \mathcal{F}_t.
\end{align*}
\]
Which is what had to be shown.