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# American option pricing under GARCH by a Markov chain approximation

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**Abstract**

We propose a numerical method for valuing American options in general and for the GARCH option pricing model in particular. The method is based on approximating the underlying asset price process by a finite-state, time-homogeneous Markov chain. Since the Markov transition probability matrix can be derived analytically, the price of an American option can be computed by simple matrix operations. The Markov transition probability matrix is typically sparse. The use of a sparse matrix representation can substantially increase the dimension of the Markov chain to obtain better numerical results. The Markov chain method works well for the GARCH option pricing framework, and it serves as an alternative to the existing numerical methods for the valuation of American options in other pricing settings. We provide a convergence proof for the Markov chain method and analyze its numerical performance for the Black–Scholes (1973) and GARCH option pricing models. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Most financial time series are subject to stochastic changes in volatility over time. Time-varying volatility has become an important and indisputable empirical fact. The ARCH family of models first proposed by Engle (1982) has increasingly gained prominence as the leading model for describing time-varying volatility. Its popularity is evident by referring to the comprehensive review of the earlier ARCH literature in Bollerslev et al. (1992). Stochastic asset volatility has, among others, an important implication on the valuation of derivative securities. Duan (1995) has developed an option pricing model in which the underlying asset follows a generalized ARCH (or GARCH) process of Bollerslev (1986). This model has, so far, experienced some empirical successes.[1] The purpose of this article is to devise a new numerical method for pricing European and American options, particularly in the GARCH framework. The basic idea involves the use of a finite-state, time-homogeneous Markov chain as the approximation device.[2]

Until very recently, Monte Carlo simulation was the only numerical method for option valuation in the GARCH framework. Unfortunately, the Monte Carlo method is inherently ill-suited for dealing with early exercise decisions required for pricing American options. Recently, three Monte Carlo simulation methods have been proposed by Tilley (1993), Barraquand and Martineau (1995) and Broadie et al. (1997) for pricing American options. These procedures are numerically feasible for the simpler pricing framework like the Black–Scholes model (1973) if the number of early exercise possibilities are restricted.

Geske and Johnson's (1984) approximation scheme is an alternative way of numerically assessing American option values in the Black–Scholes framework. In order to apply their idea in the GARCH setting, one must be able to evaluate analytically, for any intermediate time points, the maximum of the immediate exercise value and the option value if it is kept alive. This, however, poses a formidable challenge in the GARCH setting, even if the analytical formula for European options is available. Because the value of an option at any future time point (except the expiration date) is a function of both the asset price and its conditional volatility, its present value requires assessing the integral of the maximum of the live value of the option and its premature exercise value. The evaluation of this integral requires an analytical formula for the joint

[1] For examples, Amin and Ng (1993), Heynen et al. (1994), Duan (1996a), Heston and Nandi (1997) and Duan and Zhang (1999).

[2] The first article in the derivatives literature involving the use of Markov chain approximation is, to our knowledge, Zipkin (1993). He develops a model for the evaluation of mortgage-backed securities in which a Markov chain is used to approximate a continuous-time interest rate process.

distribution of the asset price and its conditional volatility over an arbitrary time interval. Such an analytical formula is, unfortunately, not yet known at this moment.

A numerical method that can handle the case of American options more efficiently than Monte Carlo simulation is the binomial tree method. Although the method works well in the constant volatility case, its generalization to time-varying volatility is not a simple task. Ritchken and Trevor (1999) have proposed a lattice scheme that is efficient for option pricing in the GARCH and bivariate stochastic volatility frameworks. They accomplished the difficult task of designing a suitable lattice approximation scheme for the GARCH model by restricting the storage of conditional variance to the maximum and minimum values at every node of the discretized asset price. Since the maximum and minimum volatilities change with nodes, the partition for the conditional volatility cannot be fixed beforehand. Leisen (1997) developed a different numerical scheme specifically for the bivariate diffusion models. He constructed a two-dimensional grid for the price and volatility that is path-independent and recombining. The convergence to the bivariate diffusion model is accomplished by adjusting the transition probabilities. Leisen's (1997) method is not applicable to the GARCH model, however, because the discretized system directly converges to the bivariate diffusion model, but not the GARCH process. In contrast, Ritchken and Trevor's (1999) method provides a two-step convergence scheme, first to the GARCH process and then to the bivariate diffusion model by relying on the result of Nelson (1990) and Duan (1996b,1997).

The Markov chain method proposed in this paper relies on making the stock price and volatility discrete. In contrast to Ritchken and Trevor (1999), we deal with a fixed set of stock prices and volatilities. This fixed partitioning of the state space simplifies dramatically the task of option pricing to a sequence of standard matrix operations. Under a Markov chain representation, the conditional expected value is simply a product of two components. For European options, the first component is the transition probability matrix raised to a power equal to the maturity of the option (measured in terms of the basic transition period), and the second is the payoff vector associated with the option. For American options, the use of the dynamic programming idea requires only repeated revaluation of the one-period discounted conditional average against the early exercise value before proceeding to the next matrix multiplication.

We show that the approximating Markov chain converges to its target GARCH process. The option value computed using the Markov chain method also converges properly to its theoretical value. Numerically, the procedure is helped by the fact that the transition probability matrix is highly sparse. The sparsity is a natural consequence resulting from the properties of the target GARCH process.

The Markov chain approximation method is applied to the Black–Scholes and GARCH pricing frameworks. Our numerical analyses suggest that the method works very well for pricing American options in the Black–Scholes setting by comparing to the values computed with a 10,000-step binomial lattice. For the GARCH pricing framework, the Markov chain method produces acceptable values, using penny accuracy, in comparison with the price estimates obtained by a 200,000 sample path Monte Carlo simulation. The convergence patterns for American options are also presented.

## 2. Option pricing under GARCH – the Markov chain approach

### 2.1. *Option pricing under* GARCH

This section briefly reviews the GARCH option pricing model of Duan (1995). This pricing model was established using an equilibrium argument, and it prescribes a dynamic under which options can be priced in a discrete-time GARCH setting. In Kallsen and Taqqu (1998), a continuous-time version of the GARCH model was proposed so that a continuous hedging argument can be invoked in providing an alternative way of arriving at the same option pricing result. We now describe the GARCH option pricing model using the standard discrete-time GARCH specification. Particularly, we restrict the model to a version of the GARCH process known as the non-linear asymmetric GARCH(1, 1), NGARCH(1, 1) for short, process that first appeared in Engle and Ng (1993). Our pricing results can, nevertheless, be extended to all GARCH(1, 1) specifications with little effort. The NGARCH(1, 1) process is chosen over the standard linear GARCH(1, 1) model because it permits the leverage effect, an important feature of asset returns. Of course, other GARCH models such as the EGARCH process by Nelson (1991) and the GJR-GARCH process by Glosten et al. (1993) can also capture the leverage effect. According to Engle and Ng (1993), the GJR-GARCH model performs best, but Duan (1997) concluded that the NGARCH model is better. Since there is no conclusive evidence in favor of a particular GARCH specification, our choice of the NGARCH model is better viewed as for the demonstration purpose for the time being. We will later describe how a critical step could be modified for other GARCH specifications.

Specifically, the asset return dynamics under the data generating probability measure $P$ is

$$\ln\frac{S_{t+1}}{S_t} = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1}, \tag{1}$$

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t(\varepsilon_t - \theta)^2, \tag{2}$$

$$\varepsilon_{t+1}|\mathscr{F}_t \overset{\mathrm{P}}{\sim} \mathrm{N}(0,1), \tag{3}$$

where $r$ is the one-period, continuously compounded return on the risk-free security, $\lambda$ is a constant unit risk premium, $h_{t+1}$ is the conditional variance of the asset return, and $\varepsilon_{t+1}$, conditional on the time-$t$ information $\mathscr{F}_t$ (the $\sigma$-field generated by $(S_0, h_0, \varepsilon_s$: $s \in \{1, 2, \ldots, t\})$, is a standard normal random variable. The conditional variance follows a non-linear asymmetric GARCH (NGARCH) process with the typical GARCH parameter restrictions: $\beta_0 > 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$. Parameter $\theta$ determines the 'leverage effect'. A positive $\theta$ signifies a negative correlation between the innovations for the asset return and its conditional volatility.

Invoking the pricing result in Duan (1995), the asset return process under the locally risk-neutralized pricing measure $Q$ can be written as

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \epsilon_{t+1}, \tag{4}$$

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t (\epsilon_t - \theta - \lambda)^2, \tag{5}$$

$$\epsilon_{t+1} | \mathscr{F}_t \overset{Q}{\sim} \mathrm{N}(0, 1), \tag{6}$$

where $\epsilon_{t+1} = \varepsilon_{t+1} + \lambda$ is a standard normal random variable under the locally risk-neutralized probability measure $Q$. Note that the risk-neutralized system remains to be an NGARCH model with only the change in the leverage parameter to $\theta + \lambda$. The stationary variance of $\epsilon_t$, under measure $Q$ and denoted by $h^*$, was shown in Duan (1995) to be $\beta_0 \{1 - \beta_1 - \beta_2 [1 + (\theta + \lambda)^2]\}^{-1}$, which will be needed later. The risk-neutralized system serves as the backbone for option pricing because the value of a European derivative contract can be computed by simply taking the discounted average under measure $Q$.

If we use the GJR-GARCH process as an alternative to the NGARCH model, the volatility dynamic in Eq. (2) is replaced by

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \varepsilon_t^2 + \beta_3 h_t \max(-\varepsilon_t, 0)^2, \tag{7}$$

where $\beta_3 > 0$ captures the leverage effect. The corresponding locally risk-neutralized volatility dynamic becomes

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t (\epsilon_t - \lambda)^2 + \beta_3 h_t \max(-\epsilon_t + \lambda, 0)^2. \tag{8}$$

Similarly, we can use the EGARCH model to replace the volatility dynamic in Eq. (2) by

$$\ln(h_{t+1}) = \beta_0 + \beta_1 \ln(h_t) + \beta_2 (|\varepsilon_t| - \gamma \varepsilon_t), \tag{9}$$

where the combination of $\beta_2 > 0$ and $\gamma > 0$ captures the leverage effect. The locally risk-neutralized volatility dynamic becomes

$$\ln(h_{t+1}) = \beta_0 + \beta_1 \ln(h_t) + \beta_2[|\epsilon_t - \lambda| - \gamma(\epsilon_t - \lambda)]. \tag{10}$$

### 2.2. The Markov chain approach

Let the payoff function at time $T$ of a European-style option be $f(S_T, K)$. Using the asset return process under the locally risk-neutralized probability measure, its time-$t$ value must be $e^{-r(T-t)}E^Q[f(S_T, K)|\mathscr{F}_t]$. If the option contract is of the American style, the valuation problem becomes much more complicated.

First, we need to represent the underlying GARCH model as a bivariate Markovian system; that is, the bivariate system for $(S_t, h_{t+1})$ is Markovian. Although the payoff function of the option at the terminal time $T$ is not a direct function of $h_{T+1}$, the early exercise decision depends on the level of volatility, because it determines the live value of an option. When one makes the early exercise decision, it is important to consider all possible paths that could be taken by the future conditional volatility. The current level of the conditional volatility therefore determines in part its future dynamic evolution. Clearly, the conditional volatility must enter the valuation function for American options. Denote the time-$t$ value of the American option by $V(S_t, h_{t+1}, t)$. Since the bivariate system is Markovian, we can invoke the following dynamic programming principle to value the American option recursively:

$$V(S_t, h_{t+1}, t) = \max\{f(S_t, K), e^{-r}E^Q[V(S_{t+1}, h_{t+2}, t+1)|\mathscr{F}_t]\}, \tag{11}$$

where

$$V(S_T, h_{T+1}, T) = f(S_T, K). \tag{12}$$

Valuation for European options can be viewed as a special case of Eq. (11) because if there is no early exercise possibility, the recursive formula simplifies to the discounted expected future option value on a period-by-period basis, and eventually becomes $e^{-r(T-t)}E^Q[f(S_T, K)|\mathscr{F}_t]$. Implementing Eq. (11) to value American options requires the computation of conditional expected option values at each point of time whenever early exercise is possible. This dynamic programming formulation presents a numerical problem even in the simple Black–Scholes framework.

To handle this problem, the method proposed in this paper uses a Markov chain process to approximate the asset price and conditional volatility under the risk-neutralized pricing measure. Formally, the stock prices are discretized into $m$ different values while the volatilities can take $n$ different values. Consider an

$mn \times 1$ vector $\bar{V}(t)$ containing the approximate option values at time $t$ for all possible states. Denote by $\Pi$ the $mn \times mn$ Markov transition probability matrix for the bivariate system. Element $\pi(i,j;k,l)$ of this matrix represents the probability of going from price and volatility states $(i,j)$ at time $t$ to price and volatility states $(k,l)$ at time $t+1$ with the elements of $\Pi$ ordered as follows:

$$\Pi = \begin{bmatrix} \pi(1,1;1,1) & \pi(1,1;2,1) & \cdots & \pi(1,1;m,1) & \pi(1,1;1,2) & \cdots & \pi(1,1;m,n) \\ \pi(2,1;1,1) & \pi(2,1;2,1) & \cdots & \pi(2,1;m,1) & \pi(2,1;1,2) & \cdots & \pi(2,1;m,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi(m,1;1,1) & \pi(m,1;2,1) & \cdots & \pi(m,1;m,1) & \pi(m,1;1,2) & \cdots & \pi(m,1;m,n) \\ \pi(1,2;1,1) & \pi(1,2;2,1) & \cdots & \pi(1,2;m,1) & \pi(1,2;1,2) & \cdots & \pi(1,2;m,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi(m,n;1,1) & \pi(m,n;2,1) & \cdots & \pi(m,n;m,1) & \pi(m,n;1,2) & \cdots & \pi(m,n;m,n) \end{bmatrix}. \tag{13}$$

Note that we have assumed the Markov chain is time-homogenous. The GARCH model is indeed such a case.

Let $\bar{S}$ be a $mn \times 1$ vector defined as

$$\bar{S}' = [\bar{s}(1), \bar{s}(2), \ldots, \bar{s}(m), \ldots, \bar{s}(1), \bar{s}(2), \ldots, \bar{s}(m)]. \tag{14}$$

In other words, $\bar{S}$ contains $m$ discretized values for the asset price, repeated for $n$ different values of the conditional volatilities.[3] Using this Markov chain representation for the underlying states simplifies the computation of the dynamic programming relationship in Eq. (11), which can now be implemented as follows:

$$\bar{V}(t) = \max[g(\bar{S}, K), \mathrm{e}^{-r}\Pi\bar{V}(t+1)] \tag{15}$$

with

$$\bar{V}(T) = g(\bar{S}, K),$$

where max is a vector-valued function applied element by element. For American equity put options $g(\bar{S}, K) = \max\{(K\mathbf{1} - \bar{S}), \mathbf{0}\}$ where $\mathbf{0}$ and $\mathbf{1}$ denote vectors of zeros and ones. For American equity call options when the underlying asset does not pay dividends, early exercise is never optimal (see Merton, 1973). Indeed, there is no need to proceed recursively in this Markov chain setting, and

[3] Although the option payoff vector is typically defined only in terms of the underlying asset price, repetition is necessary because the use of a Markovian representation for the GARCH process enlarges the relevant dimension of the system.

the solution to Eq. (15) can be simplified to

$$\bar{V}(0) = \mathrm{e}^{-r\mathrm{T}}\Pi^{\mathrm{T}}\max\{\bar{S} - K\mathbf{1}, \mathbf{0}\}. \tag{16}$$

## 3. Implementing the Markov chain approach

### 3.1. A Markov chain approximation for the GARCH process

The target of our approximation is the GARCH process under the locally risk-neutralized probability measure $Q$, with its precise description already given in the preceding section. The first difficulty in using a finite-state Markov chain approximation for the stock price process is the existence of a trend in the target process, particularly for the problem of pricing long-term options. Allowing for the target process to have an unchecked upward drift will require setting a large value for the maximum state. If the number of states is fixed, the partition will have to be coarse. To avoid this problem, we will build a Markov chain based on an adjusted asset price defined as

$$S_t^* = \mathrm{e}^{-(r-(1/2)h^*)t}S_t. \tag{17}$$

Recall that $h^*$ is the stationary variance under measure $Q$. This adjustment is used to remove the inherent growth trend in the unadjusted asset price.[4] Since the adjustment factor is deterministic, a compounding factor can be easily incorporated back later to recover the pre-adjusted asset price. Apart from numerical considerations, approximating $S_t^*$ is equivalent to approximating $S_t$.

Using this definition, the logarithm of the adjusted price and conditional variance are partitioned into $m$ and $n$ states, respectively.[5] Denote the logarithm of the adjusted asset price in state $i$ by $\bar{p}(i)$ for $i = 1, \dots, m$ and the logarithm of the conditional variance by $\bar{q}(j)$ for $j = 1, \dots, n$. The Markov transition probability from state $(i,j)$ at time $t$ to state $(k,l)$ at time $t+1$ is defined as

$$\pi(i,j;k,l) = \Pr{}^{Q}\{p_{t+1} = \bar{p}(k), q_{t+2} = \bar{q}(l) | p_t = \bar{p}(i), q_{t+1} = \bar{q}(j)\}$$
$$\text{for } t = 0, \dots, T-1, \tag{18}$$

[4] By Eq. (4), we have $\ln(S_{t+1}^*/S_t^*) = \frac{1}{2}(h^* - h_{t+1}) + \sqrt{h_{t+1}}\epsilon_{t+1}$. Since $\mathrm{E}^Q(h_{t+1}) = h^*$, the one-step continuously compounded return on the adjusted price has a long-run average of zero.

[5] We use the logarithm of the asset price and volatility because this transformation is likely to yield distributions that are more symmetrical around their respective means. For example, the adjusted asset price process has a conditionally log-normal distribution, and the use of the logarithmic transformation is thus ideal. This is not true for the conditional volatility, however. Using the logarithmic volatility is motivated in part by the fact that the conditional volatility of the GARCH process is positively skewed. Our numerical experience shows that partitioning logarithmic volatility gives a better performance than simply partitioning volatility.

where $p_t$ denotes the logarithm of the adjusted asset price at time $t$ and $q_{t+1}$ the logarithm of the conditional variance known at time $t$. Notice that $\pi(i,j;k,l)$ is not indexed by time. This is because the transition probability is time homogeneous. The dimension of this Markov transition probability matrix is $mn \times mn$.

To compute the transition probabilities, we first set the discrete values for the asset price and its conditional variance. To do this, we define an interval that covers a chosen set of discrete asset prices and variances. Our chosen discrete values are equidistant points inside this interval. Specifically, for the logarithm of the adjusted asset price, we consider a symmetric interval around $p_0$, the logarithm of the initial asset price. This interval is written as $[p_0 - I_p, p_0 + I_p]$ where the determination of $I_p$ is based on the conditional standard deviation of the logarithm of the adjusted asset price over the life of the option contract.

For the conditional variance, it might not be desirable to center the interval around $q_1$, the logarithm of the initial variance. The option valuation is performed in accordance with the locally risk-neutralized system and this system's conditional volatility reverts to its stationary level, $h^*$, when the maturity becomes longer. Centering a symmetric interval around $q_1$ might result in a bad coverage of the possible values for the variance. To allow for a better coverage, we define a symmetric interval around some value denoted by $q_1^*$. This interval is written as $[q_1^* - I_q, q_1^* + I_q]$ where the determination of $I_q$ is based on the dispersion of the logarithm of the conditional variance. A simple choice for $q_1^*$ is the logarithm of a weighted sum between $h_1$ and $h^*$ with the weights determined by the maturity of the option. For a short-maturity option, a case for which the volatility process has less time to revert to its long-run mean, more weight would be put on $h_1$. For a long-maturity option, a case where the process has more time to revert to its long-run mean, more weight will be put on $h^*$. We thus define $q_1^*$ with

$$q_1^* = \ln\left(\frac{\tau - \min(T,\tau)}{\tau} h_1 + \frac{\min(T,\tau)}{\tau} h^*\right), \tag{19}$$

where $T$ is the maturity of the option to be valued and $\tau$ is some preset value for determining weighting. For example, if $\tau$ equals three months, the volatility partitioning will be centered at the logarithm of $h^*$ for all options with a maturity longer than or equal to three months. For shorter term options, the weighting is tilted more toward the initial conditional volatility, $h_1$. It is important to ensure that $q_1 \in [q_1^* - I_q, q_1^* + I_q]$. This can always be achieved by increasing $\tau$. This strategy is also intuitively attractive, because if the current conditional volatility is far away from the stationary volatility, it is better to move the center of the interval closer to the current conditional volatility.

Once the overall interval for the logarithmic asset price is determined, we divide it equally into $m - 1$ cells to yield $m$ discrete values for the asset price.

Likewise, we have $n$ discretized conditional volatilities. We require $m$ and $n$ to be odd integers. For the logarithm of the adjusted asset price, we consider $m$ cells corresponding to $m$ asset prices, $\bar{p}(i)$, $i = 1, \ldots, m$:

$$C(i) = [c(i), c(i+1)) \quad \text{for } i = 1, \ldots, m, \tag{20}$$

where $c(1) = -\infty$, $c(i) = (\bar{p}(i) + \bar{p}(i-1))/2$ for $i = 2$ to $m$ and $c(m+1) = +\infty$. For the logarithm of the conditional variance, the intervals are defined as

$$D(i) = [d(i), d(i+1)) \quad \text{for } i = 1, \ldots, n, \tag{21}$$

where $d(1) = -\infty$, $d(i) = (\bar{q}(i) + \bar{q}(i-1))/2$ for $i = 2$ to $n$ and $d(n+1) = +\infty$. We assign the probability of $p_{t+1} = \bar{p}(k)$, conditional on $p_t = \bar{p}(i)$ and $q_{t+1} = \bar{q}(j)$, by the probability of the asset price falling into the cell $C(k)$ under the GARCH process.

By Eqs. (4) and (5), $q_{t+2}$ is a deterministic function of $q_{t+1}$, $p_{t+1}$ and $p_t$. To facilitate the discussion, we define the following function:

$$\begin{aligned}\Phi(q_{t+1}, p_{t+1}, p_t) \equiv \ln\{\beta_0 + \beta_1 \mathrm{e}^{q_{t+1}} + \beta_2 [p_{t+1} - p_t + \tfrac{1}{2}(\mathrm{e}^{q_{t+1}} - h^*) \\ - (\theta + \lambda)\mathrm{e}^{q_{t+1}/2}]^2\}.\end{aligned} \tag{22}$$

This function is obtained by substituting the expression for $\varepsilon_{t+1}$ from Eq. (4) into Eq. (5), i.e.,

$$h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1}\left(\frac{\ln S_{t+1} - \ln S_t - r + (1/2)h_{t+1}}{\sqrt{h_{t+1}}} - \theta - \lambda\right)^2$$

and using the definitions for $q_t, p_t$ and $S_t^*$.

The transition probability under measure $Q$ can be computed as

$$\pi(i,j;k,l) = \begin{cases} \Pr{}^Q\{p_{t+1} \in C(k) | p_t = \bar{p}(i), & \text{if } \Phi(\bar{q}(j), \bar{p}(k), \bar{p}(i)) \in D(l) \\ \quad q_{t+1} = \bar{q}(j)\} & \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

The conditional probability can be analytically computed as follows:

$$\begin{aligned}&\Pr{}^Q\{p_{t+1} \in C(k) | p_t = \bar{p}(i), q_{t+1} = \bar{q}(j)\} \\ &\qquad = \Pr{}^Q\{L_{ij}(k) \le Z < L_{ij}(k+1)\},\end{aligned} \tag{24}$$

where

$$L_{ij}(k) = \frac{c(k) - \bar{p}(i) + \frac{1}{2}[\exp(\bar{q}(j)) - h^*]}{\sqrt{\exp(\bar{q}(j))}} \tag{25}$$

and $Z$ is a standard normal random variable under measure $Q$.

As mentioned earlier in Section 2.1, the Markov chain method can be extended to all GARCH(1, 1) specifications without much effort. A different GARCH(1, 1) specification requires a change to Eq. (22). Specifically, we have for the GJR-GARCH model

$$\begin{aligned}\Phi(q_{t+1}, p_{t+1}, p_t) \\ \equiv \ln\{\beta_0 + \beta_1 e^{q_{t+1}} + \beta_2[p_{t+1} - p_t + \tfrac{1}{2}(e^{q_{t+1}} - h^*) - \lambda e^{q_{t+1}/2}]^2 \\ + \beta_3 \max[-p_{t+1} + p_t - \tfrac{1}{2}(e^{q_{t+1}} - h^*) + \lambda e^{q_{t+1}/2}, 0]^2\}\end{aligned} \tag{26}$$

which is derived from Eq. (8). For the EGARCH model, we can use Eq. (10) to obtain

$$\begin{aligned}\Phi(q_{t+1}, p_{t+1}, p_t) \\ \equiv \beta_0 + \beta_1 q_{t+1} + \beta_2 |e^{-0.5q_{t+1}}[p_{t+1} - p_t + \tfrac{1}{2}(e^{q_{t+1}} - h^*)] - \lambda| \\ - \beta_2\gamma\{e^{-0.5q_{t+1}}[p_{t+1} - p_t + \tfrac{1}{2}(e^{q_{t+1}} - h^*)] - \lambda\}.\end{aligned} \tag{27}$$

The Markov transition probability matrix $\Pi$ resulting from the above calculation is usually a highly sparse matrix. Sparsity comes from two facts. First, Eq. (23) makes it clear that every future stock price uniquely corresponds to a single volatility value if its preceding stock price and volatility level are given. This is, of course, due to the property of the GARCH model. As a result, there are at most $m$ non-zero probabilities for every row of $\Pi$ even though there are $mn$ entries in each row. These $m$ non-zero probabilities do not cluster together, however. This sparsity effectively reduces the size of matrix $\Pi$ from $mn \times mn$ to $mn \times m$. The second source for sparsity comes from the property of the normal distribution. The transition probability from one stock price to another is negligible if two prices are far apart, measured relatively to their standard deviation. We can take advantage of this fact by skipping over the states with negligeable probabilities. This sparsity of the transition probability matrix is a useful property in actual implementations of the method. Special numerical techniques can be implemented for these matrices in order to save on storage and computing time.

In order to apply the Markov chain method, one must decide on the range of values for the asset price and its conditional volatility. There are many ways of setting a value for $I_p$ and $I_q$. The basic idea is to find an interval that extends for $\delta$ standard deviations in each direction from the starting point of the process. For the logarithm of the adjusted asset price, we choose to set $I_p$ equal to

$$I_p = \delta_p(m)\sqrt{\sum_{t=1}^{T} E^Q(h_t|\mathscr{F}_0)}, \tag{28}$$

where $\sum_{t=1}^{T} E^Q(h_t|\mathscr{F}_0)$ is the variance, conditional on the time-0 information set, for $\sum_{t=1}^{T}\sqrt{h_t}\epsilon_t$. Since $\sum_{t=1}^{T}\sqrt{h_t}\epsilon_t$ is the key component of $\ln S_T^*$, we use

$\sum_{t=1}^{T} \mathrm{E}^Q(h_t|\mathscr{F}_0)$ to approximate the variance of $\ln S_T^*$. This sum can be evaluated analytically and the formula for $\mathrm{E}^Q(h_t|\mathscr{F}_0)$ is available in Duan (1995, p.29). Specifically,

$$\mathrm{E}^Q(h_t|\mathscr{F}_0) = h_1 v^{t-1} + \frac{\beta_0(1 - v^{t-1})}{1 - v}, \tag{29}$$

where $v = \beta_1 + \beta_2[1 + (\theta + \lambda)^2]$.[6]

The adjustment factor $\delta_p(m)$ is an increasing function of $m$, satisfying two conditions.

*Partition condition*. (1) $\delta_p(m) \to \infty$ as $m \to \infty$; (2) $\delta_p(m)/m \to 0$ as $m \to \infty$.

The first condition ensures that the interval is eventually extended to cover arbitrarily large (small) values. The second condition is required to ensure that the increase (decrease) in the upper (lower) bound does not outpace the increase in the number of subintervals. In other words, an increase in refinement is achieved as $m$ goes to infinity. This partition condition is used to guarantee that the approximating Markov chain converges to its target GARCH process, a result formally established in Proposition 1 later. We can, for example, consider a scheme that is comparable to the binomial-tree asset price range if we let $\delta_p(m) = \sqrt{(m-1)/2}$. In this case, $I_p$ approaches infinity at a rate of $\sqrt{m}$. If $h_t$ is a constant (the Black–Scholes model), it must equal $h^*$ so that $I_p = \sqrt{[(m-1)/2]Th^*}$. This $I_p$ is identical to the difference between the maximum asset price (adjusted) and the current asset price (adjusted) in a $(m-1)/2$-step binomial tree.

For the logarithm of the conditional variance, we choose to set $I_q$ equal to

$$I_q = \ln[\exp(q_1) + \delta_q(n)\sigma_h] - q_1, \tag{30}$$

where

$$\sigma_h = \sqrt{\mathrm{E}^Q(h_T^2|\mathscr{F}_0) - [\mathrm{E}^Q(h_T|\mathscr{F}_0)]^2}. \tag{31}$$

[6] The formula in (29) has corrected a minor error in Duan (1995), in which the corresponding formula amounts to $\mathrm{E}^Q(h_t|\mathscr{F}_0) = h_0 v^t + \beta_0(1 - v^t)/(1 - v)$. Since $h_1$ is measurable with respect to $\mathscr{F}_0$, the backward recursion used in solving the difference equation should be stopped at $h_1$ instead of $h_0$. If we consider the GJR-GARCH model in Eq. (8), the formula for $\mathrm{E}^Q(h_t|\mathscr{F}_0)$ stays the same except that

$$v = \beta_1 + [\beta_2 + \beta_3 N(\lambda)](1 + \lambda^2) + \beta_3 \lambda n(\lambda),$$

where $N(\cdot)$ and $n(\cdot)$ are the standard normal distribution and density functions, respectively.

Note that the formula for $\mathrm{E}^Q(h_T|\mathscr{F}_0)$ has been given in Eq. (29). The analytical expression for $\mathrm{E}^Q(h_T^2|\mathscr{F}_0)$ can be found on p. 30 of Duan (1995), which is

$$\mathrm{E}^Q(h_T^2|\mathscr{F}_0) = h_1^2 u^{T-1} + 2\beta_0 h_1 \frac{v(u^{T-1} - v^{T-1})}{u - v} + \beta_0^2 \left[ \frac{1 - u^{T-1}}{1 - u} + 2\frac{v}{u - v}\left( \frac{1 - u^{T-1}}{1 - u} - \frac{1 - v^{T-1}}{1 - v} \right) \right], \quad (32)$$

where $v$ has been defined earlier and $u = \beta_2^2[3 + 6(\theta + \lambda)^2 + (\theta + \lambda)^4] + 2\beta_1\beta_2[1 + (\theta + \lambda)^2] + \beta_1^2$.[7] The parameter $\sigma_h$ is the standard deviation, conditional on the time-0 information set, of the conditional volatility $h_T$. The logarithmic transformation simply reflects the fact that we are partitioning the logarithmic volatility, but are unable to directly obtain an analytical expression for the conditional variance of $\ln(h_T)$. The adjustment factor $\delta_q(n)$ is an increasing function of $n$, satisfying the same partition condition required of $\delta_p(m)$.[8]

Because of the logarithmic transformation in setting $I_q$, we need to check whether, under the partition condition, $I_q$ goes to infinity and $I_q/n$ goes to zero as $n$ approaches infinity. It is obvious that $I_q$ goes to infinity. We need to prove the second condition, however.

$$\begin{aligned}
\frac{I_q}{n} &= \frac{\ln[\exp(q_1) + \delta_q(n)\sigma_h] - q_1}{n} \\
&= \frac{\ln[\exp(q_1)(1 + \exp(-q_1)\delta_q(n)\sigma_h)] - q_1}{n} \\
&\leq \frac{q_1 + \exp(-q_1)\delta_q(n)\sigma_h - q_1}{n} \quad \text{(because } \ln(1 + x) \leq x\text{)} \\
&= \exp(-q_1)\sigma_h \frac{\delta_q(n)}{n} \\
&\to 0 \quad \text{(under the partition condition).}
\end{aligned} \quad (33)$$

[7] Similar to an earlier footnote, the formula in (32) has corrected the error in Duan (1995). If we consider the GJR-GARCH model in Eq. (8), the formula for $\mathrm{E}^Q(h_T^2|\mathscr{F}_0)$ stays the same except that

$$v = \beta_1 + [\beta_2 + \beta_3 N(\lambda)](1 + \lambda^2) + \beta_3 \lambda n(\lambda)$$

and

$$\begin{aligned}
u = {} & \beta_1^2 + \beta_2^2(\lambda^4 + 6\lambda^2 + 3) + [\beta_3^2 + 2\beta_2\beta_3](\lambda^4 N(\lambda) + \lambda^3 n(\lambda) + 6\lambda^2 N(\lambda) + 5\lambda n(\lambda) + 3N(\lambda)) \\
& + 2\beta_1\beta_2(1 + \lambda^2) + 2\beta_3\beta_1[\lambda^2 N(\lambda) + \lambda n(\lambda) + N(\lambda)],
\end{aligned}$$

where $N(\cdot)$ and $n(\cdot)$ are the standard normal distribution and density functions, respectively.

[8] We find convenient to set $I_q$ as in Eq. (30). Alternatively, one can set $I_q = \delta_q(n)\ln(\sigma_h)$.

This result implies that the variance partition is indeed properly refined as $n$ increases.

Does this approximating Markov chain properly converge to its target GARCH model? Since the entries of the transition probability matrix have been computed in accordance with the GARCH model, it is reasonable to think that this approximation can work. We now give a formal argument as to why this is the case. We first denote the sample path of the two-dimensional Markov process over $(0, 1, \ldots, T)$ by $(\boldsymbol{p}_T^{(m,n)}, \boldsymbol{q}_T^{(m,n)}) \equiv (p_0, q_1, p_1^{(m,n)}, q_2^{(m,n)}, \ldots, p_T^{(m,n)}, q_{T+1}^{(m,n)})$. Since we are only interested in the dynamic conditional on the pair of transformed current asset price and volatility, i.e., $(p_0, q_1)$, the relevant sample space for the approximating Markov chain and its target GARCH process is $(\boldsymbol{R}^{2T}, \mathscr{B})$ where the first entry denotes the $2T$-dimensional Euclidean space and the second the Borel $\sigma$-field generated by $\boldsymbol{R}^{2T}$. The approximating Markov chain and the target GARCH process differ in their distributions over $\boldsymbol{R}^{2T}$, however. As defined earlier, the probability law governing the GARCH process is $Q$. Its induced distribution over $\boldsymbol{R}^{2T}$ is denoted by $F$. For the approximating Markov chain, it depends on both $m$ and $n$ so that it is natural to denote the distribution over $\boldsymbol{R}^{2T}$ by $F^{(m,n)}$. With the notation in place, we can now state the first convergence result as follows.

*Proposition 1. Given the partition condition, $F^{(m,n)}$ converges to $F$ as $m$ and $n$ go to infinity.*

*Proof.* See the appendix.

Simply put, the approximating Markov chain and its target GARCH process share the same distributional characteristics when the partition of the asset price and volatility becomes finer and finer. In other words, the approximating Markov chain reproduces the probabilistic behavior of the target GARCH process.

### 3.2. Computing option prices

Using the Markov chain approximation described in the preceding section, it is now possible to compute the prices of American and European options. The argument of function $g(\cdot)$ needs to be changed because the Markov chain was constructed to approximate the logarithm of the adjusted stock price. More specifically, we can write this function $g(\bar{P}, K, t)$ where $\bar{P}$ is an $mn \times 1$ vector defined as

$$\bar{P}' = [\bar{p}(1), \bar{p}(2), \ldots, \bar{p}(m), \ldots, \bar{p}(1), \bar{p}(2), \ldots, \bar{p}(m)]. \tag{34}$$

Notice also that the option payoff function $g(\bar{P}, K, t)$ must be written as a function of time because $\bar{P}$ contains the logarithm of the adjusted asset prices.

The argument $t$ allows us to undo the earlier adjustment and recover the correct asset prices. For put options, for example, $g(\bar{P}, K, t) = \max\{K\mathbf{1} - \exp[(r - \frac{1}{2}h^*)t\mathbf{1} + \bar{P}], \mathbf{0}\}$ where $\mathbf{0}$ and $\mathbf{1}$ denote vectors of zeros and ones, respectively.

The option value vector at time 0, i.e., $\bar{V}(0)$, is $mn \times 1$ dimensional. The option price of our interest is the one corresponding to the current state. Although the current asset price is situated at the center of the price partition, the current volatility is not, because of the consideration of mean-reversion described earlier in Eq. (19). In order to deal with this complication, we identify two adjacent discretized logarithmic volatilities so that $d(j) \le \ln(h_1) \le d(j+1)$. We denote two elements of $\bar{V}(0)$ with the index numbers $[(j-1)m + (m+1)/2]$ and $[jm + (m+1)/2]$ by $v(j)$ and $v(j+1)$, respectively. The option price corresponding to the initial asset price $S_0$ and volatility $h_1$ is the linearly interpolated value of $v(j)$ and $v(j+1)$ based on the following formula:

$$C(S_0, h_1) = \frac{d(j+1) - \ln(h_1)}{d(j+1) - d(j)} v(j) + \frac{\ln(h_1) - d(j)}{d(j+1) - d(j)} v(j+1). \tag{35}$$

Option pricing in a general sense amounts to evaluating the expectation of some function defined over the whole sample path over $(0, 1, \dots, T)$. Can we be certain that the option price computed by using the approximating Markov chain converges to its theoretical value? Even if we have the weak convergence result in Proposition 1, we cannot be certain that the payoff transformation does not cause difficulties. The following proposition gives us the needed assurance.

*Proposition 2. Assume the partition condition is satisfied.* (i) *If a non-negative real-value continuous payoff function $f(\boldsymbol{p}_T, \boldsymbol{q}_T)$ is dominated by a real-valued, uniformly integrable function $g(\boldsymbol{p}_T, \boldsymbol{q}_T)$ (conditional on $\mathscr{F}_0$ and with respect to $F^{(m.n)}$ for all $(m, n)$), then*

$$\lim_{m,n \to \infty} \mathrm{E}^{F^{(m.n)}}\{f(\boldsymbol{p}_T, \boldsymbol{q}_T) | \mathscr{F}_0\} = \mathrm{E}^F\{f(\boldsymbol{p}_T, \boldsymbol{q}_T) | \mathscr{F}_0\}. \tag{36}$$

(ii) *The stock price $S_t$, for any $t \in \{1, 2, \dots, T\}$, is uniformly integrable (conditional on $\mathscr{F}_0$ and with respect to $F^{(m.n)}$ for all $(m, n)$); moreover,*

$$\lim_{m,n \to \infty} \mathrm{E}^{F^{(m.n)}}\{S_t | \mathscr{F}_0\} = S_0 \mathrm{e}^{rt}. \tag{37}$$

*Remark.* The uniform integrability condition is $\lim_{c \to \infty} \sup_{m,n} \int_{\{g > c\}} g \, \mathrm{d}F^{(m,n)} = 0$. The stock price $S_t = \exp\{p_t + (r - \frac{1}{2}h^*)t\}$ by our definition of $p_t$.

*Proof.* See the appendix.

First note that $F^{(m,n)}$ assigns zero probability to the points that are not equal to the discretized value of the Markov chain (based on $m$ asset prices and $n$ volatilities). Integrating over $\boldsymbol{R}^{2T}$ under $F^{(m,n)}$ is the same as considering only the discretized values with positive probabilities. This proposition can be used to ensure convergence for many standard contingent claims. For standard European call options, the dominating function can simply be the asset price itself, which satisfies uniform integrability by part (ii) of Proposition 2. In terms of European put options, the dominating function can continue to be the asset price itself or a constant equal to the strike price. In either case, the dominating function is uniformly integrable. Similarly, one can easily identify the dominating functions for many exotic options. Two specific types of exotic options – digital and barrier options – need further discussion because their payoff functions are discontinuous in the asset prices. The above proposition continues to apply to these cases because the GARCH model gives rise to a continuous multivariate distribution for the sample path of asset prices. It is well known that a discontinuous payoff function can be approximated by a continuous function to an arbitrary degree of accuracy except at the points of discontinuity. Since the set of discontinuous points have a zero probability, it will not change the convergence result.

Does convergence also hold true for American options? The answer is yes. One can apply the above proposition at time $T-1$ by replacing $\mathscr{F}_0$ with $\mathscr{F}_{T-1}$. For a large enough $(m,n)$-partition, the error is bounded by a small constant. Repeating the same argument, the time $T-2$ error must be bounded by another small constant, and so on. Since there are only a finite number of time steps in this recursive process, the error at time 0 can be made arbitrarily small if one increases $(m,n)$. In short, the consistency of our proposed approximation scheme is guaranteed.

Amin and Khanna (1994) used some intricate weak convergence arguments involving stopping time to show that the typical numerical solution based on a Markov chain converges to the theoretical value of the American option if the target model is a diffusion process. The simplicity of our argument for convergence of the American option value is made possible by the fact that the target GARCH model is of discrete time so that only a finite set of early exercise points is available. The result of Amin and Khanna (1994) can be conveniently combined with our result to solve the numerical valuation problem for the bivariate stochastic volatility model. Since the GARCH model is known to weakly converge to the bivariate stochastic model (Nelson, 1990; Duan, 1996b, 1997), Amin and Khanna's (1994) result implies that theoretical value of an American option under the GARCH model converges, by shrinking time step, to its counterpart under the bivariate stochastic volatility model. The fact that our Markov chain price converges to the theoretical value under the GARCH model actually completes a two-step convergence algorithm for option pricing in the bivariate stochastic volatility framework.

Proposition 2, although somewhat abstract, is a more powerful result in comparison with the indirect valuation method outlined in Duffie (1992, p. 199). The indirect method calls for first numerically valuing the put option and then converting to the price for the call via the put-call parity relationship. The indirect approach hits its limit when, for example, an American call on an asset paying dividends needs to be valued. The put-call parity relationship for American options is an inequality relationship, and thus insufficient in providing an exact conversion from the put to the corresponding call.[9] Our approach of using a uniformly integrable function to bound the payoff ensures a much wider applicability of the convergence result.

## 4. Numerical results

Our numerical analyses will be divided into two parts. First, we analyze the use of the Markov chain approximation for the Black–Scholes model. For European options, the benchmark is the Black–Scholes formula. In the case of American options, we use a 10,000-step binomial tree to produce the benchmark values. In the second part of our numerical analyses, we implement the Markov chain approximation method for the GARCH option pricing model. For European options, we compare the results to the ones obtained by Monte Carlo simulations. Since no benchmark is available for checking the pricing accuracy of American options, we thus focus on their convergence patterns.

### *4.1. The Black and Scholes model*

The Markov chain approximation method can be specialized to the Black–Scholes model. Although the familiar binomial-tree method can be interpreted as a special Markov chain method, its assignment of the transition probability has nothing to do with the Markov chain method proposed in this paper. Apart from the difference in assigning transition probabilities, a major difference is in terms of the relationship between the number of time steps and the number of discrete asset values. In the case of the binomial-tree method, the length of a time step and the number of asset prices generated by the tree are simultaneously determined for a particular maturity. For the Markov chain

[9] If the underlying asset price follows a geometric Brownian motion, the indirect valuation method is still workable for American call options with the underlying asset paying dividends. In such a case, there exists a put-call symmetry relationship between American call and put options. Broadie and Detemple (1996, p. 1220) has employed such a relationship in their study.

method, the length of a time step and the number of discrete asset prices are independently set.

Table 1 presents the numerical results for European put options by both methods. We consider three maturities (1, 3 and 9 months) and three strike-to-asset price ratios (1.1, 1.0 and 0.9). In the numerical analyses, we assume 365 days per year for the annualizing purpose. The current asset price is set at 50. The annualized risk-free rate (continuously compounded) and the asset's standard deviation are 5% and 20%, respectively. The first row of this table presents the theoretical prices for the European puts.

The first panel of this table presents the European option prices according to the binomial-tree method with different numbers of steps. Observe that a $((m-1)/2)$-step binomial tree also yields $m$ different asset prices (the prices of the last 2 steps). This allows us to interpret the binomial-tree method from the perspective of a Markov chain approximation. These values are computed with the Jarrow and Rudd (1983) version of the binomial-tree method, because their method constructs the binomial tree for the adjusted asset price. As the results

Table 1
European put options in the Black and Scholes framework[a]

| | Maturity = 1 month | | | Maturity = 3 months | | | Maturity = 9 months | | |
|---|---|---|---|---|---|---|---|---|---|
| $K/S_0$ | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 |
| *Theoretical prices* | | | | | | | | | |
| | 4.8457 | 1.0416 | 0.0293 | 4.9098 | 1.6769 | 0.2706 | 5.2318 | 2.5343 | 0.9150 |
| *Binomial tree prices* ($(m-1)/2$ *steps*) | | | | | | | | | |
| $m = 11$ | 4.8361 | 1.0987 | 0.0283 | 4.9635 | 1.7715 | 0.2846 | 5.0865 | 2.6773 | 0.9718 |
| $m = 51$ | 4.8461 | 1.0520 | 0.0274 | 4.9160 | 1.6912 | 0.2735 | 5.2614 | 2.5403 | 0.9291 |
| $m = 101$ | 4.8454 | 1.0418 | 0.0291 | 4.9066 | 1.6821 | 0.2646 | 5.2469 | 2.5503 | 0.9237 |
| $m = 201$ | 4.8446 | 1.0426 | 0.0292 | 4.9122 | 1.6811 | 0.2715 | 5.2298 | 2.5411 | 0.9181 |
| $m = 301$ | 4.8450 | 1.0427 | 0.0292 | 4.9114 | 1.6801 | 0.2690 | 5.2364 | 2.5361 | 0.9182 |
| $m = 401$ | 4.8457 | 1.0426 | 0.0292 | 4.9102 | 1.6793 | 0.2692 | 5.2274 | 2.5328 | 0.9158 |
| $m = 501$ | 4.8455 | 1.0425 | 0.0291 | 4.9079 | 1.6787 | 0.2695 | 5.2342 | 2.5315 | 0.9146 |
| *Markov chain prices* (*time step* = 1 *month*), $I_p = [2 + \ln(\ln(m))]\sigma\sqrt{T}$) | | | | | | | | | |
| $m = 11$ | 4.8478 | 1.0363 | 0.0304 | 4.9716 | 1.7270 | 0.2832 | 5.5170 | 2.8340 | 1.1487 |
| $m = 51$ | 4.8452 | 1.0427 | 0.0290 | 4.9129 | 1.6819 | 0.2729 | 5.2466 | 2.5484 | 0.9297 |
| $m = 101$ | 4.8455 | 1.0419 | 0.0292 | 4.9106 | 1.6772 | 0.2708 | 5.2348 | 2.5389 | 0.9189 |
| $m = 201$ | 4.8457 | 1.0416 | 0.0292 | 4.9098 | 1.6769 | 0.2706 | 5.2330 | 2.5357 | 0.9158 |
| $m = 301$ | 4.8457 | 1.0416 | 0.0292 | 4.9098 | 1.6768 | 0.2706 | 5.2321 | 2.5348 | 0.9154 |
| $m = 401$ | 4.8457 | 1.0416 | 0.0292 | 4.9098 | 1.6768 | 0.2706 | 5.2320 | 2.5345 | 0.9151 |
| $m = 501$ | 4.8457 | 1.0416 | 0.0293 | 4.9098 | 1.6769 | 0.2706 | 5.2319 | 2.5345 | 0.9151 |

[a] Parameters: $S_0 = 50$, $r = 0.05$ (annualized), $\sigma = 0.20$ (annualized).

indicate, the binomial-tree prices exhibit a typical jagged convergence pattern.[10] The Markov chain option prices for different numbers of states are presented in the second panel of this table. This panel uses one month as the basic transition time step. The value for $I_p$ is computed using $\delta(m) = 2 + \ln(\ln(m))$. As can be seen, the method converges to the theoretical prices as the number of states is increased. Furthermore, the Markov chain method seems to converge faster than the binomial tree method.[11]

The results for American options are presented in Table 2. These prices are compared to the binomial-tree prices computed with 10,000 steps, which, for all practical purposes, can be considered as the theoretical values on the basis of continuous early exercise. For the Markov chain method, we again consider the double logarithmic function to determine the asset price range. The values in the first panel are based on the $(m-1)/2$-step binomial tree, where $m$ is the number of discrete asset prices. Under the binomial-tree structure, the American option has $(m-1)/2$ early exercise opportunities. An increase in the number of discrete asset prices can result only from an increase in the number of binomial steps. In contrast to the binomial-tree method, an increase in the number of discrete asset prices under the Markov chain method does not require an increase in the early exercise opportunities. In Table 2, we consider the case of the number of time steps equal to the maturity (in number of days) of the option contracts. Because of premature exercises are only allowed on a daily basis, the theoretical price that the Markov chain method attempts to approximate should be lower than the theoretical price under premature exercises on a continuous time basis.

For the case of constant volatility, a fairly accurate price can be obtained in fractions of a second on a standard desktop computer. The computing time will of course depend on the number of time steps as well as the length of a time step. A smaller transition time step will make the transition probability matrix more sparse. This will result in a shorter computation time for building and multiplying the transition probability matrix. A smaller transition step will, however, require more steps and matrix multiplications in computing option prices.

[10] As discussed in Broadie and Detemple (1996), it is possible to modify the binomial-tree method so that this oscillatory pattern is removed and the convergence speed improved. The modification calls for replacing the calculation of the option values in the last time step with the value from the Black–Scholes formula. This modification, when applied to the Markov chain method, does not alter the pattern of convergence.

[11] Although not reported here, other functional forms for $\delta(m)$ were also considered. We checked the performance of the Markov chain using $\ln(m)$ and $\sqrt{(m-1)/2}$ as the factors for determining the asset price range. These alternative specifications were found to converge more slowly to the theoretical prices.

Table 2
American put options in the Black and Scholes framework[a]

| | Maturity = 1 month | | | Maturity = 3 months | | | Maturity = 9 months | | |
|---|---|---|---|---|---|---|---|---|---|
| $K/S_0$ | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 |
| *Binomial tree prices* (10,000 *steps*) | | | | | | | | | |
| | 5.0001 | 1.0567 | 0.0295 | 5.1608 | 1.7295 | 0.2758 | 5.7473 | 2.7182 | 0.9637 |
| *Binomial tree prices* ($(m-1)/2$ *steps*) | | | | | | | | | |
| $m = 11$ | 5.0000 | 1.1166 | 0.0283 | 5.2043 | 1.8249 | 0.2915 | 5.7029 | 2.8353 | 0.9919 |
| $m = 51$ | 5.0000 | 1.0682 | 0.0276 | 5.1649 | 1.7459 | 0.2783 | 5.7628 | 2.7301 | 0.9738 |
| $m = 101$ | 5.0000 | 1.0571 | 0.0293 | 5.1611 | 1.7344 | 0.2706 | 5.7565 | 2.7305 | 0.9719 |
| $m = 201$ | 5.0000 | 1.0578 | 0.0294 | 5.1625 | 1.7336 | 0.2770 | 5.7475 | 2.7245 | 0.9669 |
| $m = 301$ | 5.0000 | 1.0579 | 0.0295 | 5.1620 | 1.7327 | 0.2747 | 5.7499 | 2.7207 | 0.9665 |
| $m = 401$ | 5.0000 | 1.0578 | 0.0294 | 5.1614 | 1.7320 | 0.2747 | 5.7459 | 2.7182 | 0.9642 |
| $m = 501$ | 5.0000 | 1.0577 | 0.0293 | 5.1602 | 1.7315 | 0.2750 | 5.7490 | 2.7168 | 0.9637 |
| *Markov chain prices* (*time step* = 1 *day*, $I_p = [2 + \ln(\ln(m))]\sigma\sqrt{T}$) | | | | | | | | | |
| $m = 11$ | 5.0000 | 1.0991 | 0.0465 | 5.0000 | 0.5763 | 0.0082 | 5.0000 | 0.0013 | 0.0000 |
| $m = 51$ | 5.0000 | 1.0803 | 0.0333 | 5.2230 | 1.8465 | 0.3370 | 6.1077 | 3.1438 | 1.2918 |
| $m = 101$ | 5.0000 | 1.0627 | 0.0306 | 5.1765 | 1.7611 | 0.2921 | 5.8734 | 2.8720 | 1.0791 |
| $m = 201$ | 5.0000 | 1.0576 | 0.0297 | 5.1637 | 1.7375 | 0.2801 | 5.7809 | 2.7603 | 0.9948 |
| $m = 301$ | 5.0000 | 1.0567 | 0.0296 | 5.1612 | 1.7327 | 0.2778 | 5.7620 | 2.7372 | 0.9779 |
| $m = 401$ | 5.0000 | 1.0563 | 0.0295 | 5.1603 | 1.7310 | 0.2768 | 5.7551 | 2.7288 | 0.9716 |
| $m = 501$ | 5.0000 | 1.0561 | 0.0295 | 5.1598 | 1.7301 | 0.2764 | 5.7518 | 2.7248 | 0.9687 |

[a] Parameters: $S_0 = 50$, $r = 0.05$ (annualized), $\sigma = 0.20$ (annualized).

### 4.2. The GARCH option pricing model

To implement the GARCH option pricing model, we need to know the current asset price and its conditional volatility. For the numerical analyses of this section, the initial conditional variance is set equal to the stationary variance under the data generating probability measure $P$; that is, $h_1 = \beta_0[1 - \beta_1 - \beta_2(1 + \theta^2)]^{-1}$. Recall that due to the notational convention of the GARCH literature, $h_1$ is the conditional variance for the first asset price innovation, and it is known at time 0. We use the stationary variance under measure $P$ instead of measure $Q$ because it represents the true average variance from observing the data series. For the centering scheme in Eq. (19), $\tau$ is set equal to 3 months. Note that $h^* = \beta_0[1 - \beta_1 - \beta_2[1 + (\theta + \lambda)^2]]^{-1}$. Since typical GARCH parameter estimates yield a value of $\beta_1 + \beta_2[1 + (\theta + \lambda)^2]$ close to one, reversion to $h^*$ takes a longer time and $\tau$ should not be set too small. We assume the GARCH model for the daily frequency, and use the following set of GARCH parameter values: $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$, $\theta = 0.3$ and $\lambda = 0.2$.

For each option contract, we consider three maturities (1, 3 and 9 months) and three strike-to-asset price ratios (1.1, 1.0 and 0.9). In the numerical analyses, we assume 365 days per year for the purpose of annualization. The current asset price is set at 50. The annualized risk-free rate (continuously compounded) is fixed at 5%. The factors used to determine the asset price and volatility range are: $\delta_p = 2 + \ln(\ln(m))$ and $\delta_q = 2 + \ln(\ln(n))$. We use the double logarithmic transformation because it works well for the Black–Scholes framework. Since we have adopted the GARCH model in terms of the daily frequency, the Markov transition probability matrix over one day can be constructed in accordance with the procedure described in the earlier sections. We are, however, not restricted to using one day as the length of the time step in option valuation, because the Markov transition probability matrix over any number of days can be obtained by raising the daily Markov transition probability matrix to an appropriate power. For the values in Tables 3 and 4, the length of the time step is always kept at one day.

Table 3 presents the results for European call options. Since it is possible to compute the option prices using a Monte Carlo method, we have them as the benchmark for comparison. The benchmark European option prices are computed using a 200,000 sample path control-variate Monte Carlo simulation. The control variable is the Black–Scholes formula price using the stationary variance, under measure $P$, as the volatility. The Monte Carlo prices and their corresponding standard errors are presented in the top two rows of Table 3. The rest of the table reports the Markov chain prices under different partitioning of the states.

The results indicate that the Markov chain method with a moderate number of states can in many cases yield prices within a penny difference from the Monte Carlo prices. Typically, the convergence is slower for longer-term options. The results also suggest that a finer price partitioning seems to perform better in relation to a finer volatility partitioning.

American put option prices under the GARCH setting are reported in Table 4. Although there is no benchmark available for checking the pricing accuracy, the comparison of Tables 3 and 4 clearly indicates a consistency. The American option for any strike price and maturity has a higher value than its European counterpart if $n$ and $m$ are fixed. The convergence pattern for American options is similar to that for European ones.

As with the constant volatility case, the computing speed will depend on the number of time steps and the length of a time step. With the GARCH model examined in this paper, the matrix is usually highly sparse with only 1 or 2% of the elements being non-zero. Although the GARCH option pricing model requires more computing time than the Black–Scholes model, a reasonably accurate price estimate for a 90 days option with daily transition steps can be obtained with a standard desktop computer in under one minute.

Table 3
European put option in the GARCH framework[a]

| | Maturity = 1 month | | | Maturity = 3 months | | | Maturity = 9 months | | |
|---|---|---|---|---|---|---|---|---|---|
| $K/S_0$ | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 |
| *Monte Carlo with control variate* | | | | | | | | | |
| Price | 4.8388 | 1.0880 | 0.0778 | 4.9546 | 1.8197 | 0.4158 | 5.4773 | 2.8416 | 1.1945 |
| std. | (0.0012) | (0.0009) | (0.0007) | (0.0021) | (0.0018) | (0.0014) | (0.0033) | (0.0028) | (0.0022) |
| *Markov chain prices* (*time step* = 1 *day*): | | | | | | | | | |
| $n = 25$, $m = 25$ | 4.8756 | 1.2502 | 0.1023 | 5.2628 | 2.2142 | 0.6334 | 4.0344 | 0.6721 | 0.0878 |
| $n = 31$, $m = 31$ | 4.8599 | 1.1937 | 0.0902 | 5.3264 | 2.3070 | 0.6884 | 4.8435 | 2.0628 | 0.6681 |
| $n = 35$, $m = 35$ | 4.8535 | 1.1707 | 0.0855 | 5.2752 | 2.2371 | 0.6396 | 5.3373 | 2.6470 | 1.0516 |
| $n = 41$, $m = 41$ | 4.8482 | 1.1469 | 0.0813 | 5.2053 | 2.1544 | 0.5877 | 5.9755 | 3.3301 | 1.5539 |
| $n = 45$, $m = 45$ | 4.8472 | 1.1427 | 0.0802 | 5.1771 | 2.1117 | 0.5618 | 6.1733 | 3.5370 | 1.7238 |
| $n = 51$, $m = 51$ | 4.8443 | 1.1271 | 0.0773 | 5.1301 | 2.0542 | 0.5293 | 6.2313 | 3.6008 | 1.7689 |
| $n = 25$, $m = 75$ | 4.8417 | 1.1132 | 0.0738 | 5.0498 | 1.9456 | 0.4660 | 6.0560 | 3.4061 | 1.6048 |
| $n = 31$, $m = 93$ | 4.8412 | 1.1066 | 0.0739 | 5.0073 | 1.8915 | 0.4416 | 5.8631 | 3.2223 | 1.4687 |
| $n = 35$, $m = 105$ | 4.8394 | 1.0995 | 0.0723 | 4.9957 | 1.8743 | 0.4312 | 5.7944 | 3.1522 | 1.4124 |
| $n = 41$, $m = 123$ | 4.8376 | 1.0914 | 0.0713 | 4.9823 | 1.8549 | 0.4206 | 5.7015 | 3.0596 | 1.3433 |
| $n = 45$, $m = 135$ | 4.8394 | 1.0978 | 0.0725 | 4.9789 | 1.8524 | 0.4206 | 5.6749 | 3.0314 | 1.3229 |
| $n = 51$, $m = 153$ | 4.8379 | 1.0918 | 0.0715 | 4.9753 | 1.8469 | 0.4171 | 5.6135 | 2.9704 | 1.2773 |
| $n = 25$, $m = 125$ | 4.8425 | 1.1080 | 0.0729 | 4.9931 | 1.8678 | 0.4262 | 5.6722 | 3.0298 | 1.3201 |
| $n = 31$, $m = 155$ | 4.8386 | 1.0951 | 0.0724 | 4.9694 | 1.8399 | 0.4134 | 5.6268 | 2.9829 | 1.2857 |
| $n = 35$, $m = 175$ | 4.8389 | 1.0940 | 0.0714 | 4.9651 | 1.8321 | 0.4084 | 5.5760 | 2.9323 | 1.2478 |
| $n = 41$, $m = 205$ | 4.8379 | 1.0893 | 0.0712 | 4.9645 | 1.8316 | 0.4087 | 5.5325 | 2.8889 | 1.2162 |
| $n = 45$, $m = 225$ | 4.8386 | 1.0921 | 0.0716 | 4.9569 | 1.8218 | 0.4041 | 5.5377 | 2.8944 | 1.2209 |
| $n = 51$, $m = 255$ | 4.8373 | 1.0879 | 0.0713 | 4.9606 | 1.8268 | 0.4073 | 5.5202 | 2.8774 | 1.2086 |
| $n = 25$, $m = 175$ | 4.8392 | 1.0940 | 0.0713 | 4.9760 | 1.8453 | 0.4140 | 5.6101 | 2.9649 | 1.2711 |
| $n = 31$, $m = 217$ | 4.8385 | 1.0919 | 0.0719 | 4.9607 | 1.8284 | 0.4081 | 5.5338 | 2.8913 | 1.2193 |
| $n = 35$, $m = 245$ | 4.8387 | 1.0929 | 0.0720 | 4.9604 | 1.8269 | 0.4067 | 5.5222 | 2.8787 | 1.2090 |
| $n = 41$, $m = 287$ | 4.8380 | 1.0893 | 0.0713 | 4.9555 | 1.8207 | 0.4042 | 5.5014 | 2.8582 | 1.1946 |
| $n = 45$, $m = 315$ | 4.8381 | 1.0899 | 0.0715 | 4.9524 | 1.8170 | 0.4025 | 5.4916 | 2.8477 | 1.1860 |
| $n = 51$, $m = 357$ | 4.8377 | 1.0884 | 0.0715 | 4.9550 | 1.8197 | 0.4036 | 5.4899 | 2.8471 | 1.1867 |

[a]Parameters: $S_0 = 50$, $r = 0.05$ (annualized), $\beta_0 = 0.00001$, $\beta_1 = 0.80$, $\beta_2 = 0.10$, $\theta = 0.3$ and $\lambda = 0.2$.

Table 4
American put option in the GARCH framework[a]

| | Maturity = 1 month | | | Maturity = 3 months | | | Maturity = 9 months | | |
|---|---|---|---|---|---|---|---|---|---|
| $K/S_0$ | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 | 1.10 | 1.00 | 0.90 |
| *Markov chain prices* (*time step* = 1 *day*): | | | | | | | | | |
| $n = 25$, $m = 25$ | 5.0099 | 1.2772 | 0.1163 | 5.4688 | 2.2950 | 0.6736 | 5.0000 | 0.7594 | 0.0970 |
| $n = 31$, $m = 31$ | 5.0021 | 1.2170 | 0.1011 | 5.5373 | 2.3923 | 0.7309 | 5.2492 | 2.1886 | 0.7026 |
| $n = 35$, $m = 35$ | 5.0004 | 1.1921 | 0.0948 | 5.4843 | 2.3132 | 0.6715 | 5.7483 | 2.8044 | 1.1040 |
| $n = 41$, $m = 41$ | 5.0000 | 1.1667 | 0.0892 | 5.4169 | 2.2244 | 0.6135 | 6.4207 | 3.5394 | 1.6443 |
| $n = 45$, $m = 45$ | 5.0000 | 1.1620 | 0.0876 | 5.3877 | 2.1788 | 0.5849 | 6.6331 | 3.7644 | 1.8273 |
| $n = 51$, $m = 51$ | 5.0000 | 1.1454 | 0.0839 | 5.3434 | 2.1181 | 0.5491 | 6.7039 | 3.8351 | 1.8763 |
| $n = 25$, $m = 75$ | 5.0000 | 1.1300 | 0.0788 | 5.2688 | 2.0043 | 0.4802 | 6.5222 | 3.6239 | 1.6935 |
| $n = 31$, $m = 93$ | 5.0000 | 1.1228 | 0.0784 | 5.2319 | 1.9489 | 0.4545 | 6.3353 | 3.4338 | 1.5497 |
| $n = 35$, $m = 105$ | 5.0000 | 1.1153 | 0.0766 | 5.2215 | 1.9309 | 0.4433 | 6.2677 | 3.3598 | 1.4894 |
| $n = 41$, $m = 123$ | 5.0000 | 1.1069 | 0.0752 | 5.2101 | 1.9110 | 0.4319 | 6.1795 | 3.2648 | 1.4169 |
| $n = 45$, $m = 135$ | 5.0000 | 1.1132 | 0.0764 | 5.2064 | 1.9081 | 0.4318 | 6.1528 | 3.2355 | 1.3953 |
| $n = 51$, $m = 153$ | 5.0000 | 1.1070 | 0.0752 | 5.2035 | 1.9023 | 0.4280 | 6.0959 | 3.1733 | 1.3477 |
| $n = 25$, $m = 125$ | 5.0000 | 1.1236 | 0.0767 | 5.2190 | 1.9240 | 0.4377 | 6.1539 | 3.2350 | 1.3926 |
| $n = 31$, $m = 155$ | 5.0000 | 1.1102 | 0.0760 | 5.1993 | 1.8955 | 0.4242 | 6.1078 | 3.1856 | 1.3564 |
| $n = 35$, $m = 175$ | 5.0000 | 1.1089 | 0.0748 | 5.1954 | 1.8872 | 0.4189 | 6.0607 | 3.1337 | 1.3165 |
| $n = 41$, $m = 205$ | 5.0000 | 1.1041 | 0.0745 | 5.1945 | 1.8863 | 0.4190 | 6.0206 | 3.0895 | 1.2835 |
| $n = 45$, $m = 225$ | 5.0000 | 1.1068 | 0.0747 | 5.1881 | 1.8764 | 0.4142 | 6.0246 | 3.0949 | 1.2884 |
| $n = 51$, $m = 255$ | 5.0000 | 1.1024 | 0.0743 | 5.1907 | 1.8812 | 0.4173 | 6.0081 | 3.0773 | 1.2754 |
| $n = 25$, $m = 175$ | 5.0000 | 1.1090 | 0.0746 | 5.2051 | 1.9007 | 0.4246 | 6.0924 | 3.1669 | 1.3406 |
| $n = 31$, $m = 217$ | 5.0000 | 1.1066 | 0.0751 | 5.1914 | 1.8832 | 0.4183 | 6.0219 | 3.0923 | 1.2868 |
| $n = 35$, $m = 245$ | 5.0000 | 1.1074 | 0.0750 | 5.1907 | 1.8814 | 0.4167 | 6.0101 | 3.0787 | 1.2758 |
| $n = 41$, $m = 287$ | 5.0000 | 1.1037 | 0.0742 | 5.1869 | 1.8750 | 0.4140 | 5.9913 | 3.0580 | 1.2609 |
| $n = 45$, $m = 315$ | 5.0000 | 1.1042 | 0.0742 | 5.1845 | 1.8713 | 0.4122 | 5.9821 | 3.0470 | 1.2516 |
| $n = 51$, $m = 357$ | 5.0000 | 1.1026 | 0.0742 | 5.1861 | 1.8737 | 0.4132 | 5.9800 | 3.0463 | 1.2524 |

[a]Parameters: $S_0 = 50$, $r = 0.05$ (annualized), $\beta_0 = 0.00001$, $\beta_1 = 0.80$, $\beta_2 = 0.10$, $\theta = 0.3$ and $\lambda = 0.2$.

## 5. Conclusion

The GARCH family of models has been extensively used in the literature for modeling financial time series. Its use for derivative securities has just begun recently. We have proposed a Markov chain approximation method for American option pricing. Particularly, we have developed an explicit scheme for the GARCH option pricing model. We have provided a convergence proof for the algorithm and conducted a numerical analysis on the performance of this method. Our results suggest that the Markov chain approximation method works well in different settings. This numerical method makes it possible to apply the newly-emerged GARCH option pricing theory to a broader class of exchange-traded option contracts, and thus opens an area of empirical research that was not previously accessible.

Our Markov chain method for the GARCH model can be specialized to the Black–Scholes (1973) model. Its numerical performance has been shown to be good. An explicit Markov chain approximation scheme can also be developed for other modeling frameworks such as jump-diffusion and bivariate stochastic volatility diffusion. Its applicability appears to be limited only by whether the underlying process can be described as a lower-dimensional Markov process.

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## Appendix

*Proof for Proposition 1.* Denote the one-period conditional distribution based on the $(m,n)$-Markov chain and the target GARCH process by $F^{(m,n)}(\cdot,\cdot\,|\,\cdot,\cdot)$ and $F(\cdot,\cdot\,|\,\cdot,\cdot)$, respectively. By the construction of the transition probability matrix, $F^{(m,n)}(\bar{p}(k-1),\bar{q}(l-1))|\bar{p}(i),\bar{q}(j)) = F(c(k),d(l)|\bar{p}(i),\bar{q}(j))$ for any $i,j,k,l$. (For the definition of the notation, please refer to Section 3.) Consider the difference between $F(p_t, q_{t+1}|p_{t-1}, q_t)$ and $F^{(m,n)}(\bar{p}(k), \bar{q}(l))|\bar{p}(i), \bar{q}(j))$, where $(\bar{p}(k), \bar{q}(l))$ is the discretized values closest to $(p_t, q_{t+1})$ and similarly $(\bar{p}(i), \bar{q}(j))$ is the discretized values closest to $(p_{t-1}, q_t)$. Note that $F(p_t, q_{t+1}|p_{t-1}, q_t)$ is uniformly continuous in $(p_t, q_{t+1})$ as well as in $(p_{t-1}, q_t)$. First, the GARCH innovation has a continuous distribution and by definition the distribution function is monotonically increasing and bounded. In other words, $F(p_t, q_{t+1}|p_{t-1}, q_t)$ is uniformly continuous in $(p_t, q_{t+1})$. (This assertion is proved at the end.) Second, the kernel that determines $F(p_t, q_{t+1}|p_{t-1}, q_t)$ is the normal density function with mean, $p_{t-1} - \frac{1}{2}[\exp(q_t) - h^*]$, and variance, $\exp(q_t)$. Therefore, $F(p_t, q_{t+1}|p_{t-1}, q_t)$ is uniformly continuous in $(p_{t-1}, q_t)$ if $\exp(q_t)$ is bounded below by a positive constant, which is for the GARCH model $\beta_0$ (see Eq. (5)). The maximum difference can be made arbitrarily small (uniformly) by increasing $m$ and $n$. This is possible because of the partition condition (ensuring a finer partition as $m$ and $n$ approach infinity). The fact that the approximation errors are uniformly small ensures that the $2T$-dimensional distribution function constructing from the one-period transition probability matrix converges to the $2T$-dimensional distribution function based on the target GARCH process; that is, $F^{(m,n)} \to F$ as $m$ and $n \to \infty$.

We now prove the uniform continuity assertion made earlier. Let $G(x)$ denote a continuous distribution function mapping from $\boldsymbol{R}^k$ to $[0,1]$. Find two points, $x_u > x_d$ and both $x_u$ and $x_d$ are elements of $\boldsymbol{R}^k$. Define a compact set $A = \{x \in \boldsymbol{R}^k\colon x_d \le x \le x_u\}$ and consider any two points, $x$ and $y$, in $\boldsymbol{R}^k$. Let $\bar{x} \in A$ be the closest point to $x$, and similarly for $\bar{y}$. By construction, if $\|x - y\| < \delta$, then $\|\bar{x} - \bar{y}\| < \delta$. Since $G(x)$ is monotonically increasing and bounded, for any $\varepsilon > 0$, there exist a large enough $x_u$ and a small enough $x_d$ such that $|f(x) - f(\bar{x})| < \varepsilon/3$ for any $x \in \boldsymbol{R}^k$. Since $G(x)$ is uniformly continuous over the compact set $A$, there exists a $\delta > 0$ such that $|f(\bar{x}) - f(\bar{y})| < \varepsilon/3$ if $\|\bar{x} - \bar{y}\| < \delta$. This allows us to bound the difference of two functional values evaluated at any two points, $x$ and $y$, in $\boldsymbol{R}^k$ satisfying $\|x - y\| < \delta$.

$$
\begin{aligned}
|f(x) - f(y)| &\le |f(x) - f(\bar{x})| + |f(y) - f(\bar{y})| + |f(\bar{x}) - f(\bar{y})| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

The function $G(x)$ is therefore uniformly continuous over $\boldsymbol{R}^k$. □

*Proof for Proposition 2.* (i) We first use a positive real number $c$ to define a modified version of $f(\boldsymbol{p}_T, \boldsymbol{q}_T)$ as follows:

$$
f_c(\boldsymbol{p}_T, \boldsymbol{q}_T) = \begin{cases} f(\boldsymbol{p}_T, \boldsymbol{q}_T) & \text{if } f(\boldsymbol{p}_T, \boldsymbol{q}_T) < c, \\ 0 & \text{otherwise.} \end{cases}.
$$

Since $F^{(m.n)}$ converges to $F$ (Proposition 1) and $f_c$ is a bounded continuous function, we have

$$\lim_{m,n\to\infty} \mathrm{E}^{F^{(m.n)}}\{f_c(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} = \mathrm{E}^F\{f_c(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} \tag{A.1}$$

except for $c$ such that $\Pr^F\{f(\boldsymbol{p}_T,\boldsymbol{q}_T) = c|\mathscr{F}_0\} > 0$. If $\Pr^F\{f(\boldsymbol{p}_T,\boldsymbol{q}_T) = c|\mathscr{F}_0\} > 0$, there exists $\bar{c} > c$ such that $\Pr^F\{f(\boldsymbol{p}_T,\boldsymbol{q}_T) = \bar{c}|\mathscr{F}_0\} = 0$ and we will use $\bar{c}$ instead. We now consider

$$\begin{aligned}
&\lim_{m,n\to\infty} |\mathrm{E}^{F^{(m.n)}}\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} - \mathrm{E}^F\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\}| \\
&= \lim_{m,n\to\infty} |\mathrm{E}^{F^{(m.n)}}\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} - \mathrm{E}^{F^{(m.n)}}\{f_c(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} \\
&\quad + \mathrm{E}^F\{f_c(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} - \mathrm{E}^F\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\}| \quad \text{(by A.1)} \\
&\le \lim_{m,n\to\infty} |\mathrm{E}^{F^{(m.n)}}\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} - \mathrm{E}^{F^{(m.n)}}\{f_c(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\}| \\
&\quad + |\mathrm{E}^F\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\} - \mathrm{E}^F\{f_c(\boldsymbol{p}_T,\boldsymbol{q}_T)|\mathscr{F}_0\}| \\
&= \lim_{m,n\to\infty} |\mathrm{E}^{F^{(m.n)}}\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)1_{\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)\ge c\}}|\mathscr{F}_0\}| + |\mathrm{E}^F\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)1_{\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)\ge c\}}|\mathscr{F}_0\}| \\
&\le \lim_{m,n\to\infty} |\mathrm{E}^{F^{(m.n)}}\{g(\boldsymbol{p}_T,\boldsymbol{q}_T)1_{\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)\ge c\}}|\mathscr{F}_0\}| + |\mathrm{E}^F\{g(\boldsymbol{p}_T,\boldsymbol{q}_T)1_{\{f(\boldsymbol{p}_T,\boldsymbol{q}_T)\ge c\}}|\mathscr{F}_0\}| \\
&\le \lim_{m,n\to\infty} |\mathrm{E}^{F^{(m.n)}}\{g(\boldsymbol{p}_T,\boldsymbol{q}_T)1_{\{g(\boldsymbol{p}_T,\boldsymbol{q}_T)\ge c\}}|\mathscr{F}_0\}| + |\mathrm{E}^F\{g(\boldsymbol{p}_T,\boldsymbol{q}_T)1_{\{g(\boldsymbol{p}_T,\boldsymbol{q}_T)\ge c\}}|\mathscr{F}_0\}| \\
&\to 0 \quad \text{as } c\to\infty \quad \text{(by the uniform integrability assumption).}
\end{aligned}$$

(ii) Denote by $A^{(m,n)}_{ij,t-1}$ the event 'the Markov chain is in state $ij$ at time $t-1$', i.e.,

$$A^{(m,n)}_{ij,t-1} = \{\omega\in\Omega\colon p^{(m,n)}_{t-1}(\omega) = \bar{p}^{(m,n)}(i) \text{ and } q^{(m,n)}_t(\omega) = \bar{q}^{(m,n)}(j)\}$$

and by $A_{ij,t-1}$ the event 'the GARCH process at time $t-1$ takes on values equal to the values of the Markov chain in state $ij$', i.e.,

$$A_{ij,t-1} = \{\tilde{\omega}\in\tilde{\Omega}\colon p_{t-1}(\tilde{\omega}) = \bar{p}^{(m,n)}(i) \text{ and } q_t(\tilde{\omega}) = \bar{q}^{(m,n)}(j)\}.$$

The distribution governing the approximating Markov chain is exactly the same as the distribution obtained by first generating from $(p^{(m,n)}_{t-1}, q^{(m,n)}_t)$ to $(p_t, q_{t+1})$

using the exact one-period GARCH conditional distribution and then replacing $(p_t, q_{t+1})$ with the closest approximating values $(p_t^{(m,n)}, q_{t+1}^{(m,n)})$. That is,

$$\mathscr{L}(p_t^{(m,n)}|A_{ij,t-1}^{(m,n)}) = \mathscr{L}\left(\sum_{k=1}^{m} \bar{p}^{(m,n)}(k)1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}}|A_{ij,t-1}\right),$$

where $\mathscr{L}(X|A)$ denotes the $Q$-probability law for the random variable $X$ conditional on $A$. Let $v(m,n) = \frac{1}{2}u(m,n)$ where $u(m,n)$ denotes the maximum distance between any two adjacent discretized logarithmic transformed asset prices in the $(m,n)$-partition. The process of assigning $(p_t, q_{t+1})$ to $(p_t^{(m,n)}, q_{t+1}^{(m,n)})$ ensures that

$$\begin{aligned}\sum_{k=1}^{m} \bar{p}^{(m,n)}(k)1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}} &\le \sum_{k=1}^{m} (p_t + v(m,n))1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}} \\ &= (p_t + v(m,n))\sum_{k=1}^{m} 1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}} \\ &= (p_t + v(m,n))\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{k=1}^{m} \bar{p}^{(m,n)}(k)1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}} &\ge \sum_{k=1}^{m} (p_t - v(m,n))1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}} \\ &= (p_t - v(m,n))\sum_{k=1}^{m} 1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}} \\ &= (p_t - v(m,n)).\end{aligned}$$

Using the results above and knowing that $S_t = \exp\{p_t + (r - h^*/2)t\}$, we can bound the expected value of the Markov chain as follows:

$$\begin{aligned}&\mathrm{E}^Q\{S_t^{(m,n)}|A_{ij,t-1}^{(m,n)}\} \\ &\quad = \mathrm{E}^Q\left\{\exp\left[p_t^{(m,n)} + \left(r - \frac{h^*}{2}\right)t\right]|A_{ij,t-1}^{(m,n)}\right\} \\ &\quad = \exp\left[\left(r - \frac{h^*}{2}\right)t\right]\mathrm{E}^Q\{\exp(p_t^{(m,n)})|A_{ij,t-1}^{(m,n)}\} \\ &\quad = \exp\left[\left(r - \frac{h^*}{2}\right)t\right]\mathrm{E}^Q\left\{\exp\left[\sum_{k=1}^{m} \bar{p}^{(m,n)}(k)1_{\{c^{(m,n)}(k) \le p_t < c^{(m,n)}(k+1)\}}\right]|A_{ij,t-1}\right\} \\ &\quad \begin{smallmatrix}\le \\ \ge\end{smallmatrix} \exp\left[(r - \frac{h^*}{2})t\right]\mathrm{E}^Q\{\exp[p_t \pm v(m,n)]|A_{ij,t-1}\}\end{aligned}$$

$$= \exp\left[\left(r-\frac{h^*}{2}\right)t \pm v(m,n)\right]\mathrm{E}^Q\{\exp(p_t)|A_{ij,t-1}\}$$

$$= \exp\left[\left(r-\frac{h^*}{2}\right)t \pm v(m,n)\right]\mathrm{E}^Q\left\{S_t\exp\left[-\left(r-\frac{h^*}{2}\right)t\right]|A_{ij,t-1}\right\}$$

$$= \exp\left[\left(r-\frac{h^*}{2}\right)t \pm v(m,n)\right]\mathrm{E}^Q\left\{S_{t-1}\exp\left[r-\frac{h_t}{2}+\sqrt{h_t}\varepsilon_t\right]\right.$$

$$\left.\times\exp\left[-\left(r-\frac{h^*}{2}\right)t\right]|A_{ij,t-1}\right\}$$

$$= \exp\left[\left(r-\frac{h^*}{2}\right)t \pm v(m,n)\right]\mathrm{E}^Q\left\{\exp\left[p_{t-1}+\left(r-\frac{h^*}{2}\right)(t-1)\right]\right.$$

$$\left.\times\exp\left[r-\frac{h_t}{2}+\sqrt{h_t}\varepsilon_t\right]\exp\left[-\left(r-\frac{h^*}{2}\right)t\right]|A_{ij,t-1}\right\}$$

$$= \exp\left[\left(r-\frac{h^*}{2}\right)t \pm v(m,n)\right]\mathrm{E}^Q\left\{\exp\left[p_{t-1}+\frac{h^*}{2}-\frac{h_t}{2}+\sqrt{h_t}\epsilon_t\right]|A_{ij,t-1}\right\}$$

$$=\exp\left[\left(r-\frac{h^*}{2}\right)t \pm v(m,n)\right]\exp\left[\bar{p}^{(m,n)}(i)+\frac{h^*}{2}-\frac{1}{2}\mathrm{e}^{\bar{q}^{(m,n)}(j)}\right]\mathrm{E}^Q\left\{\exp\left[\sqrt{\mathrm{e}^{\bar{q}^{(m,n)}(j)}}\epsilon_t\right]\right\}$$

(since $\epsilon_t$ is independent of $A_{ij,t-1}$)

$$= \exp\left[\bar{p}^{(m,n)}(i)+\left(r-\frac{h^*}{2}\right)t+\frac{h^*}{2} \pm v(m,n)\right] \quad (\text{since } \epsilon_t \overset{Q}{\sim} \mathrm{N}(0,1)).$$

Using the Markov property and the above result we can write

$$\mathrm{E}^Q\{S_t^{(m,n)}|\mathscr{F}_{t-1}^{(m,n)}\}$$

$$= \mathrm{E}^Q\{S_t^{(m,n)}|\sigma\{A_{ij,t-1}^{(m,n)}: i\in\{1,\ldots,m\},\, j\in\{1,\ldots,n\}\}\}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} 1_{A_{ij,t-1}^{(m,n)}}\mathrm{E}^Q\{S_t^{(m,n)}|A_{ij,t-1}^{(m,n)}\}$$

$$\begin{matrix}\le\\ \ge\end{matrix} \sum_{i=1}^{m}\sum_{j=1}^{n} 1_{A_{ij,t-1}^{(m,n)}}\exp\left[\bar{p}^{(m,n)}(i)+\left(r-\frac{h^*}{2}\right)t+\frac{h^*}{2} \pm v(m,n)\right]$$

$$= \exp\left[p_{t-1}^{(m,n)}+\left(r-\frac{h^*}{2}\right)t+\frac{h^*}{2} \pm v(m,n)\right]$$

$$= S_{t-1}^{(m,n)}\exp\left[\left(r-\frac{h^*}{2}\right)+\frac{h^*}{2} \pm v(m,n)\right]$$

$$= S_{t-1}^{(m,n)}\exp[r \pm v(m,n)].$$

Repeating the same argument all the way back to time 0 and recognizing $\mathscr{F}_0^{(m,n)} = \mathscr{F}_0$ and $S_0^{(m,n)} = S_0$ yields

$$S_0 \exp[rt - tv(m,n)] \leq \mathrm{E}^Q[S_t^{(m,n)}|\mathscr{F}_0] \leq S_0 \exp[rt + tv(m,n)]. \tag{A.2}$$

Note that $\mathrm{E}^{F^{(m,n)}}\{S_t|\mathscr{F}_0\} = \mathrm{E}^Q\{S_t^{(m,n)}|\mathscr{F}_0\}$ by definition. The partition condition ensures that $\lim_{m,n\to\infty} v(m,n) = 0$. We thus have $\lim_{m,n\to\infty} \mathrm{E}^{F^{(m,n)}}\{S_t|\mathscr{F}_0\} = S_0\mathrm{e}^{rt}$.

We now turn to uniform integrability. Clearly, $\sup_{m,n} \mathrm{E}^{F^{(m,n)}}\{S_t|\mathscr{F}_0\} < \infty$ by (A.2) and the fact that $\sup_{m,n} v(m,v) < \infty$. Moreover, $\lim_{m,n\to\infty} \mathrm{E}^{F^{(m,n)}}\{S_t|\mathscr{F}_0\} = S_0\mathrm{e}^{rt} = \mathrm{E}^Q\{S_t|\mathscr{F}_0\}$ with the second equality due to the GARCH model in (4)–(6). These two facts combined with the distribution convergence result established earlier in Proposition 1 allow us to invoke Proposition 2.3 of Ethier and Kurtz (1986, p. 494)[12] to yield uniform integrability. □

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[12] Proposition 2.3 from Ethier and Kurtz (1986): If $X_n \Rightarrow X$ and $\{X_n\}$ is uniformly integrable, then $\lim_{n\to\infty} \mathrm{E}[X_n] = \mathrm{E}[X]$. Conversely, if the $X_n$ are integrable, $X_n \Rightarrow X$, and $\lim_{n\to\infty} \mathrm{E}[|X_n|] = \mathrm{E}[|X|]$, then $\{X_n\}$ is uniformly integrable.

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