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Johnson binomial trees

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Rubinstein developed a binomial lattice technique for pricing European and American derivatives in the context of skewed and leptokurtic asset return distributions. A drawback of this approach is the limited set of skewness and kurtosis pairs for which valid stock return distributions are possible. A solution to this problem is proposed here by extending Rubinstein's Edgeworth tree idea to the case of the Johnson system of distributions which is capable of accommodating all possible skewness and kurtosis pairs. Numerical examples showing the performance of the Johnson tree to approximate the prices of European and American options in Merton's jump diffusion framework and Duan's GARCH framework are examined.

Keywords: Edgeworth binomial tree; Skewness; Kurtosis; Johnson distribution; American option; Jump diffusion; GARCH

1. Introduction

For the purpose of pricing options, Jarrow and Rudd (1982) have shown how an Edgeworth expansion built with the first four moments of a risk-neutral asset distribution can be used for European-style claims. This approach is useful as it provides approximating formulae for cases in which the underlying asset distribution is unknown. For example, Turnbull and Wakeman (1991) and Ritchken *et al*. (1993) used this method to obtain analytical approximations for European option prices on the arithmetic mean. More recently, Duan *et al*. (1999, 2006) used it to compute European option prices in various GARCH contexts.

For American-style options, a direct application of this approach is not possible. For these claims, because of the early exercise possibilities, the entire asset price path from the time of valuation to the maturity of the option contract needs to be described. Rubinstein (1998) proposed an approach for constructing such paths from the distribution of the asset prices at the maturity of the option. Rubinstein's Edgeworth tree uses an Edgeworth expansion to approximate the risk-neutral asset price distribution at the maturity of the option. A recombining implied binomial tree is then deduced using risk-neutral principles to describe the asset price evolution over the life of the option contract. Duan *et al*. (2003) demonstrated numerically that this approach provides a useful and simple alternative for approximating the prices of American options in the GARCH framework. Although such an approach does not have the convergence property shared by other methods proposed in the literature, it represents a convenient alternative if computation time or memory requirement is an issue.

An important drawback of Edgeworth expansion-based approaches, used either for the European or American cases, is that they fails to generate valid approximating densities in many cases. As discussed by Rubinstein (1998), this approach can deliver unimodal approximating densities with positive probabilities for a limited set of skewness and kurtosis pairs. Outside of this set, the Edgeworth approach produces either multimodal densities or densities with negative probabilities. Negative probabilities are obviously undesirable. When the Edgeworth tree is used to approximate American option prices given an assumption about the return process, multimodal densities are also undesirable because most of the commonly assumed processes imply unimodal densities. In this paper, an alternative to the Edgeworth approximating density in the context of Rubinstein's approach is proposed. More specifically, we extend Rubinstein's Edgeworth tree idea to the case of the Johnson (1949) system of distributions. As with the Edgeworth approach, the Johnson system of distributions

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can be used as approximating distributions given the first four moments of an unknown target distribution. Unlike the Edgeworth approach, the Johnson system of distributions is capable of accommodating all possible skewness and kurtosis combinations with genuine distributions in all cases. In the finance literature, Posner and Milevsky (1998) have used this approach to obtain an analytical approximation in the context of Asian options while Gauthier *et al.* (2004) used it in the context of options on the variance in a GARCH setting.

Just like the Edgeworth approach, the Johnson binomial tree approach proposed here uses, as a building block, a transformation of a binomial random variable. The Edgeworth approach modifies the probabilities of this binomial random variable but leaves the values taken by the variable unchanged. The modification of the probabilities cannot guarantee positive values. The Johnson tree examined here modifies the values of the binomial random variable instead of the probabilities which remain unmodified. It therefore avoids the problem associated with negative probabilities by construction.

Section 2 briefly reviews the Edgeworth binomial tree approach. Section 3 looks at Johnson distributions and examines how they can be used to modify the Edgeworth approach. Sections 4 and 5 then present some numerical results comparing the Edgeworth and Johnson tree performance for approximating European and American option prices in Merton's (1976) jump diffusion and Duan's (1995) GARCH contexts. Section 6 concludes.

2. The Edgeworth binomial tree

Rubinstein's (1998) approach allows one to build an n time-step recombining binomial implied tree with skewed and leptokurtic returns. In a first step, this approach requires constructing a discrete random variable with zero mean and unit variance with the appropriate skewness and kurtosis. This variable will be used to build the stock prices at the end of the tree. For this purpose, Rubinstein (1998) used an Edgeworth approximation. Appendix A describes this procedure which results in a set of discrete values $x_{n,j}$ for $j=0$ to n, and associated probabilities $P_{n,j}$. Here, the first subscript is for the time step and the second subscript is for indexing the branches from the tree.

In a second step, using these discrete values and probabilities, a tree that recombines to yield $n+1$ nodes after n time steps is built as follows. To distinguish the prices of the binomial tree from the actual stock price, upper case (binomial tree) and lower case (actual stock price) letters are used. At the last time step, the asset value at the j-th node, $S_{n,j}$, is set to

$$S_{n,j} = s_0 e^{\mu\tau + v\sqrt{\tau}x_{n,j}}, \tag{1}$$

with

$$\mu = r - \delta - \frac{1}{\tau}\ln\sum_{j=0}^{n} P_{n,j}e^{v\sqrt{\tau}x_{n,j}}, \tag{2}$$

where s_0 is the initial stock price, τ is the maturity of the option, r is the annual continuously compounded risk-free rate, δ is the annual continuously compounded dividend rate and $v=\sqrt{\mathrm{Var}(\rho_\tau)/\tau}$ is the annualized volatility of the cumulative asset return $\rho_\tau = \ln(s_\tau/s_0)$.

In this setting, risk neutrality is imposed through μ which ensures that the annual expected risk-neutral asset return equals r. The skewness and kurtosis are defined as $\kappa_3 = E[\varrho_\tau^3]$ and $\kappa_4 = E[\varrho_\tau^4]$ with $\varrho_\tau = (\rho_\tau - E[\rho_\tau])/(v\sqrt{\tau})$ representing the standardized cumulative asset return. With these terminal stock prices and probabilities, a recombining binomial tree can then be deduced. Appendix B describes how this can be done. The resulting tree will be different from the usual binomial tree described in finance textbooks as the move size will not be constant in the different portions of the tree.

In the above procedure, a crucial step is the construction of the standardized skewed and leptokurtic discrete random variable with the Edgeworth expansion. As mentioned by Rubinstein (1998), this approximation produces random variables with unimodal densities and non-negative probabilities for a limited set of skewness and kurtosis pairs. Although this set can be enlarged by using a Gram–Charlier expansion, which is essentially the Edgeworth expansion with the last term omitted, the set of skewness and kurtosis pairs generating valid distributions remains limited. The top graph in figure 1 shows the set of skewness and kurtosis pairs generating acceptable Gram–Charlier expansions with $n=100$. For this set, the approximating densities are unimodal with non-negative probabilities. Outside of this set, the Gram–Charlier expansion provides either multimodal densities or densities with negative probabilities. As an example of the invalid densities generated by this approach, the top graph in figure 2 plots the approximate density generated for $n=100$, $\kappa_3=-0.9$ and $\kappa_4=7.0$. The resulting density

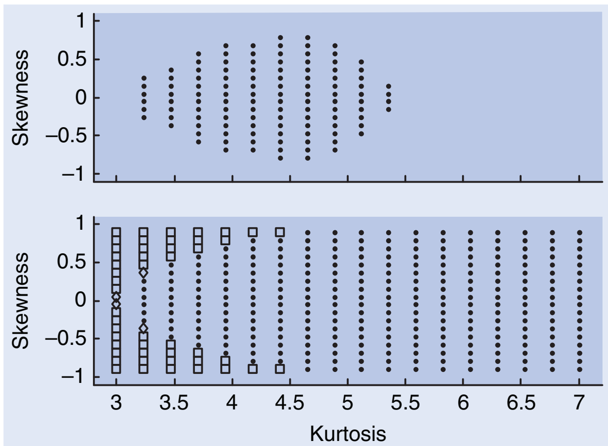

Figure 1. Locus of valid Gram–Charlier and Johnson densities. The graphs in this figure show the set of skewness and kurtosis pairs generating unimodal densities with positive probabilities for the Gram–Charlier (upper graph) and Johnson (bottom graph) approaches for a zero mean and unit variance random variable with $n=100$. In the graph showing the results with the Johnson approach, a square is a density from the bounded family, a dot is a density from the unbounded family and diamonds are for the lognormal family.

is clearly multimodal with negative probabilities around $x_{n,j} = -1.5$ and 2.0.

3. The Johnson binomial tree

For a continuous random variable X with an unknown distribution which needs to be approximated, Johnson (1949) proposed a set of 'normalizing' translations. These translations transform the continuous random variable X into a standard normal variable Z and have the general form

$$Z = a + b \cdot g\left(\frac{X-c}{d}\right),$$

where a and b are shape parameters, c is a location parameter, d is a scale parameter and $g(\cdot)$ is a function whose form defines the four families of distributions of the Johnson system:

$$g(u) = \begin{cases} \ln(u), & \text{for the lognormal family,} \\ \ln(u + \sqrt{u^2+1}), & \text{for the unbounded family,} \\ \ln(u/(1-u)), & \text{for the bounded family,} \\ u, & \text{for the normal family.} \end{cases}$$

As discussed by Johnson (1949), the above system has the flexibility to match any feasible set of values for the mean, variance, skewness, and kurtosis. With this system, the skewness and kurtosis also uniquely identify the appropriate form for the $g(\cdot)$ function. As a result, the process of using the Johnson system to approximate an unknown distribution is reduced to the problem of finding the values of a, b, c and d that will match the moments of the unknown target distribution with those of the appropriate family from the Johnson system. Hill *et al.* (1976) provide an efficient algorithm which finds, given the first four moments of a target distribution, the appropriate family (the form of the $g(\cdot)$ function) and values of the parameters required to approximate the unknown distribution. This algorithm is summarized in appendix D.

With the parameters determined as above, the Johnson random variable can be expressed as the inverse of the above normalizing translation, i.e.

$$X = c + d \cdot g^{-1}\left(\frac{Z-a}{b}\right), \quad (3)$$

where

$$g^{-1}(u) = \begin{cases} e^u, & \text{for the log normal family,} \\ (e^u - e^{-u})/2, & \text{for the unbounded family,} \\ 1/(1+e^{-u}), & \text{for the bounded family,} \\ u, & \text{for the normal family.} \end{cases}$$

Using this transformation, the probability density functions of such random variables can be found using standard calculus techniques and are given by

$$f_X(x) = \frac{b}{d(2\pi)^{1/2}} g'\left(\frac{x-c}{d}\right) e^{-(1/2)\{a+b\cdot g[(x-c)/d]\}^2},$$

where

$$g'(u) = \begin{cases} 1/u, & \text{for the lognormal family,} \\ 1/\sqrt{u^2+1}, & \text{for the unbounded family,} \\ 1/(u/(1-u)), & \text{for the bounded family,} \\ 1, & \text{for the normal family,} \end{cases}$$

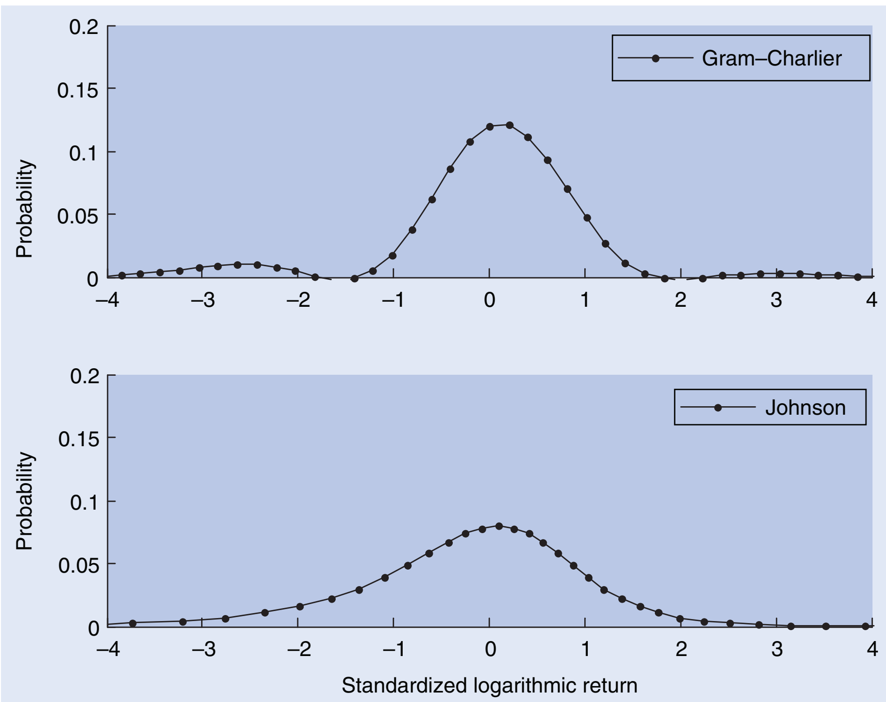

Figure 2. Gram–Charlier and Johnson densities for $\kappa_3 = -0.9$ and $\kappa_4 = 7.0$. The graphs in this figure plot the discrete densities obtained from the Gram–Charlier (upper graph) and Johnson (bottom graph) approaches for a zero mean and unit variance random variable with $n = 100$ and $\kappa_3 = -0.9$ and $\kappa_4 = 7.0$.

and support $[c, +\infty)$ for the lognormal family, $(-\infty, +\infty)$ for the unbounded family, $[c, c+d]$ for the bounded family, and $(-\infty, +\infty)$ for the normal family.

Given the above system, building an implied recombining tree can be done as follows.

- **Step 1:** With an expected value of zero, a standard deviation of one and the skewness and kurtosis of the target asset return distribution, use the Hill *et al.* (1976) algorithm to determine the appropriate form of the $g(\cdot)$ function and $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ and $\widehat{d}$, the parameter values matching the moments of the Johnson distribution with the target distributions.
- **Step 2:** Using a binomial random variable $z_{n,j} = [(2j) - n]/\sqrt{n}$ for $j = 0$ to n with probabilities $P_{n,j} = [n!/(j!(n-j)!)](1/2)^n$, compute

$$\widetilde{x}_{n,j} = \widehat{c} + \widehat{d} \cdot g^{-1}\left(\frac{z_{n,j} - \widehat{a}}{\widehat{b}}\right).$$

  As for the Edgeworth case, the $\widetilde{x}_{n,j}$ can be standardized to have mean 0 and variance 1 with $x_{n,j} = (\widetilde{x}_{n,j} - M)/V$, $M = \sum_j P_{n,j}\widetilde{x}_{n,j}$ and $V^2 = \sum_j P_{n,j}(\widetilde{x}_{n,j} - M)^2$.
- **Step 3:** Compute the terminal stock prices on the tree using equation (1). The remaining prices on the tree are then obtained recursively using

$$S_{i-1,j} = [0.5 \cdot S_{i,j+1} + 0.5 \cdot S_{i,j}]\, \mathrm{e}^{-(r-\delta)\times(\tau/n)}. \quad (4)$$

The above procedure for the stock price recursion is simpler than that required for the Edgeworth case. For the Johnson case, only the terminal values of the binomial random variable are adjusted. The probabilities are left unchanged, avoiding the need for a recursive procedure and the possibility of negative probabilities. The probabilities of jumping up or down in the branches are thus constant and equal to 0.5 throughout the tree.

It should also be noted that the Johnson tree approach can encounter overflow or underflow for extreme kurtosis values when a large number of time steps is used. These numerical problems appear when computing the discrete stock prices with equation (1). However, for sizes such as $n = 150$, the approach does not usually encounter numerical problems.

The Johnson tree procedure described above could be amended for time-dependent interest rates. Equation (2) should then be computed with the spot interest rate for the maturity τ while equation (4) can be adjusted by replacing the interest rate by the forward rate prevailing for the time step. The Johnson tree cannot, however, explicitly handle time-dependent volatilities structures. The volatility parameter is only used once when computing the terminal stock prices. The recursion with which the remaining prices are computed on the tree does not involve this parameter.

The bottom graph in figure 1 shows that, unlike the Edgeworth case, the entire set of skewness and kurtosis pairs examined generates acceptable Johnson approximating distributions, i.e. unimodal distributions with positive probabilities. As shown in this graph, the majority of the distributions are of the unbounded family (dots). The majority of the remaining cases consist of the bounded family (squares) while the lognormal family (diamonds) only appears in a few cases.

As an example of the kind of densities generated by the Johnson approach, the bottom graph in figure 2 plots the discrete density for a zero mean and unit variance random variable with $n = 100$, $\kappa_3 = -0.9$ and $\kappa_4 = 7.0$. As shown in this graph, unlike the Edgeworth case illustrated in the top graph of this figure, the resulting Johnson approximating density is clearly unimodal with positive probabilities for all values.

It is also interesting to examine how the discrete random variable $(x_{n,j}, P_{n,j})$ can approximate the target skewness and kurtosis. For this purpose, table 1 presents the root-mean-squared errors, as a percentage of the true moments, for the set of skewness and kurtosis pairs examined in figure 1. As can be seen from this table, the typical errors are in general below 2%. The two approaches also generate similar precision in computing the third and fourth moments.

The following sections examine how the Johnson tree described above can be used to approximate European and American options in frameworks generating skewed and leptokurtic distributions.

4. Approximating option prices in the jump diffusion framework

Merton (1976) proposed a model combining a return process moving continuously but which can also jump discretely according to a Poisson process. For European-style options, under the assumption of lognormal distributed jumps, this model has a closed-form solution. For American claims, many numerical approaches have been proposed in the literature. See, for example, the references in Chiarella and Ziogas (2005) and Kou (2008). However, as noted by these authors, the available approaches are numerically intensive. For example, Chiarella and Ziogas (2005) report that the different

Table 1. Third and fourth moments approximation.

	$n = 50$		$n = 100$		$n = 150$	
	3	4	3	4	3	4
Edgeworth	3.3%	3.2%	1.6%	1.6%	1.1%	1.1%
Johnson	3.0%	3.6%	1.7%	1.8%	1.3%	1.2%
Johnson all	4.1%	5.0%	2.5%	2.6%	2.0%	1.7%

This table reports the root-mean-squared error (RMSE) as a percentage of the true moment, computed with the Johnson and Edgeworth approaches for random variables with zero mean and unit variance and all combinations of skewness and kurtosis from the following sets: $\kappa_3 = \{-0.9, -0.7, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9\}$ and $\kappa_4 = \{3.0, 3.5, 3.9, 4.3, 4.8, 5.2, 5.7, 6.1, 6.6, 7.0\}$. 'Edgeworth' and 'Johnson' report the RMSE computed for the subset of skewness and kurtosis combinations for which the Edgeworth approximation generates unimodal densities and positive probabilities. 'Johnson all' reports the RMSE for all skewness and kurtosis combinations.

approaches they consider yield computing times between 4 and 95 seconds per option price, depending on the method.

In many situations, computing option values quickly is important. For example, large trading desks often need to quickly price large books of options. In such a context, a time and memory intensive method can become inappropriate and one may be willing to compromise on accuracy to gain computation speed and/or reduce memory needs. It is therefore interesting to have an approach which can compute in fractions of seconds the prices of American options in the jump diffusion framework. The Johnson tree described in the previous section can be used for such a task. It should be noted that, in this context, the univariate Johnson tree will contain an approximation error that will not vanish as the number of time step grows. A one-dimensional lattice is not sufficient to describe the stochastic evolution of the asset price which is driven by two sources of uncertainty. Despite this theoretical limitation, the Edgeworth tree is a convenient alternative when computing time is an issue. The typical computing time of a Matlab code for one American option with $n=100$ is around 0.004 seconds on a standard desktop computer. This time includes finding the Johnson parameters with the algorithm of Hill *et al.* (1976), computing the first four moments and the computations with the implied tree.

In Merton's (1976) jump diffusion framework, the stock price follows the process

$$\frac{\mathrm{d}s_t}{s_t} = (r - \delta - \lambda k)\,\mathrm{d}t + \sigma\,\mathrm{d}Z + \mathrm{d}Q,$$

where r is the annual continuously compounded risk-free rate, δ is the annual continuously compounded dividend rate, Z is a standard Brownian motion, σ the constant volatility parameter associated with the Brownian motion, λ is the arrival rate of jumps and $\mathrm{d}Q$ is the jump portion of the process. Over an interval $\mathrm{d}t$, the stock price can jump with probability $\lambda\mathrm{d}t$. The jump portion $\mathrm{d}Q$ takes a value of 0 if there is no jump and $Y-1$ if there is jump with $k=E(Y-1)$. It is assumed that the jump magnitude, Y, is lognormally distributed, i.e. $\ln Y \sim N(\alpha_J, \sigma_J^2)$, which yields $k = \mathrm{e}^{\alpha_J} - 1$.

For this model, Das and Sundaram (1999) have derived formulae for the variance, skewness and kurtosis of the stock return over a time interval of τ years. The formulae for these moments are

$$v^2 = \sigma^2 + \lambda\sigma_J^2 + \lambda\alpha_J^2, \tag{5}$$

$$\kappa_3 = \frac{1}{\sqrt{\tau}}\left(\frac{\lambda(\alpha_J^3 + 3\alpha_J\sigma_J^2)}{(\sigma^2 + \lambda\sigma_J^2 + \lambda\alpha_J^2)^{3/2}}\right), \tag{6}$$

$$\kappa_4 = 3 + \frac{1}{\tau}\left[\frac{\lambda(\alpha_J^4 + 6\alpha_J^2\sigma_J^2 + 3\sigma_J^4)}{(\sigma^2 + \lambda\sigma_J^2 + \lambda\alpha_J^2)^2}\right]. \tag{7}$$

Using these, it becomes possible to use the Johnson binomial tree approach described in section 3 to approximate the prices of European and American options.

Table 2 reports the moments of the stock returns and prices computed for European calls for various maturities and strike prices. We see here the typical structure for the

Table 2. European call option prices in the Merton (1976) jump diffusion context.

τ	20/365	30/365	60/365	90/365	270/365
v	0.3033	0.3033	0.3033	0.3033	0.3033
κ_3	−0.4654	−0.3800	−0.2687	−0.2194	−0.1267
κ_4	6.4948	5.3298	4.1649	3.7766	3.2589
			Strike price 45		
Analytical	5.2927	5.4582	5.9702	6.4607	8.8668
Edgeworth	NA	NA	5.9518	6.4372	8.8423
Johnson	5.2658	5.4304	5.9461	6.4385	8.8432
Rel. diff. Edgeworth (%)	NA	NA	0.31	0.36	0.28
Rel. diff. Johnson (%)	0.51	0.51	0.40	0.34	0.27
			Strike price 50		
Analytical	1.3426	1.7038	2.5468	3.2119	6.0041
Edgeworth	NA	NA	2.5209	3.1879	5.9737
Johnson	1.3744	1.7306	2.5592	3.2149	5.9859
Rel. diff. Edgeworth (%)	NA	NA	1.02	0.75	0.51
Rel. diff. Johnson (%)	2.37	1.57	0.49	0.09	0.30
			Strike price 55		
Analytical	0.1550	0.2936	0.7989	1.3147	3.8850
Edgeworth	NA	NA	0.7987	1.3134	3.8696
Johnson	0.1557	0.3030	0.8120	1.3251	3.8720
Rel. diff. Edgeworth (%)	NA	NA	0.03	0.10	0.40
Rel. diff. Johnson (%)	0.50	3.22	1.64	0.79	0.34

'Analytical' are the benchmark prices computed with the Merton (1976) formula. 'Edgeworth' and 'Johnson' are option prices computed with the Edgeworth and Johnson binomial trees with $n=150$ and a year defined as 365 days. 'Rel. diff.' is the relative difference computed as $|(C_{\text{lattice}} - C_{\text{analytical}})|/C_{\text{analytical}}$ where C is a call option price. 'NA' indicates the cases for which the Edgeworth prices are not available because of non-valid densities. Parameters: $r=0.05$, $\delta=0$, $\sigma=0.2$, $s_0=50$, $\lambda=5$, $\alpha_J=-0.02$ and $\sigma_J=0.1$.

skewness and kurtosis for this model which are decreasing as the maturity is increasing. Since this table examines European prices, the analytical prices obtained with the Merton (1976) formula can be used as benchmarks. The table also reports the prices obtained with the Edgeworth tree approach. As seen in this table, the Edgeworth tree approach cannot compute the prices for 20 and 30 days to maturity cases because it cannot generate valid densities. For these cases, it is clear that the skewness and kurtosis pairs are outside of the locus indicated in figure 1. For the other maturities, the Johnson and Edgeworth option prices are usually close and provide adequate approximations. The relative errors computed with the Johnson tree are almost all under 2% and in many cases below 1%. The errors are in general decreasing as maturity is increasing and the moments become closer to the normal case. These relative errors are less than the typical bid–ask spread, which is about 5–10% for exchange traded options.

Tables 3, 4 and 5 report American call prices with a non-zero dividend rate. For such cases, early exercise becomes a possibility and numerical procedures must be used. We therefore use benchmark values taken from Chiarella and Ziogas (2005), who report the prices of American options computed with the Kim (1990) integral equation. For a fixed maturity, these tables report the Edgeworth and Johnson prices for various parameter values. For both approaches, the majority of the relative errors are below 1%.

Table 6 reports American option prices for different maturities. For the first panel, which illustrates a high volatility of the geometric Brownian motion portion of the process, the precision is again very good. The largest errors are associated with small maturities for which the kurtosis and skewness are further away from the Gaussian case. For the second panel of this table, which illustrate cases with extreme kurtosis values for the shorter maturities, we see that the Johnson approach can generate large pricing errors in the range of 3–4%.

For a general assessment of the relative precision of the Edgeworth and Johnson approaches for approximating option prices, table 7 presents root-mean-squared errors (RMSE) measuring the aggregate pricing errors on different test pools of American and European options. The RMSE is computed as

$$RMSE = \sqrt{\frac{1}{m}\sum_{i=1}^{m} e_i^2},$$

with $e_i = |C_{i,\text{lattice}} - C_{i,\text{benchmark}}|/C_{i,\text{benchmark}}$, where $C_{i,\text{benchmark}}$ is the benchmark call price obtained for the ith option, $C_{i,\text{lattice}}$ is the ith call price obtained with the Edgeworth or Johnson tree and m is the size of the test pool. For the European test pool, 1000 different contracts are generated by a simulation which randomly selects the values of the following parameters: τ, r, σ, α_J, σ_J, λ and the strike price. See appendix E for a description of the simulation procedure. The option contracts in the American test pool come from the 50 cases examined in tables 3 to 6.

As shown in table 7, the results for the European options are in the same direction and, in most cases, similar in magnitude to those obtained for the American cases. For the whole sample, the European and American cases show that the Johnson approach is more precise with a smaller RMSE. These results also point out that

Table 3. American call option prices in the Merton (1976) jump diffusion context.

Stock price	80	90	100	110	120
			$v=0.4463$, $\kappa_3=0.0000$, $\kappa_4=3.2324$, $\alpha_J=0.0000$ and $\sigma_J=0.1980$		
Benchmark	3.6600	7.0400	11.8000	17.8400	24.9600
Edgeworth	3.6983	7.0987	11.8483	17.8646	24.9755
Johnson	3.7057	7.0955	11.8529	17.8762	24.9805
Rel. diff. Edgeworth (%)	1.05	0.83	0.41	0.14	0.06
Rel. diff. Johnson (%)	1.25	0.79	0.45	0.20	0.08
			$v=0.4450$, $\kappa_3=0.0856$, $\kappa_4=3.2207$, $\alpha_J=0.0488$ and $\sigma_J=0.1888$		
Benchmark	3.7400	7.1000	11.8200	17.8200	24.9100
Edgeworth	3.8184	7.1986	11.9055	17.8697	24.9328
Johnson	3.8212	7.1879	11.9038	17.8734	24.9483
Rel. diff. Edgeworth (%)	2.10	1.39	0.72	0.28	0.09
Rel. diff. Johnson (%)	2.17	1.24	0.71	0.30	0.15
			$v=0.4538$, $\kappa_3=-0.1030$, $\kappa_4=3.2983$, $\alpha_J=-0.0513$ and $\sigma_J=0.2082$		
Benchmark	3.6700	7.1100	11.9200	18.0000	25.1500
Edgeworth	3.6716	7.1095	11.9165	17.9889	25.1559
Johnson	3.6720	7.1190	11.9373	18.0217	25.1578
Rel. diff. Edgeworth (%)	0.04	0.01	0.03	0.06	0.02
Rel. diff. Johnson (%)	0.06	0.13	0.15	0.12	0.03

'Benchmark' are the benchmark prices computed with the Kim (1990) integral equation reported by Chiarella and Ziogas (2005). 'Edgeworth' and 'Johnson' are option prices computed with the Edgeworth and Johnson binomial trees with $n=150$. 'Rel. diff.' is computed as $|(C_{\text{lattice}} - C_{\text{benchmark}})|/C_{\text{benchmark}}$ where C is a call option price. Parameters: $r=0.03$, $\delta=0.05$, $\tau=0.5$ year, $\sigma=0.4$, strike price 100 and $\lambda=1$.

the Johnson approximation is more precise for smaller maturities, out-of-the-money options and low volatility contracts. In-the-money options are however handled better by the Edgeworth tree.

5. Approximating option prices in the GARCH framework

We examine here how the Johnson tree performs for approximating the prices of European and American options in the GARCH framework. In this framework, analytical expressions for the first four moments of the cumulative return are available for various GARCH specifications. See, for example, Duan *et al*. (1999, 2006).

When the asset's conditional expected return is specified to have a constant risk premium per unit of conditional standard deviation, the asset return dynamic with respect to the risk-neutral measure can be written as

$$\ln\left(\frac{s_{t+1}}{s_t}\right) = r_p - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\epsilon_{t+1}, \quad \text{for } t = 0, 1, 2, \ldots,$$

where $\epsilon_{t+1} \mid \Phi_t \sim N(0, 1)$ with Φ_t denoting the information set at time t, h_t is the conditional variance, and r_p is the one-period risk-free rate (continuously compounded). If the length of one period is one calendar day, then

Table 4. American call option prices in the Merton (1976) jump diffusion context.

Stock price	80	90	100	110	120
			$v=0.4463$, $\kappa_3=0.0000$, $\kappa_4=3.2324$, $\alpha_J=0.0000$ and $\sigma_J=0.1980$		
Benchmark	4.0500	7.6700	12.6800	18.9400	26.2200
Edgeworth	4.0885	7.7069	12.6996	18.9205	26.2092
Johnson	4.0966	7.7026	12.7026	18.9305	26.2072
Rel. diff. Edgeworth (%)	0.95	0.48	0.15	0.10	0.04
Rel. diff. Johnson (%)	1.15	0.43	0.18	0.05	0.05
			$v=0.4450$, $\kappa_3=0.0856$, $\kappa_4=3.2207$, $\alpha_J=0.0488$ and $\sigma_J=0.1888$		
Benchmark	4.1200	7.7100	12.6800	18.8900	26.1400
Edgeworth	4.2048	7.7973	12.7445	18.9149	26.1601
Johnson	4.2107	7.7854	12.7409	18.9158	26.1668
Rel. diff. Edgeworth (%)	2.06	1.13	0.51	0.13	0.08
Rel. diff. Johnson (%)	2.20	0.98	0.48	0.14	0.10
			$v=0.4538$, $\kappa_3=-0.1030$, $\kappa_4=3.2983$, $\alpha_J=-0.0513$ and $\sigma_J=0.2082$		
Benchmark	4.0700	7.7600	12.8300	19.1400	26.4600
Edgeworth	4.0561	7.7295	12.7804	19.0539	26.3958
Johnson	4.0685	7.7386	12.8002	19.0859	26.3915
Rel. diff. Edgeworth (%)	0.34	0.39	0.39	0.45	0.24
Rel. diff. Johnson (%)	0.04	0.28	0.23	0.28	0.26

'Benchmark' are the benchmark prices computed with the Kim (1990) integral equation reported by Chiarella and Ziogas (2005). 'Edgeworth' and 'Johnson' are option prices computed with the Edgeworth and Johnson binomial trees with $n=150$. 'Rel. diff.' is the relative difference computed as $|(C_{\text{lattice}} - C_{\text{benchmark}})|/C_{\text{benchmark}}$ where C is a call option price. Parameters: $r=0.05$, $\delta=0.03$, $\tau=0.5$ year, $\sigma=0.4$, strike price 100 and $\lambda=1$.

Table 5. American call option prices in the Merton (1976) jump diffusion context.

Stock price	80	90	100	110	120
			$v=0.2814$, $\kappa_3=0.0000$, $\kappa_4=4.4700$, $\lambda=1.00$ and $\sigma=0.20$		
Benchmark	1.1000	3.0300	6.9500	13.1100	21.0600
Edgeworth	1.2378	3.1146	6.9580	13.1041	21.1697
Johnson	1.1621	3.1599	7.0850	13.2595	21.2125
Rel. diff. Edgeworth (%)	12.52	2.79	0.11	0.04	0.52
Rel. diff. Johnson (%)	5.65	4.29	1.94	1.14	0.72
			$v=0.3311$, $\kappa_3=0.0000$, $\kappa_4=3.3838$, $\lambda=0.50$ and $\sigma=0.3000$		
Benchmark	1.7200	4.3000	8.6300	14.7000	22.2200
Edgeworth	1.7725	4.3373	8.6228	14.7146	22.2481
Johnson	1.7565	4.3332	8.6376	14.7323	22.2547
Rel. diff. Edgeworth (%)	3.05	0.87	0.08	0.10	0.13
Rel. diff. Johnson (%)	2.12	0.77	0.09	0.22	0.16

'Benchmark' are the benchmark prices computed with the Kim (1990) integral equation reported by Chiarella and Ziogas (2005). 'Edgeworth' and 'Johnson' are option prices computed with the Edgeworth and Johnson binomial trees with $n=150$. 'Rel. diff.' is the relative difference computed as $|(C_{\text{lattice}} - C_{\text{benchmark}})|/C_{\text{benchmark}}$ where C is a call option price. Parameters: $r=0.03$, $\delta=0.05$, $\tau=0.5$ year, strike price 100, $\alpha_J=0.0$ and $\sigma_J=0.1980$.

Table 6. American call option prices in the Merton (1976) jump diffusion context.

	Years to maturity				
	0.05	0.10	0.15	0.20	0.25
			$\alpha_J = 0.0000$ and $\sigma_J = 0.1988$		
v	0.4467	0.4467	0.4467	0.4467	0.4467
κ_3	0.0000	0.0000	0.0000	0.0000	0.0000
κ_4	5.3542	4.1771	3.7847	3.5885	3.4708
Benchmark	3.7600	5.3500	6.5600	7.5700	8.4500
Edgeworth	3.5613	5.2740	6.5315	7.5641	8.4554
Johnson	3.7003	5.3337	6.5655	7.5860	8.4705
Rel. diff. Edgeworth (%)	5.28	1.42	0.44	0.08	0.06
Rel. diff. Johnson (%)	1.59	0.30	0.08	0.21	0.24
	Years to maturity				
	0.10	0.20	0.30	0.40	0.50
			$\alpha_J = 0.0488$ and $\sigma_J = 0.1888$		
v	0.2793	0.2793	0.2793	0.2793	0.2793
κ_3	0.7740	0.5473	0.4469	0.3870	0.3461
κ_4	10.1069	6.5534	5.3690	4.7767	4.4214
Benchmark	2.9900	4.3500	5.3900	6.2600	7.0100
Edgeworth	NA	NA	NA	6.2896	7.0746
Johnson	3.1044	4.4988	5.5529	6.4411	7.2009
Rel. diff. Edgeworth (%)	NA	NA	NA	0.47	0.92
Rel. diff. Johnson (%)	3.83	3.42	3.02	2.89	2.72

'Benchmark' are the benchmark prices computed with the Kim (1990) integral equation reported by Chiarella and Ziogas (2005). 'Edgeworth' and 'Johnson' are option prices computed with the Edgeworth and Johnson binomial trees with $n = 150$. 'Rel. diff.' is the relative difference computed as $|(C_{\text{lattice}} - C_{\text{benchmark}})|/C_{\text{benchmark}}$ where C is a call option price. 'NA' indicates the cases for which the Edgeworth approach generates non-valid densities. Parameters: $r = 0.03$, $\delta = 0.05$, $\sigma = 0.4$, the strike price $s_0 = 100$ and $\lambda = 1$. In the second panel, prices with maturities of 0.1, 0.2 and 0.3 years have been computed with $n = 80$, 100 and 100 instead of 150 because of overflow problems.

Table 7. RMSE for call option prices in the Merton (1976) jump diffusion context.

	All sample	$\tau < 0.5$	$\tau \geq 0.5$	At-money	Out-money	$v < 0.35$	$v \geq 0.35$
				European call prices			
Edgeworth	0.0251	0.0339	0.0163	0.0100	0.0299	0.0314	0.0188
Johnson	0.0159	0.0182	0.0140	0.0123	0.0174	0.0171	0.0148
Ratio	1.5807	1.8612	1.1621	0.8082	1.7206	1.8332	1.2698
				American call prices			
Edgeworth	0.0218	0.0225	0.0217	0.0121	0.0333	0.0383	0.0116
Johnson	0.0143	0.0136	0.0145	0.0124	0.0176	0.0253	0.0076
Ratio	1.5174	1.6539	1.4988	0.9813	1.8927	1.5141	1.5297

This table presents the root-mean-squared errors (RMSE) computed for test pools of option contracts in the context of the Merton (1976) model. The RMSE are computed as $\sqrt{1/m\sum_i e_i^2}$ with $e_i = |C_{i,\text{lattice}} - C_{i,\text{benchmark}}|/C_{i,\text{benchmark}}$ where $C_{i,\text{benchmark}}$ is the benchmark obtained for the ith option, $C_{i,\text{lattice}}$ is the ith price obtained with the corresponding lattice with 100 time steps and m is the number of options in the test pool. 'Ratio' is the Edgeworth RMSE divided by the Johnson RMSE. 'At-money' is for the subsample with $0.9 \leq s_0/K \leq 1.1$ while 'Out-money' is for $s_0/K > 1.1$ or $s_0/K < 0.9$ where K denotes the strike price.

$r_p = r/365$. Different versions of the GARCH model have their specific dynamic for h_t. We examine here the NGARCH model (Engle and Ng 1993), with the following risk-neutral volatility dynamic:

$$h_{t+1} = \beta_0 + h_t[\beta_1 + \beta_2(\epsilon_t - \theta - \psi)^2],$$

where ψ is the risk premium and β_0, β_1, β_2 and θ are parameters governing the volatility under the NGARCH specification. These parameters are subject to different restrictions to ensure stationarity. For details, readers are referred to Engle and Ng (1993) where this model is developed.

Tables 8 and 9 present the results for European and American put option prices. This case uses parameter values different from those found in Duan *et al.* (2006) which examine the performance of the Edgeworth tree for approximating American options prices in the GARCH context. In that paper, the examples examined were limited by the set of skewness and kurtosis acceptable for the Edgeworth approach. In the example shown here, for many cases, the Edgeworth approach does not allow the computation of prices because the skewness and kurtosis pairs are in the invalid region. The Johnson approach gives a reasonable approximation with small relative differences which are in most cases around 1% of the benchmark prices. The large relative errors reported in these tables are mostly caused by small benchmark prices. The benchmark prices for the European options are

Table 8. European put option prices in the NGARCH context.

τ	30/365	90/365	150/365	200/365	270/365
v	0.2719	0.2723	0.2724	0.2725	0.2725
κ_3	−0.3470	−0.2496	−0.2013	−0.1770	−0.1542
κ_4	3.5547	3.1914	3.0659	3.0094	2.9578
			Strike price 45		
Monte Carlo	0.1722	0.7051	1.1449	1.4544	1.8132
Standard error	0.0014	0.0039	0.0057	0.0069	0.0082
Edgeworth	0.1828	0.7174	NA	NA	NA
Johnson	0.1770	0.7145	1.1588	1.4736	1.8444
Rel. diff. Edgeworth (%)	6.19	1.74	NA	NA	NA
Rel. diff. Johnson (%)	2.79	1.33	1.21	1.32	1.72
			Strike price 50		
Monte Carlo	1.4168	2.3459	2.9172	3.2800	3.6788
Standard error	0.0053	0.0086	0.0106	0.0118	0.0132
Edgeworth	1.4019	2.3497	NA	NA	NA
Johnson	1.4250	2.3631	2.9438	3.3119	3.7220
Rel. diff. Edgeworth (%)	1.05	0.16	NA	NA	NA
Rel. diff. Johnson (%)	0.58	0.73	0.91	0.97	1.17
			Strike price 55		
Monte Carlo	4.9691	5.4037	5.7889	6.0558	6.3587
Standard error	0.0097	0.0137	0.0158	0.0170	0.0184
Edgeworth	4.9705	5.4035	NA	NA	NA
Johnson	4.9687	5.3991	5.7949	6.0721	6.3902
Rel. diff. Edgeworth (%)	0.03	0.00	NA	NA	NA
Rel. diff. Johnson (%)	0.01	0.09	0.10	0.27	0.49

'Monte Carlo' are the benchmark prices for European put options computed with a Monte Carlo simulation using 500 000 sample paths and the Empirical Martingale approach described by Duan and Simonato (1998). 'Standard error' are the standard error estimates of the Monte Carlo prices. 'Edgeworth' and 'Johnson' are option prices computed with the Edgeworth and Johnson binomial trees with n equal to the maturity of the option in days. 'Rel. diff.' is the relative difference computed as $|(P_{\text{lattice}} - P_{\text{MonteCarlo}})|/P_{\text{MonteCarlo}}$ where P is a put option price. 'NA' indicates the cases for which the Edgeworth prices are not available because of non-valid densities. Parameters: $s_0 = 50$, $r = 0.05$, $\beta_0 = 0.000025$, $\beta_1 = 0.75$, $\beta_2 = 0.10$, $\psi = 0.3$, $\theta = 0.2$ and h_1 set to the stationary variance computed as $\beta_0/(1 - \beta_2(1 + (\theta + \psi)^2) - \beta_1)$.

Table 9. American put option prices in the NGARCH context.

τ	30/365	90/365	150/365	200/365	270/365
v	0.2719	0.2723	0.2724	0.2725	0.2725
κ_3	−0.3470	−0.2496	−0.2013	−0.1770	−0.1542
κ_4	3.5547	3.1914	3.0659	3.0094	2.9578
			Strike price 45		
Markov chain	0.1673	0.7067	1.1683	1.4975	1.8958
Edgeworth	0.1843	0.7291	NA	NA	NA
Johnson	0.1780	0.7249	1.1844	1.5149	1.9117
Rel. diff. Edgeworth (%)	10.11	3.17	NA	NA	NA
Rel. diff. Johnson (%)	6.39	2.58	1.38	1.16	0.84
			Strike price 50		
Markov Chain	1.4292	2.3928	3.0137	3.4194	3.8876
Edgeworth	1.4170	2.3988	NA	NA	NA
Johnson	1.4385	2.4106	3.0304	3.4334	3.8950
Rel. diff. Edgeworth (%)	0.86	0.25	NA	NA	NA
Rel. diff. Johnson (%)	0.65	0.74	0.56	0.41	0.19
			Strike price 55		
Markov chain	5.0609	5.5735	6.0462	6.3834	6.7897
Edgeworth	5.0641	5.5656	NA	NA	NA
Johnson	5.0666	5.5688	6.0365	6.3705	6.7710
Rel. diff. Edgeworth (%)	0.06	0.14	NA	NA	NA
Rel. diff. Johnson (%)	0.11	0.08	0.16	0.20	0.28

'Markov chain' are the benchmark prices for American put options computed with a Markov chain approach described by Duan and Simonato (2001) with 301 states for the stock price and 101 states for the variance. 'Edgeworth' and 'Johnson' are option prices computed with the Edgeworth and Johnson binomial trees with n equal to the maturity of the option in days. 'Rel. diff.' is the relative difference computed as $|(P_{\text{lattice}} - P_{\text{MarkovChain}})|/P_{\text{MarkovChain}}$ where P is a put option price. 'NA' indicates the cases for which the Edgeworth prices are not available because of non-valid densities. Parameters: $s_0 = 50$, $r = 0.05$, $\beta_0 = 0.000025$, $\beta_1 = 0.75$, $\beta_2 = 0.10$, $\psi = 0.3$, $\theta = 0.2$ and h_1 set to the stationary variance computed as $\beta_0/(1 - \beta_2(1 + (\theta + \psi)^2) - \beta_1)$.

computed with a precise Monte Carlo simulation using 500 000 paths with an empirical martingale variance reduction approach suggested by Duan and Simonato (1998). For American-style options, the Markov chain approach described by Duan and Simonato (2001) is used with 301 states for the stock price and 101 states for the variance. In these tables, for the 12 cases for which the Johnson and Edgeworth both provide prices, the Johnson approach yields eight cases for which the computed prices are as good or better approximations.

6. Conclusion

We have introduced here the Johnson binomial tree approach for pricing options in the context of skewed and leptokurtic asset return distributions. This approach modifies the Rubinstein Edgeworth tree approach with the use of the Johnson system of distributions as an approximation for obtaining discrete densities. Unlike the Edgeworth approach, which generates valid distributions for a limited set of skewness and kurtosis pairs, the Johnson approach can yield valid distributions for all combinations of skewness and kurtosis. Examples from Merton's (1976) jump diffusion framework and Duan's (1995) GARCH framework show that the Edgeworth approach cannot generate valid distributions in many cases which are handled appropriately by the Johnson approach. In many cases, the Johnson approach is also more precise than the Edgeworth approach.

For the numerical examples examined here, as is the case with the Edgeworth approach, the Johnson binomial tree contains an approximation error that will not vanish as the number of time step grows. It is used here to approximate option prices in frameworks with two sources of uncertainty while the Johnson tree is clearly a univariate lattice. However, for situations where computing time is an issue, the Johnson approach can yield American price approximations that will, in general, be accurate enough for many practical applications.

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Appendix A: The Edgeworth approximation

We describe here Rubinstein's (1998) approach to the construction of the skewed and leptokurtic standardized returns $x_{n,j}$. The construction starts by considering an n-step binomial distribution with $n+1$ possible values. Denote these values by $z_{n,j} = [(2j) - n]/\sqrt{n}$ for $j = 0$ to n and associated probabilities $\pi_{n,j} = [n!/j!(n-j)!](1/2)^n$. Given pre-specified skewness and kurtosis values, the binomial distribution is modified by the Edgeworth expansion up to the fourth moment to yield

$$f_{n,j} = \left[1 + \frac{1}{6}\kappa_3(z_{n,j}^3 - 3z_{n,j}) + \frac{1}{24}(\kappa_4 - 3)(z_{n,j}^4 - 6z_{n,j}^2 + 3) + \frac{1}{72}\kappa_3^2(z_{n,j}^5 - 10z_{n,j}^3 + 15z_{n,j})\right]\pi_{n,j},$$

where $\kappa_3 = E[\varrho_\tau^3]$ is the skewness and $\kappa_4 = E[\varrho_\tau^4]$ the kurtosis of the cumulative return for the option's maturity under the risk-neutral measure with $\varrho_\tau = (\rho_\tau - E[\rho_\tau])/v\sqrt{\tau}$, v the annualized standard deviation of the return, $\rho_\tau = \ln(s_\tau/s_0)$ the cumulative return, and τ the maturity of the option. Finally, scaling is required to ensure that the probabilities sum up to one because the Edgeworth expansion only approximates a probability distribution. The scaling operation is

$$P_{n,j} = f_{n,j} \Big/ \sum_i f_{n,i}.$$

The variable $z_{n,j}$ based on the probability $P_{n,j}$ is no longer a binomial random variable and can be standardized to have mean 0 and variance 1 with

$$x_{n,j} = (z_{n,j} - M)/V,$$

where $M = \sum_j P_{n,j} z_{n,j}$ and $V^2 = \sum_j P_{n,j}(z_{n,j} - M)^2$. The variable $x_{n,j}$ can then be used in equation (1) in conjunction with the procedure in appendix B to create an implied recombining binomial tree for pricing options.

Appendix B: Edgeworth implied binomial tree construction

Using the terminal stock prices and probabilities obtained from equation (1) with the procedure described in appendix A, the recombining Edgeworth implied tree can be constructed recursively following Rubinstein's procedure which is summarized here. Denote a branch of the recombining tree as

$$\begin{array}{ccc} & & (S_{i,j+1}, p_{i,j+1}) \\ & \nearrow & \\ (S_{i-1,j}, p_{i-1,j}) & & \\ & \searrow & \\ & & (S_{i,j}, p_{i,j}) \end{array}$$

where $p_{i,j}$ is the risk-neutral probability of a single path to node j. For the last time step, this risk-neutral probability can be computed from

$$p_{n,j} = \frac{P_{n,j}}{n!/j!(n-j)!},$$

since we require the binomial tree to be recombining. The values for $S_{i-1,j}$ and $p_{i-1,j}$ can be deduced recursively from $(S_{i,j+1}, p_{i,j+1})$ and $(S_{i,j}, p_{i,j})$ using the following procedure.

- Compute the risk-neutral probability of an up jump with $q_{i,j+1} = p_{i,j+1}/p_{i-1,j}$, where $p_{i-1,j}$ is computed with $p_{i,j} + p_{i,j+1}$. This last relation can be understood by noting that computing the terminal probabilities of a single path to a node can be written as $p_{i,j+1} = p_{i-1,j} \times q_{i,j+1}$ and $p_{i,j} = p_{i-1,j} \times (1 - q_{i,j+1})$ where $q_{i,j+1}$ is the risk-neutral probability of jumping up at this branch. Note that $q_{i,j+1}$ and $(1 - q_{i,j})$ add up to one and yields the desired result.
- Under the risk-neutral probabilities, today's stock price is the expected value of the stock price next period discounted at the risk-free rate. We compute the stock price at the prior node using

$$S_{i-1,j} = [q_{i,j+1} \cdot S_{i,j+1} + (1 - q_{i,j+1}) \cdot S_{i,j}]\mathrm{e}^{-(r-\delta)\times(\tau/n)},$$

where δ is the annual continuously compounded dividend yield, which is assumed to be reinvested in the stock.

Appendix C: An example of the Johnson binomial tree construction

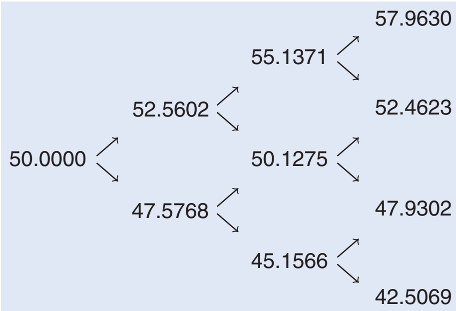
This appendix presents a detailed example of the Johnson binomial tree construction with $n = 3$ and the Merton (1976) jump-diffusion model presented in section 4 with the following parameter values: $s_0 = 50$, $\tau = 30/365$, $r = 0.05$, $\delta = 0$, $\sigma = 0.2$, $\lambda = 5$, $\alpha_J = -0.02$, and $\sigma_J = 0.1$. The three steps described in section 3 are as follows.

- **Step 1:** The goal of this step is to determine the family of the Johnson system of distributions and its associated parameter values. To do this, one first computes the third and fourth moments using equations (6) and (7) to obtain $\kappa_3 = -0.3800$ and $\kappa_4 = 5.3298$. Using the Hill *et al.* (1976) algorithm with $0, 1, -0.3800$ and 5.3298 as input values for the first four moments, the approximating distribution is determined to be from the unbounded family with $\widehat{a} = 0.2934$, $\widehat{b} = 1.7862$, $\widehat{c} = 0.2878$ and $\widehat{d} = 1.4914$.
- **Step 2:** This step computes $x_{n,j}$ for $j = 0$–3 with which the stock price tree will be built in the next step. These $x_{n,j}$ can be computed using the data from table C1. These numbers are computed using $z_{n,j} = [(2j) - n]/\sqrt{n}$, $P_{n,j} = [n!/(j!(n-j)!)]$ $(1/2)^n$, $\widetilde{x}_{n,j} = \widehat{c} + \widehat{d} \times g^{-1}[(z_{n,j} - \widehat{a})/\widehat{b}]$, and $g^{-1}(u) = (\mathrm{e}^u - \mathrm{e}^{-u})/2$, the unbounded transformation. With $M = \sum_j P_{n,j}\widetilde{x}_{n,j} = 0.00076$ and $V^2 = \sum_j P_{n,j}(\widetilde{x}_{n,j} - M)^2 = 0.9569^2$, we obtain $x_{n,0} = -1.8711$, $x_{n,1} = -0.4902$ and $x_{n,2} = 0.5488$ and $x_{n,3} = 1.6954$ with $x_{n,j} = (\widetilde{x}_{n,j} - M)/V$.
- **Step 3:** This step computes the stock price tree, starting from the end, and obtains recursively the stock prices for the remaining periods. To compute the prices at the end of the tree, one first computes the adjustment factor ensuring that the expected return is equal to the risk-free rate using equation (2) to obtain $\mu = 0.0043$. Note that, for the case of the Merton (1976) model, this adjustment factor is found with a volatility $v = 0.3033$ computed using equation (5).

Table C1. Data used to compute $x_{n,j}$.

j	$z_{n,j}$	$P_{n,j}$	$\widetilde{x}_{n,j}$
3	1.7321	0.125	1.6232
2	0.5774	0.375	0.5259
1	−0.5774	0.375	−0.4684
0	−1.7321	0.125	−1.7898

Equation (1) can then be used to compute the four prices appearing at the end of the following tree:



The remaining prices are obtained recursively, starting from the end of the tree, using equation (4).

Appendix D: Hill *et al.* (1976) algorithm

We provide here a brief sketch of the Hill *et al.* (1976) algorithm which determines the parameter values and family of the Johnson system given an expected value of zero, a standard deviation of one and κ_3 and κ_4, the third and fourth moments of the target distribution.

Determining the family of the Johnson system is done by first computing

$$\gamma = w^4 + 2w^3 + 3w^2 - 3,$$

with

$$w = \frac{1}{2}\left(8 + 4\kappa_3^2 + 4\sqrt{4\kappa_3^2 + \kappa_3^4}\right)^{1/3} + \frac{1}{2}\left(8 + 4\kappa_3^2 + 4\sqrt{4\kappa_3^2 + \kappa_3^4}\right)^{-1/3} - 1.$$

If γ is close to κ_4, the lognormal family should be used. If γ is smaller than κ_4, the unbounded family is appropriate while the bounded case is for γ greater than κ_4.

The procedure used to determine the parameter values then depends on the identified family. For the lognormal case, the parameters are obtained from

$$\widehat{b} = (\ln w)^{-1/2}, \qquad \widehat{a} = \frac{1}{2}\widehat{b}\ln(w(w-1)),$$

$$\widehat{d} = \mathrm{sign}(\kappa_3), \qquad \widehat{c} = -\exp\left(\frac{\frac{1}{2}\widehat{b} - \widehat{a}}{\widehat{b}}\right).$$

For the unbounded family, the values of $\widehat{a}$ and $\widehat{b}$ are first determined and then used to find $\widehat{c}$ and $\widehat{d}$. If κ_3 is close to zero, $\widehat{a}$ and $\widehat{b}$ are given by $\widehat{a} = 0$ and $\widehat{b} = (\ln \zeta)^{-1/2}$ with $\zeta = \sqrt{(2\kappa_4 - 2)^{1/2} - 1}$. If κ_3 is different from zero, the iterative approach described by Hill *et al.* (1976) is required to find ζ, $\widehat{a}$ and $\widehat{b}$. Given ζ, $\widehat{a}$ and $\widehat{b}$ the values of $\widehat{c}$ and $\widehat{d}$ are then found with $\widehat{d} = (0.5(\zeta - 1)(\zeta\cosh(2\widehat{a}/\widehat{b}) + 1))^{-1/2}$ and $\widehat{c} = \widehat{d}\sqrt{\zeta}\sinh(\widehat{a}/\widehat{b})$.

For the bounded family, the iterative approach described by Hill *et al.* (1976) is required.

Appendix E: European option test pool simulation

The following distributions are used for simulating the parameter values of the call options contracts in the European test pool. Each parameter value is drawn independently of the others: τ is uniformly distributed between 0.05 and 1.0 years; K, the strike price, is uniformly distributed between 70 and 130; r is uniformly distributed between 0 and 0.1; σ is uniformly distributed between 0.1 and 0.4; λ is uniformly distributed between 0 and 5; α_J is uniformly distributed between -0.05 and 0.05; σ_J is uniformly distributed between 0.1 and 0.2. The values of s_0 and δ are fixed at 100 and zero. The cases for which the theoretical option prices are smaller than 0.5 are removed to avoid large relative errors caused by small dividers.