Heterogeneous Basket Options Pricing Using Analytical Approximations*

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Abstract

This paper proposes the use of analytical approximations to price an heterogeneous basket option combining commodity prices, exchange rates and zero-coupon bonds. We examine the performance of three moment matching approximations: inverse gamma, Edgeworth expansion around the lognormal and Johnson family distributions. Since there is no closed-form formula for basket options, we carry out Monte Carlo simulations with quadratic resampling technique to generate the benchmark values. We perform a simulation experiment on a whole set of options based on a random choice of parameters. Our results show that the lognormal and Johnson distributions give the most accurate results.

Keywords : Basket Options, Options Pricing, Analytical Approximations, Monte Carlo Simulation.

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1 Introduction

Basket options are options on a portfolio of several underlying assets. With a growing diversification in investors’ portfolios, basket options are a very efficient tool to hedge multiple risk exposures simultaneously. Moreover, basket options are cheaper than portfolios of standard options but also less liquid and less flexible.

The pricing of basket options is more challenging than that of standard options because there is no explicit analytical solution for the density function of a weighted sum of correlated assets. Several approaches are proposed in literature to price homogeneous basket options where all underlying assets are geometric Brownian motions and interest rates are constant. They can be categorized as follows:


This article proposes to derive analytical approximations to price heterogeneous basket options when interest rates are stochastic,1 and to compare the accuracy and the performance of these approximations for different sets of parameters. Three distributions based on the moments matching technique will be used to approximate the basket density function: inverse gamma distribution, Edgeworth expansion around the lognormal distribution and Johnson distribution. Because of the heterogeneity of our basket, it is found that, unlike the literature on homogeneous basket options, the Edgeworth-lognormal and Johnson approximations perform better than the inverse gamma approximation.

The next section presents the pricing model under the forward measure. Section 3 derives the inverse gamma, the Edgeworth-lognormal and the Johnson approximations. Section 4 compares the performance of the three approximations. Section 5 computes the hedging ratios and Section 6 will conclude.

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1 The heterogeneous basket model will be briefly presented in Section 2. For details, refer to Dionne, Gauthier and Ouertani (2005).
2 Pricing of a European heterogeneous basket option

Let us consider a non-financial firm looking for an alternative way to simultaneously hedge its different financial risk exposures: commodity risk, exchange rate risk and interest rate risk. We assume that the firm’s financial assets dynamics are given by the following stochastic differential equations (SDE hereafter) under the historical probability measure $P$:

\begin{align*}
    dS_t &= \alpha_s (S_t - \delta_t) dt + \sigma_s S_t dW_t^{(1)}, \\
    d\delta_t &= \kappa (\theta - \delta_t) dt + \sigma_\delta dW_t^{(1)}, \\
    dC_t &= \alpha_c C_t dt + \sigma_c C_t dW_t^{(2)}, \\
    dP(t, T) &= P(t, T) \left[ (r_t - \sigma_f \gamma_t (T - t)) dt - \sigma_f (T - t) dW_t^{(3)} \right], \\
    dK(t, T) &= K(t, T) \left[ (u_t - \sigma_g \lambda_t (T - t)) dt - \sigma_g (T - t) dW_t^{(4)} \right], \\
    dD_t &= r_t D_t dt, \\
    dF_t &= u_t F_t dt,
\end{align*}

where $S$ is the commodity price in domestic currency units; $\delta$ is the continuously-compounded convenience yield; $C$ is the exchange rate (the value of one foreign currency unit expressed in domestic currency units); $P$ and $K$ are respectively the domestic and the foreign zero-coupon bonds prices; $r$ and $u$ are respectively the continuously-compounded domestic and foreign risk-free interest rates; and $D$ and $F$ are respectively the values of the domestic and the foreign risk-free money-market accounts. $\gamma_t$ and $\lambda_t$ correspond respectively to the domestic and the foreign interest rate risk premiums. The four-dimensional Brownian motion $\left\{ (W_t^{(1)}, W_t^{(2)}, W_t^{(3)}, W_t^{(4)}) : 0 \leq t \leq T \right\}$ is constructed on a filtered probability space $(\Omega, F, \{F_t : t \geq 0\}, P)$ with the following correlation structure:

\[ Corr^P (W_t^{(i)}, W_t^{(j)}) = \rho_{ij}, \text{ for each } i, j = \{1, 2, 3, 4\} \text{ and } 0 \leq t \leq T. \]

We consider a European call option that gives the firm the right to buy a basket consisting in the commodity, the domestic zero-coupon bond and the foreign zero-coupon bond expressed in domestic currency units, at a strike price $K_B$. The payoff at maturity $T$ is given by the $\mathcal{F}_T$-measurable random variable defined by:

\[ X_B = \max \left[ w_1 S_T + w_2 P(T, T_1) + w_3 Y(T, T_1) - K_B, 0 \right], \]

where $w_1$, $w_2$ and $w_3$ correspond to the proportions invested respectively in the commodity $S_T$, the domestic zero-coupon bond $P(T, T_1)$ and the foreign zero-coupon bond expressed in domestic currency units as $Y(T, T_1) = C_T K(T, T_1)$.

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\footnote{In this setting, the convenience yield and the commodity price share the same source of risk to ensure the market completeness. For more details, refer to Dionne, Gauthier and Ouertani (2005).}
Since the proposed market model is complete,\textsuperscript{3} Harisson and Pliska (1981) allows the pricing of any contingent claim as the expectation of the discounted payoff under the risk-neutral measure $Q$. Consequently, the price of the European call option at time $t$ is given by:

$$
V_t^B = \mathbb{E}_Q \left[ \exp \left( - \int_t^T r_v dv \right) X_B \mid \mathcal{F}_t \right] \\
= P(t,T) \mathbb{E}_{Q_T} [X_B \mid \mathcal{F}_t] \\
= P(t,T) \mathbb{E}_{Q_T} [\max (B_T - K_B, 0) \mid \mathcal{F}_t] \\
= P(t,T) \int_{-\infty}^{+\infty} \max (x - K_B, 0) v(x) dx,
$$

where $B_T = w_1 S_T + w_2 P(T, T_1) + w_3 Y(T, T_1)$ is the value of the basket at maturity $T$ and $v(x)$ is the true (unknown) density function of $B_T$ under the $T$-forward measure $Q_T$. This forward measure has a Radon-Nikodym derivative w.r.t. $Q$ equal to $dQ_T/dQ$. The associated $Q$-martingale is given by:

$$
\zeta_t = \mathbb{E}_Q \left( \frac{dQ_T}{dQ} \mid \mathcal{F}_t \right) = \exp \left( - \int_t^T r_v dv \right) / P(t,T).
$$

Details on the $T$-forward measure $Q_T$ are presented in Appendix A. The SDEs satisfied by the basket underlying assets under the $T$-forward measure $Q_T$ can be written as:

1. $dS_t = \left[ r_t - \delta_t - \sigma_s \sigma_f \rho_{13} (T - t) \right] S_t dt + \sigma_s S_t \hat{W}_t^{(1)}$,
2. $d\delta_t = a \left[ m_t - \delta_t - \frac{\sigma_s \sigma_f \rho_{13} (T - t)}{a} \right] dt + \sigma_\delta \hat{W}_t^{(1)}$,
3. $dC_t = \left[ r_t - \sigma_c \sigma_f \rho_{23} (T - t) \right] C_t dt + \sigma_c C_t \hat{W}_t^{(2)}$,
4. $dP(t,T) = \left[ r_t + \sigma_f^2 (T - t)^2 \right] P(t,T) dt - \sigma_f (T - t) P(t,T) d\hat{W}_t^{(3)}$,
5. $dK(t,T) = \left[ u_t + \sigma_g \sigma_f \rho_{24} (T - t) + \sigma_f \sigma_g \rho_{34} (T - t)^2 \right] K(t,T) dt - \sigma_g (T - t) K(t,T) d\hat{W}_t^{(4)}$,

where $a = \left( \kappa + \frac{\sigma_s}{\sigma_\delta} \right), m_t = \frac{1}{a} \left[ \kappa \theta - \frac{\sigma_s}{\sigma_\delta} (\alpha_s - r_t) \right]$,

and $\hat{W} = \left( \hat{W}^{(1)}, \hat{W}^{(2)}, \hat{W}^{(3)}, \hat{W}^{(4)} \right)' = A\hat{B}$. $\hat{B}$ represents a vector of independent Brownian motions under the $T$-forward measure $Q_T$. The strong solution of SDEs (3) under $Q_T$ exists and is

\textsuperscript{3}Refer to Dionne, Gauthier and Ouertani (2005) for the derivation of the unique risk-neutral measure.
given by:

\[
S_T = S_t \exp \left[ \int_t^T r_v dv - \int_t^T \delta_v dv - \frac{1}{2} \sigma_s^2 (T-t) - \frac{1}{2} \sigma_s \sigma_f \rho_{13} (T-t)^2 \right],
\]

\[
\delta_T = \left[ \delta_t e^{-\alpha(T-t)} + \left( b - \frac{\sigma_s \sigma_f \rho_{13}}{\alpha} \right) \left( 1 - e^{-\alpha(T-t)} \right) + \frac{\sigma_s}{\alpha} \int_t^T r_v e^{-\alpha(T-v)} dv \right],
\]

\[
C_T = C_t \exp \left[ \int_t^T \delta_v dv + \int_t^T u_v dv - \frac{1}{2} \sigma_c^2 (T-t) - \frac{1}{2} \sigma_c \sigma_f \rho_{23} (T-t)^2 \right],
\]

\[
P(T, T_1) = \exp \left[ - \int_T^{T_1} f(0, v) dv + \frac{1}{2} \sigma_f^2 (T_1 - T)^2 - \sigma_f (T_1 - T) \tilde{W}^{(3)}_T \right],
\]

\[
K(T, T_1) = \exp \left[ - \int_T^{T_1} g(0, v) dv - \frac{1}{2} \sigma_g^2 T_1 (T_1 - T) + \sigma_g \sigma_f \rho_{24} T_1 (T_1 - T) \right]
\]

\[
- \sigma_g (T_1 - T) \tilde{W}^{(4)}_T
\]

where \( b = \frac{1}{\alpha} \left( \kappa \theta - \frac{\sigma_s \sigma_f}{\sigma_s} \right) \).

Details on the solution are presented in Appendix A.

The evaluation of the call basket option is complicated by the absence of a closed form equation for the density function \( v(x) \) in Equation (2). Among the different approaches proposed in the basket options literature, we find some numerical techniques such as Monte Carlo, Quasi-Monte Carlo methods and lattice-based methods, the upper and lower bound computations,\(^4\) and some analytical approximations.

Lattice-based approaches are widely used for options on a single asset. They are, however, exponentially complicated and computationally expensive for options on multiple assets. For example, our three-asset basket option needs \((n+1)^3\) terminal nodes on an \(n\)-step trinomial tree. On the other hand, Monte Carlo and Quasi-Monte Carlo methods can be used for multi-assets options and are less time-consuming than lattice-based approaches. The estimates can be as accurate as needed at a computational cost however: to improve the accuracy of an \(n\)-path simulation by one half, one needs to simulate \(4n\) paths and thus needs \(4\) times more computing time.

A practitioner might be interested in slightly less accurate but very fast methods such as analytical approximations. These methods consist in approximating the unknown basket density function with an alternative and easy-to-compute distribution. In this article, we will extend three well-known analytical approximations to heterogeneous basket options: the inverse gamma distribution, the Edgeworth expansion around the lognormal distribution and Johnson distribution.

\(^4\)The upper and lower bounds methods are not useful in our case since the market model proposed here is complete and thus a unique option price can be computed.
3 Analytical approximations

In order to apply these moments matching-based approximations, we need to calculate the first four moments of the weighted sum underlying the European option under the $T$-forward measure $Q_T$.

Lemma 1 The first four non-centered moments of the true distribution of the weighted sum $B_T$, under the $T$-forward measure $Q_T$, are given for $n = \{1, 2, 3, 4\}$ by:

\[
\mu'_n (v) = E_{Q_T} (B_n^v) = E_{Q_T} ((w_1 S_T + w_2 P(T, T_1) + w_3 Y(T, T_1))^n)
\]

\[
= \left[ \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!}{(n-k)! j!(k-j)!} w_1^j w_2^{(k-j)} w_3^{(n-k)} \right] 
\times E_{Q_T} \left( S_T^j P(T, T_1)^{(k-j)} Y(T, T_1)^{(n-k)} \right).
\]

The derivation of these moments is based on the following identity:

\[
E[\exp (\mu + \sigma Z)] = \exp \left( \mu + \frac{\sigma^2}{2} \right) \quad \text{where} \quad Z \sim N(0, 1).
\]

Details are given in Appendix B. We will adopt the following notations:

\[
\mu'_n (h) = \int_{-\infty}^{+\infty} x^n h (x) \, dx \quad (5a)
\]

\[
\mu_n (h) = \int_{-\infty}^{+\infty} (x - \mu'_1 (h))^n h (x) \, dx, \quad (5b)
\]

$\mu'_n (h)$ and $\mu_n (h)$ represent respectively the $n^{th}$ non-centered and centered moments of the density function $h \in \{v, a\}$, where $h = v$ corresponds to the exact density while $h = a$ corresponds to the approximate density. The first four cumulants of distribution $h$, i.e. the mean, the variance, the skewness and the kurtosis are defined as:

\[
\kappa_1 (h) = \mu'_1 (h) \quad (6a)
\]

\[
\kappa_2 (h) = \mu_2 (h) \quad (6b)
\]

\[
\kappa_3 (h) = \mu_3 (h) \quad (6c)
\]

\[
\kappa_4 (h) = \mu_4 (h) - 3 \mu_2 (h). \quad (6d)
\]

3.1 Inverse gamma approximation

In this section, we use the inverse gamma distribution to approximate the sum of correlated lognormal variables. This approximation was first used by Milevsky and Posner (1998a, 1998b) to price Asian and basket options. In fact, a finite sum of correlated lognormal variables converges asymptotically to an inverse gamma variable. Under an inverse gamma distribution for the underlying basket, an European basket call option has a closed-form solution that looks like Black and Scholes (1973) (B&S hereafter) formula:

\[
V^B_{gamma} = P(t, T) \left( \mu'_1 (v) G \left( \frac{1}{K_B} \mid \alpha - 1, \beta \right) - K_B G \left( \frac{1}{K_B} \mid \alpha, \beta \right) \right), \quad (7)
\]
where $G(\bullet | \alpha, \beta)$ is the cumulative function of a gamma distribution with parameters $(\alpha, \beta)$. These parameters are determined by matching the first two moments of the exact and the approximate distributions to obtain:

$$
\alpha = \frac{\mu_1^2(v) - 2\mu_1'(v)}{\mu_1^2(v) - \mu_2'(v)}, \quad (8a)
$$

$$
\beta = \frac{\mu_2'(v) - \mu_1^2(v)}{\mu_1'(v) \mu_2'(v)}. \quad (8b)
$$

Mathematical details of the pricing formula are provided in Appendix C.

### 3.2 Edgeworth expansion around the lognormal distribution

This section presents an analytical approximation based on a generalized Edgeworth expansion around the lognormal distribution. This approach, introduced by Jarrow and Rudd (1982) in option pricing, consists in substituting an unknown density function by a Taylor-like expansion around an easy-to-use density function. Notice however that Edgeworth expansion usually lead to a function which is not a true density function. Barton and Denis (1952) derive some conditions on the third and fourth moments of the unknown distribution to guarantee that the approximation obtained with a truncated Edgeworth expansion is positive and unimodal.

Moreover, Ju (2002) points out that Edgeworth expansion may diverge for some parameters values, which consequently can give incorrect prices for high volatility and long maturity options. However, in this paper we do not have this problem in the application of the Edgeworth expansion.

Following Huynh (1994) who uses this approach for basket options, we will use an Edgeworth expansion of order 4 and we will match the first and second moments of the exact and the lognormal distributions.$^5$ Under this approximation, an European basket call option can be obtained as a B&S price adjusted for the excess skewness and the excess kurtosis from the lognormal density:

$$
V_{lognormal}^B = \left[ V_1 - P(t, T) \frac{\kappa_3(v) - \kappa_3(a) da(K_B)}{3!} + P(t, T) \frac{\kappa_4(v) - \kappa_4(a) d^2a(K_B)}{4!} \right], \quad (9)
$$

where

$$
V_1 = P(t, T) \left( \mu_1'(v) N(d_1) - K_B N(d_2) \right), \quad (10a)
$$

$$
d_1 = d_2 + \beta, \quad (10b)
$$

$$
d_2 = \frac{\alpha - \ln(K_B)}{\beta}, \quad (10c)
$$

---

$^5$Jarrow and Rudd (1982) state that, in general, there is no bound on the error term resulting from an Edgeworth expansion. Consequently, the error does not necessarily decrease with the expansion’s order.
\[ \alpha = \ln \left( \mu'_1(v)^2 \right) - \frac{1}{2} \ln \left( \mu'_1(v)^2 + \mu_2(v) \right), \quad (10d) \]

\[ \beta = \sqrt{\ln \left( 1 + \frac{\mu_2(v)}{\mu'_1(v)^2} \right)}, \quad (10e) \]

and \( N(\bullet) \) represents the cumulative function of the standard normal distribution. The third and fourth moments of the lognormal distribution needed for the Edgeworth expansion depend only on the first and second moments of the exact distribution and are given by:

\[ \mu'_3(a) = \left( \frac{\mu_2(v)}{\mu'_1(v)} \right)^3, \quad (11a) \]

\[ \mu'_4(a) = \frac{\mu_2(v)^6}{\mu'_1(v)^8}. \quad (11b) \]

Details about the Edgeworth approximation formula are given in Appendix D.

### 3.3 Johnson approximation

Johnson (1949) proposes a family of density functions, obtained via a transformation of a standard normal variable, that can be used to approximate unknown distributions. Let \( Z \) be a standard normal variable and \( X \) be a random variable with an unknown density function, Johnson (1949) suggests the following form of transformation between \( Z \) and \( X \):

\[ Z = \gamma + \delta \psi \left( \frac{X - \varepsilon}{\lambda} \right), \quad (12a) \]

\[ X = \varepsilon + \lambda \psi^{-1} \left( \frac{Z - \gamma}{\delta} \right), \quad (12b) \]

where \( \gamma \) and \( \delta \) are the shape parameters of Johnson distribution, \( \lambda \) is the scale parameter, \( \varepsilon \) is the threshold parameter, and \( \psi(\bullet) \) is one of the following Johnson functions:

\[ S_L(x) = \ln(x), \quad \text{Lognormal system} \quad (13a) \]

\[ S_U(x) = \ln \left( x + \sqrt{x^2 + 1} \right), \quad \text{Unbounded system} \quad (13b) \]

\[ S_B(x) = \ln \left( \frac{x}{1-x} \right), \quad \text{Bounded system.} \quad (13c) \]

The choice of the system and fitting parameters provides a great flexibility in adjusting the curve to match the first four moments of the unknown distribution. We will use the lognormal \((S_L)\) and the unbounded \((S_U)\) systems that are largely used in literature. We will apply Hill, Hill and Holder (1976) algorithm, based on the true skewness and kurtosis of the basket distribution, to determine which of Johnson systems \((S_L \text{ or } S_U)\) should be used in the approximation. Unlike approximations obtained with a truncated Edgeworth expansion, approximations based on Johnson (1949) systems correspond to true density functions with a perfect match of the first four moments.
Following Posner and Milevsky (1999), we will substitute the unknown distribution of the underlying basket by the lognormal and the unbounded Johnson functions where the system parameters are calculated by matching the four moments. Under a Jonhson density function, an European basket call option can be priced by:

\[
V_{Johnson}^B = P(t, T) \int_{K_B}^{+\infty} (x - K_B) \psi(x) \, dx
\]

\[
= P(t, T) \left( \int_{0}^{+\infty} x \psi(x) \, dx - K_B \int_{0}^{+\infty} \psi(x) \, dx - \int_{0}^{K_B} (x - K_B) \psi(x) \, dx \right)
\]

\[
= P(t, T) \left( \mu_1(v) - K_B + \int_{0}^{K_B} \left[ \int_{0}^{x} \psi(y) \, dy \right] \, dx \right)
\]

\[
\leq P(t, T) \left( \mu_1(v) - K_B + \int_{-\infty}^{K_B} \left[ \int_{0}^{x} \psi(y) \, dy \right] \, dx \right).
\]  

(14)

The third line in the equation is obtained using an integration by parts. Milevsky and Posner (1999) show that the last double integral, involving a Johnson density function, can be computed as follows:

1. Lognormal system \( S_L \): \( X = \varepsilon + \lambda \exp \left( \frac{Z - \gamma}{\delta} \right) \)

\[
\int_{-\infty}^{K_B} \left[ \int_{0}^{x} \psi(y) \, dy \right] \, dx = (K_B - \varepsilon) N \left( \gamma + \delta \ln \left( \frac{K_B - \varepsilon}{\lambda} \right) \right) - \lambda \exp \left( \frac{1 - 2\gamma\delta}{2\delta^2} \right) N \left( \gamma + \delta \ln \left( \frac{K_B - \varepsilon}{\lambda} \right) - \frac{1}{\delta} \right).
\]  

(15)

2. Unbounded system \( S_U \): \( X = \varepsilon + \lambda \sinh \left( \frac{Z - \gamma}{\delta} \right) \)

\[
\int_{-\infty}^{K_B} \left[ \int_{0}^{x} \psi(y) \, dy \right] \, dx = (K_B - \varepsilon) N(q) + \frac{\lambda}{2} \exp \left( \frac{1}{2\delta^2} \right) \exp \left( \frac{\gamma}{\delta} \right) N \left( q + \frac{1}{\delta} \right) - \frac{\lambda}{2} \exp \left( \frac{1}{2\delta^2} \right) \exp \left( -\frac{\gamma}{\delta} \right) N \left( q - \frac{1}{\delta} \right).
\]  

(16)

where \( q = \gamma + \delta \sinh^{-1} \left( \frac{K_B - \varepsilon}{\lambda} \right) \).  

4 Performance analysis of approximations

In this section, we analyze the performance of the three approximations presented previously. We will use prices obtained by Monte Carlo simulations as benchmarks since there is no closed-form

\[6\] Notice that \( \sinh(x) = \frac{\exp(x) - \exp(-x)}{2} \) and thus \( \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) = S_U(x) \).
solution for basket options. Following Barraquand (1995), we will use a variance reduction technique based on the matching of the first and second moments. This will ensure that sample mean and variance are equal to the their theoretical counterparts. The Monte Carlo basket option price combined with the variance reduction technique is given by:

\[ V^B_0 = \frac{1}{N} \sum_{i=1}^{N} P(0,T) \max \left( B^*_i - K_B, 0 \right) \]

where \( B^*_i = (B_{i,T} - B) \frac{\mu_2(v)}{S} + \mu_1(v) \),

\( B_{i,T} \) is the time \( T \) basket value obtained with sample path \( i \), and \( B \) and \( S \) are respectively the sample basket mean and standard deviation.

Our performance study will be conducted with two analyses. In the first sensitivity analysis, we will compare the basket option price obtained with the analytical approximations to the Monte Carlo price obtained with 1,000,000 paths repeated 40 times for different maturities, different moneynesses and different levels for the basket volatility. The second sensitivity analysis is a more detailed analysis based on works done by Broadie and Detemple (1996). It computes the option prices over a wide range of parameters chosen randomly from a realistic set of values in order to generalise our previous results independently of the model parameters. For each combination, we compare the prices obtained with the approximations and Monte Carlo simulations.

Table 1 presents the set of parameters used in the first analysis. These values are based on the estimation done in Dionne, Gauthier and Ouertani (2005) using real data on gold prices, CAD/USD exchange rate and Canadian and American 3-month zero-coupon bonds. Although we have positive and negative correlations in the set of parameters, we ensure that the volatility of the basket will increase when individual assets volatilities increase.
Table 1: Parameters used in the first sensitivity analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domestic risk-free rate, $r_D$</td>
<td>0.06</td>
</tr>
<tr>
<td>Foreign risk-free rate, $r_f$</td>
<td>0.05</td>
</tr>
<tr>
<td>Commodity drift, $\alpha_s$</td>
<td>0.15</td>
</tr>
<tr>
<td>Exchange rate drift, $\alpha_c$</td>
<td>0.04</td>
</tr>
<tr>
<td>Commodity volatility, $\sigma_s$</td>
<td>0.15</td>
</tr>
<tr>
<td>Exchange rate volatility, $\sigma_c$</td>
<td>0.06</td>
</tr>
<tr>
<td>Domestic instantaneous forward rate volatility, $\sigma_f$</td>
<td>0.01</td>
</tr>
<tr>
<td>Foreign instantaneous forward rate volatility, $\sigma_g$</td>
<td>0.01</td>
</tr>
<tr>
<td>Convenience yield volatility, $\sigma_\delta$</td>
<td>0.3</td>
</tr>
<tr>
<td>Convenience yield mean reversion parameter, $\kappa$</td>
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</tr>
<tr>
<td>Convenience yield long-run mean, $\theta$</td>
<td>0.15</td>
</tr>
<tr>
<td>Commodity and exchange rate correlation, $\rho_{12}$</td>
<td>0.1</td>
</tr>
<tr>
<td>Commodity and foreign instantaneous forward rate correlation, $\rho_{14}$</td>
<td>-0.25</td>
</tr>
<tr>
<td>Commodity and domestic instantaneous forward rate correlation, $\rho_{13}$</td>
<td>-0.2</td>
</tr>
<tr>
<td>Exchange rate and domestic instantaneous forward rate correlation, $\rho_{23}$</td>
<td>0.05</td>
</tr>
<tr>
<td>Exchange rate and foreign instantaneous forward rate correlation, $\rho_{24}$</td>
<td>0.1</td>
</tr>
<tr>
<td>Foreign and domestic instantaneous forward rates correlation, $\rho_{34}$</td>
<td>0.85</td>
</tr>
<tr>
<td>Basket weights (commodity, domestic and foreign zero-coupon bonds)</td>
<td>0.5; 0.25; 0.25</td>
</tr>
</tbody>
</table>

Table 2 presents sensitivity analysis of the basket option price w.r.t. the basket volatility and option maturity. The average level volatility corresponds to the values in Table 1, while high and low levels correspond respectively to an increase and a decrease of 50% in the volatility values given in Table 1.
Table 2: Sensitivity analysis w.r.t. the basket volatility and option maturity

<table>
<thead>
<tr>
<th>Basket Volatility</th>
<th>Monte Carlo Price</th>
<th>Inverse Gamma Approximation Price</th>
<th>Lognormal Approximation Price</th>
<th>Johnson Approximation Price</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Relative error*</td>
<td>Relative error*</td>
<td>Relative error*</td>
<td>Relative error*</td>
</tr>
<tr>
<td>90-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>5.3103</td>
<td>5.3044</td>
<td>5.3102</td>
<td>5.3102</td>
</tr>
<tr>
<td>Average</td>
<td>6.6406</td>
<td>6.6204</td>
<td>6.6404</td>
<td>6.6403</td>
</tr>
<tr>
<td>High</td>
<td>8.2140</td>
<td>8.1753</td>
<td>8.2135</td>
<td>8.2132</td>
</tr>
<tr>
<td>180-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>5.7015</td>
<td>5.6933</td>
<td>5.7014</td>
<td>5.7013</td>
</tr>
<tr>
<td>Average</td>
<td>7.1630</td>
<td>7.1369</td>
<td>7.1627</td>
<td>7.1625</td>
</tr>
<tr>
<td>High</td>
<td>8.7899</td>
<td>8.7410</td>
<td>8.7892</td>
<td>8.7889</td>
</tr>
<tr>
<td>270-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>7.7413</td>
<td>7.7332</td>
<td>7.7411</td>
<td>7.7411</td>
</tr>
<tr>
<td>Average</td>
<td>8.7173</td>
<td>8.6834</td>
<td>8.7170</td>
<td>8.7167</td>
</tr>
<tr>
<td>360-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>11.0230</td>
<td>11.0185</td>
<td>11.0229</td>
<td>11.0229</td>
</tr>
<tr>
<td>Average</td>
<td>11.1912</td>
<td>11.1518</td>
<td>11.1907</td>
<td>11.1904</td>
</tr>
<tr>
<td>High</td>
<td>11.7935</td>
<td>11.7156</td>
<td>11.7927</td>
<td>11.7922</td>
</tr>
</tbody>
</table>

* Relative errors are computed as $\epsilon = \frac{|V^B(a) - \bar{V}^B|}{\bar{V}^B}$, where $V^B(a)$ and $\bar{V}^B$ are the option prices obtained respectively with the analytical approximation and the Monte Carlo simulation with the variance reduction technique presented previously.

The results show that the approximation based on the Edgeworth-lognormal expansion is slightly more accurate than that based on Johnson distribution. Their relative errors presented in Table 2 are very low with a magnitude between $10^{-6}$ and $10^{-5}$. For low volatility levels, Edgeworth-lognormal and Johnson approximate prices and Monte Carlo prices are very close. As for the inverse gamma approximation, our results show that it is much less accurate with relative errors around $10^{-3}$.

Notice that all three approximations are biased downwards for all maturities and volatility levels. Their accuracy decreases for longer maturities and higher volatilities, which supports the findings in Ju (2002).\(^7\) Also for all approximations, option prices increase with maturity and volatility.

---

\(^7\)Ju (2002) proposes an analytical approximation to price Asian and basket options based on a Taylor expansion of the ratio of the characteristic function of the average of lognormal variables to that of the approximating lognormal random variable around zero volatility.
Table 3 presents sensitivity analysis of the basket option price w.r.t. moneyness and option maturity. The findings are similar to those in Table 2. The Edgeworth-lognormal and Johnson approximations are much more accurate, with relative errors between $10^{-7}$ and $10^{-4}$, than the inverse gamma approximation with a relative error between $10^{-4}$ and $10^{-2}$. In-the-money options are underpriced while at-the-money and out-of-the-money options are overpriced by the Edgeworth-lognormal and Johnson approximations. Notice that for all approximations, option prices decrease with moneyness and accuracy is lower for out-of-the-money options.

Table 3: Sensitivity analysis w.r.t. the moneyness and option maturity

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Monte Carlo</th>
<th>Inverse Gamma Approximation</th>
<th>Lognormal Approximation</th>
<th>Johnson Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price</td>
<td>Relative error*</td>
<td>Price</td>
<td>Relative error*</td>
</tr>
<tr>
<td>90-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>6.6406</td>
<td>6.6204</td>
<td>3.05e-3</td>
<td>6.6404</td>
</tr>
<tr>
<td>1</td>
<td>2.3995</td>
<td>2.4082</td>
<td>3.62e-3</td>
<td>2.3996</td>
</tr>
<tr>
<td>1.05</td>
<td>0.5925</td>
<td>0.6138</td>
<td>3.59e-2</td>
<td>0.5927</td>
</tr>
<tr>
<td>180-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>7.1630</td>
<td>7.1369</td>
<td>3.64e-3</td>
<td>7.1627</td>
</tr>
<tr>
<td>1</td>
<td>3.0151</td>
<td>3.0232</td>
<td>2.68e-3</td>
<td>3.0151</td>
</tr>
<tr>
<td>1.05</td>
<td>0.9717</td>
<td>1.0005</td>
<td>2.96e-2</td>
<td>0.9725</td>
</tr>
<tr>
<td>270-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>8.7172</td>
<td>8.6834</td>
<td>3.87e-3</td>
<td>8.7170</td>
</tr>
<tr>
<td>1</td>
<td>4.0985</td>
<td>4.0968</td>
<td>4.34e-4</td>
<td>4.0988</td>
</tr>
<tr>
<td>1.05</td>
<td>1.5353</td>
<td>1.5629</td>
<td>1.79e-2</td>
<td>1.5356</td>
</tr>
<tr>
<td>360-day maturity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>11.1913</td>
<td>11.1518</td>
<td>3.52e-3</td>
<td>11.1907</td>
</tr>
<tr>
<td>1</td>
<td>5.8639</td>
<td>5.8455</td>
<td>3.14e-3</td>
<td>5.8637</td>
</tr>
<tr>
<td>1.05</td>
<td>2.5144</td>
<td>2.5324</td>
<td>7.15e-3</td>
<td>2.5145</td>
</tr>
</tbody>
</table>

* Relative errors are computed as $e = \frac{|V^B(a) - \bar{V}^B|}{\bar{V}^B}$, where $V^B(a)$ and $\bar{V}^B$ are the option prices obtained respectively with the analytical approximation and the Monte Carlo simulation with the variance reduction technique presented previously.

Along with the previous analysis, we also conduct a sensitivity analysis of the basket option price by changing only one volatility or one correlation parameter at a time. We find that only volatilities of the commodity, of the convenience yield and of the domestic zero-coupon bond as well as the correlation between the commodity and the domestic zero-coupon bond have a considerable impact on the option price. The other parameters included in the basket volatility calculation have no significant impact.
The previous analysis is only local, our findings depend on the set of parameters used and may change if we modify the parameters. To confirm our conclusions, a more carefully designed numerical study where the parameters are randomly chosen is conducted. Following Broadie and Detemple (1996), the significant model parameters are chosen randomly from continuous or discrete uniform distributions. These uniform distributions are based on the estimation of the model parameters using real data. For the non-significant parameters, we use some intuitive values. The correlation between the commodity and the exchange rate is expected to be positive, so we assume that $\rho_{12} \in [0.01, 0.55]$. Based on Schwartz (1997), the convenience yield long-run mean is positive for gold and the mean reversion parameter is small and less than 1, we assume thus that $\theta \in [0.05, 0.5]$ and $\kappa \in [0.05, 0.8]$. All parameters distributions used in the pricing are presented in Table 5.

We compare the three previous analytical approximations and Monte Carlo simulation combined with the moments matching technique for 5000 random sets of parameters. First, we examine the accuracy of each approximation by calculating its root mean square error (RMSE). Secondly, we calculate the maximum relative error (MRE) for each approximation to examine the worst case scenario. More precisely, we define,

$$RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( \frac{V_{i}^{B} (a) - \overline{V}_{i}^{B}}{\overline{V}_{i}^{B}} \right)^{2}}$$

$$MRE = \max \frac{|V_{i}^{B} (a) - \overline{V}_{i}^{B}|}{\overline{V}_{i}^{B}},$$

where $n$ is the number of different sets of parameters, i.e. 4780, and $V_{i}^{B} (a)$ and $\overline{V}_{i}^{B}$ correspond respectively to the approximate and Monte Carlo prices for parameters set $i$. Monte Carlo prices are obtained with 1,000,000 paths combined with the moments matching technique.

Table 4: RMSE and MRE for the three approximations

<table>
<thead>
<tr>
<th>Approximation</th>
<th>RMSE</th>
<th>MRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse Gamma</td>
<td>1.351%</td>
<td>13.50%</td>
</tr>
<tr>
<td>Lognormal Approximation</td>
<td>0.103%</td>
<td>1.97%</td>
</tr>
<tr>
<td>Johnson Approximation</td>
<td>0.104%</td>
<td>2.01%</td>
</tr>
</tbody>
</table>

The results in table 4 confirm those obtained with the first sensitivity analysis and show that the Edgeworth-lognormal and Johnson approximations are much more accurate than the inverse gamma approximation. It is also found that for the Edgeworth-lognormal and Johnson approximations,

---

*See Dionne, Gauthier and Ouertani (2005) for the estimation of the parameters using real data.

*Only 4780 sets of parameters provide positive definite correlation matrices.
all options have relative errors below 2%, while for the inverse gamma approximation, a small proportion, 9.50%, of options have a relative error above 2%.
Table 5: Parameters distributions

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domestic risk free rate, $r_D$</td>
<td>Randomly chosen between 0.02 and 0.08</td>
</tr>
<tr>
<td>Foreign risk free rate, $r_f$</td>
<td>Randomly chosen between 0.02 and 0.08</td>
</tr>
<tr>
<td>Commodity drift, $\alpha_s$</td>
<td>Randomly chosen between 0.05 and 0.35</td>
</tr>
<tr>
<td>Exchange rate drift, $\alpha_c$</td>
<td>Randomly chosen between 0.01 and 0.1</td>
</tr>
<tr>
<td>Commodity volatility, $\sigma_s$</td>
<td>Randomly chosen between 0.1 and 0.35</td>
</tr>
<tr>
<td>Exchange rate volatility, $\sigma_c$</td>
<td>Randomly chosen between 0.02 and 0.15</td>
</tr>
<tr>
<td>Domestic instantaneous forward rate volatility, $\sigma_f$</td>
<td>Randomly chosen between 0.001 and 0.06</td>
</tr>
<tr>
<td>Foreign instantaneous forward rate volatility, $\sigma_g$</td>
<td>Randomly chosen between 0.001 and 0.06</td>
</tr>
<tr>
<td>Convenience yield volatility, $\sigma_\delta$</td>
<td>Randomly chosen between 0.1 and 0.4</td>
</tr>
<tr>
<td>Convenience yield mean reversion parameter, $\kappa$</td>
<td>Randomly chosen between 0.05 and 0.8</td>
</tr>
<tr>
<td>Convenience yield long-run mean, $\theta$</td>
<td>Randomly chosen between 0.05 and 0.5</td>
</tr>
<tr>
<td>Commodity and exchange rate correlation, $\rho_{12}$</td>
<td>Randomly chosen between 0.01 and 0.55</td>
</tr>
<tr>
<td>Commodity and domestic instantaneous forward rate correlation, $\rho_{13}$</td>
<td>Randomly chosen between -0.5 and 0.25</td>
</tr>
<tr>
<td>Commodity and foreign instantaneous forward rate correlation, $\rho_{14}$</td>
<td>Randomly chosen between -0.5 and 0.25</td>
</tr>
<tr>
<td>Exchange rate and domestic instantaneous forward rate correlation, $\rho_{23}$</td>
<td>Randomly chosen between -0.35 and 0.35</td>
</tr>
<tr>
<td>Exchange rate and foreign instantaneous forward rate correlation, $\rho_{24}$</td>
<td>Randomly chosen between -0.35 and 0.35</td>
</tr>
<tr>
<td>Foreign and domestic instantaneous forward rates correlation, $\rho_{34}$</td>
<td>Randomly chosen between 0.15 and 0.9</td>
</tr>
<tr>
<td>Option maturity (in days)</td>
<td>Randomly chosen in {30; 90; 180; 270}</td>
</tr>
<tr>
<td>Moneyness</td>
<td>Randomly chosen in {0.95; 1; 1.05}</td>
</tr>
<tr>
<td>Basket weights (commodity, domestic and foreign zero-coupon bonds), $w_1$, $w_2$, $w_3$</td>
<td>Fixed respectively at 0.5; 0.25; 0.25</td>
</tr>
</tbody>
</table>

A more detailed look at the results shows that out-of-the-money and high volatility options have the largest relative errors, which confirms our findings in the first sensitivity analysis. Figures 1, 2 and 3 present respectively the histograms of the relative errors of the inverse gamma, the Edgeworth-lognormal and Johnson approximations. Table 6 shows a summary of descriptive statistics of the relative errors.
Figure 1: Histogram of relative errors of the inverse gamma approximation

X-axis represents relative errors
Y-axis represents the number of observations
Figure 2: Histogram of relative errors of the Edgeworth approximation

X-axis represents relative errors
Y-axis represents the number of observations
Figure 3: Histogram of relative errors of Johnson approximation

X-axis represents relative errors
Y-axis represents the number of observations
Table 6: Descriptive statistics of relative errors

<table>
<thead>
<tr>
<th></th>
<th>Inverse Gamma Approximation</th>
<th>Lognormal Approximation</th>
<th>Johnson Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>0.1350</td>
<td>0.0197</td>
<td>0.0201</td>
</tr>
<tr>
<td>Minimum</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mean</td>
<td>0.0087</td>
<td>4.16e-4</td>
<td>4.18e-4</td>
</tr>
<tr>
<td>Median</td>
<td>0.0054</td>
<td>1.29e-4</td>
<td>1.27e-4</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0104</td>
<td>9.47e-4</td>
<td>9.50e-4</td>
</tr>
<tr>
<td>Nb of observations</td>
<td>4780</td>
<td>4780</td>
<td>4780</td>
</tr>
</tbody>
</table>

Histograms show that for the Edgeworth-lognormal and Johnson approximations, 94% of parameters sets (4500 out of 4780) have relative errors less than 0.002. This proves that these approximations are very accurate and that they give prices very close to Monte Carlo benchmarks. The inverse gamma approximation has 86% (4100 out of 4780) cases where the relative errors are less than 0.02, which is still acceptable from a practitioner point of view. We reach the same results by using larger and more general intervals for the parameters distributions.

To conclude this section regarding the pricing of heterogeneous basket options, we attest that Edgeworth expansion around the lognormal distribution at order 4 and the Johnson distribution are equally accurate and very acceptable for practitioners. However, we suggest the use of the Edgeworth-lognormal distribution for two reasons, first it is a slightly more accurate, and secondly the algorithm to calibrate Johnson distributions may not converge in a few cases, which can lead to mispriced options.

5 Hedging ratios

The analytical approximations allow for deriving the option price in a functional form which can also provide analytical expressions for the sensitivities w.r.t. the underlying parameters, such as deltas, vegas and theta, known as the hedging ratios or the Greeks. However, due to the complexity of the moments involved in our analytical approximations, deriving the hedging ratios analytically is beyond the scope of this article. Instead, we can compute them numerically as follows:

\[
\text{Ratio}^B_{\text{App}} = \frac{V^B(\varepsilon) - V^B_{\text{App}}}{\varepsilon},
\]

where \( \varepsilon > 0 \) is a small number, \( V^B(\varepsilon) \) and \( V^B_{\text{App}} \) are respectively the approximate \( \varepsilon \)-disturbed and non-disturbed option prices. As an application, we compute numerically the delta w.r.t. the commodity price for different values of \( \varepsilon \). We base our calculations on the parameters in Table 1. We consider an in-the-money call basket option with a 6-month maturity. The deltas are presented in Table 7.
Table 7: Basket option Delta w.r.t. the commodity price

<table>
<thead>
<tr>
<th>Commodity Price</th>
<th>Inverse Gamma Approximation</th>
<th>Lognormal Approximation</th>
<th>Johnson Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Price</td>
<td>Delta*</td>
<td>Price</td>
</tr>
<tr>
<td>330 $</td>
<td>7.1369</td>
<td></td>
<td>7.1627</td>
</tr>
<tr>
<td>330.33 $</td>
<td>7.1440 0.02151</td>
<td></td>
<td>7.1698 0.02151</td>
</tr>
<tr>
<td>331.65 $</td>
<td>7.1724 0.02151</td>
<td></td>
<td>7.1984 0.02164</td>
</tr>
<tr>
<td>333.30 $</td>
<td>7.2080 0.02154</td>
<td></td>
<td>7.2341 0.02164</td>
</tr>
<tr>
<td>334.95 $</td>
<td>7.2436 0.02155</td>
<td></td>
<td>7.2698 0.02164</td>
</tr>
</tbody>
</table>

* Delta corresponds to the sensitivity w.r.t. the commodity price and is given by Equation 17.

Table 7 shows that for the three approximations, the delta is very stable over different values of ε. Indeed, an increase of 1$ in the commodity price leads to an increase of approximately 0.02$ in the option price. The positiveness of delta is expected but its value depends on the set of parameters used. Following the same procedure, one can compute the other sensitivities w.r.t. other parameters of interest.

6 Conclusion

Basket options can be used by firms to hedge their exposure to different risks, such as commodity risk, interest rate risk and exchange rate risk. However, pricing this kind of options is not an easy task since no closed-form solution can be derived for the basket density function. Consequently, a standard pricing formula such as B&S cannot be derived. The main contribution of this article is the comparison of the performance of three analytical approximations to price heterogeneous basket options, consisting in a commodity, a domestic and a foreign zero-coupon bonds, when interest rates are stochastic. The three approximations used are the inverse gamma proposed in Milevsky and Posner (1998a, 1998b), the Edgeworth expansion around the lognormal distribution as well as Johnson distribution developped in Posner and Milevsky (1999).

In order to assess and compare the accuracy of the approximations, we use two analyses. The first one is a local sensitivity analysis where the parameters of the model are fixed arbitrarily. Our findings show that both the Edgeworth-lognormal and Johnson approximations are very accurate while the inverse gamma approximation is much less accurate.

These results are confirmed with the second analysis where 5000 sets of parameters are chosen randomly from different uniform distributions. It is also found that the pricing relative errors,
computed with Monte Carlo benchmark prices, are small and of an acceptable magnitude for practitioners. The accuracy of all approximations deteriorates for out-of-the-money as well as for high volatility options.
A Derivation of the forward measure

In this appendix, we will derive the model under the $T$-forward measure. Let $A_i$ be the $i^{th}$ row of matrix $A$, where $A = [a_{ij}]_{i,j=1,2,3,4}$ is the Cholesky decomposition of the correlation matrix of the four-dimensional $P$-Brownian motions $\mathbf{W} = (W^{(1)}, W^{(2)}, W^{(3)}, W^{(4)})'$, and $\mathbf{B}$ corresponds to the vector of independent Brownian motions under $Q$. The SDEs satisfied by all underlying assets under $Q$ can be written as:

\[ dS_t = (r_t - \delta_t)S_t dt + \sigma_s S_t A_1 d\mathbf{B}_t, \]
\[ d\delta_t = a(m_t - \delta_t)dt + \sigma_\delta A_1 d\mathbf{B}_t, \]
\[ dC_t = (r_t - u_t)C_t dt + \sigma_c A_2 d\mathbf{B}_t, \]
\[ dP(t, T) = P(t, T) \left[ r_t dt - \sigma_f (T - t) A_3 d\mathbf{B}_t \right], \]
\[ dK(t, T) = K(t, T) \left[ (u_t + \sigma_\rho_{24} (T - t)) dt - \sigma_g (T - t) A_4 d\mathbf{B}_t \right]. \]

The risk-neutral measure $Q$ corresponds to a numeraire equal to the domestic bank account $e^{\int_0^T r_s ds}$, while the $T$-forward measure $Q_T$ corresponds to a numeraire equal to the domestic zero-coupon bond, $P(t, T)$. Following Baxter and Rennie (1996) and using Girsanov theorem, we have that:

\[ d\mathbf{B}_t = d\mathbf{B}_t + \sigma_f (T - t) A_3' dt, \]

where $\mathbf{B}$ are independent Brownian motions under $Q_T$. The correlation structure is the same under $Q$ and $Q_T$, and the previous SDEs can be rewritten as:

\[ dS_t = \left[ r_t - \delta_t - \sigma_s \sigma_f \rho_{13} (T - t) \right] S_t dt + \sigma_s S_t d\mathbf{\tilde{W}}_t^{(1)}, \quad \text{(18a)} \]
\[ d\delta_t = a \left[ m_t - \delta_t - \frac{\sigma_\delta \sigma_f \rho_{13} (T - t)}{a} \right] dt + \sigma_\delta d\mathbf{\tilde{W}}_t^{(1)}, \quad \text{(18b)} \]
\[ dC_t = \left[ r_t - u_t - \sigma_c \sigma_f \rho_{23} (T - t) \right] C_t dt + \sigma_c C_t d\mathbf{\tilde{W}}_t^{(2)}, \quad \text{(18c)} \]
\[ dP(t, T) = \left[ r_t + \sigma_f^2 (T - t)^2 \right] P(t, T) dt - \sigma_f (T - t) P(t, T) d\mathbf{\tilde{W}}_t^{(3)}, \quad \text{(18d)} \]
\[ dK(t, T) = \left[ \left( u_t + \sigma_\rho_{24} (T - t) + \sigma_f \sigma_\rho_{24} (T - t)^2 \right) K(t, T) dt - \sigma_g (T - t) K(t, T) d\mathbf{\tilde{W}}_t^{(4)} \right], \quad \text{(18e)} \]

where $\mathbf{\tilde{W}} = \left( \mathbf{\tilde{W}}^{(1)}, \mathbf{\tilde{W}}^{(2)}, \mathbf{\tilde{W}}^{(3)}, \mathbf{\tilde{W}}^{(4)} \right)' = \mathbf{A} \mathbf{B}$. The strong solution of the system of Equations (18) under $Q_T$ exists and is given by the system of Equations (4).

A.1 Solution of the system of Equations (4)

Recall that under the risk-neutral measure $Q$, the domestic and foreign short rates are given by:

\[ r_t = f(0, t) + \frac{1}{2} \sigma_f^2 t^2 + \sigma_f \mathbf{\tilde{W}}_t^{(3)}, \]
\[ u_t = g(0, t) + \frac{1}{2} \sigma_g^2 t^2 - \sigma_c \sigma_\rho_{24} t + \sigma_g \mathbf{\tilde{W}}_t^{(4)}. \]
Using the fact that
\[ \hat{B}_t = \hat{B}_t + \sigma_f A_f \int_0^t (T - v) \, dv, \]
we can rewrite the short rate dynamics under \( Q_T \) as:
\[
\begin{align*}
  r_t &= f (0, t) - \sigma_f^2 t (T - t) + \sigma_f \tilde{W}_t^{(3)}, \\
  u_t &= g (0, t) + \frac{1}{2} \left( \sigma_g^2 + \sigma_f \sigma_g \rho_{g3} \right) t^2 - \sigma_c \sigma_g \rho_{g2} t - \sigma_g \rho_{g3} T + \sigma_g \tilde{W}_t^{(4)}.
\end{align*}
\]
(19a) (19b)

Integrating between \( t \) and \( T \), we obtain:
\[
\begin{align*}
  \int_t^T r_v \, dv &= \left[ \int_t^T f (0, v) \, dv - \frac{1}{2} \sigma_f^2 T (T^2 - t^2) + \frac{1}{3} \sigma_f^2 (T^3 - t^3) + \sigma_f \int_t^T \tilde{W}_v^{(3)} \, dv \right], \\
  \int_t^T u_v \, dv &= \left[ \int_t^T g (0, v) \, dv - \frac{1}{2} \sigma_c \sigma_g \rho_{g2} + \frac{1}{2} \sigma_f \sigma_g \rho_{g3} T \right] (T^2 - t^2) + \sigma_g \int_t^T \tilde{W}_v^{(4)} \, dv.
\end{align*}
\]
(20a) (20b)

A.1.1 Solution of the SDE (4b)

Under \( Q_T \), the convenience yield dynamic is given by:
\[ d\delta_t = a \left[ m_t - \delta_t - \frac{\sigma \delta \sigma_f \rho_{g2}}{a} (T - t) \right] \, dt + \sigma \delta \tilde{W}_t^{(1)}, \]
where \( m_t = \frac{1}{a} \left[ \kappa \theta - \frac{\sigma \delta \sigma_f \rho_{g2}}{a} (\alpha_s - r_t) \right] \). Let \( k_t = m_t - \frac{\sigma \delta \sigma_f \rho_{g2}}{a} (T - t) \), so that:
\[ d\delta_t = a \left[ k_t - \delta_t \right] \, dt + \sigma \delta \tilde{W}_t^{(1)}. \]

Applying Itô’s lemma, the convenience yield is solved for by:
\[
\begin{align*}
  \delta_T &= \delta_t e^{-a(T-t)} + a \int_t^T k_v e^{-a(T-v)} \, dv + \sigma \delta \int_t^T e^{-a(T-v)} \, d\tilde{W}_v^{(1)} \\
  &= \left[ \delta_t e^{-a(T-t)} + \left( \kappa \theta - \frac{\sigma \delta \sigma_f \rho_{g2}}{a} - \sigma \delta \sigma_f \rho_{g3} T \right) \int_t^T e^{-a(T-v)} \, dv \\
  &\quad + \sigma \delta \int_t^T r_v e^{-a(T-v)} \, dv + \sigma \delta \sigma_f \rho_{g2} \int_t^T v e^{-a(T-v)} \, dv + \sigma \delta \int_t^T e^{-a(T-v)} \, d\tilde{W}_v^{(1)} \right] \\
  &= \left[ \delta_t e^{-a(T-t)} + \left( b' - \frac{\sigma \delta \sigma_f \rho_{g2} T}{a} \right) (1 - e^{-a(T-t)}) + \frac{\sigma \delta}{a} \int_t^T r_v e^{-a(T-v)} \, dv \\
  &\quad + \sigma \delta \sigma_f \rho_{g2} \left( \frac{T t e^{-a(T-t)}}{a} - \frac{1 - e^{-a(T-t)}}{a^2} \right) + \sigma \delta \int_t^T e^{-a(T-v)} \, d\tilde{W}_v^{(1)} \right],
\end{align*}
\]

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where \( b = \frac{1}{a} (\kappa_\theta - \frac{\sigma \delta \mu_s}{\sigma_s}) \), and \( \int_t^T r_v e^{-a(T-v)} dv \) can be computed by:

\[
\int_t^T r_v e^{-a(T-v)} dv = \left[ \int_t^T f (0, v) e^{-a(T-v)} dv - \int_t^T v (T - v) e^{-a(T-v)} dv \right] + \sigma_f \int_t^T \widehat{W}_v^3 e^{-a(T-v)} dv
\]

\[
= \left[ \int_t^T f (0, v) e^{-a(T-v)} dv + \sigma_f \int_t^T v^2 e^{-a(T-v)} dv \right] - \sigma_f \int_t^T T e^{-a(T-v)} dv + \sigma_f \int_t^T \widehat{W}_v^3 e^{-a(T-v)} dv
\]

\[
= \left[ \int_t^T f (0, v) e^{-a(T-v)} dv + \sigma_f \left( \frac{a^2 T^2}{2} - a e^{-a(T-v)} - a \int_t^T e^{-a(T-v)} dv \right) \right] - \sigma_f \int_t^T T e^{-a(T-v)} dv + \sigma_f \int_t^T \widehat{W}_v^3 e^{-a(T-v)} dv
\]

\[
(21)
\]

**A.1.2 Solution of the SDE (4a)**

The SDE satisfied by the commodity price is:

\[
dS_t = [r_t - \delta_t - \sigma_s \sigma_f \rho_{13} (T - t)] S_t dt + \sigma_s S_t d\widehat{W}_t^{(1)}
\]

which has the following solution:

\[
S_T = S_t \exp \left[ \int_t^T r_v dv - \int_t^T \delta_v dv - \frac{1}{2} \sigma_s^2 (T - t) - \frac{1}{2} \sigma_s \sigma_f \rho_{13} (T - t)^2 + \sigma_s \int_t^T d\widehat{W}_v^{(1)} \right]
\]

Integrating \( \delta \) between \( t \) and \( T \), we obtain:

\[
\int_t^T \delta_v dv = \left[ \int_t^T \delta_v e^{-a(T-v)} dv + \int_t^T \left( b - \frac{\sigma \delta \rho_{13} \rho_{13}}{\sigma_s} \right) \left( 1 - e^{-a(T-v)} \right) dv \right]
\]

\[
+ \frac{\sigma_f}{\sigma_s} \int_t^T \left( \int_t^T r_v e^{-a(T-v)} dv \right) \int_t^T e^{-a(T-v)} dv + \sigma_s \int_t^T \left( \frac{a^2 T^2}{2} - a e^{-a(T-v)} - a \int_t^T e^{-a(T-v)} dv \right)
\]

\[
= \left( b - \frac{\sigma \delta \rho_{13} \rho_{13}}{\sigma_s} \right) (T - t) + \frac{\sigma_f}{\sigma_s} \int_t^T \left( \int_t^T r_v e^{-a(T-v)} dv + \frac{\sigma_s}{\sigma_s} \int_t^T \left( \frac{a^2 T^2}{2} - a e^{-a(T-v)} - a \int_t^T e^{-a(T-v)} dv \right) \right)
\]

\[
+ \frac{\delta}{\sigma_s} \int_t^T \left( \int_t^T r_v e^{-a(T-v)} dv + \frac{\sigma_s}{\sigma_s} \int_t^T \left( \frac{a^2 T^2}{2} - a e^{-a(T-v)} - a \int_t^T e^{-a(T-v)} dv \right) \right)
\]

\[
(22)
\]
Substituting \( \int_t^T r_v dv \) and \( \int_t^T \delta_v dv \) by the expressions in Equations (20a) and (22), we can write that:

\[
S_T = S_t \exp \left[ \left( 1 - \frac{\sigma_s}{\sigma_s} \right) \int_t^T f \left( 0, v \right) dv + \frac{\sigma_s}{\sigma_s} \int_t^T f \left( 0, v \right) e^{-a(T-v)} dv - \frac{1}{2} \sigma_s \sigma_f \rho_{13} \left( T - t \right)^2 
- \left( \frac{1}{2} \sigma_s^2 + b - \frac{\sigma_s \sigma_f \rho_{13}}{a} \right) \left( T - t \right) 
- \frac{1}{2} \sigma_f (\sigma_f - h) \left( T^2 - t^2 \right) 
+ \frac{1}{3} \sigma_f (\sigma_f - h) \left( T^3 - t^3 \right) 
- \left( \delta_t - b - \frac{\sigma_s \sigma_f \rho_{23}}{a} + \frac{\sigma_s \sigma_f \rho_{13}}{a^2} \right) \left( \frac{1}{2} - e^{-a(T-t)} \right) \right]
\]

where \( b = \frac{1}{a} (\kappa \theta - \frac{\sigma_s \sigma_f}{\sigma_s}) \) and \( h = \frac{\sigma_s \sigma_f \rho_{13}}{\sigma_s} \).

A.1.3 Solution of the SDE (4c)

The exchange rate dynamics is given by:

\[
dC_t = \left[ r_t - \sigma_c \sigma_f \rho_{23} (T - t) \right] C_t dt + \sigma_c C_t d\bar{W}^{(2)}_t.
\]

Applying Itô’s lemma, we have:

\[
C_T = C_t \exp \left[ \int_t^T r_v dv - \int_t^T u_v dv - \frac{1}{2} \sigma_c^2 (T - t) - \frac{1}{2} \sigma_c \sigma_f \rho_{23} (T - t)^2 + \sigma_c \int_t^T d\bar{W}^{(2)}_v \right].
\]

Substituting \( \int_t^T r_v dv \) and \( \int_t^T u_v dv \) again using Equations (20a) and (20b), we obtain:

\[
C_T = C_t \exp \left[ \int_t^T g \left( 0, v \right) dv - \int_t^T h \left( 0, v \right) dv - \frac{1}{2} \sigma_c \sigma_f \rho_{23} (T - t)^2 - \frac{1}{2} \sigma_c^2 (T - t) 
+ \frac{1}{2} \left( \sigma_c^2 \sigma_f \rho_{24} + \sigma_f \sigma_c \sigma_f \sigma_{23} T - \sigma_f^2 T \right) \left( T^2 - t^2 \right) + \sigma_c \int_t^T d\bar{W}^{(2)}_v 
+ \sigma_f \int_t^T \bar{W}^{(3)}_v dv - \sigma_f \int_t^T \bar{W}^{(4)}_v dv + \frac{1}{6} \left( 2 \sigma_f^2 - \sigma_f^2 - \sigma_f \sigma_f \sigma_{23} T^3 - t^3 \right) \right].
\]

A.1.4 Solution of the SDE (4d)

For the domestic zero-coupon bonds, recall that under \( Q_T \), we have:

\[
dP(t, T) = \left[ r_t + \sigma_f^2 (T - t)^2 \right] P(t, T) dt - \sigma_f (T - t) P(t, T) d\bar{W}^{(3)}_t.
\]

which gives:

\[
P(t, T) = \exp \left[ - \int_t^T r_v dv - \frac{1}{6} \sigma_f^2 (T^3 - t^3) + \frac{1}{2} \sigma_f^2 T T (T - t) - \sigma_f (T - t) \bar{W}^{(3)}_t + \sigma_f \int_t^T \bar{W}^{(3)}_v dv \right].
\]
For the foreign zero-coupon bond, solving:

\[ P(T, T_1) = \exp \left[ - \int_T^{T_1} f(0, v) dv + \frac{1}{2} \sigma_f^2 T (T_1 - T)^2 - \frac{1}{6} \sigma_f^2 (T_1^3 - T^3) - \frac{1}{2} \sigma_f^2 (T_1^3 - T^3) \right] \]

\[ + \frac{1}{2} \sigma_f^2 T (T_1 - T) \sigma_f \int_T^{T_1} \tilde{W}_v^{(3)} dv - \sigma_f (T_1 - T) \tilde{W}_T^{(3)} + \sigma_f \int_T^{T_1} \tilde{W}_v^{(3)} dv \]

\[ = \exp \left[ - \int_T^{T_1} f(0, v) dv + \frac{1}{2} \sigma_f^2 T (T_1 - T)^2 - \sigma_f (T_1 - T) \tilde{W}_T^{(3)} \right]. \tag{25} \]

A.1.5 Solution of the SDE (4e)

For the foreign zero-coupon bond, solving:

\[ dK(t, T) = \left[ \left( u_t + \sigma_c \sigma_g \rho_{24} (T - t) + \sigma_f \sigma_g \rho_{34} (T - t)^2 \right) K(t, T) dt - \sigma_g (T - t) K(t, T) d\tilde{W}_t^{(4)} \right], \]

we obtain:

\[ K(t, T) = \exp \left[ - \int_0^T f(0, v) dv + \frac{1}{2} \left( \sigma_f^2 + \sigma_f \rho_{24} \rho_{34} (T_1 - T)^2 \right) - \frac{1}{6} \sigma_f^2 \rho_{24} \rho_{34} (T_1 - T)^3 \right] \]

\[ - \sigma_g \int_0^T \tilde{W}_v^{(4)} dv - \sigma_g (T_1 - T) \tilde{W}_T^{(4)} + \sigma_g \int_0^T \tilde{W}_v^{(4)} dv \]

\[ = \exp \left[ - \int_0^T f(0, v) dv + \frac{1}{2} \sigma_g^2 T (T_1 - T)^2 + \sigma_c \sigma_g \rho_{24} T (T_1 - T) \right] \]

\[ + \sigma_f \sigma_g \rho_{34} \left( \frac{1}{2} T_1 (T_1^2 - T^2) - \frac{1}{6} \left( T_1^3 - T^3 \right) - \frac{1}{3} (T_1 - T)^3 \right) \tilde{W}_T^{(4)} \]. \tag{26} \]

B Derivation of moments

This appendix shows the calculation details of the first four moments of the basket value at maturity \( T \) under the \( T \)-forward measure \( Q_T \):

\[ B_T = w_1 S_T + w_2 P(T, T_1) + w_3 Y(T, T_1). \]
Using the system of Equations (4), and for \( j, k \) and \( n \in \{1, 2, 3, 4\} \), we have:

\[
(S_T)^j = (S_t)^j \exp \left[ j \chi + j \int_T^t \left\{ \sigma_s - \sigma_\delta \left( \frac{1-e^{-a(T-t)}}{a} \right) \right\} d\tilde{W}_v^{(1)} + j \int_T^t \frac{W_{T_1}^{(3)} e^{-a(T-t)}}{a} d\tilde{W}_v^{(3)} \right] ,
\]

\[ P(T, T_1)^{(k-j)} = \exp \left[ (k-j) \psi - (k-j) \sigma_f (T_1 - T) \tilde{W}_T^{(3)} \right] , \]

\[ Y(T, T_1)^{(n-k)} = (C_t)^{(n-k)} \exp \left[ (n-k) \omega + (n-k) \sigma_g \int_T^t d\tilde{W}_v^{(2)} + (n-k) \sigma_f \int_T^t \tilde{W}_v^{(3)} dv - (n-k) \sigma_g \int_T^t \tilde{W}_v^{(4)} dv \right] , \]

where:

\[
\chi = \left[ \left( 1 - \frac{\sigma_s}{\alpha \sigma_s} \right) \int_T^t f(0, v) dv + \frac{\sigma_s}{\alpha \sigma_s} \int_T^t f(0, v) e^{-a(T-v)} dv - \frac{1}{2} \sigma_s \sigma_f \rho_{13} (T-t)^2 \right]
- \frac{1}{2} \sigma_f (\sigma_f - h) T (T^2 - t^2) - \left( \frac{3}{2} \sigma_s^2 + b - \frac{\sigma_f \sigma_{f \rho_{13}}}{a} \right) (T-t)
+ \left( h \sigma_f T + \frac{\sigma_f \sigma_{f \rho_{13}}}{a} \right) \left( \frac{T - e^{-a(T-t)}}{a} - 1 - e^{-a(T-t)} \right) \left( \frac{1 - e^{-a(T-t)}}{a} \right)
+ h \sigma_f \frac{T^2 - 2 e^{-a(T-t)}}{a^2}
\]

\[ \psi = - \int_T^{T_1} f(0, v) dv + \frac{1}{2} \sigma_f^2 T (T_1 - T)^2 , \]

\[ \omega = \left[ \int_T^{T_1} f(0, v) dv + \frac{1}{2} \left( \sigma_c \sigma_f \rho_{24} + \sigma_f \sigma_{g \rho_{34}} T - \sigma_f^2 T \right) (T^2 - t^2) - \int_T^{T_1} g(0, v) dv \right]
- \frac{1}{2} \sigma_c \sigma_f \rho_{23} (T-t)^2 - \frac{1}{2} \sigma_f^2 (T-t) + \frac{1}{6} \left( 2 \sigma_f^2 - \sigma_f^2 - \sigma_f \sigma_{g \rho_{34}} (T^3 - t^3) + \sigma_f \sigma_{g \rho_{34}} (T_1^3 - T^3) - \frac{1}{3} (T_1 - T)^3 \right)
+ \frac{1}{2} \sigma_f^2 TT_1 (T_1 - T) + \sigma_c \sigma_{g \rho_{24}} T (T_1 - T) \]

with \( b = \frac{1}{a} \left( \kappa \theta - \frac{\sigma_f \sigma_s}{\sigma_s} \right) \) and \( h = \frac{\sigma_f \sigma}{a \sigma_s} \).

Using the following formula:

\[
E \left[ \exp (\mu + \sigma Z) \right] = \exp \left( \mu + \frac{\sigma^2}{2} \right) \quad \text{where} \ Z \sim N(0, 1) ,
\]

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we then have:

\[
\mu_n' = E_{Q_T}(B_T^n) = E_{Q_T}((w_1 T_s + w_2 P(T, T_1) + w_3 Y(T, T_1))^n) = \\
\sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!}{k!(n-k)!} \frac{k!}{j!(k-j)!} w_1^j w_2^{(k-j)} w_3^{(n-k)} S_j C_t^{(n-k)} \\
\] 

\[
\begin{aligned}
j \chi + (k-j) \psi + (n-k) \omega + \frac{1}{2} \left( \frac{\psi}{2a} - (k-j) \sigma_f (T_1 - T) \right)^2 (T-t) \\
+ \frac{1}{2} \left( \frac{\psi}{2a} - \left( k-j \right) \sigma_f (T_1 - T) \right) \left( \frac{\psi}{2a} - \left( k-j \right) \sigma_f (T_1 - T) \right) (T-t) \\
+ \left( \frac{\psi}{2a} - \left( k-j \right) \sigma_f (T_1 - T) \right) \sigma_x (T_1 - T) (T-t) \\
+ \left( \frac{\psi}{2a} - \left( k-j \right) \sigma_f (T_1 - T) \right) \sigma_y (T_1 - T) (T-t) \\
+ \left( \frac{\psi}{2a} - \left( k-j \right) \sigma_f (T_1 - T) \right) \sigma_z (T_1 - T) (T-t) \\
\end{aligned}
\]

\[
\times \exp \left[ \\
+ \frac{1}{2} \left( j (\sigma_f - h) + (n-k) \sigma_f \right)^2 + (n-k)^2 \sigma_y^2 (T-t)^3 \\
- \frac{1}{2} (n-k) \sigma_y (j (\sigma_f - h) + (n-k) \sigma_f) (T-t)^3 \\
+ \frac{j^2}{2a^2} \left( \sigma^2_y + h^2 - 2h \sigma \rho_{13} \right) \left( 1-e^{-a(T-t)} \right) \\
+ \left( \frac{\sigma^2_y}{2a^2} \right) \left( \sigma_y - \frac{\sigma_y}{a} \right) \left( \sigma_y \sigma \rho_{12} - h \sigma c \rho_{23} \right) \left( 1-e^{-a(T-t)} \right) \\
+ \left( \frac{j^2}{2a^2} \frac{(n-k)}{a} \right) \left( h \sigma \rho_{34} - \sigma_y \sigma \rho_{14} \right) \left( T_1 - T \right) \left( 1-e^{-a(T-t)} \right) \\
+ \left( \frac{j^2}{2a^2} \frac{(n-k)}{a} \right) \left( h \sigma \rho_{34} - \sigma_y \sigma \rho_{14} \right) \left( 1-e^{-a(T-t)} \right) \\
+ \left( \frac{j^2}{2a^2} \frac{(n-k)}{a} \right) \left( h \sigma \rho_{34} - \sigma_y \sigma \rho_{14} \right) \left( 1-e^{-a(T-t)} \right) \\
+ \left( \frac{j^2}{2a^2} \frac{(n-k)}{a} \right) \left( h \sigma \rho_{34} - \sigma_y \sigma \rho_{14} \right) \left( 1-e^{-a(T-t)} \right) \\
\right]
\]

Table 8 presents the theoretical first four moments, calculated as explained previously, and those obtained by Monte Carlo simulation. This ensures that our theoretical formulas give the exact moments.
Table 8: Comparison of theoretical and simulated moments

<table>
<thead>
<tr>
<th>Order</th>
<th>Theoretical Moments</th>
<th>Simulated Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>162.13</td>
<td>162.14</td>
</tr>
<tr>
<td>2</td>
<td>26414</td>
<td>26415</td>
</tr>
<tr>
<td>3</td>
<td>4.3243e+6</td>
<td>4.3243e+6</td>
</tr>
<tr>
<td>4</td>
<td>7.1141e+8</td>
<td>7.1131e+8</td>
</tr>
<tr>
<td>Mean</td>
<td>162.13</td>
<td>162.14</td>
</tr>
<tr>
<td>Variance</td>
<td>128.3</td>
<td>126.64</td>
</tr>
</tbody>
</table>

C Inverse gamma approximation

This appendix shows how we obtain the pricing formula using the inverse gamma approximation. The density function of a gamma random variable $X$ of parameters $(\alpha, \beta)$, $X \sim G(\alpha, \beta)$, is given by:

$$g(x \mid \alpha, \beta) = \frac{x^{\alpha-1} \exp \left( \frac{-x}{\beta} \right)}{\beta^\alpha \Gamma(\alpha)}, \quad x \geq 0$$

where $\alpha > 0$, $\beta > 0$ and $\Gamma(\alpha)$ is the Gamma function.

**Proposition 1** Let $X$ be a gamma random variable with parameters $(\alpha, \beta)$. Then, the random variable given by $Y = \frac{1}{X}$, follows an inverse gamma distribution $Y \sim G_R(\alpha, \beta)$ and its density function is given by:

$$g_R(y \mid \alpha, \beta) = \frac{\frac{1}{y} \mid \alpha, \beta}{y^2} = \frac{\exp \left( \frac{-1}{y\beta} \right)}{y^{\alpha+1} \beta^\alpha \Gamma(\alpha)}, \quad y > 0, \quad \alpha, \beta > 0.$$  

**Proposition 2** The non-centered moments of the random variable $Y \sim G_R(\alpha, \beta)$ are given by:

$$\mu'_n(g_R) = \frac{1}{\beta^n (\alpha - 1)(\alpha - 2) \ldots (\alpha - n)}, \quad n = 1, 2, 3, ...$$  

We price the basket option by approximating the sum of lognormal variables by an inverse gamma distribution. We match the two first moments such as:

$$\mu'_1(g_R) = \mu'_1(v) = \frac{1}{\beta (\alpha - 1)},$$
$$\mu'_2(g_R) = \mu'_2(v) = \frac{1}{\beta^2 (\alpha - 1)(\alpha - 2)}.$$
to get the two parameters of the inverse gamma density:

\[ \alpha = \frac{\mu_0^2(v) - 2\mu_2'(v)}{\mu_1^2(v) - \mu_2'(v)}, \]

\[ \beta = \frac{\mu_2'(v) - \mu_1^2(v)}{\mu_1'(v) \mu_2'(v)}. \]

Using the density given by Equation (28), the option price is given by:

\[ V_{\text{gamma}}^B = P(t, T) \left( \mu_1'(v) G \left( \frac{1}{K_B} | \alpha - 1, \beta \right) - K_B G \left( \frac{1}{K_B} | \alpha, \beta \right) \right). \]

**Proof.**

\[ V_{\text{gamma}}^B = P(t, T) \int_{-\infty}^{\infty} \max(x - K_B, 0) g_R(x | \alpha, \beta) dx \]

\[ = P(t, T) \int_{-\infty}^{\infty} (x - K_B) g_R(x | \alpha, \beta) dx \]

\[ = P(t, T) \int_{K_B}^{\infty} (x - K_B) g_R \left( \frac{1}{x} | \alpha, \beta \right) dx. \]

Let \( y = \frac{1}{x} \),

\[ V_{\text{gamma}}^B = P(t, T) \int_{\frac{1}{K_B}}^{\infty} \left( \frac{1}{y} - K_B \right) g(y | \alpha, \beta) \frac{y}{(y)^2} \left( \frac{1}{y^2} \right) dy \]

\[ = P(t, T) \int_{0}^{\frac{1}{K_B}} g(y | \alpha, \beta) dy - P(t, T) K_B \int_{0}^{\frac{1}{K_B}} g(y | \alpha, \beta) dy. \]

We have:

\[ g(y | \alpha, \beta) = \left( \frac{y}{\beta} \right)^{\alpha - 1} \exp \left( -\frac{y}{\beta} \right) = \left( \frac{y}{\beta} \right)^{\alpha - 2} \exp \left( -\frac{y}{\beta} \right) \]

\[ = \frac{\left( \frac{y}{\beta} \right)^{\alpha - 2} \exp \left( -\frac{y}{\beta} \right)}{\beta^2 (\alpha - 1) \Gamma(\alpha - 1)} \]

\[ = 1 \left( \frac{y}{\beta} \right)^{\alpha - 2} \exp \left( -\frac{y}{\beta} \right) \]

\[ = \frac{1}{\beta (\alpha - 1)} g(y | \alpha - 1, \beta). \]
The basket option price can thus be approximated by:

\[
V_{\text{gamma}} = P(t, T) \left( \frac{1}{\beta (\alpha - 1)} \int_0^{\frac{1}{\beta \mu_B}} g(y \mid \alpha - 1, \beta) \, dy - K_B \int_0^{\frac{1}{\beta \mu_B}} g(y \mid \alpha, \beta) \, dy \right)
\]

= \left. P(t, T) \left( \mu'_1(v) G \left( \frac{1}{K_B} \mid \alpha - 1, \beta \right) \right) - K_B G \left( \frac{1}{K_B} \mid \alpha, \beta \right) \right).

D  Edgeworth-lognormal expansion

This appendix shows how we obtain the pricing formula using an Edgeworth expansion around the lognormal distribution. Matching the first two moments, the lognormal density used is given by:

\[
a(x) = \frac{1}{\sqrt{2\pi \ln (1 + \frac{\mu_2(v)}{\mu'_1(v)^2})}} \exp \left\{ -\frac{1}{2} \left( \ln(x) - \ln(\mu'_1(v)^2) - \frac{\mu_2(v)}{\mu'_1(v)^2} \right)^2 \right\}
\]

where:

\[
\mu'_1(a) = \mu'_1(v),
\mu_2(a) = \mu_2(v).
\]

Following Jarrow and Rudd (1982), the unknown basket density function can be approximated by:

\[
v(x) = a(x) + \frac{\kappa_2(v) - \kappa_2(a)}{2!} \frac{d^2a}{dx^2}(x) - \frac{\kappa_3(v) - \kappa_3(a)}{3!} \frac{d^3a}{dx^3}(K) + \frac{\kappa_4(v) - \kappa_4(a)}{4!} \frac{d^4a}{dx^4}(x) + \xi(x),
\]

where \( \xi(x) \) is an error term and \( \kappa_i(h), i = \{1, 2, 3, 4\} \) are the first four cumulants of the density function \( h = \{v, a\} \) defined by the system of Equations (6). Given that the two first moments are the same for the true density and for the approximated density, Equation (31) becomes:

\[
v(x) = a(x) - \frac{\kappa_3(v) - \kappa_3(a)}{3!} \frac{d^3a}{dx^3}(K) + \frac{\kappa_4(v) - \kappa_4(a)}{4!} \frac{d^4a}{dx^4}(x) + \xi(x).
\]

The basket option price can thus be approximated by:

\[
V_{\text{log normal}}^B = \left. P(t, T) \int_{-\infty}^{+\infty} \max(x - K_B, 0) v(x) \, dx \right.
\]

\[
\simeq \left. P(t, T) \int_{-\infty}^{+\infty} \max(x - K_B, 0) \left( a(x) - \frac{\kappa_3(v) - \kappa_3(a)}{3!} \frac{d^3a}{dx^3}(x) + \frac{\kappa_4(v) - \kappa_4(a)}{4!} \frac{d^4a}{dx^4}(x) + \xi(x) \right) \, dx \right.
\]

\[
= \left[ P(t, T) \int_{K_B}^{+\infty} (x - K_B) a(x) \, dx - P(t, T) \left( \frac{\kappa_3(v) - \kappa_3(a)}{3!} \int_{K_B}^{+\infty} (x - K_B) \frac{d^3a}{dx^3}(x) \, dx \right) \right.
\]

\[
+ P(t, T) \left( \frac{\kappa_4(v) - \kappa_4(a)}{4!} \int_{K_B}^{+\infty} (x - K_B) \frac{d^4a}{dx^4}(x) \, dx \right)
\]

\[
= \left[ P(t, T) \int_{K_B}^{+\infty} (x - K_B) a(x) \, dx - P(t, T) \left( \frac{\kappa_3(v) - \kappa_3(a)}{3!} \right) \int_{K_B}^{+\infty} (x - K_B) \frac{d^3a}{dx^3}(x) \, dx \right.
\]

\[
+ P(t, T) \left( \frac{\kappa_4(v) - \kappa_4(a)}{4!} \right) \int_{K_B}^{+\infty} (x - K_B) \frac{d^4a}{dx^4}(x) \, dx \right].
\]
Using the fact that:

$$\int_{K_B}^{+\infty} (x - K_B) \frac{d^j a(x)}{dx^j} dx = \frac{d^{j-2} a}{dx^{j-2}} (K_B) \quad \text{for} \quad j \geq 2,$$

we obtain:

$$V_{\log\text{normal}}^B = P(t, T) \int_{K_B}^{+\infty} (x - K_B) a(x) dx - P(t, T) \frac{\kappa_3(v) - \kappa_3(a) da}{3!} (K_B)$$

$$+ P(t, T) \frac{\kappa_4(v) - \kappa_4(a) d^2 a}{4!} dx^2 (K_B). \quad (33)$$

Notice that the first integral in Equation (33) corresponds to B&S price.

Edgeworth expansion depends on the first four moments of $X$. Recall that $X \sim LN(\alpha, \beta^2)$, where $\alpha$ and $\beta$ are given by Equations (10d) and (10e). Given that we match the first two moments of the true and the lognormal distributions, we only need to compute the third and fourth moments of the lognormal distribution:

$$\mu'_3(a) = E_Q(v) (X^3) = \exp \left( 3 \alpha + \frac{9 \beta^2}{2} \right)$$

$$= \exp \left[ 3 \left( \ln \left( \mu'_1(v)^2 \right) - \frac{1}{2} \ln \left( \mu'_2(v) \right) \right) + \frac{9}{2} \ln \left( 1 + \frac{\mu_2(v)}{\mu'_1(v)^2} \right) \right]$$

$$= \exp \left[ \ln \left( \mu'_1(v)^6 \right) + \ln \left( \mu'_2(v)^{-\frac{3}{2}} \right) + \ln \left( 1 + \frac{\mu_2(v)}{\mu'_1(v)^2} \right) \right]$$

$$= \exp \left[ \ln \left( \mu'_1(v)^6 \mu'_2(v)^{-\frac{3}{2}} \left( 1 + \frac{\mu_2(v)}{\mu'_1(v)^2} \right) \right) \right]$$

$$= \mu'_1(v)^6 \mu'_2(v)^{-\frac{3}{2}} \left( 1 + \frac{\mu_2(v)}{\mu'_1(v)^2} \right) = \mu'_1(v)^6 \mu'_2(v)^{-\frac{3}{2}} \left( \frac{\mu_2(v)}{\mu'_1(v)^2} \right)^{\frac{9}{2}}$$

$$= \left( \frac{\mu'_2(v)}{\mu'_1(v)} \right)^3.$$
\[ \mu'_4(a) = E_{Q_f}(X^4) = \exp \left( 4\alpha + \frac{16\beta^2}{2} \right) \]

\[ = \exp \left[ 4 \left( \ln (\mu'_1(v)^2) - \frac{1}{2} \ln (\mu'_2(v)) \right) + 8 \ln \left( 1 + \frac{\mu'_2(v)}{\mu'_1(v)^2} \right) \right] \]

\[ = \exp \left[ \ln (\mu'_1(v)^8) + \ln (\mu'_2(v)^{-2}) + \ln \left( 1 + \frac{\mu'_2(v)}{\mu'_1(v)^2} \right)^8 \right] \]

\[ = \mu'_1(v)^8 \mu'_2(v)^{-2} \left( \frac{\mu'_2(v)}{\mu'_1(v)^2} \right)^8 \]

\[ = \frac{\mu'_2(v)^6}{\mu'_1(v)^8}. \]
References


