A Regime-Switching Term Structure Model With Observable State Variables

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Abstract

This paper proposes an arbitrage-free term structure model based on regime-shifts with the target and Fed Funds rates as observable state variables. A brief empirical investigation of their time series properties suggests that a three-state regime-shift environment associated with FOMC monetary actions is justified. Based on these findings, a closed-form solution for zero-coupon bonds is provided which includes priced regime-shift risk. The solution is flexible enough to incorporate additional state variables.

Keywords : Regime-switch, Target rate, Risk-neutral valuation, FOMC regime, Hidden Markov chain.

JEL classification codes : E43 and G12.

1. INTRODUCTION

The affine framework of Duffie and Kan (1996) is the dominant methodological paradigm in the area of arbitrage-free yield curve modeling. In this framework, researchers usually model latent state variables drawn from probability distributions insensitive to regime changes. Extensions of this approach are proposed by Ang, Piazzesi and Wei (2006), who specify term structure models without the help of latent variables and by Wu and Zeng (2008), Wu and Zeng (2006), Dai, Singleton and Yang (2007), and Bansal and Zhou (2002), who resort to Markov chains to capture regime-shifts displayed by the state variables. In this paper, a term structure model merging these two strands of research is proposed. While the inclusion of observable state variables sheds light onto the mechanisms driving the yield curve in the absence of arbitrage, the explanatory power of the model is improved when the state variables are confined to a regime-switching structure. A closed-form solution for zero-coupon bond pricing is derived. This solution accounts for priced regime-shift risk.

Among the various pervasive factors and market movers that influence the yield curve, the target rate of the Federal Reserve seems of primary importance (Piazzesi (2004)). Given this perspective, the Fed Funds rate

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(henceforth FF) appears as a good candidate to model the short rate. The model captures the dynamics of these two interest rate series within a hidden Markov chain environment that reflects monetary regimes.

2. STATE VARIABLE DYNAMICS

Monetary actions of the Federal Reserve are reflected in the movements of the target rate. These actions are communicated to financial markets on FOMC meetings. Since the target and FF rate series are the observed state variables in the model, their characteristics are being assessed outside the no-arbitrage framework. They are sampled from the first week of August, 1998 to the last week of December 2007. Since FOMC meetings are mostly held on Tuesdays, but sometimes on Wednesdays or Thursdays, and to ensure that the information conveyed by movements of the target rate is fully disseminated, the FF rate is extracted on the Fridays of each week. Figure 1 shows these two series.

Figure 1. Target and Fed Funds rate series

A visual inspection of Figure 1 suggests that the target rate can be modeled as a Markov chain that fluctuates only on FOMC meetings by multiples of 25 basis points within an upward, stable or a downward regime. Therefore, let \( \rho_t \) denote the unobserved regime of the Fed at time \( t \), and \( \theta_t \) the value of the target rate at time \( t \). The next two assumptions characterize their behavior.

Assumption 1 \( \theta_t \) behaves as a time-inhomogeneous Markov chain. Outside FOMC meeting, the transition probability matrix is the identity matrix; otherwise, it takes the form

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1 Concerns over the time series properties of the Fed Funds rate are raised by Hamilton (1996) and Piazzesi (2003) given the so-called "settlement Wednesday effect" which resulted from the settlement process imposed to banks. However, since August 1998, a "lagged reserve maintenance system" has been implemented. Kotomin and Winters (2007) provide empirical evidence on the time series properties of the FF rate within the old and new settlement mechanisms. They contend that the FF rate demonstrates a much smoother behavior after July, 1998 and notice that the "settlement Wednesday effect" has vanished.

2 These series are obtained from www.federalreserve.gov.
\[ P((\theta, \rho), (\theta', \rho')) = \mathbb{P}[\rho_{t_{n+1}} = \rho', \theta_{t_{n+1}} = \theta' | \rho_{t_n} = \rho, \theta_{t_n} = \theta] = \mathbb{P}[\theta_{t_{n+1}} = \theta' | \rho_{t_n} = \rho, \theta_{t_n} = \theta] \mathbb{P}[\rho_{t_{n+1}} = \rho' | \theta_{t_{n+1}} = \theta', \rho_{t_n} = \rho, \theta_{t_n} = \theta] = \mathbb{P}[\theta_{t_{n+1}} = \theta' | \rho_{t_n} = \rho, \theta_{t_n} = \theta] \mathbb{P}[\rho_{t_{n+1}} = \rho' | \rho_{t_n} = \rho] = M_{p}(\theta, \theta') M(\rho, \rho') \] (1)

where \( t_n \) designates a meeting date.

**Assumption 2** \( \rho_t \) behaves as a time-inhomogeneous discrete-state Markov chain. Outside FOMC meeting, the transition probability matrix is the identity matrix; otherwise, it takes the form

\[ M^{\rho} = \begin{pmatrix} p_{dd} & p_{ds} & p_{du} \\ p_{sd} & p_{ss} & p_{su} \\ p_{ud} & p_{us} & p_{uu} \end{pmatrix} \] (2)

where \( u, d \) and \( s \) respectively denotes the FOMC upward, stable and downward regime.

The previous assumption presumes that \( \rho_t \) follows a discrete three-state Markov chain.

**Assumption 3** At each FOMC meeting date \( t_n \), \( \rho_{t_n} \) moves according to the upward, stable or downward regime. In the upward (downward) regime, \( \theta_t \) increases (decreases) by 0, 25 or 50 basis points. In the stable regime, \( \theta_t \) varies up or down by a maximum of 50 basis points. The three corresponding transition matrices are \( M_u, M_d \) and \( M_s \). The state space of \( \theta \) is restricted to \( \{0.5\%, 0.75\%, \ldots, 6.75\%, 7.0\%\} \) while the Markov chain reflects at the boundaries.

**Assumption 4** Consistently with the concept of monetary regimes, the dynamic of the FF rate, \( r_t \), under \( \mathbb{P} \) exhibits a reverting behavior toward the target rate

\[ dr_t = \kappa_{\rho_{t_n}}^{\rho} (\theta_{t_n} - r_t) dt + \sigma_{\rho_{t_n}} \sigma^\rho_{\rho_{t_n}} dW_t^\rho, \quad t_n < t < t_{n+1} \] (3)

where \( W^\rho \) is a \( \mathbb{P} \)-Brownian motion. The parameters \( \kappa_{u}^{\rho}, \kappa_{s}^{\rho}, \kappa_{d}^{\rho} \) and \( \sigma_u, \sigma_s, \sigma_d \) are the speed of reversion and the volatilities observed in each regime.

### 3. Empirical Results

The discrete version of the strong solution of equation (3) gives

\[ r_{t_{n+1}} = (1 - \phi_{\rho_{t_n}}) r_{t_n} + \phi_{\rho_{t_n}} \theta_{t_n} + \sigma_{\rho_{t_n}}^s Z_{n+1} \] (4)

where \( \{Z_n : n \in \mathbb{N}\} \) are independent standard normal random variables, and where \( \phi_{\rho_{t_n}} = 1 - e^{-\kappa_{\rho_{t_n}} \Delta t} \) and \( \sigma_{\rho_{t_n}}^s = \sigma_{\rho_{t_n}} \sqrt{\frac{1-e^{-2\kappa_{\rho_{t_n}} \Delta t}}{2\kappa_{\rho_{t_n}}}} \), with \( \Delta t = t_{n+1} - t_n \).

The model is estimated via maximum likelihood in two steps. First, the matrices \( M^\rho, M_d, M_s \) and \( M_u \) are estimated following Hamilton (1989) using only \( \theta_t \). Then, the parameters of equation (4) are estimated by maximizing the log-likelihood function

\[ \ln L = \sum_{t=1}^{T} \ln f(r_t, \theta_t | \tilde{r}_{t-1}, \tilde{\theta}_{t-1}) \] (5)

where \( \tilde{r}_t = (r_t, r_{t-1}, \ldots, r_0) \) and \( \tilde{\theta}_t = (\theta_t, \theta_{t-1}, \ldots, \theta_0) \). Using the filtered probabilities

\[ \pi_t(\rho) = \Pr \left\{ \rho_t = \rho | \tilde{r}_t, \tilde{\theta}_t \right\}, \] (6)

the log-likelihood is written as

---

3
Table 1: Estimation of the regime's transition matrix

\[
\hat{M} = \begin{bmatrix}
\hat{p}_{uu} & \hat{p}_{us} & \hat{p}_{ud} \\
\hat{p}_{su} & \hat{p}_{ss} & \hat{p}_{sd} \\
\hat{p}_{du} & \hat{p}_{ds} & \hat{p}_{dd}
\end{bmatrix} = \begin{bmatrix}
0.8969 & 0.1031 & 0.0000 \\
(0.0645) & - & - \\
0.0524 & 0.8994 & 0.0483 \\
(0.0396) & (0.0757) & - \\
0.0000 & 0.1605 & 0.8395 \\
- & - & (0.1495)
\end{bmatrix}
\] (9)

The numbers inside the parenthesis are the estimated standard deviation of the estimator.

Table 2: Parameter's estimation

<table>
<thead>
<tr>
<th>Regime ( \rho ) : Downward</th>
<th>Stable</th>
<th>Upward</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel 1: Target rate’s transition probabilities</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( p_{\rho}(-50) )</td>
<td>0.5745 (0.1432)</td>
<td>0.0215 (0.0397)</td>
</tr>
<tr>
<td>( p_{\rho}(-25) )</td>
<td>0.2378 (0.1164)</td>
<td>0.0271 (0.0349)</td>
</tr>
<tr>
<td>( p_{\rho}(0) )</td>
<td>0.1877</td>
<td>0.9403</td>
</tr>
<tr>
<td>( p_{\rho}(25) )</td>
<td>-</td>
<td>0.0111 (0.0262)</td>
</tr>
<tr>
<td>( p_{\rho}(50) )</td>
<td>-</td>
<td>0.0000</td>
</tr>
<tr>
<td><strong>Panel 2: Discrete time short rate’s parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \phi_{\rho} )</td>
<td>0.561717 (0.116831)</td>
<td>1.0 (NaN)</td>
</tr>
<tr>
<td>( \sigma^*_{\rho} )</td>
<td>0.00236480 (0.00015928)</td>
<td>0.00038581 (0.00002119)</td>
</tr>
<tr>
<td><strong>Panel 3: Continuous time short rate’s parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \kappa_{\rho} )</td>
<td>41.245</td>
<td>230.26</td>
</tr>
<tr>
<td>( \sigma_{\rho} )</td>
<td>0.023895</td>
<td>0.0082798</td>
</tr>
</tbody>
</table>

Estimated standard deviations are inside the parenthesis. The transition from the discrete time parameters of Panel 2 to their continuous counterparts is reported at Panel 3. Since \( \phi \rightarrow 1 \) implies that \( \kappa \rightarrow \infty, \kappa_s \) and \( \sigma_s \) are based on the assumption \( \phi_s = 0.99 \).

\[
\ln L = \sum_{t=1}^{T} \ln \left( \sum_{\rho} \sum_{\rho'} v_{t-1}(\rho) P_{t-1}( (\theta_{t-1}, \rho), (\theta_t, \rho') ) p(r_t \mid r_{t-1}, \theta_{t-1}, \rho) \right)
\] (7)

where

\[
p(y \mid x, \theta, \rho) = \frac{1}{\sqrt{2\pi\sigma^2_{\rho}}} \exp \left\{ -\frac{[y - \mu(x; \theta, \rho)]^2}{2\sigma^2_{\rho}} \right\}
\] (8)

and \( \mu(x; \theta, \rho) = x(1 - \phi_{\rho}) + \phi_{\rho} \theta \).

Table 1 reports the estimated transition probabilities of \( \rho_t \). The estimates suggest that \( \rho_t \) tends to stay in the same state for long periods since the diagonal probabilities are large. This is consistent with the so-called "policy inertia".

The first panel of Table 2 contains the transition probabilities of the target rate in each regime. As expected, the target rate tends to increase (decrease) in the upward (downward) regime. Interestingly, the target rate does not seem to fluctuate much in the stable regime as the probability of no movement is 94.03%. The second panel shows the estimated parameters for the FF rate model while the third presents their continuous equivalent. All estimates are significant. Clearly, the reversion and volatility coefficients change across regimes.

4. BOND PRICING

In this section, a closed-form solution for bond pricing is presented.
Assumption 5 Under the $Q$ measure, the regime transition matrix of $\rho_t$ becomes $M^Q$ (whose entries are labeled as $q_{\rho_n, \rho}$) on FOMC meetings and the identity matrix otherwise. The transitions probabilities of the target become $q_d(k), k \in \{-50, -25, 0\}$; $q_s(k), k \in \{-50, -25, 0, 25, 50\}$; and $q_u(k), k \in \{0, 25, 50\}$ under the downward, stable and upward regimes, respectively.

Assumption 6 The dynamic of $r_t, \kappa$ under $Q$ is
\[
\begin{align*}
dr_t &= \kappa^Q \left( \theta^Q_{tn} - r_t \right) \, dt + \sigma_{\rho_{tn}} \, dW_t^Q, \quad t_n < t < t_{n+1} \\
\end{align*}
\] where $\theta^Q_{tn} = \theta_{tn} + \delta_{\rho_{tn}}$.

The parameter $\delta_{\rho_{tn}}$ is a risk premium whose value depends on the FOMC regime $\rho$. On the other hand, it is required to keep $\kappa^Q$ constant under $Q$ as in Dai, Singleton and Yang (2007), in order to produce a closed-form solution.

Assumption 7 An additional state variable $\{x_t : t \geq 0\}$, correlated with $r_t$, evolves according to
\[
\begin{align*}
dx_t &= b^Q \left( a^Q_{\rho_{tn}} - x_t \right) \, dt + \eta_{\rho_{tn}} \left( c_{\rho_{tn}} \, dW_t^Q + \sqrt{1 - e^{2 \, c_{\rho_{tn}} \, dB_t^Q}} \right), \quad t_n < t < t_{n+1} \\
\end{align*}
\] where $B^Q$ and $W^Q$ are independent Brownian motions.

This assumption presumes that the fluctuations exhibited by $x_t$ are influenced by FOMC regime-shifts. Given the empirical findings of the previous section and the complementary assumptions, Theorem 1 below shows the closed-form solution for zero-coupon bonds.

Theorem 1 Let $t_1 < t_2 < \ldots < t_n < \ldots$ be the time discretization where $t_n = n \Delta_t$. Assuming that the FF rate, $r_{tn}$; the target rate, $\theta_{tn}$; its current regime, $\rho_{tn}$; and the state variable, $x_{tn}$; are all observed, the time $t_n$ value of the zero-coupon bond paying one dollar at time $t_{n+m}$ is
\[
P(t_n, t_{n+m}; r_{tn}, x_{tn}, \theta_{tn}, \rho_{tn}) = \exp \left( A^r_m r_{tn} + A^x_m x_{tn} + B_m (\theta_{tn}, \rho_{tn}) \right)
\] where the coefficients $A^r$, $A^x$ and $B$ are determined recursively. Indeed, if $\phi(x) = x^{-1} \left(1 - e^{-x \Delta_t}\right)$, then
\[
\begin{align*}
A^r_m &= A^r_{m-1} \left(1 - \kappa^Q\phi(\kappa^Q)\right) - \phi(\kappa^Q) = -\frac{1 - e^{-m \kappa^Q \Delta_t}}{\kappa^Q}, \\
A^x_m &= A^x_{m-1} \left(1 - b^Q\phi(b^Q)\right) - \phi(b^Q) = -\frac{1 - e^{-mb^Q \Delta_t}}{b^Q}, \\
B_m (\theta_{tn}, \rho_{tn}) &= \ln \sum_{\rho, \theta} \exp \left( B_{m-1}(\theta, \rho) \right) q_{\rho_{tn}} (\theta - \theta_{tn}) q_{\rho_{tn}, \theta} + \tau_1 (\theta_{tn}, \rho_{tn}) + \tau_2 (\rho_{tn}) + \tau_3 (\rho_{tn})
\end{align*}
\] with
\[
\begin{align*}
\tau_1 (\theta, \rho) &= \left( -\Delta_t + e^{-(m-1)\kappa^Q\Delta_t} \phi(\kappa^Q) \right) (\theta - \delta_{\rho}) + \sigma_{\rho} \frac{\Delta_t - 2e^{-(m-1)\kappa^Q\Delta_t} \phi(\kappa^Q) + e^{-2(m-1)\kappa^Q\Delta_t} \phi(2\kappa^Q)}{2(\kappa^Q)^2}, \\
\tau_2 (\rho) &= \left( -\Delta_t + e^{-(m-1)b^Q\Delta_t} \phi(\kappa^Q) \right) a_{\rho} + \eta_{\rho} \frac{\Delta_t - 2e^{-(m-1)b^Q\Delta_t} \phi(\kappa^Q) + e^{-2(m-1)b^Q\Delta_t} \phi(2b^Q)}{2(b^Q)^2}, \\
\tau_3 (\rho) &= \frac{\sigma_{\rho} a_{\rho} \eta_{\rho}}{\kappa b^Q} \left[ \Delta_t + e^{-(m-1)\kappa^Q\Delta_t} \phi(\kappa^Q) - e^{-(m-1)b^Q\Delta_t} \phi(\kappa^Q) - e^{-(m-1)b^Q\Delta_t} \phi(2b^Q) \right].
\end{align*}
\]

The recursive nature of the solution provided in Theorem 1 allows for priced regime-shift risk as in Dai, Singleton and Yang (2007) and Wu and Zeng (2006). This is reflected explicitly (implicitly) by the risk-neutral probabilities $q_{\rho_{tn}, \rho}$ which influence the term $B_m (\theta_{tn}, \rho_{tn})$ (the risk-neutral trajectories of $\theta_t$ and $r_t$). On the other hand, the concept of FOMC regime-shifts, $\rho_{tn}$, plays a central role in Theorem 1. It drives
bond prices via the parameters $\delta_\rho$, the target rate risk premium; $a_\rho$, the "mean value" of $x_t$; $\sigma_\rho$ ($\eta_\rho$), the instantaneous volatility parameter of $r_t$ ($x_t$); $c_\rho$, a parameter at the source of correlated trajectories between $r_t$ and $x_t$; and the transition probabilities $q_{tn_\rho} (\cdot)$ and $q_{tn_\rho,\rho}$. The importance of the parameter $c_\rho$ must be underlined as the term $\tau_3(\rho)$ vanishes if orthogonality is imposed between the state variables $r_t$ and $x_t$. 
References


