New warrant issues valuation with leverage and noisy equity values

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Abstract

The empirical analysis of new warrant issues in the context of a structural model of the firm typically assumes the absence of debt and a perfect equity pricing model. We examine here an approach relaxing these two assumptions. The proposed approach develops simple analytical expressions for the prices of warrant, debt and equity in the presence of leverage. An empirical strategy is proposed to implement the model with equity prices containing model errors. An illustration with a recent warrant issue deal between Bank of America and Berkshire Athaway is provided.

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1 Introduction

Warrants are commonly issued and traded derivatives securities. For example, in 2009-2010, the US Department of the Treasury received nearly $4.4 billion by disposing of warrants it purchased through the Capital Purchase Program as part of Troubled Assets Relief Program following the 2008-09 subprime crisis. More recently, in August 2011, Bank of America issued warrants giving Berkshire Athaway the right to buy 700 millions of common shares at a strike price of $7.14 for the next 10 years. Warrants have characteristics close to those of standard options. However, when exercised, new common shares must be issued and given to the call warrant holder in exchange of the agreed price. Such a characteristic potentially creates a dilution effect that might impact on common share prices and debt values.

The theoretical approaches used to model warrant prices can be classified in two broad categories: the structural and the reduced form approaches. The structural approach assumes full information about the asset value process and the liability structure of the firm. In a reduced form model, the available information is about the stock price process and the outstanding warrant. As argued in Jarrow and Trautmann (2011), for outstanding warrant issues, a reduced form approach may be more relevant. In this case, under the lognormality of the future stock price, the Black-Scholes formula applies for European style warrants without any modification or dilution adjustments since the information embedded in the stock price already contains the anticipated dilution. However, for new warrant issues that have yet to be announced, the reduced form model cannot provide answers about how debt and stock prices will be affected and about the value at which the new issue should be priced. In such a situation, despite many shortcomings, a structural approach can provide a useful tool to perform what-if type analysis about the impact of warrants introduction on the debt and equity values. It can also provide a price estimate, given some strong assumptions about the asset value process, default structure and capital structure of the firm. With this purpose in mind, this paper proposes a practical approach to implement a structural model relaxing two standard assumptions used in the empirical literature on warrants.

A first assumption is the absence of debt. See for example Galai and Schneller (1978) where closed form solutions for warrants and equity prices are derived for firms without leverage. Such a closed form solution is convenient for empirical implementation and tests as in Schulz and Trautmann (1994). In

Crouhy and Galai (1994), and Bensoussan, Crouhy and Galai (1995), leverage is introduced in the model. However, their model is not solved in closed form and numerical or analytical approximations are used in their analysis. Here, for the purpose of relaxing the all equity assumption typically found in empirical implementations of structural warrant models, we examine a restricted version of the Crouhy and Galai (1994) model. The key restriction allowing these closed form solutions is the assumption of an identical maturity for the new European style warrant issue and the debt. This assumption can be justified on the ground that firms have much more complex debt structures than the assumed zero-coupon bond of the model. In this context, someone wishing to implement the model must make an ad-hoc choice about the maturity of the debt. Letting this maturity to be the same as the maturity of the warrant at the issue date is thus a convenient choice with the important advantage of analytical solutions for the financial asset prices. With the identical maturities assumption, the stock, warrant and debt values can be written as simple combinations of Black-Scholes call option formulas. With these formulas, and the empirical approach described below, it is then possible to assess the impact of warrants introduction on the financial assets of the firm.

A second assumption typically used in the empirical implementation of structural model warrants is the absence of pricing errors for the equity. For example, in Schulz and Trautmann (1994) and Ukhov (2004), a non-linear system of equation using equity prices and volatility is solved to obtain estimates of the unobserved asset value and its volatility under the assumption that equity is perfectly priced by the structural model. In Duan (1994), a maximum likelihood approach is proposed to obtain coherent statistical estimates for the unobserved asset values and its associated parameters in the context of a structural model of the firm. This method corrects some of the shortcomings associated with the non-linear equation system approach. When applied to warrant pricing, as with the non-linear equation system, the maximum likelihood approach needs to assume the pricing of equity without errors. In an empirical implementation of Merton's (1974) structural debt model, Duan and Fulop (2009) introduce the possibility of trading noises in equity values. This explicitly admits the overly simplified nature of the structural debt model, which is now appropriately considered as an approximation. Here, for relaxing the perfect equity pricing assumption, we adopt this hypothesis from Duan and Fulop (2009). Our approach differs however from theirs as we use a simpler method, which avoids the simulation based filtering approach proposed in their study. Our results show that the approach proposed here, despite its theoretical non-optimality, produces reliable results. An application on a recent warrant issue of Bank of America shows that adding pricing errors to the model obtains estimation results that are more robust to a noisy stock pricing environment such as the one provided by the recent subprime crisis of 2008-09.

Clearly, applying the model examined above and computing warrant prices and their impact on the existing equity and debt values requires some strong assumptions in addition of the equivalent maturity hypothesis. For example, it is also assumed that the American style warrant issue examined here will be exercised in block at maturity. There is a large literature examining the optimal exercise of warrants. The exercise before maturity depends on factors such as the nature of warrant holders (price takers or large traders), dividends and magnitude of interest rates. See Jarrow and Trautmann (2011) for a short review. The approach proposed here should thus be considered with this important caveat in mind.

The rest of the paper is organized as follows. Section 2 describes the structural model proposed here, and derives analytical expressions for the stock, warrants and debt values. Section 3 describes the estimation approach. Section 4 provides an illustration of the proposed approach with a recent warrant issue deal between Berkshire Athaway and Bank of America. Section 5 concludes.

2 The model

2.1 Assumptions

As in Crouhy and Galai (1994), we consider a firm that has a zero-coupon debt B_t with a face value of $\overline{B}$ and maturing at time T. The total equity of the firm, E_t, is split into m shares of common stock denoted by S_t. At $t = 0$, the firm considers introducing n European warrants in its capital structure, each with value W_0 and exercise price K. This creates a cash inflow of nW_0 dollars that are invested in projects of similar risks to those already undertaken. Denoting the asset value just prior to the warrants issue by A_{0-}, this cash inflow increases the assets to $A_0 = A_{0-} + nW_0$.

The asset value process follows a geometric Brownian motion given by:

$$\frac{dA_t}{A_t} = (\mu - \delta)\, dt + \sigma dZ_t$$

where μ is the expected rate of return, δ is the rate of cash payout made to claim holders on the firm, and σ is the volatility. As with all models from the structural approach, we assume full information about the asset value process and the liability structure of the firm.

With such assumptions, the stock, debt and warrant pricing formulas are special cases of the generic formulas outlined as in Crouhy and Galai (1994) and Jarrow and Trautmann (2011). However, these papers do not provide the closed form expressions for the financial asset values. Hence, for the purpose of generating simple closed form solutions, we adopt the assumption of identical maturities for the debt and warrants. As argued in the introduction, this assumption is justified by the oversimplified nature of the model whose implementation requires an ad-hoc choice about the maturity of the assumed zero-coupon bond for the debt. Letting this maturity to be the same as the maturity of the warrant at issue date is a choice allowing analytical solutions for the financial asset prices. The next subsection describes these solutions.

2.2 Valuation formulas for equity, warrants and debt

The value of one share of stock at maturity depends if the warrants are exercised or not. If the warrants are not exercised, the equity holders have a payoff as described in Merton (1974). The payoff is the difference between the asset value and the face value of debt, if the asset value is larger than the face value. Otherwise, a value of zero is given to these claimants. If the warrants are exercised, there is an additional cash inflow of $n \times K$ dollars that is added to the asset value. However, this cash flow is exchanged for equity on the form of n new shares of stocks. For exercise to take place, these new shares must have higher value. This creates a loss or dilution in value for the equity holders. The cash flow function of one share of common stock at maturity can thus be written as:

$$S_T = \begin{cases} \frac{\max\left(A_T - \overline{B}, 0\right)}{m} & \text{if } A_T < \overline{A} \\ \\ \frac{\max\left(A_T + n \times K - \overline{B}, 0\right)}{n+m} & \text{if } A_T \geq \overline{A} \end{cases} \tag{1}$$

where $\overline{A}$ is the level of the firm's assets for which a warrant holder is indifferent between exercising or not. When the warrants and the debt have the same maturity, this level is the stock price equating the strike price i.e.

$$\frac{\overline{A} + n \times K - \overline{B}}{n + m} = K \quad \longrightarrow \quad \overline{A} = \overline{B} + m \times K.$$

The payoff of a warrant at maturity is null if they are not exercised. If exercised, their payoff is that of a standard call. Substituting the share price payoff described in equation (1) obtains:

$$W_T = \begin{cases} 0 & \text{if } A_T < \overline{A} \\ \frac{\max\left(A_T + n \times K - \overline{B}, 0\right)}{m+n} - K & \text{if } A_T \geq \overline{A} \end{cases} . \tag{2}$$

As shown in Appendix B, computing the discounted expected cash flow with respect to the risk neutral measure yields the following expression for the share price:

$$S_0 = \frac{1}{m}\left[BSCall\left(A_0, \overline{B}, r, T, \sigma, \delta\right) - \frac{n}{m+n} BSCall\left(A_0, \overline{A}, r, T, \sigma, \delta\right)\right] \tag{3}$$

where $BSCall(\cdot)$ is the Black-Scholes call option pricing formula described in Appendix A. The expression between brackets is the total value of equity. This expression is composed of two terms. The first term is a call on the asset value, striked at the face value of debt. This is the standard equity formula for the Merton (1974) model. It represents the expected cash flow going both to equity and warrant holders. This first term is reduced by a second one representing the discounted expected cash flow going to the warrant holders. This discounted expected cash flow is also formed of a Black-Scholes call function, but striked at $\overline{A}$, the asset level at which a warrants would start to be exercised. This call is multiplied by the proportion of shares represented by the warrants if these were exercised. When there are no warrants in the capital structure, $n = 0$ and only the first term remains.

As shown in Appendix C, computing the discounted expected cash flow with respect to the risk neutral measure yields the following expression for an individual warrant value:

$$W_0 = \frac{1}{(m+n)} \times BSCall\left(A_0, \overline{A}, r, T, \sigma, \delta\right). \tag{4}$$

This expression is a call option on the asset price, striked at $\overline{A}$, the asset level at which a warrant would start to be exercised. Multiplying the expression for W_0 by n gives the total value of the warrants, which is identical to the second part in the bracket of equation (3). It is easy to see that a special case of the above equation is the all equity case examined in Galai and Schneller (1978).

Finally, the bond price can be deduced from the accounting identity which states that the asset value is the sum of the financial asset values i.e.

$$B_0 = A_0 - mS_0 - nW_0$$

which obtains, when substituting equations (3) and (4):

$$B_0 = A_0 - BSCall\left(A_0, \overline{B}, r, T, \sigma, \delta\right) \tag{5}$$

which is the standard expression for the debt value in Merton (1974) model. This shows that the discounted expected cash flow for the debt is independent of the warrants being exercised or not.

As noticed earlier, introducing warrants in the capital structure produces a cash inflow of $n \times W_0$ dollars and $A_0 = A_{0-} + nW_0$ where A_{0-} is the asset value just prior to the warrant issue. Substituting this relation in the warrant price equation (4) obtains a formula for W_0 that depends on W_0. Given values for A_{0-}, K, n, m, T, r and σ, the warrant value can be solved for iteratively. The pre-warrant debt and equity valuation should be performed with A_{0-}. The post-warrant debt and equity valuation should be made with A_0, once the value for W_0 is solved for.

2.3 Impact of warrants introduction on debt and equity values

With the above formulas, the impact of introducing warrants in the capital structure can be evaluated numerically. It is also possible to obtain convenient analytical approximations giving some intuition about the nature of these changes.

As show in Appendix D, for a firm without warrants in the capital structure before the issue, the change in equity can be approximated in closed form with:

$$E_0 - E_{0-} \simeq -nW_0 \times \left[1 - \Delta BSCall\left(A_{0-}, \overline{B}, r, T, \sigma, \delta\right)\right],$$

where E_{0-} denotes the theoretical equity before the warrants introduction in the capital structure and $\Delta BSCall\left(A_{0-}, \overline{B}, r, T, \sigma, \delta\right)$ is the Black-Scholes delta function described in Appendix A. This approximation shows that, within the context of this model and its assumptions, introducing warrants reduces the equity value. This reduction is the result of two effects. A first effect is the increase in asset value resulting from the warrant introduction, which in turns increases the expected cash flow going both to equity and warrant holders by $nW_0 \times \Delta BSCall(\cdot)$. However, this cash flow increase is reduced by the portion going to the warrant holders i.e. $-nW_0$, which always dominates the previous increase because $0 < \Delta BSCall(\cdot) < 1$.

For the bond holders, the introduction of the warrants is beneficial because the increase in asset value decreases the probability of default and increases the expected amount recovered in case of default. For this model, the impact is identical in magnitude to the one on the equity, but with an opposite sign. Indeed, from the accounting identity, the changes resulting from the warrant introduction can be described as:

$$(A_{0-} + nW_0 - A_{0-}) = (B_0 - B_{0-}) + (E_0 - E_{0-}) + (nW_0 - 0)$$

which implies

$$B_0 - B_{0-} = -(E_0 - E_{0-}).$$

3 Empirical implementation

The practical use of the above model is confronted to two important problems. A first problem is the assumption that the asset value and its volatility are observed. Such values are clearly unavailable and in many cases, only the equity prices are observed at regular intervals. A second problem is the simplistic nature of the capital structure. Firms have typically much more complex structures than the assumed zero-coupon debt.

These problems can be addressed by relying on observed equity prices to infer the unobserved asset and volatility values. In the structural model literature, the non-linear equation system approach was the first method to use equity prices to infer the unobservable values. See for example Ronn and Verma (1984) for an application in Banking and Schulz and Trautmann (1994) for an application to warrants. In Duan (1994), maximum likelihood estimation is proposed as an alternative. Both approaches assume that the models are pricing observed equity without errors. In Duan and Fulop (2009) this assumption is relaxed with the addition of trading noises in equity values. Relaxing this assumption explicitly admits the overly simplified nature of structural models. However, introducing errors in the equity prices considerably complicates the estimation and requires a filtering based method. To tackle this problem, Duan and Fulop (2009) proposed a simulation based method which yields an optimal solution to the filtering and estimation problem. We examine here an alternative to this simulation based approach. We implement the structural model above with an extended Kalman filter. The main advantage of this approach is its simplicity. However, unlike the Duan and Fulop method, this approach does not yield an optimal solution to the filtering and estimation problem. The extended Kalman filter must rely on a first order Taylor series expansion to make the non-linear model amenable to the Kalman filter recursion. With this linearization, the likelihood function computed with the filter is an approximate one. However, this can yield a valid approximation if the non-linear function is sufficiently smooth and well behaved. A Monte Carlo simulation experiment, presented at the end of this section, shows that this is indeed the case. The next subsection shows how to cast the model in the extended Kalman filter framework. The details of the Monte Carlo study follows.

3.1 Estimation approach

The equity values are observed at regular time points, and we denote the time series of $M+1$ observations by $\{E_0, E_h, E_{2h}, \cdots, E_{Mh}\}$ where h is the length of time, measured in years, between two observations. In this section, observation M is the equity value at the date of the warrant introduction.

The Kalman filter is a recursive tool allowing to estimate the time series of unobserved asset values $\{A_0, A_h, A_{2h}, \cdots, A_{Mh}\}$ from the time series of observed equities, given the theoretical pricing relation and known values for the parameters of the model. In addition, the Kalman filter gives, as a by-product, the log likelihood function of the model. Maximizing this log likelihood function with respect to the unknown parameters obtains consistent estimates with which the model can be implemented. To use a Kalman filtering based approach, one must first formulate the model in a state-space representation. The state-space representation consists of two equations: a measurement and a transition equation. The measurement equation relates the unobserved dynamic factor (the asset value) with an observed variable (the equity value). The transition equation specifies the dynamics of the unobserved factor.

In our context, the measurement equation is provided by the theoretical value of the equity i.e.

$$E_{ih} = f\left(\exp\left(\alpha_{ih}\right)\right) + v_{ih}$$

where $\alpha_{ih} = \ln A_{ih}$ and v_{ih} are i.i.d. $Normal(0, \sigma_v^2)$ noises and $f(\cdot)$ is equation (3) multiplied by m. A first point to notice is that $f(\cdot)$ is written in terms of the logarithm of the asset price. This is required since the Kalman filter works with normally distributed unobserved factors. Because the geometric Brownian motion process for the asset prices yields lognormally distributed prices, rewriting the pricing model in terms of logarithms of asset prices obtains a normally distributed unobserved factor. Second, for a time series sample of equity values of a firm without warrants, $f(\cdot)$ is a restricted version of equation (3) without the last term i.e. it is a Black-Scholes call pricing function. For the rest of this section, we will use this restricted form for $f(\cdot)$. A third observation is that v_{ih} is added to the theoretical relation to account for model imperfections and/or trading noises. Without this noise, maximum likelihood estimation is possible without a Kalman filtering framework (see Duan (1994) and (2000)). However, with a measurement error, a filtering based approach must be used. Finally, a fourth point to notice is the nonlinear relation between the unobserved factor (the log of the asset price) and the stock price. As noticed earlier, this equation must be linearized with a Taylor series in order to use the Kalman filter. Using the linearization described in Appendix E, the measurement equation of the extended Kalman filter is written as

$$E_{ih} = \nabla \times \alpha_{ih} + g_{ih} + v_{ih} \tag{6}$$

where expressions for ∇ and g_{ih} are given in appendix.

The second equation of the state-space framework is the transition equation which describes the dynamics of the log of the asset price. For a geometric Brownian motion process, the logarithm of the asset price can be written as a discrete-time random walk with drift and normal errors:

$$\alpha_{ih} = \alpha_{(i-1)h} + \left(\mu - \delta - \frac{1}{2}\sigma^2\right) h + \epsilon_{ih} \tag{7}$$

where $\epsilon_{ih} \sim Normal\left(0, \sigma^2 h\right)$. With this state-space representation, given some numerical values for $\mu, \sigma,$ and σ_v, a set of recursive equations defining the extended Kalman filter is given in Appendix E. As mentioned above, this recursion allows to compute estimates of the unobserved asset values with the time series of equity values. A by-product of the extended Kalman filter recursion is the approximate likelihood function given by:

$$\log L\left(\mu, \sigma, \sigma_v\right) = -\frac{M}{2} \log 2\pi - \frac{1}{2} \sum_{i=1}^{M} \log F_{ih} - \frac{1}{2} \sum_{i=1}^{M} \frac{e_{ih|(i-1)h}^2}{F_{ih}}$$

where F_{ih} and e_{ih} are obtained from the extended Kalman filter recursion. Maximizing this function with respect to μ, σ, σ_v provides an approximate maximum likelihood estimator for these parameters.

As indicated in appendix, genuine maximum likelihood parameter estimates are (asymptotically) normally distributed. Confidence intervals about these parameters can thus be constructed for statistical inference. Continuously differentiable functions of these parameters are also normally distributed. It is thus possible to compute confidence intervals for these functions. For example, the warrant price formula is a function of the parameter estimates σ and σ_v, and maximum likelihood estimators could yield confidence intervals for such values. However, as mentioned above, the estimation approach only obtains an approximate maximum likelihood function. Although we expect the above estimator to have properties close to those of a genuine maximum likelihood estimator, it is important to verify that this is indeed the case. The next subsection examines this issue with a Monte Carlo experiment.

3.2 A Monte Carlo study

The data for this experiment is simulated as follows. Let $\widetilde{A}_t$ denote the simulated values of the asset at time t. Using the strong solution of the Geometric Brownian motion, we simulate a time series of asset prices with

$$\widetilde{A}_t = \widetilde{A}_{t-h} \exp\left(\left(\mu - \delta - \frac{1}{2}\sigma^2\right) h + \sigma\sqrt{h}\widetilde{\varepsilon}_t\right)$$

where $\widetilde{\varepsilon}_t$ is a $Normal\,(0,1)$ pseudo random number. The equity value is then simulated with

$$\widetilde{E}_t = BSCall\left(\widetilde{A}_t, \overline{B}, r, T-t, \sigma, \delta\right) + \widetilde{v}_t$$

where $BSCall\,(\cdot)$ is the standard Black-Scholes call option pricing formula and $\widetilde{v}_t$ is a $Normal\left(0, \sigma_v^2\right)$ pseudo random number. To be consistent with the use of daily data we set $h = 1/250$. The parameter values used in the above equations are: $\widetilde{A}_0 = 500$, $\mu = 0.1$, $\sigma = 0.3$, $\delta = 0$, $\sigma_v = 2.0$, $\overline{B} = 1,800$, $r = 0.05$, $T = 10$. These parameter values are similar to those obtained in the empirical section looking at the Bank of America case. Starting at $\widetilde{A}_0$, a time series of 250 asset values is simulated to end up with a time series $\{\widetilde{E}_0, \widetilde{E}_h, \widetilde{E}_{2h}, \cdots, \widetilde{E}_{250h}\}$ of equity. The maturity of debt for these equity values is $T, T-h, T-2h, ..., T-250h$.

Table 1 reports the result of the Monte Carlo study where 5,000 simulated time series are used to obtain 5,000 maximum likelihood estimates. The approximate maximum likelihood estimates and approximate standard errors are obtained with the extended Kalman filter approach described in the above subsection. Rows one, two, three and four of this table report the true parameter values, the mean, the medians and the standard deviations of the parameter estimates for μ, σ and σ_v. As the results indicate, the averages of the estimated parameters are close to their true values.

The remaining four rows of Table 1 report the 25%, 50%, 75% and 95% coverage rates. The coverage rate is the percentage of the 5,000 parameter estimates for which the theoretical value is contained in the $\beta\%$ confidence interval implied by the approximate asymptotic normal distribution. The computed values are close to the theoretical values. This indicates that, for this sample size of 250, the normal distribution is a reasonable approximation to the sampling distribution of the estimator.

4 Application

On August 25, 2011, an agreement between Berkshire Athaway and Bank of America was announced. Bank of America sold 50,000 perpetual preferred shares with a face value of \$100,000 per share and a 6% dividend. In addition, Berkshire Athaway also received warrants on 700 million ordinary shares of Bank of America with an exercise price of \$7.142 per share. These warrants can be exercised early and have a maturity of 10 years. The global price received by Bank of America for the preferred shares and warrants was 5 billion dollars.

We analyze here this warrant issue using the model developed above. We perform the analysis approximately one week before this announcement i.e. on August 19, 2011. The model examined above assumes European style warrants. The Bank of America issue is of American style. As mentioned in the introduction, we therefore use the assumption that these warrants will be exercised in block at their maturity.

To implement the model, we first build a daily time series of equity values from January 1, 2008 to August 19, 2010. This obtains $M = 917$ data points. For a given day, the equity value is the closing stock price multiplied by the number of ordinary shares, obtained from the most recent available balance sheet at that date. We also build a daily time series of face values of debt. As argued in Crouhy et al. (2000), the probability of the asset value falling below the total face value of liabilities may not be an accurate measure of the actual default probability. Default tends to occur when the asset value reaches a level somewhere between the face value of total liabilities and the face value of the short-term debt.

Therefore, at a given day, we compute the face value of debt as the short term liabilities (short term debt + domestic deposits + foreign deposits) plus one half of the long term liabilities (long term debt + preferred shares) from the most recent available balance sheet at that date. Because balance sheet data is available on a quarterly basis, the face value of debt is constant within a quarter. Figure 1 plots both time series and the prices of a common share. This period contains the volatile events associated with the subprime crisis in 2008-09. The common share prices dropped sharply in the end of 2008 and remained about half of the January 2008 level after. The equity also dropped sharply in the end of 2008 but increased in the end of 2009 to the level observed in January 2008. This increase was caused by an increase in the number of common shares. This number roughly doubled over the sample period. The estimated face values of debt increased sharply in 2009, largely due to an increase in the long term debt.

Table 2 gives some balance sheet data numbers at the warrant issue date. It also gives the input values used to implement the model and estimation techniques. The maturity for January 1, 2008 is set to 13.635 years and decreases gradually (by steps of $h = 1/252$) to ten years at the evaluation date i.e. august 19, 2010 (one week before the announcement date).

In order to compare the approach proposed in this paper with other approaches from the literature, two alternative estimation methods are examined and compared with the proposed approximate maximum likelihood method (hereafter AML). A first approach is the non-linear system of equations (NLSE hereafter), which assumes that equity are priced without errors by the model. This approach uses the equity pricing formula of Merton (1974) as a first equation, using the inputs described as above. This first equation, given the observed equity value, has two unobserved inputs: the asset value and is volatility. A second equation is the theoretical value for the equity volatility, obtained from Ito's lemma. This equation, given an historical estimate of the standard deviation for the daily log equity value, has the same two unobserved input values as the previous equation. Solving this two-equation system for the two-unknown parameters obtains the estimates for the asset value and its volatility. Appendix F gives the details of this method. A second estimation method used for comparison is the maximum likelihood (ML hereafter) approach described in Duan (1994) and (2000). This approach also assumes a perfect equity pricing formula. Given numerical values for the volatility of the asset returns and the observed equity, this method inverts the equity pricing formula of Merton (1974), to obtain an implied asset value. Using the distributional assumption of the asset returns, the likelihood function of the equity value can be computed and maximized with respect to the drift and volatility parameters. Appendix F gives the details of this method.

Table 3 reports the results of the estimation. Columns 2 to 4 present the results for the whole sample i.e. January 1, 2008 to August 19, 2011 for the NLSE, ML and AML. Columns 5 to 7 are for the subsample from January

1, 2009 to August 19, 2011, while columns 8 to 10 are for January 1, 2010 to August 19, 2011. The drift parameter estimates are only available for the maximum likelihood based methods. The estimates for this parameter vary greatly with the methods and sample periods. As shown by their estimated standard deviations, these are not estimated with great accuracy. Such a result is expected as it well known that expected returns are difficult to estimate with precision from a time series of prices. The estimates of the volatility of the asset returns are also very different for each method. The methods with the perfect equity pricing assumption produces more variations when the sample is changed. For example, the NLSE estimates for the different subperiods range from 0.58 to 0.05 while those of the ML are from 0.70 to 0.13. The subsamples containing the volatile period of the subprime crisis obtain high estimates of the volatility of the asset return. This is not surprising given that these models transfer the whole equity volatility into the estimated parameters. The AML approach is more stable through the sample periods and obtains volatility ranging from 0.37 to 0.14. The volatility of the equity pricing error σ_v, only available for the AML method, is significant in all the subsamples, and higher for the samples containing the subprime crisis. Part of the volatility of the equity is absorbed by the model errors. Finally, for the likelihood based approach, we have available the log likelihood function. The AML nests the ML and obtains a significantly higher likelihood with one additional parameter.

Table 4 reports the estimation results for the elements of the capital structure. The estimated asset values before the introduction of the warrants in the capital structure ($\widehat{A}_{M-}$ in the table) show opposite patterns to those of the estimated volatility. It is lower during volatile periods and higher during the quieter period. Again, the AML approach yields more stable estimates with a smaller range for the estimated values. For the theoretical value of the equity just before the warrant introduction ($\widehat{E}t_{M-}$ in the table), both the NLSE and ML approach obtain values that are exactly equal to the observed equity value at that date. This is not the case for the AML approach. However, the theoretical value is close to the observed one in all cases. Finally, the theoretical debt values before warrant introduction are reported as $\widehat{B}_{M-}$. Again, these estimates show greater variability for the NLSE and ML approaches.

The estimated warrant value is also reported for each approach and subsamples. This value is computed iteratively with equation (4) using an estimated asset value of $\widehat{A}_M = \widehat{A}_{M-} + n \times \widehat{W}$ and a face value of debt of 1836.6 billion. This face value of debt now contains the preferred shares that were introduced simultaneously with the warrants. As explained earlier, when expressed this way, the warrant price appears on the left and the right of the equation. A trial and error approach is required to find the value $\widehat{W}$ satisfying the equality. The estimated individual warrant value is higher for the NLSE and ML approaches for the volatile subsamples, reflecting the high value of the volatility estimate. Again the AML approach yields more stable estimates. The differences between the estimated prices can be substantial. For example, the 2010-2011 subsample produces a NLSE warrant price of 4.82 while the maximum likelihood approaches are around 5.75 and 5.80. In the 2009-2011 subsample, the AML approach produces smaller prices by about \$0.25 with the other prices, a statistically significant difference.

The estimates for the assets, equity and debt with the warrants in the capital structure are also reported with their standard errors for the maximum likelihood based approaches. With these, the impact on the equity and debt values can be computed as the difference between the after and before warrant values. We see that the AML provides an estimated impact on equity for the 2008-11 and 2009-11 subsamples which is higher than the other two methods, despite a much lower estimate of the volatility parameter. This can be explained by the higher values of the asset estimated with the AML. Indeed, the other approaches, with the absence of model error, attribute the drop in equity to a drop in asset value. This is not the case for the AML, which has the possibility to attribute, in part, drops in equity to pricing errors.

5 Conclusion

A practical approach is proposed here for evaluating new warrant issues in the presence of debt in the context of a structural model of the firm. For the purpose of developing analytical expressions, the approach uses strong assumptions such as the identical maturities of the debt and warrants. It is also assumed that the warrants holders are price takers that will exercise these securities in block at their maturity. With these, simple analytical expressions are developed for the debt, equity and warrants which become combinations of Black-Scholes call pricing formula.

The empirical approach specifically takes into account the oversimplified nature of the proposed model by allowing equity to be priced imperfectly. As shown by a case study of a recent warrant issue by Bank of America, the proposed estimator, when compared to alternative approaches proposed in the literature, is shown to provide more stable results in turbulent market conditions such those associated to the 2008-09 subprime crisis.

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A Black-Scholes call formula

The Black-Scholes call formula is:

$$BSCall\,(A,B,r,T,\sigma,\delta) = Ae^{-\delta T}N\,(b_1) - e^{-rT}BN\,(b_2)$$

with

$$b_1 = \frac{\ln\,(A/B) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad b_2 = b_1 - \sigma\sqrt{T},$$

where $N\,(\cdot)$ is the standard normal distribution function, A is the underlying asset price, B is the strike price, r is the constant risk-free rate, T is the maturity, σ is the asset return volatility, and δ is the dividend yield. The delta of the Black-Scholes call is given by:

$$\Delta BSCall\,(A,B,r,T,\sigma,\delta) = e^{-\delta T}N\,(b_1)\,.$$

B Equity value in the presence of debt and warrants

We want to compute $S_0 = e^{-rT}E^*\,[S_T]\,,$ where the star denotes an expectation taken with respect to the risk neutral measure for the asset value. Under this measure, the strong solution for the asset value is given by

$$\ln A_T - \ln A_0 = \left(r - \delta - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon$$

where $\varepsilon \sim Normal\,(0,1)\,,$ which implies a lognormal distribution for the asset level at T. Using the payoff function given by equation (1) we obtain

$$\begin{aligned} S_0 = {} & \frac{e^{-rT}}{m}\int_0^{\overline{A}} \max\left(A_T - \overline{B}, 0\right) \times \phi\,(A_T)\,dA_T + \\ & \frac{e^{-rT}}{n+m}\int_{\overline{A}}^{\infty} \max\left(A_T - \left(\overline{B} - nK\right), 0\right) \times \phi\,(A_T)\,dA_T \end{aligned}$$

where $\phi\,(\cdot)$ is the lognormal risk-neutral density of the asset value at time T. We develop below the formulas for the each parts of the above expression.

B.1 First part

Because $\overline{B}$ is always smaller than $\overline{A} = \overline{B} + m \times K$, the $\max\,(\cdot)$ operator can be removed by changing the integration limits and the first part of the above expression can be written as

$$\frac{e^{-rT}}{m}\int_{\overline{B}}^{\overline{A}} A_T\phi\,(A_T)\,dA_T - e^{-rT}\frac{\overline{B}}{m}\int_{\overline{B}}^{\overline{A}} \phi\,(A_T)\,dA_T$$

which can be expanded into

$$\frac{e^{-rT}}{m}\left[\int_{\overline{B}}^{\infty} A_T\phi\left(A_T\right)dA_T - \int_{\overline{A}}^{\infty} A_T\phi\left(A_T\right)dA_T\right] -$$
$$e^{-rT}\frac{\overline{B}}{m}\left[\int_{\overline{B}}^{\infty} \phi\left(A_T\right)dA_T - \int_{\overline{A}}^{\infty} \phi\left(A_T\right)dA_T\right].$$

Substituting the lognormal density expression and integrating obtains:

$$\frac{1}{m}A_0e^{-\delta T}\left(N\left(b_{1,\overline{B}}\right) - N\left(b_{1,\overline{A}}\right)\right) - e^{-rT}\frac{\overline{B}}{m}\left(N\left(b_{2,\overline{B}}\right) - N\left(b_{2,\overline{A}}\right)\right)$$

where

$$b_{1,x} = \frac{\ln\left(A_0/x\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad b_{2,x} = b_{1,x} - \sigma\sqrt{T},$$

with $N(\cdot)$ denoting the standard normal distribution function.

B.2 Second part

For the second part, because $\overline{B}$ is always smaller than $\overline{A}$, $\overline{B} - nK$ is also smaller than $\overline{A}$. Hence the max $(\cdot)$ operator can be removed from the second part of the stock price formula to get

$$\frac{e^{-rT}}{n+m}\int_{\overline{A}}^{\infty}\left(A_T - \widetilde{B}\right)\times\phi\left(A_T\right)dA_T$$

where $\widetilde{B} = \overline{B} - nK$. Substituting the lognormal density expression and integrating obtains:

$$\frac{1}{n+m}\left(A_0e^{-\delta T}N\left(b_{1,\overline{A}}\right) - \widetilde{B}e^{-rT}N\left(b_{2,\overline{A}}\right)\right)$$

B.3 The stock price

Adding up the expressions obtained for the first and second parts obtains

$$S_0 = \frac{A_0e^{-\delta T}}{m}N\left(b_{1,\overline{B}}\right) - \frac{e^{-rT}}{m}\overline{B}N\left(b_{2,\overline{B}}\right) +$$
$$\left[\frac{1}{n+m} - \frac{1}{m}\right]A_0e^{-\delta T}N\left(b_{1,\overline{A}}\right) + \left[\frac{1}{m}\overline{B} - \frac{1}{n+m}\widetilde{B}\right]e^{-rT}N\left(b_{2,\overline{A}}\right)$$

which can be rearranged and simplified to

$$S_0 = \frac{1}{m}\left\{\begin{array}{c}\left[A_0e^{-\delta T}N\left(b_{1,\overline{B}}\right) - e^{-rT}\overline{B}N\left(b_{2,\overline{B}}\right)\right] - \\ \frac{n}{(m+n)}\left[A_0e^{-\delta T}N\left(b_{1,\overline{A}}\right) - \overline{A}e^{-rT}N\left(b_{2,\overline{A}}\right)\right]\end{array}\right\}.$$

The first and second term between curly brackets are standard Black-Scholes call functions. Hence

$$S_0 = \frac{1}{m}\left[BSCall\left(A_0, \overline{B}, r, T, \sigma, \delta\right) - \frac{n}{m+n}BSCall\left(A_0, \overline{A}, r, T, \sigma, \delta\right)\right].$$

C Warrant value

We want to compute $W_0 = e^{-rT} E^* \left[W_T \right]$, where the star denotes an expectation taken with respect to the risk neutral measure for the asset value. Using the payoff function given by equation (2) , we have

$$W_0 = e^{-rT} \int_{\overline{A}}^{\infty} \left(\frac{\max\left(A_T - \left(\overline{B} - nK\right), 0\right)}{m+n} - K \right) \times \phi\left(A_T\right) dA_T$$

Because $\overline{B}$ is always smaller than $\overline{A}$, $\overline{B} - nK$ is also smaller than $\overline{A}$ and the $\max(\cdot)$ operator can be removed, to obtain after some simplifications:

$$W_0 = \frac{e^{-rT}}{m+n} \int_{\overline{A}}^{\infty} \left(A_T - \left(\overline{B} + Km\right)\right) \times \phi\left(A_T\right) dA_T$$

Substituting the lognormal density and integrating yields

$$W_0 = \frac{A_0 e^{-\delta T} N\left(b_{1,\overline{A}}\right) - \overline{A} \times e^{-rT} N\left(b_{2,\overline{A}}\right)}{m+n}.$$

This expression is a standard Black-Scholes call price i.e.

$$W_0 = \frac{1}{m+n} BSCall\left(A_0, \overline{A}, r, T, \sigma, \delta\right).$$

D Impact of warrant introduction on equity

For a firm initially without warrants, the impact of the warrant introduction on the equity, for small changes in asset value, can be approximated with:

$$E_0 - E_{0-} = \left[BSCall\left(A_0, \overline{B}, r, T, \sigma, \delta\right) - nW_0\right] - BSCall\left(A_{0-}, \overline{B}, r, T, \sigma, \delta\right),$$

where E_{0-} denotes the theoretical equity value before the warrant introduction in the capital structure. Using a first order approximation to find the change in option values triggered by the increase in asset value from A_{0-} to A_0 caused by the warrant introduction obtains:

$$E_0 - E_{0-} \simeq \Delta BSCall\left(A_{0-}, \overline{B}, r, T, \sigma, \delta\right) \times \left(A_0 - A_{0-}\right) - nW_0,$$

where $\Delta BSCall(\cdot)$ is the delta of the Black-Scholes call function. Given that the change in asset value is nW_0, we obtain:

$$E_0 - E_{0-} \simeq -nW_0 \times \left[1 - \Delta BSCall\left(A_{0-}, \overline{B}, r, T, \sigma, \delta\right)\right].$$

E Extended Kalman filter recursion

The theoretical relation between the observed equity value and the unobserved asset value is the Black-Scholes call pricing function since we assume a sample of equity value for a firm without any warrants. This relation for observation i is denoted by $E_{ih} = f(\exp(\alpha_{ih})) + v_{ih}$ where $\alpha_{ih} = \ln A_{ih}$ and v_{ih} are i.i.d. $Normal(0, \sigma_v^2)$ noises with $f(\exp(\alpha_{ih}))$ a Black-Scholes call function price. Because the Kalman filter works with linear functions of the unobserved variables, this relation needs to be linearized in the log of the asset value. Denote the linearized theoretical relation by

$$f(\exp(\alpha_{ih})) \simeq f\left(\exp\left(a_{ih|(i-1)h}\right)\right) + \nabla \times \left(\alpha_{ih} - a_{ih|(i-1)h}\right)$$

where $a_{ih|(i-1)h}$ is an estimated value of the unobserved log-asset value α_{ih} knowing the information available at $t = (i-1)h$ and

$$\nabla = \left.\frac{\partial f(\exp(\alpha_{ih}))}{\partial \alpha_{ih}}\right|_{\alpha_{ih} = a_{ih|(i-1)h}}$$

is the first derivative of the nonlinear function, also evaluated at the estimated value. Here, because f is the Black-Scholes call function,

$$\nabla = \frac{\partial f(\cdot)}{\partial \exp(\alpha)} \times \frac{\partial \exp(\alpha)}{\partial \alpha} = \Delta BSCall(\cdot) \times \exp(\alpha)$$

where $\Delta BSCall(\cdot)$ is the delta of a Black-Scholes call function described in Appendix A. With this linearization, we obtain

$$E_{ih} = \nabla \times \alpha_{ih} + g_{ih} + v_{ih}$$

with $g_{ih} = f\left(\exp\left(a_{ih|(i-1)h}\right)\right) - \nabla \times a_{ih|(i-1)h}$. With the above measurement equation, the transition equation (7) and some numerical values for μ, σ, and σ_ε, a first set of equations defining the extended Kalman filter recursion is given by:

$$a_{ih|(i-1)h} = a_{(i-1)h} + \left(\mu - \delta - \frac{1}{2}\sigma^2\right) h$$

and

$$P_{ih|(i-1)h} = P_{(i-1)h} + \sigma^2 h.$$

This first set of equations are the prediction equations, providing estimates of the unobserved factors α and their variance P given the information available at time $(i-1)h$. The second set of equations, called the updating equations, is

$$\begin{aligned}
e_{ih|(i-1)h} &= E_{ih} - \left(\nabla \times a_{ih|ih-h} + g_{ih}\right), \\
F_{ih} &= \nabla^2 P_{ih|(i-1)h} + \sigma_v^2, \\
a_{ih} &= a_{ih|(i-1)h} + P_{ih|(i-1)h} \nabla F_{ih}^{-1} e_{ih|(i-1)h}, \\
P_{ih} &= P_{ih|(i-1)h} - P_{ih|(i-1)h}^2 \nabla^2 F_{ih}^{-1}.
\end{aligned}$$

These obtain the updated estimates which now contain the information available at time $t = ih$ Because the transition equation is non-stationary, the extended Kalman filter recursion can be initialized with $P_0 = \kappa$ where κ is a large number. We use here $\kappa = 1,000$. As explained in Harvey (1989), this corresponds to a diffuse prior for the initial observation. The initial value a_0 is arbitrarily set equal to the logarithm of implied asset value obtained from the Black-Scholes call function, i.e. $a_0 = \ln\left(f^{-1}\left(E_0\right)\right)$. Computing this value is similar to computing an implied volatility. Here the numerical value of the volatility is known, but the asset value needs to be estimated. Using a bisection algorithm, one can recover the asset value which makes the theoretical call function value equal to the observed stock price. This initialization procedure for a_0 is equivalent to assume a noise of zero for the first stock price. The approximate likelihood function can be computed with the quantities obtained from the filter. The logarithm of this function is written as:

$$\ln L\left(\mu, \sigma, \sigma_v\right) = -\frac{M}{2}\ln 2\pi - \frac{1}{2}\sum_{i=1}^{M}\ln F_{ih} - \frac{1}{2}\sum_{i=1}^{M}\frac{e_{ih|(i-1)h}^2}{F_{ih}}$$

For inference purposes, we will apply the standard results of maximum likelihood estimators. Let $\widehat{\theta}_M$ denote genuine maximum likelihood parameter estimator for $\theta_0 = [\mu, \sigma, \sigma_v]$ based on the sample size M. For such parameters, the standard asymptotic theory yields the following asymptotic distribution

$$\sqrt{M}\left(\widehat{\theta}_M - \theta_0\right) \sim Normal\left(0, I^{-1}\right)$$

where I is the 3×3 asymptotic information matrix whose sample estimate is given by

$$\widehat{I}_M = -\frac{1}{M} \times \left.\frac{\partial L\left(\cdot\right)}{\partial\theta_i\partial\theta_j}\right|_{\theta=\widehat{\theta}_M}.$$

Continuously differentiable functions of the parameters are also asymptotically normally distributed. Hence, for a function $G\left(\widehat{\theta}\right)$, we have

$$\sqrt{M}\left(G\left(\widehat{\theta}\right) - G\left(\theta_0\right)\right) \sim Normal\left(0, \left.\frac{\partial G'}{\partial\theta} I^{-1}\frac{\partial G}{\partial\theta}\right|_{\theta=\widehat{\theta}_M}\right).$$

We will use these results to approximate the small sample distribution of our approximate maximum likelihood estimator.

F Alternative estimation approaches

F.1 Non-linear system of equations

Here, observation M corresponds to the issue date. The non-linear system of equation approach uses the following two relations:

$$E_{Mh} = BSCall\left(A_{Mh-}, \overline{B}, r, T, \sigma, \delta\right)$$

$$\widehat{\sigma}_E = \frac{e^{-\delta T}N(d_{1,\overline{B}})\sigma A_{Mh-}}{E_{Mh}}$$

where E_{Mh} is the equity value observed at the warrant issue date, A_{Mh-} is the unobserved asset value just prior to the warrant issue and $\widehat{\sigma}_E$ is the annualized (multiplied by $\sqrt{1/h}$) estimated historical standard error of daily equity log returns for the sample under consideration. In this system, the only unknown quantities are A_{Mh-} and σ. Solving numerically this non-linear system obtains the estimates for these quantities.

F.2 Maximum likelihood

The maximum likelihood approach described in Duan (1994) and (2000) uses the following log likelihood function to compute the estimated values for A_{Mh-} and σ:

$$L(\mu, \sigma) = -\frac{M}{2} \ln\left(2\pi\sigma^2 h\right) - \frac{1}{2} \sum_{k=1}^{M} \frac{\left(R_k - \left(\mu - \frac{\sigma^2}{2}\right) h\right)^2}{\sigma^2 h} - \sum_{k=1}^{M} \ln \widehat{A}_{kh}(\sigma) - \sum_{k=1}^{M} \ln\left(\Phi\left(\hat{d}_{1,\overline{B}}(\sigma)\right)\right)$$

where $R_k = \ln \frac{E_{kh}}{E_{(k-1)h}}$, $\widehat{A}_{kh}(\sigma) = f^{-1}(E_{kh}; \sigma)$ where $f(E_{kh}; \sigma)$ is the equity pricing formula. Maximizing numerically this function with respect to μ and σ obtains maximum likelihood parameter estimates under the assumption of no pricing errors for the equity values. The asset value just prior to the warrant issue can be computed as $\widehat{A}_{Mh-}(\widehat{\sigma})$ where $\widehat{\sigma}$ is the estimated parameter value.

Table 1: A Monte Carlo analysis of the approximate maximum likelihood estimator.

	μ	σ	σ_v
true value	0.100	0.300	2.000
mean	0.103	0.311	1.945
median	0.102	0.304	1.982
std	0.307	0.086	0.442
25% cov. rate	0.257	0.257	0.276
50% cov. rate	0.509	0.510	0.540
75% cov. rate	0.763	0.751	0.795
95% cov. rate	0.967	0.930	0.962

This table reports the results of a Monte-Carlo experiment with 5,000 sample paths of 250 data points and input values as described in section 3. *true value* is the true parameter value; *mean*, *median* and *std* are the statistics computed with the 5,000 parameter estimates. *cov. rate* is the percentage of the 5,000 parameter estimates for which the theoretical value is contained in the β% confidence interval implied by the approximate asymptotic normal distribution.

Table 2: Balance sheet data and parameter values for Bank of America at the warrant issue date.

Balance sheet, equity and warrant data (in billion)	
Short term debt	575.0000
Long term debt	427.0000
Domestic deposits	949.0000
Foreign deposits	89.0000
Prefered shares	16.5620
Number of common shares (m)	10.1332
Equity value (E)	70.6284
Face value of debt ($\overline{B}$)	1834.0500
Number of warrants (n)	0.7000

Other parameters	
Length of a discrete time interval (h)	1/252
Ten years to maturity risk free rate (r)	0.0350
Dividend yield (δ)	0.0050
Historical volatility of equity log-returns (σ_E)	0.9153
Maturity of debt and warrants at the issue date (T)	10.0000
Strike price of warrants (K)	7.1429

Table 3: Estimation results

	Jan2008-July2011			Jan2009-July2011			Jan2010-July2011		
	non-lin eq. sys.	mle no errors	mle with errors	non-lin eq. sys.	mle no errors	mle with errors	non-lin eq. sys.	mle no errors	mle with errors
$\hat{\mu}$	na	0.1140	-0.0319	na	0.2742	0.0943	na	-0.0384	-0.0528
	(na)	(0.3692)	(0.2130)	(na)	(0.3894)	(0.2115)	(na)	(0.1271)	(0.1427)
$\hat{\sigma}$	0.5893	0.7021	0.3774	0.5205	0.6115	0.2875	0.0514	0.1345	0.1404
	(na)	(0.0117)	(0.0147)	(na)	(0.0101)	(0.0106)	(na)	(0.0027)	(0.0038)
$\hat{\sigma}_v$	na	na	2.3391	na	na	2.1918	na	na	0.6005
	(na)	(na)	(0.0864)	(na)	(na)	(0.0715)	(na)	(na)	(0.2155)
$\log L$	na	-3081	-2934	na	-2166	-2027	na	-1163	-1152

This table shows the estimated values for a new warrant issue made by Bank of America in July 2011. Numbers in parenthesis are the standard errors of the estimated quantities. *non-lin. eq. sys.* is the non-linear equation system estimation approach described in Appendix F. *mle no errors* is the maximum likelihood estimation for equity with no model error described in Appendix F. *mle with errors* is the approximate maximum likelihood for the equity with model errors computed with the extended Kalman filter approach. na indicates a non-available quantity.

Table 4: Estimation results

	Jan2008-July2011			Jan2009-July2011			Jan2010-July2011		
	non-lin eq. sys.	mle no errors	mle with errors	non-lin eq. sys.	mle no errors	mle with errors	non-lin eq. sys.	mle no errors	mle with errors
$\hat{A}_{M-}$	233.2	179.0	435.0	280.4	220.6	591.0	1331.7	1021.6	1000.4
$\hat{E}t_{M-}$	70.6	70.6	71.0	70.6	70.6	71.0	70.6	70.6	70.7
$\hat{B}_{M-}$	162.6	108.4	364.0	209.8	149.9	520.0	1261.0	951.0	929.8
$\hat{W}$	6.5731	6.6514	6.4044	6.5178	6.5896	6.2751	4.8260	5.7653	5.8002
	(na)	(0.0075)	(0.0159)	(na)	(0.0074)	(0.0161)	(na)	(0.0153)	(0.0184)
$\hat{A}_M$	237.8	183.7	439.5	285.0	225.2	595.4	1335.0	1025.6	1004.5
	(na)	(4.47)	(21.02)	(na)	(5.52)	(22.15)	(na)	(9.70)	(13.39)
$\hat{E}t_M$	68.1	68.5	67.9	67.9	68.2	67.8	68.1	67.5	67.5
	(na)	(0.04)	(0.04)	(na)	(0.03)	(0.05)	(na)	(0.01)	(0.03)
$\hat{B}_M$	165.1	110.5	367.1	212.5	152.4	523.2	1263.6	954.1	932.9
	(na)	(4.52)	(21.03)	(na)	(5.55)	(22.13)	(na)	(9.70)	(13.37)
$\hat{E}t_M - \hat{E}t_{M-}$	-2.4823	-2.1242	-3.0478	-2.6858	-2.4135	-3.1914	-2.5616	-3.1235	-3.1399

This table shows the estimated values for a new warrant issue made by Bank of America in July 2011. Numbers in parenthesis are the standard errors of the estimated quantities. *non-lin. eq. sys.* is the non-linear equation system estimation approach described in Appendix F. *mle no errors* is the maximum likelihood estimation for equity with no model error described in Appendix F. *mle with errors* is the approximate maximum likelihood for the equity with model errors computed with the extended Kalman filter approach. A_{M-} and A_M are the asset values (in billion) at the last data point of the sample (the date of the warrant introduction) just before and after warrants introduction. Et_{M-} and Et_M are the theoretical equity values (in billion) computed with equation (3) with the estimated asset values just before and after warrants introduction. B_{M-} and B_M are the theoretical debt values (in billion) computed with equation (5) with the estimated asset values just before and after warrants introduction. W is the individual warrant price estimate computed with equation (4). na indicates a non-available quantity.

Figure 1: Share prices, equity values and face values of debt for Bank of America