The Performance of Johnson Distributions for Computing Value at Risk and Expected Shortfall

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Abstract

The Gram-Charlier and Cornish-Fisher expansions are tools often used to compute value at risk (VaR) in the context of skewed and leptokurtic return distributions. These approximations use the first four moments of the unknown target distribution to compute approximate distribution and quantile functions. A drawback of these approaches is the limited set of skewness and kurtosis pairs for which valid approximations are possible. We examine here an alternative to these methods with the Johnson [1949] system of distributions which also uses the first four moments as main inputs but is capable of accommodating all possible skewness and kurtosis pairs. Formulas for the expected shortfall are derived. The performance of the Cornish-Fisher, Gram-Charlier and Johnson approaches for computing value at risk and expected shortfall are compared and documented. The results reveal that Johnson distributions yield smaller approximation errors than the Gram-Charlier and Cornish-Fisher approaches when used with exact or estimated moments.

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Value at risk (VaR hereafter) and expected shortfall (ES hereafter) are risk measures that are now routinely computed by portfolio managers, firms and financial institutions. These measures evaluate the potential of large losses on portfolios and guide managers in their decisions about the control of acceptable risk level or the required capital for potential losses.

When the distribution of the portfolio return is known, computing such quantities is straightforward. In most applications, the distribution is unknown, changes as the assets in the portfolio are varied, and can often be skewed with fat tails. The Gram-Charlier and Cornish-Fisher approximations are tools that can deal with such characteristics in many circumstances. These approaches only require estimating the first four moments of the portfolio return to compute the VaR or ES analytically. Hence, with a time series of portfolio returns, which can be constructed from the time series of the individual asset prices, estimates of the moments can be obtained and used as inputs in these procedures. Such moments can be estimated unconditionally or conditionally in the context of dynamic time series models such as those from the GARCH family. In such settings, analytical formulas are available for the moments of the returns at an arbitrary horizon. See for example Duan, Gauthier and Simonato [1999] and Duan, Gauthier, Sasseville and Simonato [2006] for the GARCH, NGARCH, E-GARCH and GJR-GARCH. See also, Mazzoni [2010] for the Heston-Nandi [2000] GARCH.

An important drawback of Gram-Charlier and Cornish-Fisher approaches is that they fail to generate valid quantile or density functions for some skewness and kurtosis pairs. They may generate densities with negative values or non-monotone quantile functions. Negative density values are clearly undesirable and can bring an unaware user to multiple solutions when computing VaR and ES values for a given probability level. Non-monotone quantile functions are also undesirable since they could, for example, measure a larger potential loss for a 95% VaR than the one of a 99% VaR. We examine here the performance of an alternative to the Gram-Charlier and Cornish-Fisher methods with Johnson distributions (Johnson [1949]). As with the Gram-Charlier and Cornish-Fisher approximations, Johnson distributions only require estimates of the first four moments to compute an approximate density of the portfolio return. Unlike these approaches, the Johnson system can generate genuine distributions for all possible skewness and kurtosis combinations. In the VaR literature, Johnson distributions are suggested in Zangari [1996], Mina and Ulmer [1999] and in RiskMetrics Technical Document [1996]. However, none of these studies has documented the performance of the Johnson system relative to close competing approaches, such as the Gram-Charlier and Cornish-Fisher approximations. They also do not consider the case of ES computation. Choi [2001] is a notable exception and examines empirically a GARCH model with Johnson innovations to compute VaR and ES. His study considers a limited set of Johnson distributions and leaves aside a wide range of Johnson distributions that are relevant for VaR and ES computation.

The contribution of this paper to the VaR literature is as follows. First, with the first four moments of an unknown target distribution used as inputs, the performance of the Johnson approach for the whole range of possible Johnson distributions to compute approximate VaR is documented. A detailed comparison with the Gram-Charlier and Cornish-Fisher is also done with exact benchmark values. A second contribution examines the performance of the ES computation with the Gram-Charlier, Cornish-Fisher, and Johnson approaches. The formulas are developed for each case and their relative performance with respect to an exact benchmark is documented.

To compare the performance of these methods, we use the Merton [1976] jump-diffusion framework with lognormal jumps to generate test pools of portfolio return distributions. Such a framework can generate a wide variety of skewed and leptokurtic distributions. Exact benchmark values can be obtained analytically and exact moments of the returns can be computed for any horizon. Time series of returns can also be easily simulated to assess the impact of estimated moments on the approximation quality.

The next section describes the Cornish-Fisher, Gram-Charlier and Johnson approaches for computing VaR and ES. This section also documents the locus of skewness and kurtosis generating valid approximations.

VaR and expected shortfall computations

The VaR of a portfolio, expressed as a return, is the answer to the following question: what return level is such that a lower return will occur only in a proportion p of the cases over the next time interval of h year i.e.

$$\Pr(r_h < VaR) = p$$

where $r_h = \ln(V_{t+h}/V_t)$ is the continuously compounded portfolio return over the h year horizon and V_t the portfolio value at time t. Note that this VaR is typically a negative number. In many papers, the sign is switched so that the VaR is a positive number. This convention will not be followed here. As shown in Appendix A, given a known distribution function for the standardized return, denoted as $\phi(\bullet)$, and known values for p and h, the VaR can be written as

$$VaR = \alpha_h + \sigma_h \times \phi^{-1}(p) \tag{1}$$

where α_h and σ_h are the expected value and standard deviation of the return over the horizon h, and $\phi^{-1}(p)$ is the quantile function of the standardized returns. Hence, a VaR of -20% computed with a probability $p = 5\%$ and a 5 days horizon ($h = 5/250$) indicates that there is a 5% probability that the 5 days portfolio log return will be lower than -20%.

The expected shortfall is a risk measure which considers all possible return values below the VaR level. It has been proposed as an alternative to the VaR which fails some coherence properties (see Artzner et al. [1999]). It is defined as the expected return, conditional on it being worse than the VaR i.e. $E[r_h \mid r_h < VaR]$. As shown in Appendix A, this quantity can be written as:

$$ES = \alpha_h + \sigma_h \frac{EP[z \mid z < k]}{p} \tag{2}$$

where z is the standardized return defined as

$$z = (r_h - \alpha_h)/\sigma_h$$

with $k = (VaR - \alpha_h)/\sigma_h$ and $EP[z \mid z < k] = \int_{-\infty}^{k} z \times f(z)\,\mathrm{d}z$ is a partial conditional expected value for the standardized return. Hence, an ES of -25% computed with a probability $p = 5\%$, a 5 days horizon and a VaR of -20% indicates that we can expect the 5 days portfolio log return to be around -25% if it falls in the tail of the distribution which is below the -20% VaR.

As discussed above, financial portfolios can have returns with skewed and leptokurtic distributions. The distribution is also, in most cases, unknown and changing as the horizon is varied. In this study, we examine the performance of the Gram-Charlier, Cornish-Fisher, and Johnson approaches which all use the first four moments of the returns to provide approximate quantities related with the unknown distribution of a portfolio return. For the VaR formula (1), this approximate quantity will be for $\phi^{-1}(p)$, the quantile function of the return distribution. For the ES formula (2), the approximate quantity is for $EP[z \mid z < k]$, the partial conditional expectation of the standardized return. This partial expectation can be computed with the Gram-Charlier, Cornish-Fisher, and Johnson approximations of the true density function. The next subsections examine how to obtain these approximate values for each of the three considered cases.

Gram-Charlier

A first approach to VaR and ES computation with equations (1) and (2) is the use of the Gram-Charlier approximate density, given in Appendix B.

The VaR formula in equation (1) requires the quantile function for the Gram-Charlier approximate density. This quantity can be obtained with the distribution function corresponding to the Gram-Charlier approximate density. This distribution function can then be inverted numerically to compute the VaR. Appendix B derives the expression for the Gram-Charlier approximate distribution function which is given by:

$$\phi_{GC}\left(k;\kappa_3,\kappa_4\right)=\phi_N-\frac{\kappa_3}{6}\left[f_N\times\left(k^2-1\right)\right]-\frac{\left(\kappa_4-3\right)}{24}\left[f_N\times k\left(k^2-3\right)\right] \tag{3}$$

where ϕ_N and f_N are the standard Normal distribution and density functions evaluated at k, $\kappa_3=E\left[z^3\right]$ is the skewness coefficient and $\kappa_4=E\left[z^4\right]$ is the kurtosis coefficient. The Gram-Charlier VaR is then computed with:

$$VaR_{GC}=\alpha_h+\sigma_h\times\phi_{GC}^{-1}\left(p;\kappa_3,\kappa_4\right) \tag{4}$$

where $\phi_{GC}^{-1}\left(\cdot\right)$ can be computed by finding, with a numerical search, the value $k_{GC}=k$ such that $\phi_{GC}\left(k;\kappa_3,\kappa_4\right)=p$. Such a numerical search is fast. For example, with a bisection algorithm, the search takes about 0.001 second of execution time.

In this context, an approximating formula for the expected shortfall can be developed in closed form. The partial conditional expected value, $EP\left[z\mid z<k\right]=\int_{-\infty}^{k}z\times f\left(z\right)\mathrm{d}z$, required to compute the expected shortfall formula (2), is obtained by integrating the Gram-Charlier approximate density. As shown in Appendix B, this partial conditional expected value can be computed with:

$$EP_{GC}\left[z\mid z<k_{GC}\right]=f_N\times\left[-1-\frac{\kappa_3}{6}k^3+\frac{\left(\kappa_4-3\right)}{24}\left(-k^4+2k^2+1\right)\right]. \tag{5}$$

The ES formula (2) for the Gram-Charlier case is then given by:

$$ES_{GC}=\alpha_h+\sigma_h\frac{EP_{GC}\left[z\mid z<k_{GC}\right]}{p} \tag{6}$$

with $k_{GC}=\left(VaR_{GC}-\alpha_h\right)/\sigma_h$.

As discussed in the introduction, the Gram-Charlier approximate density function does not always show desirable properties. The top graph in Exhibit 1 plots the approximate density generated for $\kappa_3=0$ and $\kappa_4=10$. The resulting density clearly shows negative values. Besides being theoretically unappealing, a consequence of negative density values for VaR computation is the possibility of multiple solutions. This can be seen in Exhibit 2, which plots the approximate distribution obtained by integrating the previous approximate Gram-Charlier distribution. In this graph, the 5% VaR are at the intersections between the dotted line (the probability level) and the continuous line (the distribution function). It is clear that the VaR is not unique : three distinct values intersecting the horizontal 5% probability line can be found. Hence, three distinct expected shortfall values can also be computed from these VaR.

The top graph in Exhibit 3 shows the set of skewness and kurtosis pairs generating acceptable Gram-Charlier expansions. For this set, the approximating densities have non-negative values as required. Outside of this set, the Gram-Charlier expansion provides approximate densities with negative values. These failures can occur not only for large kurtosis values but also for moderate kurtosis below four with skewness around plus or minus one.

Cornish-Fisher

A second approach to VaR and ES computation with equations (1) and (2) is provided by the Cornish-Fisher approximation (see for example, Zangari [1996]). This approach provides a closed form approximation for the unknown quantile function required for the VaR computation, given values for the skewness and kurtosis coefficients. This approximation is given by:

$$\phi_{CF}^{-1}(p;\kappa_3,\kappa_4) = \phi_N^{-1} + \frac{\kappa_3}{6}\left[\left(\phi_N^{-1}\right)^2 - 1\right] + \frac{\kappa_4 - 3}{24}\left[\left(\phi_N^{-1}\right)^3 - 3\phi_N^{-1}\right] - \frac{\kappa_3^2}{36}\left[2\left(\phi_N^{-1}\right)^3 - 5\phi_N^{-1}\right] \quad (7)$$

where ϕ_N^{-1} is the quantile function of a standard normal random variable evaluated at p. Using this quantile function, the Cornish-Fisher VaR is then written as:

$$VaR_{CF} = \alpha_h + \sigma_h \times \phi_{CF}^{-1}(p;\kappa_3,\kappa_4).$$

For the ES computation, there is no closed form formula. It is however possible to compute this quantity numerically. This approach requires to invert numerically the Cornish-Fisher approximation with a bisection procedure to get a probability, given a quantile value. Repeating this operations for an evenly spaced grid of quantile values, a series of points on the distribution function corresponding to the Cornish-Fisher quantile function are obtained. With this set of points, numerical differentiation and integration procedures can be used to obtain the expected shortfall. Appendix F describes the details of this procedure. It should be noted that these computations involve a double layer of numerical work and require a non-negligible execution time. For example, computing an expected shortfall value takes around ten seconds of computer time when a 10,001 point grid is used. The expected shortfall values computed with this method will be labeled as ES_{CF}.

As with the Gram-Charlier approximate density, the approximate quantile functions generated by the Cornish-Fisher approach do not always have desirable properties. It does not always generate a monotone function for all skewness and kurtosis pairs. An example of such a problem is given in the top graph of Exhibit 4 which shows the distribution function (the inverse of the quantile function) obtained with $\kappa_3 = 0.85$ and $\kappa_4 = 3.5$. The function is clearly non monotone and may generate incoherent VaR computations for different required probabilities. The top graph in Exhibit 5 shows the set of skewness and kurtosis pairs generating acceptable Cornish-Fisher approximations. For this set, the approximating quantile functions are monotone. Outside of this set, the Cornish-Fisher expansion provides non-monotone quantiles in either tail of the distributions. These failures can occur not only for large kurtosis values but also for moderate kurtosis below four with skewness around plus or minus one.

Johnson distributions

A third approach to VaR and ES computation with equations (1) and (2) is the Johnson system of densities. For a continuous random variable z with an unknown target distribution which needs to be approximated, Johnson [1949] proposed a set of "normalizing" translations. These translations transform the continuous random variable z into a standard normal variable y and have the general form:

$$y = a + b \times g\left(\frac{z-c}{d}\right)$$

where a and b are shape parameters, c is a location parameter, d is a scale parameter and $g(\cdot)$ is a function defining the four families of distributions of the Johnson system:

$$g(u) = \begin{cases} \ln(u) & \text{for the lognormal family,} \\ \ln\left(u + \sqrt{u^2+1}\right) & \text{for the unbounded family,} \\ \ln\left(u/(1-u)\right) & \text{for the bounded family,} \\ u & \text{for the normal family.} \end{cases}$$

As discussed in Johnson [1949], the above system has the flexibility to match any feasible set of values for the mean, variance, skewness, and kurtosis. With this system, the skewness and kurtosis also uniquely identify the appropriate form for the $g(\cdot)$ function. As a result, the process of using the

Johnson system to approximate an unknown distribution is reduced to the problem of finding the values of a, b, c and d that will match the moments of the unknown target distribution with those of the appropriate family from the Johnson system. Hill et al. [1976] provide an algorithm which finds, given the first four moments of a target distribution, the appropriate family (the form of the $g(\cdot)$ function) and values of the parameters required to approximate the unknown distribution. This algorithm is efficient and typically takes a fraction of a second to compute the required quantities. This algorithm does not guarantee convergence. However, failure to converge occurs in very few cases. An alternative robust approach which uses the estimated quantiles of a time series of return is also available in Wheeler [1980] while Tuenter [2001] develops an iterative method based on a root-finding procedure to obtain the parameters for the unbounded family.

With the parameters determined as above, the Johnson random variable can be expressed as the inverse of the above normalizing translation:

$$z = c + d \times g^{-1}\left(\frac{y-a}{b}\right) \tag{8}$$

where

$$g^{-1}(u) = \begin{cases} e^u & \text{for the lognormal family,} \\ \left(e^u - e^{-u}\right)/2 & \text{for the unbounded family,} \\ 1/\left(1+e^{-u}\right) & \text{for the bounded family,} \\ u & \text{for the normal family.} \end{cases}$$

As in the Gram-Charlier and Cornish-Fisher cases, the quantities required to implement equations (1) and (2) can be obtained from the skewness and kurtosis of the standardized return distribution. Hence, using a mean of zero, a standard deviation of one and the target skewness and kurtosis, one first finds the a, b, c, and d parameters using the Hill et al. [1976] algorithm. The distributional quantities related to equations (1) and (2) can then be obtained as follows. For computing the VaR, the required quantile function is given by:

$$\phi_J^{-1}\left(p; a, b, c, d\right) = c + d \times g^{-1}\left(\frac{\phi_N^{-1} - a}{b}\right),$$

where ϕ_N^{-1} is the quantile function of a standard normal random variable evaluated at p. The Johnson VaR is then defined as:

$$VaR_J = \alpha_h + \sigma_h \times \phi_J^{-1}\left(p; a, b, c, d\right).$$

For the expected shortfall, the required expressions for the partial conditional expectation $\int_{-\infty}^{k} z \times f(z)\,\mathrm{d}z$ can also be found by rewriting the Johnson random variable into a unit random normal and integrating, with an appropriate change of variable, with respect to the Gaussian density:

$$EP_J\left[z \mid z < k_J\right] = \int_{-\infty}^{K} \left(c + d \cdot g^{-1}\left(\frac{y-a}{b}\right)\right) f_N\left(y\right) \times \mathrm{d}y \tag{9}$$

where $K = a + b \cdot g\left(\frac{k_J - c}{d}\right)$ and $k_J = \left(VaR_J - \alpha_h\right)/\sigma_h$. Substituting the appropriate $g^{-1}\left(\cdot\right)$ function and the expression for the standard normal density, solutions for this partial expected value can be found by completing the square in the terms of the exponential function. Analytical solutions can then be derived for the lognormal, unbounded and normal cases. Appendix C derives the following expressions for $EP_J\left[z \mid z < k_J\right]$:

- lognormal family:
$$c \times \phi_N(K) + de^{\frac{1}{2b^2} - \frac{a}{b}} \times \phi_N\left(K - \frac{1}{b}\right),$$

- unbounded family:
$$c \times \phi_N\left(K\right) + \frac{d}{2}e^{-\frac{a}{b} + \frac{1}{2b^2}} \times \phi_N\left(K - \frac{1}{b}\right) - \frac{d}{2}e^{\frac{a}{b} + \frac{1}{2b^2}} \times \phi_N\left(K + \frac{1}{b}\right),$$

- normal family:
$$\phi_N\left(K\right) \times \left[c - \frac{da}{b}\right] - \frac{d}{b} \times f_N(K)$$

where $\phi_N\left(\cdot\right)$ and $f_N\left(\cdot\right)$ are the standard Normal distribution and density functions. For the bounded case, a numerical integration of equation (9) is required since no analytical expressions can be found. This numerical integration typically takes about 0.001 second to compute on a standard laptop computer. The Johnson ES formula is

$$ES_J = \alpha_h + \sigma_h \frac{EP_J\left[z \mid z < k_J\right]}{p}.$$

Unlike the methods examined in the previous section, the Johnson system generates genuine distributions to approximate the target distribution. An example of the approximate density function generated by the Johnson system is examined in the bottom graph of Exhibit 1. Unlike the Gram-Charlier case illustrated in the top graph, the resulting Johnson approximating density clearly shows non-negative probabilities for all values. The bottom graph of Exhibit 3 shows the set of skewness and kurtosis pairs generating acceptable approximations. Unlike the Gram-Charlier case illustrated in the top graph, the resulting Johnson approximating density has positive probabilities for all combinations of skewness and kurtosis.

Exhibit 4 shows the distribution function obtained for the tail of the distribution of a zero mean and unit variance with $\kappa_3 = 0.85$ and $\kappa_4 = 3.5$. As shown in this graph, the distribution function is clearly monotone as required, unlike the Cornish-Fisher case in the upper graph. The bottom graph in Exhibit 5 shows that the entire set of skewness and kurtosis pairs examined generated valid quantile approximations.

A simulation framework

We describe here the framework that will be used to assess the performance of the three approximations considered for computing VaR and ES. For this purpose, we use the Merton [1976] jump-diffusion framework. Such a framework is interesting since it can generate a wide range of skewed and leptokurtic return distributions. Furthermore, the moments of the returns, which are the main inputs for the approximate formulas examined here, can be computed with closed form formulas. Return series can also be quickly and efficiently simulated for an arbitrary horizon. This will be convenient to assess the effect of estimated inputs in the formulas. Finally, exact VaR and ES can be computed with this data generating process.

The jump diffusion framework generates a return process moving continuously but which can also jump discretely according to a Poisson process. This process is written as:

$$\frac{\mathrm{d}V_t}{V_t} = (\alpha - \lambda l)\, dt + \sigma \mathrm{d}W + \mathrm{d}Q$$

where α is the annual expected return on the portfolio, W is a standard Brownian motion, σ the annual volatility parameter associated to the Brownian motion, λ is the annual arrival rate of jumps and $\mathrm{d}Q$ is the jump portion of the process. Over an interval $\mathrm{d}t$, the stock price can jump with probability $\lambda \mathrm{d}t$. The jump portion $\mathrm{d}Q$ takes a value of 0 if there is no jump and $Y - 1$ if there is jump with $l = E(Y-1)$. It is assumed that the jump magnitude, Y, is lognormally distributed with $\ln Y \sim N\left(\alpha_J,\ \sigma_J^2\right)$ which yields $l = e^{\left(\alpha_J + \frac{1}{2}\sigma_J^2\right)} - 1$. Depending on the values of the parameters $\alpha, \sigma, \lambda, \alpha_J$ and σ_J, this process can generate distributions with different skewness and kurtosis. These skewness and kurtosis change as the horizon over which the returns are measured is changing and can be computed, given values for $\alpha, \sigma, \lambda, \alpha_J, \sigma_J$ and h, using the formulas provided in Appendix D.

For this process, the distribution function of the h year return can be computed analytically and is given in Appendix D. It is also possible to compute exact benchmark quantities for the

VaR and ES. More formally, the exact benchmark VaR for the jump case can be computed with a bisection numerical search over the values of k generating a target probability p i.e. :

$$VaR_B = k \text{ such that } p = \phi_{jump}(k)$$

where $\phi_{jump}(\cdot)$ is the return distribution for the jump process described in Appendix D. The expected shortfall can be computed with:

$$ES_B = \frac{\sum_{n=0}^{\infty} \frac{e^{-\lambda h}(\lambda h)^n}{n!} \left(\mu_n \times \phi_N\left(k_{B,n}\right) - \sigma_n \times f_N\right)}{p}$$

where f_N is the standard Normal density function evaluated at $k_{B,n} = \frac{VaR_B - \mu_n}{\sigma_n}$ where $\mu_n = \left(\widetilde{\alpha} - \frac{1}{2}\sigma^2\right) h + n\alpha_J$ and $\sigma_n^2 = \sigma^2 h + n\sigma_J^2$ with $\widetilde{\alpha} = \alpha - \lambda l$.

Besides the above benchmark values, in the tables presented in the next sections, two different VaR and ES will be computed for the Cornish-Fisher, Gram-Charlier and Johnson cases. The first set of reported VaR and ES will be computed with the true moments and are obtained with the following steps:

1. With a set of parameter values for h, α, σ, λ, α_J and σ_J, use the moments formulas of Appendix D to compute α_h, σ_h^2, κ_3, κ_4, the first four moments for the jump diffusion return process.

2. Compute the VaR and ES for the Cornish-Fisher, Gram-Charlier and Johnson cases with the moments obtained in step 1 and the procedures described earlier.

Because the VaR and ES computed with this procedure use the true moments, the difference between the benchmark values and those provided by the Cornish-Fisher, Gram-Charlier or Johnson cases will be caused by the approximation quality only. In a more realistic setup, the moments of the portfolio are unknown and have to be estimated. The second set of VaR and ES numbers that will be examined will include this feature and will be computed with the following procedure:

1. With a set of parameter values for $h, \alpha, \sigma, \lambda, \alpha_J$ and σ_J, simulate a time-series $\{r_h,\ r_{2h},\ \ldots r_{nh}\}$ of n returns with the procedure described in Appendix D.

2. Compute $\hat{\alpha}_h, \hat{\sigma}_h^2, \hat{\kappa}_3, \hat{\kappa}_4$, the usual estimates of these quantities with the time series of returns simulated in step 1.

3. Compute the VaR and ES for the Cornish-Fisher, Gram-Charlier and Johnson cases with the estimates obtained in step 2 and the procedures described earlier.

4. Repeat N times steps 1 to 3 and compute the average of the VaR and ES for the Cornish-Fisher, Gram-Charlier and Johnson cases.

These VaR and ES will now contain an additional source of error caused by the estimation of the moments and can be compared with the numbers without estimation errors to assess the impact of this additional feature.

Results

This section presents numerical results regarding the approximation quality of the three methods. The focus is first put on the cases with exact moments used as inputs. The results with estimated moments for simulated time series of return are then presented next.

Exact moments

Exhibit 6 presents a first set of results for VaR computations performed with the theoretical moments used as inputs. Three different horizons are examined: 5, 10 and 15 days, with a year defined as 250 days. For each maturity, three different probability levels are examined. The first line of each panel reports the skewness and kurtosis associated with returns of a given horizon. The first panel examines a case with zero skewness. The following parameter values of the jump diffusion process are used to generate the moments used as inputs : $\alpha = 0.05$, $\sigma = 0.2$, $\lambda = 5$ and $\alpha_J = 0$ and $\sigma_J = 0.1$. The kurtosis reported on the top of this panel present the typical behavior associated to the jump-diffusion process with moments becoming closer to the normal case as the return horizon is increased. The second line reports the benchmark VaR. The third line reports the differences between the Gram-Charlier VaR and the Benchmark VaR. The fourth and fifth lines report similar quantities for the Cornish-Fisher and Johnson VaR.

A first point to notice is the non-available values (na) for the Cornish-Fisher VaR for the 5 day case. The skewness and kurtosis combination produces a non-monotone quantile function for the Cornish-Fisher approximation and are thus non-available. For the Gram-Charlier case, the 5 and 10 day cases are also not available since the approximate densities show negative probabilities. In this panel, the approximation quality of all three approaches seems to be roughly equivalent when looking at probabilities of 1% and 5%. For the 0.1% probability level, the Gram-Charlier and Cornish-Fisher cases each show larger errors when compared to those of the Johnson approach. For example, the Cornish-Fisher 10 days VaR for the 0.1% probability is off with an error close to half of the exact VaR. The other two panels of this table examine negative and positive skewness cases. The parameter values generating the moments used as inputs are the same as those of the previous case except for α_j and σ_J which are now equal to ± 0.05 and 0.07. Again, the Gram-Charlier and Cornish-Fisher approximations tend to show larger errors for low probability levels.

Exhibit 7 presents the results for the ES computation with the theoretical moments identical to those from the previous table. Here and in the tables to follow, a grid with 1001 points is used to computed the ES for the Cornish-Fisher cases. The results are roughly similar to those from the VaR. The Gram-Charlier and Cornish-Fisher produce larger errors for low probabilities. This is expected since these approaches produce large errors for the VaR which is used as an input in the ES computation. For larger probabilities, the errors seem to be roughly of similar magnitude for all methods.

The results presented in the above tables are for specific cases and might not be representative of the wide variety of cases that are possible for different probability levels, horizon and combination of moments. For a general assessment of the relative precision of the Cornish-Fisher, Gram-Charlier and Johnson approaches for approximating VaR and ES, a large test pool of cases is generated randomly as follows. Each parameter value is drawn independently from the others with a uniform distribution and the following upper and lower limits: h, the horizon, between 1/250 and 20/250 years; p, the probability, between 0.001 and 0.05; α between 0.01 and 0.1; σ between 0.1 and 0.5; λ between 1 and 5; α_J between -0.1 and 0.1; σ_J between 0.01 and 0.1. Exhibit 8 presents some basic statistics about the first four theoretical moments of this random pool of cases. 5,000 cases are drawn. Among these, 12 cases are eliminated since they obtained kurtosis larger than 50.

Exhibit 9 presents root-mean-squared-errors ($rmse$) computed with the test pool of randomly generated cases. For a given method, the $rmse$ is computed as:

$$rmse = \sqrt{\frac{1}{m}\sum_{i=1}^{m} e_i^2},$$

where m is the size of the test pool and e_i is the difference between the VaR or ES obtained with one of the three approaches and the benchmark VaR or ES. In this table, the $rmse$ is computed for different subsets of the test pool to allow for meaningful comparisons. For example, VaR_{GC} refers to a Gram-Charlier VaR from the subset of cases for which the skewness and kurtosis coefficients are within the locus of valid Gram-Charlier approximations. VaR_{JGC} refers to a Johnson VaR from the subset of cases for which the skewness and kurtosis coefficients are within the locus of valid Gram-Charlier approximations. Hence, the *rmse* computed for these two subsets can be compared. VaR_{CF} and VaR_{JCF} have similar interpretations while VaR_{Jall} refers to the Johnson VaR computed within the whole sample from the test pool. The results from the table show that the Johnson VaR is more precise in terms of *rmse* than the VaR computed with the Gram-Charlier and Cornish-Fisher approaches. The table examines different sub cases such as small and large probabilities, small and long horizon and negative or positive skewness. Again the Johnson VaR consistently shows smaller *rmse*. Finally, the table reports the proportion of cases of the test pool for which we have invalid Gram-Charlier or Cornish-Fisher approximations. These proportions are 21.6% and 6.8%. The proportion of the cases for which the Hill and al [1976] algorithm did not converge is also reported. Recall that this algorithm is used to identify the appropriate family of the Johnson system and the parameters a, b, c, and d required by the approach. This proportion is very small and represents only 0.1% of the cases. For these cases, the alternative algorithm proposed in Wheeler [1980] is used to find the appropriate family and the a, b, c and d parameters. The second panel of Exhibit 9 reports the *rmse* computed for the ES. Again, the Johnson distributions dominates the other two methods.

In all the above computations, the Gram-Charlier and Cornish-Fisher are used with skewness and kurtosis pairs within the valid regions. One can wonder whether calculating VaR and ES with skewness and kurtosis combinations outside of the valid region can produce large errors. For this purpose, Exhibit 10 shows some VaR computations for the Gram-Charlier approach for pairs of skewness and kurtosis inside and outside the locus of positive densities (values in italics are outside the locus)[1]. It is clear from these numbers that the quality of the Gram-Charlier approximation deteriorates as we move outside the locus of valid densities.

Estimated moments

In most cases, the true moments of the portfolio return distribution are unknown. We therefore examine here the situation for which estimated moments are used as inputs. Exhibit 11 presents a first set of results obtained with the simulation procedure described earlier. The following parameter

[1]Because multiple VaR and ES can be computed when densities with negative values are used, only the most conservative VaR and ES are reported.

values of the jump diffusion process are used to generate the time series of return with which the moments are estimated: $\alpha = 0.05$, $\sigma = 0.2$, $\lambda = 5$, $\alpha_J = 0$ and $\sigma_J = 0.1$. Each simulated time series has $n = 250$ observations and $N = 1,000$ paths are simulated i.e. 1,000 sets of moments are estimated. The table presents summary statistics of the VaR estimation errors obtained with these estimated moments. In this table, the VaR is computed only when the estimated moments are within the locus of admissible quantile and density functions for the Gram-Charlier and Cornish-Fisher cases. For example, the 5 day case has exact skewness and kurtosis coefficients equal to 0.0 and 12.3. These moments are outside the locus of valid quantile and density functions. However, the estimated skewness and kurtosis obtained with a simulated time series can fall inside the locus. When this is the case, the VaR is estimated. The proportions of estimated VaR are reported at the bottom of the table. For the 5 day cases, 48 percent of the 1,000 sets of estimated moments are outside the locus of valid Gram-Charlier densities while 29 percent are outside the locus of valid Cornish-Fisher approximation. To allow for meaningful comparisons, we again report the statistics of the Johnson VaR for the skewness and kurtosis coefficients that are within the locus of valid Gram-Charlier and Cornish-Fisher approximations. VaR_{JGC} refers to a Johnson VaR from the subset for which the skewness and kurtosis coefficients are within the locus of valid Gram-Charlier approximations. A similar procedure and notation is adopted for the Cornish-Fisher case. The subscript *all* refers to the whole sample.

Comparing the Cornish-Fisher and Johnson cases, the panels identified with $(VaR_{CF} - VaR_B)$ and $(VaR_{JCF} - VaR_B)$ show average errors that are roughly of similar magnitudes for all probabilities and horizons. The Johnson VaR errors show smaller standard deviations for all cases. The Cornish-Fisher also shows minimum values that are large in absolute term with respect to the benchmark VaR. Comparing the Gram-Charlier and Johnson cases, the panels identified with $(VaR_{GC} - VaR_B)$ and $(VaR_{JGC} - VaR_B)$ show again average errors of similar magnitudes. The standard deviations are smaller for the Gram-Charlier cases for low probabilities while they are larger for larger probabilities. The minimum and maximum values are roughly equivalent.

An interesting point to notice is that the average errors for the Cornish-Fisher cases can be much smaller in absolute terms than the errors got with the true moments. For example, the 10 day, 0.1% case obtains an error of -6.3% (on average) with the estimated moment and -12.8% with the true moment. A possible explanation for this difference is associated with the distribution of the kurtosis estimate. A detailed examination of the results shows that the estimator's distribution for the first three moments are symmetric and close to a normal. For the kurtosis, the distribution can be skewed with a large proportion of the estimates smaller than the true quantity. For example, the 10 day, 0.1% case obtains 72% of the simulated sample below the true value of 7.6. This underestimation of the kurtosis seem's to favor the Cornish-Fisher approach because the magnitude of the VaR estimation errors are linked to the magnitude of kurtosis values used as inputs. This can be seen in Exhibit 13 which plots the VaR estimation error as a function of the estimated kurtosis for the 10 day, 0.1% case. The Cornish-Fisher approach (upper graph) shows errors that are on average smaller in absolute terms for lower values of the kurtosis. The bottom graph shows that the Johnson approach is less influenced by such an underestimation.

Exhibit 12 presents the results of the ES estimates. Comparing the Cornish-Fisher and Johnson cases, in the panels identified with $(ES_{CF} - ES_B)$ and $(ES_{JCF} - ES_B)$, we see average errors of similar magnitudes for all probabilities and horizons. Again the Johnson case has smaller standard deviations. The minimum values extremely low for the Cornish-Fisher cases. Thus, even if the Cornish-Fisher ES produces on average reasonable estimates, it can give unreasonable numbers in some cases. The Gram-Charlier and Johnson cases, in the panels identified with $(ES_{GC} - ES_B)$ and $(ES_{JGC} - ES_B)$, show similar patterns to those of the VaR from the previous table.

The results presented in the two preceding tables are for specific cases and might not be representative of the wide variety of possible cases. Therefore, we again use a simulated test pool to assess the relative precision of all three methods. The test pool is simulated with the same distributions that we previously used and 5,000 cases are generated. Exhibit 14 reports the *rmse*. Note that, in this table, for each simulated set of jump diffusion parameters, only one time series is simulated. Accordingly, one set of moments is obtained for each case. The results show that Johnson distributions dominate for the VaR or ES in terms of *rmse*. The differences are smaller than those got with the true moments. As discussed above, a likely explanation for these smaller differences is the skewed distribution for the kurtosis estimate. The more frequent underestimation of the kurtosis tends to favor the Cornish-Fisher and Gram-Charlier approaches which are typically associated with lower VaR estimation errors for smaller kurtosis values.

Conclusion

We have examined here the performance of Johnson distributions for computing value at risk and expected shortfall in the context of skewed and leptokurtic portfolio return distributions. This approach uses the first four moments of the portfolio returns to yield an approximating distribution. The performance of the Johnson system was compared to similar methods which also use the first four moments of an unknown target distribution to compute value at risk and expected shortfall. Unlike these methods, which generate valid distributions for a limited set of skewness and kurtosis pairs, the Johnson system yields valid approximating distributions for all combinations of skewness and kurtosis. An extensive simulation study using the Merton's [1976] jump diffusion framework shows that the Johnson approach is more precise in terms of root mean squared errors. Using exact or simulated moments, the Johnson system generates value-at-risk and expected shortfall estimates that are, on average, closer to the true values. Unlike the other approximations, using the Johnson distributions requires a preliminary numerical step to find the parameters and the appropriate family of distributions from the Johnson system. Using the algorithm of Hill et al. [1976], we find that the numerical work is minimal and robust. Apart from this preliminary step, value at risk computation is done analytically. For the expected shortfall, analytical approximations are available for three out of the four families of distribution composing the Johnson system. A numerical integration must be performed to compute the expected shortfall for the last family. This numerical integration is however fast and precise given the smoothness of the function to integrate. The Johnson approach can thus yield approximate risk measures that are fast and easy to compute, more robust to all inputs combinations, and more accurate on average than the commonly employed Cornish-Fisher and Gram-Charlier approximations.

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Appendix A

VaR can be defined in terms of the dollar loss of a portfolio or in terms of the returns. This appendix clarifies our VaR and expected shortfall definitions and shows how to obtain equations (1) and (2).

The dollar loss of a portfolio over an horizon of h years can be written as $L_h = V_t - V_{t+h}$ where V_t is the portfolio value at date t. The dollar VaR is then defined as the loss level such that only $p\%$ de cases will be larger in the next h years:

$$\Pr\left(L_h > \$VaR\right) = p.$$

Using this dollar VaR definition, we can obtain the VaR expressed as a return by noticing the continuously compounded return of the portfolio can be written as $r_h = \ln V_{t+h}/V_t$ which obtains the following expression for the loss: $L_h = \left(1 - e^{r_h}\right) V_t$. The dollar VaR can then be written as

$$\Pr\left(r_h < \ln\left(1 - \frac{\$VaR}{V_t}\right)\right) = p \iff \Pr\left(r_h < VaR\right) = p$$

where VaR is the VaR expressed as a continuously compounded return. Defining the standardized return z with $r_h = \alpha_h + \sigma_h z$ where α_h and σ_h are the expected value and standard deviation of the return, the probability can be rewritten in terms of the distribution function of the return $p = \phi\left(\left[VaR - \alpha_h\right]/\sigma_h\right)$. Using the inverse of the distribution function (the quantile function), the VaR can then be written as

$$VaR = \alpha_h + \sigma_h\ \phi^{-1}\left(p\right).$$

The expected shortfall measures the expected return conditional on it being worse than the VaR i.e. $E\left[r_h \mid r_h < VaR\right]$. Using this definition, it can be written as:

$$ES = \frac{EP\left[r_h \mid r_h < VaR\right]}{\Pr\left(r_h < VaR\right)} \tag{10}$$

where EP is the partial conditional expected value defined as

$$\int_{-\infty}^{VaR} r_h \times f\left(r_h\right)\ \mathrm{d}r_h.$$

Using the standardized return definition, this partial expected conditional expectation is written as:

$$\alpha_h \times \phi\left(k\right) + \sigma_h \int_{-\infty}^{k} z \times f\left(z\right)\ \mathrm{d}z$$

where $k = \frac{VaR - \alpha_h}{\sigma_h}$. The expected shortfall is then

$$ES = \alpha_h + \sigma_h \frac{\int_{-\infty}^{k} z \times f(z) \;\mathrm{d}z}{p}.$$

Appendix B

The Gram-Charlier expansion for approximating and unknown density for the standardized return z, using the skewness and kurtosis coefficients κ_3 and κ_4 is given by

$$f_{GC}(z; \kappa_3, \kappa_4) = \left[1 + \frac{\kappa_3}{6}(z^3 - 3z) + \frac{(\kappa_4 - 3)}{24}(z^4 - 6z^2 + 3)\right] \times f_N(z) \tag{11}$$

where $f_N(\bullet)$ is the standard normal density. The distribution function for the Gram-Charlier density can be obtained by integrating the above approximate density:

$$\begin{aligned}\phi_{GC}(k; \kappa_3, \kappa_4) = &\int_{-\infty}^{k} f_N(z)\,\mathrm{d}z + \frac{\kappa_3}{6}\left[\int_{-\infty}^{k} z^3 f_N(z)\,\mathrm{d}z - 3\int_{-\infty}^{k} z f_N(z)\,\mathrm{d}z\right] \\ &+ \frac{(\kappa_4 - 3)}{24}\left[\int_{-\infty}^{k} z^4 f_N(z)\,\mathrm{d}z - 6\int_{-\infty}^{k} z^2 f_N(z)\,\mathrm{d}z + 3\int_{-\infty}^{k} f_N(z)\,\mathrm{d}z\right].\end{aligned}$$

Given that $\int_{-\infty}^{k} z^0 f_N(z)\mathrm{d}z = \phi_N(k)$ and $\int_{-\infty}^{k} z f_N(z)\mathrm{d}z = -f_N(k)$ we can obtain the terms in the above expression by recursion with

$$\int_{-\infty}^{k} z^j f_N(z)\,\mathrm{d}z = -k^{j-1} f_N(k) + (j-1)\int_{-\infty}^{k} z^{j-2} f_N(z)\,\mathrm{d}z$$

which obtains, after simplification, equation (3).

The partial conditional expected value is obtained in a similar way by computing

$$\begin{aligned}EP[z \mid z < k] = &\int_{-\infty}^{k} z f_N(z)\,\mathrm{d}z + \frac{\kappa_3}{6}\left[\int_{-\infty}^{k} z^4 f_N(z)\,\mathrm{d}z - 3\int_{-\infty}^{k} z^2 f_N(z)\,\mathrm{d}z\right] \\ &+ \frac{(\kappa_4 - 3)}{24}\left[\int_{-\infty}^{k} z^5 f_N(z)\,\mathrm{d}z - 6\int_{-\infty}^{k} z^3 f_N(z)\,\mathrm{d}z + 3\int_{-\infty}^{k} z f_N(z)\,\mathrm{d}z\right]\end{aligned}$$

which can also be solved, using the recursive expression above, to yield equation (5).

Appendix C

The lognormal family

Substituting the $g^{-1}(\cdot)$ function and the expression for the standard normal density in equation (9) yields the following expression :

$$EP_J[z \mid z < k_J] = c\int_{-\infty}^{K} f_N(y) \times \mathrm{d}y + de^{\frac{-a}{b}}\int_{-\infty}^{K} \frac{1}{\sqrt{2\pi}} e^{\frac{y}{b} - \frac{y^2}{2}}\mathrm{d}y.$$

Completing the square in the terms of the exponential obtains:

$$\frac{y}{b} - \frac{y^2}{2} = -\frac{1}{2}\left(y - \frac{1}{b}\right)^2 + \frac{1}{2b^2}.$$

Substituting and simplifying yields

$$EP_J\left[z \mid z < k_J\right] = c\phi_N(K) + de^{\frac{1}{2b^2} - \frac{a}{b}}\phi_N\left(K - \frac{1}{b}\right).$$

The unbounded family

Substituting the $g^{-1}(\cdot)$ function and the expression for the standard normal density in equation (9) yields the following expression :

$$\begin{aligned} EP_J\left[z \mid z < k_J\right] = c\int_{-\infty}^{K} f_N(y) \times \mathrm{d}y + \frac{d}{2}\int_{-\infty}^{K} e^{\frac{y-a}{b} - \frac{y^2}{2}}\frac{1}{\sqrt{2\pi}}\mathrm{d}y \\ - \frac{d}{2}\int_{-\infty}^{K} e^{-\frac{y-a}{b} - \frac{y^2}{2}}\frac{1}{\sqrt{2\pi}}\mathrm{d}y. \end{aligned}$$

The terms in the exponential of the second and third terms can be rewritten as:

$$\frac{y-a}{b} - \frac{y^2}{2} = -\frac{a}{b} + \frac{1}{2b^2} - \frac{1}{2}(y - \frac{1}{b})^2,$$

$$-\frac{y-a}{b} - \frac{y^2}{2} = \frac{a}{b} + \frac{1}{2b^2} - \frac{1}{2}(y + \frac{1}{b})^2.$$

Substituting and simplifying yields

$$EP_J\left[z \mid z < k_J\right] = c\phi_N(K) + \frac{d}{2}e^{-\frac{a}{b} + \frac{1}{2b^2}}\phi_N\left(K - \frac{1}{b}\right) - \frac{d}{2}e^{\frac{a}{b} + \frac{1}{2b^2}}\phi_N\left(K + \frac{1}{b}\right).$$

The normal family

Substituting the $g^{-1}(\cdot)$ function and the expression for the standard normal density in equation (9) yields the following expression :

$$\begin{aligned} EP_J\left[z \mid z < k_J\right] = c\int_{-\infty}^{K} f_N(y) \times \mathrm{d}y + \frac{d}{b}\int_{-\infty}^{K} y\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}} \times \mathrm{d}y \\ - \frac{da}{b}\int_{-\infty}^{K} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}} \times \mathrm{d}y, \end{aligned}$$

which obtains

$$EP_J\left[z \mid z < k_J\right] = c\phi_N(K) - \frac{d}{b}f_N(K) - \frac{da}{b}\phi_N(K).$$

Appendix D

For the Merton [1976] jump diffusion process with lognormal jumps, the distribution function of the log return, for an horizon of h years, is known in closed form and is given by:

$$\phi_{Jump}(k) = \sum_{n=0}^{\infty} \frac{e^{-\lambda h} (\lambda h)^n}{n!} \phi_N \left(\frac{k - \mu_n}{\sigma_n} \right)$$

where $\phi_N(\cdot)$ is the standard normal distribution function , $\mu_n = \left(\widetilde{\alpha} - \frac{1}{2}\sigma^2\right) h + n\alpha_J$, $\sigma_n^2 = \sigma^2 h + n\sigma_J^2$ and where $\widetilde{\alpha} = \alpha - \lambda l$ and $l = e^{\left(\alpha_J + \frac{1}{2}\sigma_J^2\right)} - 1$. Although the above function requires computing an infinite sum, the magnitude of the elements quickly decay to negligible values for all practical purposes. Hence, only the first few elements are used in the computations.

In this framework, the benchmark VaR can be computed with a bisection numerical search over the values of k generating a target probability p i.e. $VaR_B = k$ such that $p = \phi_{jump}(k)$. For the expected shortfall, conditional on n jumps over the time interval h, the partial conditional expectation $EP[r_h \mid r_h < VaR]$ appearing in the numerator of equation (10) can be computed with:

$$\begin{aligned}
\int_{-\infty}^{VaR} r_h \times f(r_h) \ \mathrm{d}r_h &= \int_{-\infty}^{\frac{VaR - \mu_n}{\sigma_n}} (\mu_n + \sigma_n \times z) \times f(z) \ \mathrm{d}z \\
&= \mu_n \times \phi_N(k_{B,n}) + \sigma_n \int_{-\infty}^{k_{B,n}} z \times f_N(z) \times \mathrm{d}z \\
&= \mu_n \times \phi_N(k_{B,n}) - \sigma_n \times f_N(k_{B,n})
\end{aligned}$$

where $f_N(\cdot)$ is the standard normal density function and $k_{B,n} = \frac{VaR_B - \mu_n}{\sigma_n}$. The expected shortfall formula, is then

$$ES_B = \frac{\sum_{n=0}^{\infty} \frac{e^{-\lambda h}(\lambda h)^n}{n!} \left(\mu_n \times \phi_N(k_{B,n}) - \sigma_n \times f_N(k_{B,n})\right)}{p}.$$

The formulas to compute the expected value, variance, skewness and kurtosis over a time interval of h years are available from Das and Sundaram [1999] and are given by:

$$\alpha_h = \left(\widetilde{\alpha} - \frac{1}{2}\sigma^2\right) h + \lambda h \alpha_J,$$

$$\sigma_h^2 = \left(\sigma^2 + \lambda\sigma_J^2 + \lambda\alpha_J^2\right) \times h,$$

$$\kappa_3 = \frac{1}{\sqrt{h}} \left(\frac{\lambda\left(\alpha_J^3 + 3\alpha_J\sigma_J^2\right)}{\left(\sigma^2 + \lambda\sigma_J^2 + \lambda\alpha_J^2\right)^{3/2}} \right),$$

and

$$\kappa_4 = 3 + \frac{1}{h}\left[\frac{\lambda\left(\alpha_J^4 + 6\alpha_J^2\sigma_J^2 + 3\sigma_J^4\right)}{\left(\sigma^2 + \lambda\sigma_J^2 + \lambda\alpha_J^2\right)^2}\right].$$

Finally, a portfolio value V_{t+h} can be simulated using numerical values for $\alpha, \sigma, \lambda, \alpha_J$, σ_J and V_t, the current portfolio value, with the following procedure:

1. Simulate a lognormal portfolio value with
$$\ln \widetilde{V}_{t+h} \sim N\left(\ln V_t + \left[\widetilde{\alpha} - \frac{1}{2}\sigma^2\right] h, \sigma^2 h\right).$$
2. Simulate m, the number of jumps over h years, with a Poisson(λh) random variate.
3. Simulate m normal random variates with $\ln Y_j \sim N\left(\alpha_J,\ \sigma_J^2\right)$ with $j = 1$ to m and compute the portfolio value with:
$$V_{t+h} = \widetilde{V}_{t+h} \times Y_{i,1} \times ... \times Y_{i,m}.$$

A time series of n simulated returns can the be obtained with a simulated path of portfolio values $\{V_t, V_{t+h}, \ldots, V_{t+nh}\}$ and returns r_{ih} computed with

$$r_{ih} = \ln \frac{V_{t+ih}}{V_{t+(i-1)h}}.$$

Appendix E

This appendix provides a brief description of the Hill et al. [1976] algorithm. In a first step, this algorithm determines the appropriate family of distribution from the Johnson system. In a second step, it finds the numerical parameter values of the family identified in the first step. It does so by matching the first four moments of the appropriate Johnson distribution with those from the target distribution. In the present context, the required inputs for the algorithm are: i) an expected value of zero; ii) a standard deviation of one; iii) κ_3, the skewness of the target distribution; iv) κ_4, the kurtosis of the target distribution. The Fortran code of this algorithm is available in Hill et al. [1976]. An adaptation of this code in C (usable from Matlab) is available upon request.

Determining the family of the Johnson system is done by first computing

$$\gamma = w^4 + 2w^3 + 3w^2 - 3$$

with

$$w = \frac{1}{2}\left(8 + 4\kappa_3^2 + 4\sqrt{4\kappa_3^2 + \kappa_3^4}\right)^{\frac{1}{3}} + \frac{1}{2}\left(8 + 4\kappa_3^2 + 4\sqrt{4\kappa_3^2 + \kappa_3^4}\right)^{-\frac{1}{3}} - 1.$$

If γ is close to κ_4, the lognormal family should be used. If γ is smaller than κ_4, the unbounded family is appropriate while the bounded case is for γ greater than κ_4. The procedure used to determine the parameter values then depends on the identified family. For the lognormal case, the parameters are obtained with:

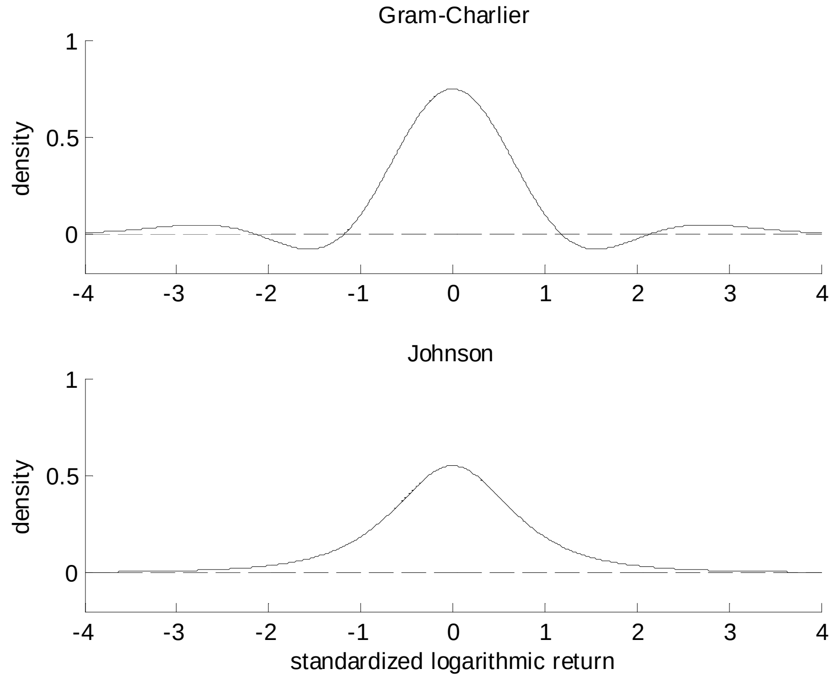
$$\widehat{b} = (\ln w)^{-\frac{1}{2}}, \quad \widehat{a} = \frac{1}{2}\widehat{b}\ln\left(w\left(w-1\right)\right), \quad \widehat{d} = sign(\kappa_3), \quad \widehat{c} = -\exp\left(\frac{\frac{1}{2}\widehat{b}-\widehat{a}}{\widehat{b}}\right).$$

For the unbounded family, the values of $\widehat{a}$ and $\widehat{b}$ are first determined and then used to find $\widehat{c}$ and $\widehat{d}$. If κ_3 is close to zero, $\widehat{a}$ and $\widehat{b}$ are given by $\widehat{a} = 0$ and $\widehat{b} = (\ln\zeta)^{-\frac{1}{2}}$ with $\zeta = \sqrt{(2\kappa_4 - 2)^{1/2} - 1}$. If κ_3 is different from zero, the iterative approach described in Hill et al. [1976] is required to find ζ, $\widehat{a}$ and $\widehat{b}$. Given ζ, $\widehat{a}$ and $\widehat{b}$ the values of $\widehat{c}$ and $\widehat{d}$ are then found with $\widehat{d} = \left(0.5\left(\zeta-1\right)\left(\zeta\cosh\left(2\widehat{a}/\widehat{b}\right)+1\right)\right)^{-\frac{1}{2}}$ and $\widehat{c} = \widehat{d}\sqrt{\zeta}\sinh\left(\widehat{a}/\widehat{b}\right)$. For the bounded family, the iterative approach described in Hill et al. [1976] is required.

Appendix F

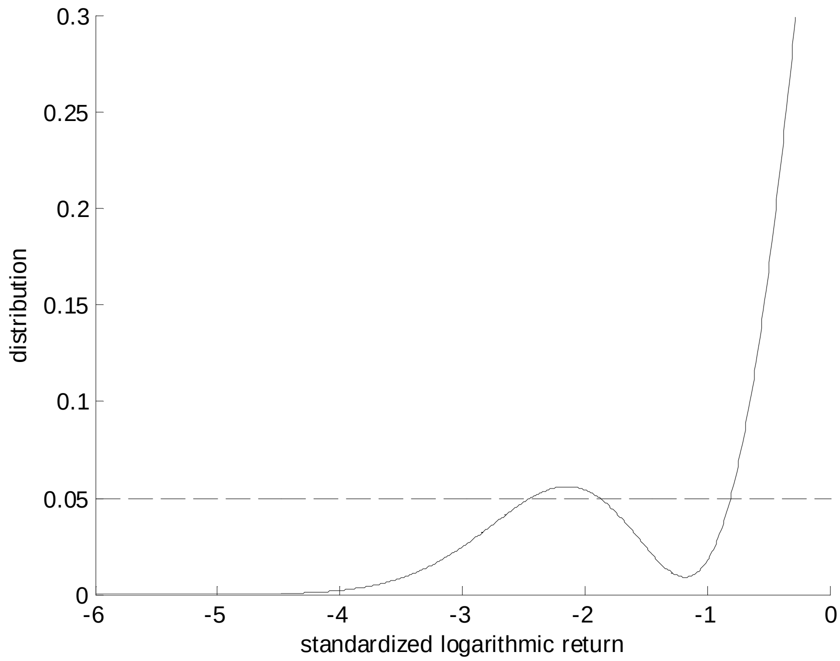
The numerical computation of the Cornish-Fisher expected shortfall is described here. In a first step, a lower and upper bound for the relevant portion of the support of the unknown Cornish-Fisher distribution function are computed as $x_{low} = \phi_{CF}^{-1}\left(0.000001; \kappa_3, \kappa_4\right)$ and $x_{high} = \phi_{CF}^{-1}\left(p; \kappa_3, \kappa_4\right)$ where p is the probability level of the expected shortfall. With m an odd integer, the interval $[x_{high}\ x_{low}]$ is divided equally into $m+1$ cells to obtain m discrete values with $x_i = x_{low} + (i - 1)/\left(m-1\right) \times \left(x_{high} - x_{low}\right)$ for $i = 1$ to m. With a bisection algorithm, for each x_i, a probability p_i is then computed as $p_i = p$ such that $x_i = \phi_{CF}^{-1}\left(p; \kappa_3, \kappa_4\right)$. The m pairs (x_i, p_i) are points on the Cornish-Fisher distribution function. Computing the expected shortfall requires the density function, the first derivative of the distribution function. Points on this function are computed using central differences $y_i = \frac{p_{i+1} - p_{i-1}}{2\times\delta x}$ for $i = 2$ to $m-1$ with $\delta x = x_i - x_{i-1}$. This obtains $m-2$ points on the Cornish-Fisher density function. With these $m-2$ evenly spaced points, the integral appearing in the expected shortfall formula can be computed numerically with the extended Simpson's rule.

Exhibit 1: Gram-Charlier and Johnson densities for $\kappa_3 = 0$ and $\kappa_4 = 10$



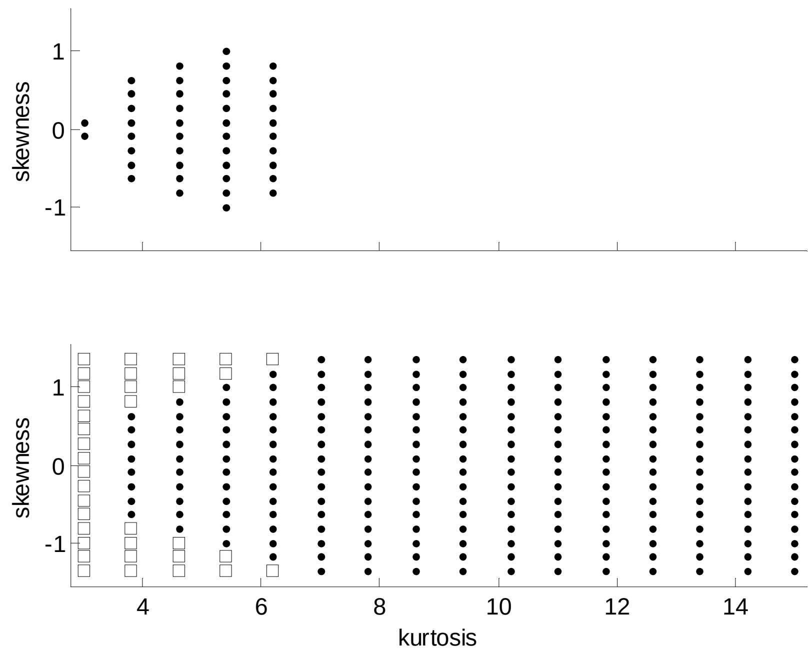
The graphs in this figure plot the densities obtained from the Gram-Charlier (upper graph) and Johnson (bottom graph) approaches for a zero mean and unit variance random variable.

Exhibit 2: Gram-Charlier distribution for $\kappa_3 = 0$ and $\kappa_4 = 10$



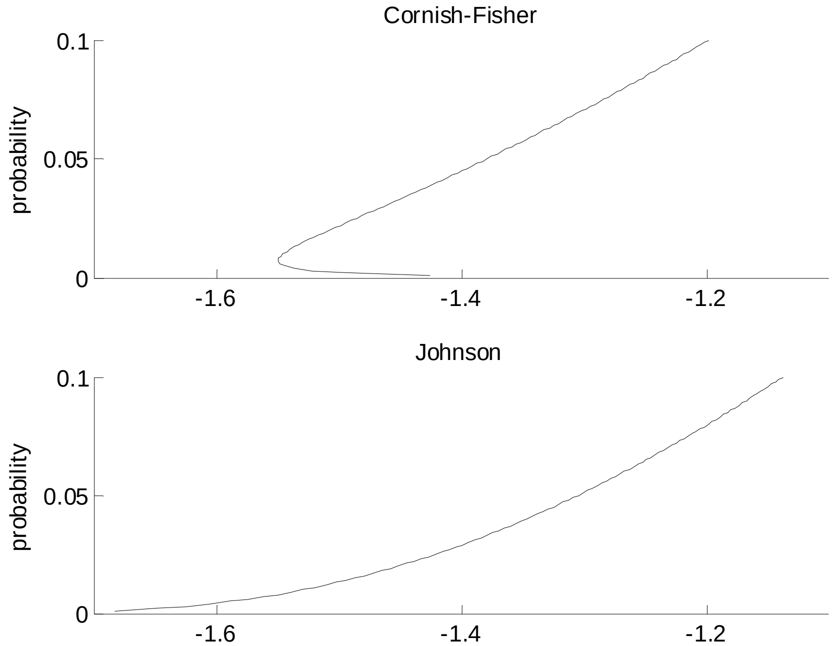
The graph in this figure plots the distribution tail obtained from a Gram-Charlier density for a zero mean and unit variance random variable.

Exhibit 3: Locus of valid Gram-Charlier and Johnson densities



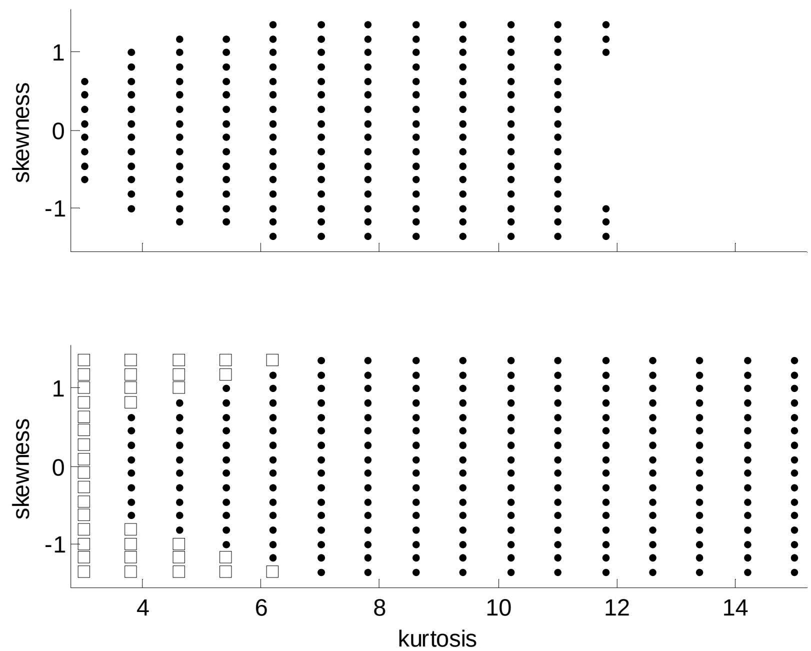
The graphs in this figure show the set of skewness and kurtosis pairs generating densities with positive probabilities for the Gram-Charlier (upper graph) and Johnson (bottom graph) approaches for a zero mean and unit variance random variable. In the graph showing the results with the Johnson approach, a square is a density from the bounded family, a dot is a density from the unbounded family.

Exhibit 4: Cornish-Fisher and Johnson distribution functions for $\kappa_3 = 0.85$ and $\kappa_4 = 3.5$



The graphs in this figure plot the tail of (approximate) distribution functions obtained with the Cornish-Fisher (upper graph) and Johnson (bottom graph) approaches for a zero mean and unit variance random variable.

Exhibit 5: Locus of valid Cornish-Fisher and Johnson quantile functions



The graphs in this figure show the set of skewness and kurtosis pairs generating monotone quantile functions with the Cornish-Fisher (upper graph) and Johnson (bottom graph) approaches for a zero mean and unit variance random variable. In the graph showing the results with the Johnson approach, a square is for the bounded family, a dot is for the unbounded family.

Exhibit 6: VaR with exact moments

	5 days			10 days			15 days		
Prob. (%)	0.1	1.0	5.0	0.1	1.0	5.0	0.1	1.0	5.0
	κ_3 = 0.0, κ_4 = 12.3			κ_3 = 0.0, κ_4 = 7.6			κ_3 = 0.0, κ_4 = 6.1		
VaR_B (%)	-24.7	-13.2	-5.6	-29.1	-17.8	-8.8	-32.6	-20.8	-11.3
$VaR_{GC} - VaR_B$ (%)	na	na	na	na	na	na	3.4	-1.8	0.7
$VaR_{CF} - VaR_B$ (%)	na	na	na	-12.8	-2.6	-0.5	-9.2	-1.6	-0.3
$Var_J - VaR_B$ (%)	2.5	1.5	-0.8	1.0	1.8	-0.6	0.3	1.5	-0.3
	κ_3 = 1.4, κ_4 = 9.4			κ_3 = 1.0, κ_4 = 6.2			κ_3 = 0.8, κ_4 = 5.1		
VaR_B (%)	-13.0	-7.7	-5.3	-16.7	-11.1	-7.7	-19.7	-13.7	-9.7
$VaR_{GC} - VaR_B$ (%)	na	na	na	na	na	na	-5.0	0.1	0.7
$VaR_{CF} - VaR_B$ (%)	-2.7	-0.2	1.1	-0.6	0.3	0.7	0.0	0.4	0.5
$Var_J - VaR_B$ (%)	1.9	0.3	0.2	1.4	0.3	0.1	1.2	0.3	0.1
	κ_3 = -1.4, κ_4 = 9.4			κ_3 = -1.0, κ_4 = 6.2			κ_3 = -0.8, κ_4 = 5.1		
VaR_B (%)	-23.4	-14.4	-6.0	-28.0	-18.2	-9.7	-31.8	-20.9	-12.2
$VaR_{GC} - VaR_B$ (%)	na	na	na	na	na	na	4.8	-0.4	-2.0
$VaR_{CF} - VaR_B$ (%)	-8.3	-1.7	-1.3	-5.2	-0.8	-0.5	-3.8	-0.6	-0.1
$Var_J - VaR_B$ (%)	0.3	1.7	-0.9	-0.5	1.2	-0.1	-0.6	0.8	0.2

VaR_B, VaR_{GC}, VaR_{CF}, and VaR_J are the VaR values computed for the benchmark, Gram-Charlier, Cornish-Fisher, and Johnson approaches. "na"indicates the cases for which the Gram-Charlier or Cornish-Fisher approaches are not available because of invalid density or quantile functions.

Exhibit 7: Expected shortfall with exact moments

	5 days			10 days			15 days		
Prob. (%)	0.1	1.0	5.0	0.1	1.0	5.0	0.1	1.0	5.0
	κ_3 = 0.0, κ_4 = 12.3			κ_3 = 0.0, κ_4 = 7.6			κ_3 = 0.0, κ_4 = 6.1		
ES_B (%)	-28.7	-18.4	-9.9	-33.5	-22.9	-14.1	-37.4	-26.0	-17.1
$ES_{GC} - ES_B$ (%)	na	na	na	na	na	na	6.0	0.4	-1.3
$ES_{CF} - ES_B$ (%)	na	na	na	-19.7	-6.6	-2.3	-14.4	-4.6	-1.4
$ES_J - ES_B$ (%)	0.2	2.3	0.0	-1.4	1.7	0.5	-1.8	1.2	0.6
	κ_3 = 1.4, κ_4 = 9.4			κ_3 = 1.0, κ_4 = 6.2			κ_3 = 0.8, κ_4 = 5.1		
ES_B (%)	-15.6	-9.7	-6.9	-19.3	-13.4	-9.9	-22.3	-16.3	-12.2
$ES_{GC} - ES_B$ (%)	na	na	na	na	na	na	-4.7	-2.8	-0.0
$ES_{CF} - ES_B$ (%)	-4.5	-1.4	0.3	-1.3	-0.1	0.5	-0.3	0.3	0.5
$ES_J - ES_B$ (%)	2.6	0.8	0.3	1.9	0.7	0.3	1.6	0.6	0.2
	κ_3 = -1.4, κ_4 = 9.4			κ_3 = -1.0, κ_4 = 6.2			κ_3 = -0.8, κ_4 = 5.1		
ES_B (%)	-26.9	-18.4	-10.9	-32.2	-22.5	-14.9	-36.3	-25.6	-17.6
$ES_{GC} - ES_B$ (%)	na	na	na	na	na	na	7.4	1.8	-1.1
$ES_{CF} - ES_B$ (%)	-12.5	-4.2	-1.9	-7.7	-2.6	-0.8	-5.7	-1.9	-0.5
$ES_J - ES_B$ (%)	-1.7	1.3	0.3	-2.1	0.5	0.6	-2.0	0.2	0.5

ES_B, ES_{GC}, ES_{CF}, and ES_J are the ES values computed for the benchmark, Gram-Charlier, Cornish-Fisher, and Johnson approaches. "na"indicates the cases for which the Gram-Charlier or Cornish-Fisher approaches are not available because of invalid density or quantile functions.

Exhibit 8: Summary statistics about the exact moments associated with the random test pool cases

	$\alpha_h \times 1/h$	$\sigma_h \times \sqrt{1/h}$	κ_3	κ_4
mean	-0.01	0.33	-0.03	5.48
std	0.04	0.11	0.85	4.83
min	-0.13	0.11	-5.13	3.00
max	0.09	0.58	4.61	48.64

Exhibit 9: Root-mean-squared errors for VaR and ES computed with exact moments for the test pool of jump diffusion parameters

	all	$p <= 0.01$	$p > 0.01$	$h <= 5/250$	$h > 5/250$	$\kappa_3 < 0$	$\kappa_3 > 0$
				Value at risk			
rmse $VaR_{GC} - VaR_B$ (%)	0.82	1.10	0.74	0.91	0.80	0.96	0.65
rmse $VaR_{JGC} - VaR_B$ (%)	0.26	0.39	0.21	0.34	0.23	0.32	0.17
rmse $VaR_{CF} - VaR_B$ (%)	0.85	1.45	0.63	1.02	0.80	0.99	0.66
rmse $VaR_{JCF} - VaR_B$ (%)	0.51	0.65	0.48	0.48	0.52	0.64	0.34
rmse $VaR_{Jall} - VaR_B$ (%)	0.61	0.81	0.56	0.74	0.57	0.75	0.43
				Expected shortfall			
rmse $ES_{GC} - ES_B$ (%)	0.93	1.80	0.56	0.99	0.92	0.97	0.89
rmse $ES_{JGC} - ES_B$ (%)	0.29	0.52	0.21	0.38	0.27	0.32	0.27
rmse $ES_{CF} - ES_B$ (%)	1.31	2.27	0.96	1.44	1.27	1.51	1.05
rmse $ES_{JCF} - ES_B$ (%)	0.63	0.81	0.58	0.49	0.66	0.68	0.58
rmse $ES_{Jall} - ES_B$ (%)	0.74	1.02	0.66	0.78	0.73	0.75	0.73
Failure GC (%)	21.6						
Failure CF (%)	6.8						
Failure hhh (%)	0.1						

Var_B and ES_B are the Benchmark VaR and ES. VaR_{GC} and ES_{GC} are the Gram-Charlier VaR and ES for the skewness and kurtosis pairs within the locus of valid Gram-Charlier density functions. VaR_{JGC} and ES_{JGC} are the Johnson VaR and ES for the skewness and kurtosis pairs within the locus of valid Gram-Charlier density functions. VaR_{CF} and ES_{CF} are the Cornish-Fisher VaR and ES for the skewness and kurtosis pairs within the locus of valid Cornish-Fisher quantile functions. VaR_{JCF} and ES_{JCF} are the Johnson VaR and ES for the skewness and kurtosis pairs within the locus of valid Cornish-Fisher quantile functions.VaR_{Jall} and ES_{Jall} are the Johnson VaR for the whole sample. Failure GC is the proportion of skewness and kurtosis pairs that are outside the locus of valid Gram-Charlier density functions. Failure CF is the proportion of skewness and kurtosis pairs that are outside the locus of valid Cornish-Fisher quantile functions. Failure hhh are the proportion of cases for which the Hill et al. [1976] algorithm has failed to converge when computing the VaR or ES with the Johnson approach.

Exhibit 10: Gram-Charlier VaR and ES computations for skewness and kurtosis pairs outside the locus of positive density region

	VaR_B	VaR_G	VaR_J	ES_B	ES_G	ES_J
$\kappa_4 = 3.00$	-4.62	-4.62	-4.62	-5.81	-5.81	-5.81
$\kappa_4 = 3.31$	-4.82	-4.83	-4.83	-6.18	-6.23	-6.21
$\kappa_4 = 4.70$	-5.07	-5.10	-5.16	-6.95	-7.44	-7.08
$\kappa_4 = 7.33$	-5.30	*-4.25*	-5.59	-8.00	*-9.41*	-8.13
$\kappa_4 = 10.59$	-5.50	*-10.13*	-6.11	-9.22	*-12.51*	-9.26
$\kappa_4 = 13.87$	-5.67	*-12.63*	-6.72	-10.54	*-14.82*	-10.45

Var_B and ES_B are the Benchmark VaR and ES. VaR_{GC} and ES_{GC} are the Gram-Charlier VaR and ES. VaR_J and ES_J are the Johnson VaR and ES. Values in italics are for skewness and kurtosis pairs outside the locus of valid Gram-Charlier density functions. For all cases, the skewness is zero. Parameter values : $\sigma = 0.2$, $\alpha = 0.05$, $\lambda = 5$, $\alpha_J = 0.0$. σ_J takes values of 0.01, 0.03, 0.05, 0.07, 0.09 and 0.11. The VaR and ES are for an horizon of 5 days and a probability of 5%.

Exhibit 11: VaR with estimated moments

	5 days $\kappa_3 = 0.0$, $\kappa_4 = 12.3$			10 days $\kappa_3 = 0.0$, $\kappa_4 = 7.6$			15 days $\kappa_3 = 0.0$, $\kappa_4 = 6.1$		
Prob. (%)	0.1	1.0	5.0	0.1	1.0	5.0	0.1	1.0	5.0
VaR_B (%)	-24.7	-13.2	-5.6	-29.1	-17.8	-8.8	-32.6	-20.8	-11.3
				$VaR_{GC} - VaR_B$					
mean (%)	8.0	0.0	-0.7	5.4	-0.5	-0.2	4.1	-0.8	-0.3
std (%)	1.9	1.8	3.0	2.5	2.6	3.0	2.8	2.9	2.7
min (%)	1.8	-5.5	-8.3	-2.3	-7.5	-10.6	-4.2	-9.3	-11.4
max (%)	12.7	7.6	2.9	13.9	9.6	4.5	12.3	8.0	5.2
				$VaR_{JGC} - VaR_B$					
mean (%)	5.6	2.3	-0.8	3.0	2.2	-0.6	1.9	1.8	-0.3
std (%)	3.7	1.7	0.8	4.9	2.3	1.1	5.2	2.4	1.2
min (%)	-4.5	-2.6	-3.3	-12.5	-5.1	-3.8	-15.9	-5.4	-3.9
max (%)	17.1	7.2	1.3	18.3	9.1	2.4	15.9	9.0	3.0
				$VaR_{CF} - VaR_B$					
mean (%)	-5.2	-0.9	-0.6	-6.3	-0.7	-0.5	-5.6	-0.5	-0.3
std (%)	8.7	3.6	1.2	10.4	4.1	1.5	10.2	4.0	1.4
min (%)	-31.5	-12.2	-4.0	-44.0	-15.9	-4.5	-51.9	-17.4	-4.3
max (%)	16.5	8.2	2.9	17.5	10.6	4.3	13.6	9.2	3.9
				$VaR_{JCF} - VaR_B$					
mean (%)	5.6	2.3	-0.8	3.0	2.3	-0.6	1.8	1.8	-0.3
std (%)	4.3	2.0	0.9	5.2	2.4	1.1	5.3	2.4	1.2
min (%)	-7.5	-3.7	-3.4	-14.1	-5.2	-3.8	-15.9	-5.4	-3.9
max (%)	17.1	7.2	1.4	20.1	10.0	3.0	15.9	9.0	3.0
				$VaR_{Jall} - VaR_B$					
mean (%)	4.5	2.0	-0.8	2.8	2.2	-0.6	1.7	1.8	-0.3
std (%)	5.2	2.3	1.0	5.4	2.5	1.1	5.4	2.5	1.2
min (%)	-21.6	-7.5	-4.1	-23.2	-7.2	-3.8	-19.0	-6.1	-3.9
max (%)	20.8	9.5	2.4	20.1	10.0	3.0	15.9	9.0	3.0
Failure GC (%)		48.1			8.9			1.9	
Failure CF (%)		29.2			3.2			0.7	
Failure hhh (%)		0.2			0.0			0.0	

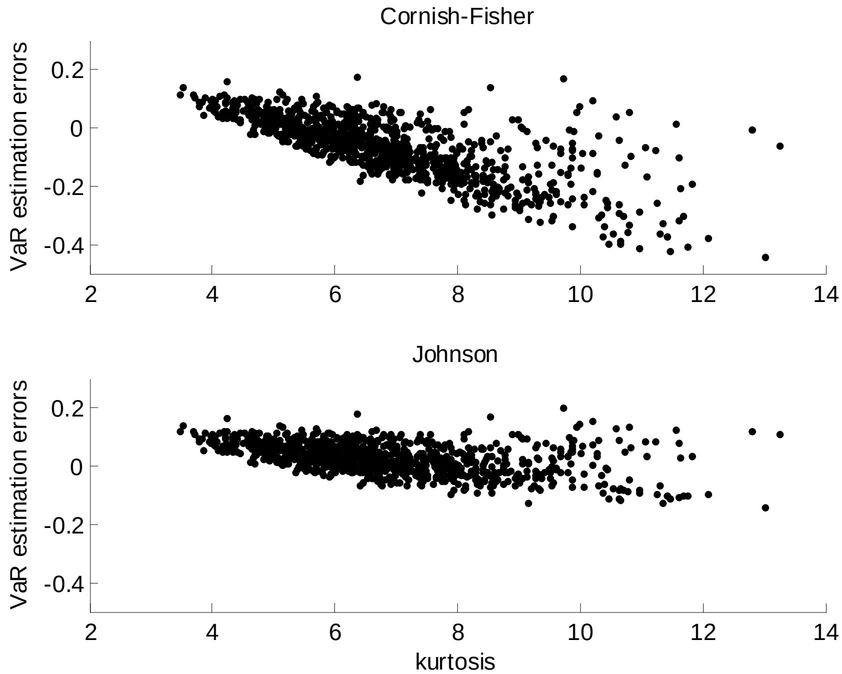
Var_B are the Benchmark VaR. VaR_{GC} are the Gram-Charlier VaR for the estimated skewness and kurtosis within the locus of valid Gram-Charlier density functions. VaR_{JGC} are the Johnson VaR for the estimated skewness and kurtosis within the locus of valid Gram-Charlier density functions. VaR_{CF} are the Cornish-Fisher VaR for the estimated skewness and kurtosis within the locus of valid Cornish-Fisher quantile functions. VaR_{JCF} are the Johnson VaR for the estimated skewness and kurtosis within the locus of valid Cornish-Fisher quantile functions. VaR_{Jall} are the Johnson VaR for the whole sample of estimated skewness and kurtosis. Failure GC is the proportion of time-series for which the estimated skewness and kurtosis are outside the locus of valid Gram-Charlier density functions. Failure CF is the proportion of time-series for which the estimated skewness and kurtosis are outside the locus of valid Cornish-Fisher quantile functions. Failure hhh are the proportion of cases for which the Hill et al. [1976] algorithm has failed to converge when computing the VaR with the Johnson approach.

Exhibit 12: ES with estimated moments

	5 days $\kappa_3 = 0.0$, $\kappa_4 = 12.3$			10 days $\kappa_3 = 0.0$, $\kappa_4 = 7.6$			15 days $\kappa_3 = 0.0$, $\kappa_4 = 6.1$		
Prob. (%)	0.1	1.0	5.0	0.1	1.0	5.0	0.1	1.0	5.0
ES_B (%)	-28.7	-18.4	-9.9	-33.5	-22.9	-14.1	-37.4	-26.0	-17.1
				$ES_{GC} - ES_B$					
mean (%)	10.9	3.6	-0.8	8.1	2.1	-0.8	6.7	1.2	-0.6
std (%)	2.0	1.8	2.2	2.6	2.5	2.8	2.9	2.8	2.8
min (%)	4.4	-2.2	-7.1	-0.1	-4.9	-9.0	-1.8	-7.0	-10.2
max (%)	15.7	8.3	4.4	16.7	11.2	5.9	15.0	9.4	6.4
				$ES_{JGC} - ES_B$					
mean (%)	5.0	4.0	0.6	1.8	2.8	0.8	0.6	2.0	0.8
std (%)	5.0	2.6	1.4	6.6	3.4	1.9	6.9	3.6	1.9
min (%)	-8.2	-3.1	-3.4	-18.5	-8.1	-5.1	-23.9	-9.6	-5.0
max (%)	20.6	11.7	4.4	21.9	13.2	6.1	18.5	12.1	6.5
				$ES_{CF} - ES_B$					
mean (%)	-9.3	-2.2	-1.3	-10.5	-2.6	-0.9	-9.0	-2.3	-0.6
std (%)	11.4	5.6	2.7	13.8	6.7	3.1	13.6	6.5	2.9
min (%)	-43.2	-19.7	-9.6	-59.7	-27.1	-12.0	-71.0	-31.2	-12.3
max (%)	19.4	11.7	5.7	20.4	13.8	7.8	15.8	11.3	7.1
				$ES_{JCF} - ES_B$					
mean (%)	5.0	4.0	0.6	1.8	2.8	0.8	0.5	2.0	0.8
std (%)	5.7	3.0	1.6	7.0	3.6	2.0	7.1	3.6	2.0
min (%)	-12.5	-5.1	-4.3	-21.5	-8.8	-5.2	-24.4	-9.6	-5.0
max (%)	20.6	11.7	4.4	24.0	14.5	6.8	18.5	12.1	6.5
				$ES_{Jall} - ES_B$					
mean (%)	3.3	3.4	0.3	1.5	2.7	0.8	0.3	1.9	0.7
std (%)	7.1	3.5	1.8	7.3	3.7	2.0	7.3	3.7	2.0
min (%)	-34.5	-13.2	-6.9	-36.7	-13.8	-6.6	-28.8	-11.4	-5.4
max (%)	24.7	14.6	6.2	24.0	14.5	6.8	18.5	12.1	6.5
Failure GC (%)		48.1			8.9			1.9	
Failure CF (%)		29.2			3.2			0.7	
Failure hhh (%)		0.2			0.0			0.0	

ES_B are the Benchmark ES. ES_{GC} are the Gram-Charlier ES for the estimated skewness and kurtosis within the locus of valid Gram-Charlier density functions. ES_{JGC} are the Johnson ES for the estimated skewness and kurtosis within the locus of valid Gram-Charlier density functions. ES_{CF} are the Cornish-Fisher ES for the estimated skewness and kurtosis within the locus of valid Cornish-Fisher quantile functions. ES_{JCF} are the Johnson ES for the estimated skewness and kurtosis within the locus of valid Cornish-Fisher quantile functions. ES_{Jall} are the Johnson ES for the whole sample of estimated skewness and kurtosis. Failure CF is the proportion of time-series for which the estimated skewness and kurtosis are outside the locus of valid Cornish-Fisher quantile functions. Failure GC is the proportion of time-series for which the estimated skewness and kurtosis are outside the locus of valid Gram-Charlier density functions. Failure hhh are the proportion of cases for which the Hill et al. [1976] algorithm has failed to converge when computing the ES with the Johnson approach.

Exhibit 13: VaR estimation errors



The graphs in this figure plot the VaR estimation errors for the 10 day, 0.1% case with $\kappa_3 = 0.0$ and $\kappa_4 = 7.6$ and estimated moments from 1000 simulated time series.

Exhibit 14: Root-mean-squared errors for VaR and ES computed with estimated moments for the test pool of jump diffusion parameters

	all	$p <= 0.01$	$p > 0.01$	$n <= 5$	$n > 5$	$\kappa_3 < 0$	$\kappa_3 > 0$
				Value at risk			
rmse $VaR_{GC} - VaR_B$ (%)	2.00	2.62	1.84	1.80	2.05	2.33	1.44
rmse $VaR_{JGC} - VaR_B$ (%)	1.48	2.35	1.22	1.10	1.58	1.70	1.13
rmse $VaR_{CF} - VaR_B$ (%)	1.81	3.15	1.37	1.34	1.93	2.18	1.39
rmse $VaR_{JCF} - VaR_B$ (%)	1.51	2.46	1.22	1.08	1.61	1.69	1.32
rmse $VaR_{Jall} - VaR_B$ (%)	1.57	2.56	1.25	1.25	1.65	1.72	1.41
				Expected shortfall			
rmse $ES_{GC} - ES_B$ (%)	2.32	3.62	1.94	1.95	2.42	2.65	1.80
rmse $ES_{JGC} - ES_B$ (%)	2.13	3.19	1.83	1.67	2.24	2.44	1.62
rmse $ES_{CF} - ES_B$ (%)	3.06	4.91	2.49	2.14	3.27	3.32	2.80
rmse $ES_{JCF} - ES_B$ (%)	2.21	3.38	1.86	1.66	2.34	2.43	1.98
rmse $ES_{Jall} - ES_B$ (%)	2.29	3.54	1.91	1.89	2.40	2.46	2.12
Failure GC (%)	33.8						
Failure CF (%)	5.1						
Failure hhh (%)	0.7						

Var_B and ES_B are the Benchmark VaR and ES. VaR_{GC} and ES_{GC} are the Gram-Charlier VaR and ES for the estimated skewness and kurtosis within the locus of valid Gram-Charlier density functions. VaR_{JGC} and ES_{JGC} are the Johnson VaR and ES for the estimated skewness and kurtosis within the locus of valid Gram-Charlier density functions. VaR_{CF} and ES_{CF} are the Cornish-Fisher VaR and ES for the estimated skewness and kurtosis within the locus of valid Cornish-Fisher quantile functions. VaR_{JCF} and ES_{JCF} are the Johnson VaR and ES for the estimated skewness and kurtosis within the locus of valid Cornish-Fisher quantile functions. VaR_{Jall} and ES_{Jall} are the Johnson VaR for the whole sample of estimated skewness and kurtosis. Failure CF is the proportion of time-series for which the estimated skewness and kurtosis are outside the locus of valid Cornish-Fisher quantile functions. Failure GC is the proportion of time-series for which the estimated skewness and kurtosis are outside the locus of valid Gram-Charlier density functions. Failure hhh are the proportion of cases for which the Hill et al. [1976] algorithm has failed to converge when computing the VaR or ES with the Johnson approach.