Probability spaces and random variables
Exercises

Exercise 1.1. This exercise refers to the random variables described in the slides from the course.
$\Omega = \left\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}\right\}$. We already established that it is not possible to build a probability measure on the sample space Ω such that both random variables X and W have a uniform distribution. Is it possible to build a probability measure on the sample space Ω such that both random variables X and W have the same distribution ?

Exercice 1.2. Let $Card(\Omega) < \infty$. Show that if $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are $\mathcal{F}-$measurable then $\min\{X, Y\}$, $\max\{X, Y\}$ and XY are also $\mathcal{F}-$measurable. In addition, show that if $\forall \omega \in \Omega$, $Y(\omega) \neq 0$ then X/Y is $\mathcal{F}-$measurable.

Exercice 1.3. The indicator function of an event $C \subseteq \Omega$ is defined as

$$\begin{aligned} \mathbb{I}_C \quad &: \quad \Omega \to \{0, 1\} \\ &\omega \mapsto \mathbb{I}_C(\omega) = \begin{cases} 1 & \text{if } \omega \in C \\ 0 & \text{if } \omega \notin C \end{cases} \end{aligned}$$

Show that

$$\begin{array}{ll} (I1) & \text{If } A \text{ belongs to the } \sigma\text{-algebra } \mathcal{G} \text{ then } \mathbb{I}_A \text{ is } \mathcal{G} - \text{measurable.} \\ (I2) & \mathbb{I}_{A\cap B} = \mathbb{I}_A \mathbb{I}_B. \\ (I3) & \mathbb{I}_{A\cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_{A\cap B}. \end{array}$$

Now, let $A_1, ..., A_n$ be a finite partition of the sample space Ω. Show that

$$(I4) \quad \sum_{i=1}^{n} \mathbb{I}_{A_i}(\omega) = 1 \text{ for every } \omega \in \Omega.$$

Exercise 1.4. Consider the following sample space Ω and function $X : \Omega \to \mathbb{R}$.

	X
ω_1	0
ω_2	0
ω_3	-1
ω_4	1
ω_5	1
ω_6	2
ω_7	-2

a) What is the smallest σ-algebra $\mathcal{F}$ such that X is a random variable ?
b) What is the smallest σ-algebra $\mathcal{G}$ such that $|X|$ is a random variable ?
c) Is $|X|$ a $\mathcal{F}-$random variable ? Justify your answer.
d) Is X a $\mathcal{G}-$random variable ? Justify your answer.

Exercise 1.5. Let $\Omega = \{1, ..., 6\}$, $A = \{\{1, 3, 5\}, \{1, 2, 3\}\}$ and $B = \{\{2, 4, 6\}, \{4, 5, 6\}\}$.
a) Describe $\mathcal{F} = \sigma(A)$, the σ-algebra generated by A.
b) List the atoms of $\mathcal{F}$.
c) Describe $\mathcal{G} = \sigma(B)$, the σ-algebra generated by B.
d) List the atoms of $\mathcal{G}$.

Exercise 1.6. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $Y(\omega) = \exp[X(\omega)]$, $\omega \in \Omega$
a) Show that Y is also a random variable, and express its cumulative distribution function F_Y as a function of F_X.
b) Assume X has a continuous distribution. Then, Y is continuous too. Express f_Y, the density function of Y, as a function of f_X, the density function of X.
c) Assume $X \sim N(0, 1)$. Calculate f_Y and $\mathbb{P}(Y \leq 1)$.
d) What is the name of the distribution of Y ?

Solutions

1 Exercise 1.1

Yes, it is possible. Indeed, let $p_i = \mathbb{P}\left\{\boxed{i}\right\}$. We are looking for values of $p_1, ..., p_6$ such that : firstly, $\mathbb{P}$ is a probability measure, meaning

$$\begin{array}{cc} (i) & 0 \leq p_i \leq 1 \\ (ii) & p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1 \end{array};$$

secondly, the distribution of the random variable X is arbitrary, meaning that

$$\begin{array}{ll} (X1) & \mathbb{P}\{X = 0\} = \mathbb{P}\left\{\boxed{1}, \boxed{2}\right\} = p_1 + p_2 = q_0 \\ (X2) & \mathbb{P}\{X = 5\} = \mathbb{P}\left\{\boxed{3}, \boxed{4}\right\} = p_3 + p_4 = q_5 \\ (X3) & \mathbb{P}\{X = 10\} = \mathbb{P}\left\{\boxed{5}, \boxed{6}\right\} = p_5 + p_6 = q_{10} \end{array}$$

and finally, the distribution of the random variable W is the same as X, meaning

$$\begin{array}{ll} (W1) & \mathbb{P}\{W = 0\} = \mathbb{P}\left\{\boxed{5}\right\} = p_5 = q_0 \\ (W2) & \mathbb{P}\{W = 5\} = \mathbb{P}\left\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}\right\} = p_1 + p_2 + p_3 + p_4 = q_5 \\ (W3) & \mathbb{P}\{W = 10\} = \mathbb{P}\left\{\boxed{6}\right\} = p_6 = q_{10} \end{array}$$

The constraints $(X3)$ and $(W3)$ imply that $p_5 = 0$ and $p_6 = q_{10}$. Then, from $(W1)$ we obtain that $q_0 = p_5 = 0$. This implies $q_5 = 1 - q_0 - q_{10} = 1 - q_{10}$. Since p_1 and p_2 are non negative and because their sum is 0 (see $X1$), we conclude that $p_1 = p_2 = 0$. The constraint $(X2)$ yields $p_3 = q_5 - p_4 = 1 - q_{10} - p_4$. Hence the solution is

x	$\mathbb{P}\{X = x\} = \mathbb{P}\{W = x\}$
0	0
5	$1 - q_{10}$
10	$q_{10} \in [0; 1]$

.

ω	$\mathbb{P}(\omega)$	ω	$\mathbb{P}(\omega)$
$\boxed{1}$	0	$\boxed{4}$	$p_4 \in [0; 1 - q_{10}]$
$\boxed{2}$	0	$\boxed{5}$	0
$\boxed{3}$	$1 - q_{10} - p_4$	$\boxed{6}$	q_{10}

2 Exercise 1.2

Since $Card(\Omega) < \infty$, the random variables X and Y can only take a finite number of values, say $x_1 < ... < x_m$ and $y_1 < ... < y_n$ respectively.

Let us show that $\min\{X, Y\}$ is $\mathcal{F}-$measurable $: \forall z \in \mathbb{R}$,

$$\begin{aligned} & \{\omega \in \Omega \,|\min\{X(\omega), Y(\omega)\} \leq z\} \\ = \; & \{\omega \in \Omega \,|X(\omega) \leq z \text{ or } Y(\omega) \leq z\} \\ = \; & \underbrace{\{\omega \in \Omega \,|X(\omega) \leq z\}}_{\in \mathcal{F}} \cup \underbrace{\{\omega \in \Omega \,|Y(\omega) \leq z\}}_{\in \mathcal{F}} \in \mathcal{F}. \end{aligned}$$

Let us show that $\max\{X, Y\}$ is $\mathcal{F}-$measurable $: \forall z \in \mathbb{R}$,

$$\begin{aligned} & \{\omega \in \Omega \,|\max\{X(\omega), Y(\omega)\} \leq z\} \\ = \; & \{\omega \in \Omega \,|X(\omega) \leq z \text{ and } Y(\omega) \leq z\} \\ = \; & \underbrace{\{\omega \in \Omega \,|X(\omega) \leq z\}}_{\in \mathcal{F}} \cap \underbrace{\{\omega \in \Omega \,|Y(\omega) \leq z\}}_{\in \mathcal{F}} \in \mathcal{F}. \end{aligned}$$

Let us show that XY is $\mathcal{F}-$measurable $: \forall z \in \mathbb{R}$,

$$\begin{aligned} & \{\omega \in \Omega \,|X(\omega) Y(\omega) \leq z\} \\ = \; & \bigcup_{x_i y_j \leq z} \{\omega \in \Omega \,|X(\omega) = x_i \text{ and } Y(\omega) = y_j\} \\ = \; & \bigcup_{x_i y_j \leq z} \underbrace{\underbrace{\{\omega \in \Omega \,|X(\omega) = x_i\}}_{\in \mathcal{F}} \cap \underbrace{\{\omega \in \Omega \,|Y(\omega) = y_j\}}_{\in \mathcal{F}}}_{\in \mathcal{F}} \in \mathcal{F}. \end{aligned}$$

Let us show that if $Y \neq 0$, X/Y is $\mathcal{F}-$measurable $: \forall z \in \mathbb{R}$,

$$\begin{aligned} & \left\{\omega \in \Omega \left| \frac{X(\omega)}{Y(\omega)} \leq z \right.\right\} \\ = \; & \bigcup_{\frac{x_i}{y_j} \leq z} \{\omega \in \Omega \,|X(\omega) = x_i \text{ and } Y(\omega) = y_j\} \\ = \; & \bigcup_{\frac{x_i}{y_j} \leq z} \underbrace{\underbrace{\{\omega \in \Omega \,|X(\omega) = x_i\}}_{\in \mathcal{F}} \cap \underbrace{\{\omega \in \Omega \,|Y(\omega) = y_j\}}_{\in \mathcal{F}}}_{\in \mathcal{F}} \in \mathcal{F}. \; \blacksquare \end{aligned}$$

3 Exercise 1.3

Proof of $(I1)$. To show : *If A belongs to the σ-algebra $\mathcal{G}$ then $\mathbb{I}_A$ is $\mathcal{G}-$measurable.*

We must show that $\forall x \in \mathbb{R}$, $\{\omega \in \Omega : \mathbb{I}_A(\omega) = x\} \in \mathcal{G}$. However,

$$\begin{aligned} &\{\omega \in \Omega : \mathbb{I}_A(\omega) = 1\} = A \in \mathcal{G}, \\ &\{\omega \in \Omega : \mathbb{I}_A(\omega) = 0\} = A^c \in \mathcal{G} \\ \text{and } \forall x \notin \{0,1\}, \quad &\{\omega \in \Omega : \mathbb{I}_A(\omega) = x\} = \varnothing \in \mathcal{G}. \blacksquare \end{aligned}$$

Proof of $(I2)$. To show : $\mathbb{I}_{A\cap B} = \mathbb{I}_A \mathbb{I}_B$.

$$\mathbb{I}_{A\cap B}(\omega) = \begin{cases} 1 & \omega \in A \cap B \\ 0 & \omega \notin A \cap B \end{cases}$$

On the other hand,

$$\begin{aligned} \mathbb{I}_A(\omega)\,\mathbb{I}_B(\omega) = 1 \quad &\Leftrightarrow \quad \mathbb{I}_A(\omega) = 1 \text{ and } \mathbb{I}_B(\omega) = 1 \\ &\Leftrightarrow \quad \omega \in A \text{ and } \omega \in B \\ &\Leftrightarrow \quad \omega \in A \cap B. \blacksquare \end{aligned}$$

Proof of $(I3)$. To show : $\mathbb{I}_{A\cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_{A\cap B}$.

Let us consider four cases :

$$\begin{array}{lll} \text{if } \omega \in A \cap B \text{ then} & \mathbb{I}_{A\cup B}(\omega) = 1 \text{ and} & \begin{array}{c} \mathbb{I}_A(\omega) + \mathbb{I}_B(\omega) - \mathbb{I}_{A\cap B}(\omega) \\ = 1 + 1 - 1 = 1 \end{array} \\ \text{if } \omega \in A \cap B^c \text{ then} & \mathbb{I}_{A\cup B}(\omega) = 1 \text{ and} & \begin{array}{c} \mathbb{I}_A(\omega) + \mathbb{I}_B(\omega) - \mathbb{I}_{A\cap B}(\omega) \\ = 1 + 0 - 0 = 1 \end{array} \\ \text{if } \omega \in A^c \cap B \text{ then} & \mathbb{I}_{A\cup B}(\omega) = 1 \text{ and} & \begin{array}{c} \mathbb{I}_A(\omega) + \mathbb{I}_B(\omega) - \mathbb{I}_{A\cap B}(\omega) \\ = 0 + 1 - 0 = 1 \end{array} \\ \text{if } \omega \in A^c \cap B^c \text{ then} & \mathbb{I}_{A\cup B}(\omega) = 0 \text{ and} & \begin{array}{c} \mathbb{I}_A(\omega) + \mathbb{I}_B(\omega) - \mathbb{I}_{A\cap B}(\omega) \\ = 0 + 0 - 0 = 0 \end{array} \end{array}$$

Proof of $(I4)$. To show : *Let $A_1, ..., A_n$ be a finite partition of the sample space Ω. Show that $\sum_{i=1}^n \mathbb{I}_{A_i}(\omega) = 1$ for every $\omega \in \Omega$.*

Let $\omega \in A_j$ be an arbitrary element from the sample space. Since $A_1, ..., A_n$ is a partition we have

$$\omega \in A_j \text{ and } \omega \notin A_i \text{ for every } i \neq j.$$

Then,

$$\mathbb{I}_{A_j}(\omega) = 1 \text{ and } \mathbb{I}_{A_i}(\omega) = 0 \text{ for every } i \neq j.$$

Therefore,

$$\sum_{i=1}^n \mathbb{I}_{A_i}(\omega) = 1.$$

4 Exercise 1.4

a) What is the smallest σ-algebra $\mathcal{F}$ such that X is a random variable ?

$$\left\{\begin{array}{c} \varnothing, \Omega, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}, \{\omega_7\} \\ \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_4, \omega_5\}, \{\omega_1, \omega_2, \omega_6\}, \{\omega_1, \omega_2, \omega_7\} \\ \{\omega_3, \omega_4, \omega_5\}, \{\omega_3, \omega_6\}, \{\omega_3, \omega_7\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_4, \omega_5, \omega_7\}, \{\omega_6, \omega_7\} \\ \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1, \omega_2, \omega_3, \omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_7\} \\ \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_7\}, \{\omega_1, \omega_2, \omega_6, \omega_7\} \\ \{\omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_3, \omega_4, \omega_5, \omega_7\}, \{\omega_3, \omega_6, \omega_7\}, \{\omega_4, \omega_5, \omega_6, \omega_7\} \\ \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_7\}, \{\omega_1, \omega_2, \omega_3, \omega_6, \omega_7\} \\ \{\omega_1, \omega_2, \omega_4, \omega_5, \omega_6, \omega_7\}, \{\omega_3, \omega_4, \omega_5, \omega_6, \omega_7\} \end{array}\right\}$$

b) What is the smallest σ-algebra $\mathcal{G}$ such that $|X|$ is a random variable ?

$$\left\{\begin{array}{c} \varnothing, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \{\omega_6, \omega_7\} \\ \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1, \omega_2, \omega_6, \omega_7\}, \{\omega_3, \omega_4, \omega_5, \omega_6, \omega_7\} \end{array}\right\}$$

c) Is $|X|$ a $\mathcal{F}-$random variable ? Justify your answer.
Yes since

$$\{\omega \in \Omega : |X(\omega)| \leq x\} = \left\{\begin{array}{cc} \varnothing \in \mathcal{F} & \text{if } x < 0 \\ \{\omega_1, \omega_2\} \in \mathcal{F} & \text{if } 0 \leq x < 1 \\ \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \in \mathcal{F} & \text{if } 1 \leq x < 2 \\ \Omega \in \mathcal{F} & \text{if } x \geq 2 \end{array}\right.$$

d) Is X a $\mathcal{G}-$random variable ? Justify your answer.
No since

$$\{\omega \in \Omega : X(\omega) \leq -2\} = \{\omega_7\} \notin \mathcal{G}.$$