Asymptotic Distribution of the EMS Option Price Estimator

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Monte Carlo simulation is commonly used for computing prices of derivative securities when an analytical solution does not exist. Recently, a new simulation technique known as empirical martingale simulation (EMS) has been proposed by Duan and Simonato (1998) as a way of improving simulation accuracy. EMS has one drawback however. Because of the dependency among sample paths created by the EMS adjustment, the standard error of the price estimate is not readily available from using one simulation sample. In this paper, we develop a scheme to estimate the EMS accuracy. The EMS price estimator is first shown to have an asymptotically normal distribution. Through a simulation study, we then find that the asymptotic normal distribution serves as a good approximation for samples consisting of as few as 500 simulation paths.

(*Monte Carlo; Empirical Martingale Simulation; Linear Control Variate; Options*)

1. Introduction

In the finance literature the price dynamics of primary securities are often modeled as continuous-time stochastic processes. Such a device has long been the main tool for valuing derivative securities whose payoffs depend directly on the values of the primary securities. As shown in Harrison and Kreps (1979) and Harrison and Pliska (1981), a derivative security can be priced as the discounted expected value of its contingent payoff, where the expectation is taken with respect to an equivalent martingale measure. Expectations can sometimes be difficult to compute because of the lack of an analytical formula. Monte Carlo simulation lends itself naturally to the evaluation of expectations. Since its introduction to derivatives pricing by Boyle (1977), it has become an indispensable valuation tool in finance. A comprehensive discussion on this subject can be found in the survey paper by Boyle et al. (1997).

Although Monte Carlo methods are flexible in dealing with more complex valuation problems, they are numerically demanding if a higher degree of pricing accuracy is desired. The relatively poor numerical performance of Monte Carlo methods motivates the continuing search for more efficient error reduction techniques. Many methods have been proposed in the literature over the years. The older and more familiar variance reduction methods are, for example, antithetic and control-variate simulations. The newer ones include the quadratic resampling method by Barraquand (1995) and the empirical martingale simulation (EMS) by Duan and Simonato (1998). These methods set out to match the simulated moments of the underlying asset price to their theoretical counterparts, and are hence given the name

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of moment-matching simulation methods in Boyle et al. (1997). Other variance reduction techniques, which seek to introduce regularity into simulation, have been proposed; for example, stratified and Latin hypercube samplings force empirical probabilities to match more closely with theoretical probabilities.

The EMS utilizes a fundamental property of derivatives pricing theory, which stipulates that asset prices (after discounting by the risk-free rate of interest) are martingales. By making a multiplicative adjustment to the sample paths, the martingale property is reproduced in the simulated sample. Duan and Simonato (1998) proved that the EMS method yields price estimates that are consistent; that is, the EMS price converges to the theoretical price as the sample size increases. They have also demonstrated that EMS substantially reduces simulation errors.

The benefit of EMS does not come without costs. The first disadvantage is that it results in biased estimates in finite samples, although the biases have been shown by Duan and Simonato (1998) to be numerically small. The second disadvantage is that, like many of the new variance reduction methods seeking to introduce regularity into simulation, the EMS adjustment creates dependency among sample paths. Because of the lack of independence, one can no longer directly apply the central limit theorem to infer the distribution of the price estimate using just one simulation sample. In other words, the asymptotic distribution of the EMS price estimator is unknown.

In this paper, we set out to derive the asymptotic distribution for the EMS price estimator. The distribution is derived under the assumption that contingent payoffs are piecewise linear continuous functions of the underlying asset price, although some of the results are valid for a broader class of functions. Note that derivative contracts such as call and put options are indeed piecewise linear and continuous. Even if a derivative contract does not have a piecewise linear continuous payoff function, it is likely that its payoff function can still be approximated closely by some piecewise linear continuous function.

Our theoretical results suggest that the EMS price estimator is asymptotically equivalent to a linear control-variate estimator with the underlying asset price as the control. This result implies that the EMS price estimator is consistent and asymptotically normal. We have also derived an analytical expression for the asymptotic variance of the EMS price estimator. This asymptotic variance can be easily estimated using the information from one simulation sample. Asymptotic normality in conjunction with the variance formula allow us to obtain confidence intervals for the EMS price estimate. Our numerical analysis shows that the asymptotic distribution is a good approximation to the actual distribution obtained by repeating (independently) the EMS calculation 1,000 times. Interestingly, the quality of the EMS estimator is found to be equivalent to the optimal linear control-variate estimator.

The remainder of this paper is organized as follows. In §2 we describe the standard Monte Carlo and the EMS simulation methods. The asymptotic properties of the EMS price estimator are derived in §3. The numerical results for European options under two pricing frameworks (Black-Scholes and GARCH) are presented in §4. Section 5 concludes the paper. Proofs are provided in the Appendix.

2. Two Simulation Methods

A typical option pricing model is constructed using a filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, P),$$

where P stands for the data generating probability measure and $\mathcal{F}_0 = \{\varnothing, \Omega\}$. The general option pricing theory implies that there exists a risk-neutral probability measure Q, equivalent to P, under which the discounted asset price process is a martingale. Let r be the continuously compounded risk-free interest rate. In the following, all expectations will be taken under the probability measure Q. Specifically, for any $0 \le u \le t \le T$, we have $\mathrm{E}[e^{-rt} S_t | \mathcal{F}_u] = e^{-ru} S_u$, and in particular,

$$\mathrm{E}[e^{-rT} S_T] = S_0. \tag{1}$$

A derivative contract with the payoff contingent upon S_T can be described by a payoff function $g(S_T)$ where g: $(\mathcal{R}_+, \mathcal{B}(\mathcal{R}_+)) \to (\mathcal{R}_+, \mathcal{B}(\mathcal{R}_+))$ is a real-valued function measurable with respect to the Borel

σ-algebra $\mathscr{B}(\mathscr{R}_+)$. The option pricing theory suggests that the time-0 value of this derivative security is

$$C = e^{-rT}\mathrm{E}[g(S_T)]. \tag{2}$$

If an analytical solution to Equation (2) is unavailable, Monte Carlo simulation becomes a natural method for evaluating the expectation. Two simulation methods are described below.

2.1. Monte Carlo Simulation (MCS)

The asset price dynamic under the probability measure Q can be used to generate a random sample of future asset prices at time T. Denote the independent and identically distributed random sample by $\{S_{i,T}\colon\ i = 1, \ldots, n\}$. The standard Monte Carlo price estimate for the option at time 0, denoted by $C_{MCS}^{(n)}$, is defined by

$$C_{MCS}^{(n)} = e^{-rT}\frac{1}{n}\sum_{i=1}^{n} g(S_{i,T}). \tag{3}$$

It is clear that $C_{MCS}^{(n)}$ is unbiased; that is, $\mathrm{E}[C_{MCS}^{(n)}] = C$. The strong law of large numbers also implies that $C_{MCS}^{(n)}$ converges almost surely to C as n tends to infinity provided that $g(S_T)$ is square-integrable. Moreover, we have, by invoking the central limit theorem,

$$\sqrt{n}\left(C_{MCS}^{(n)} - C\right) \overset{n\to\infty}{\Longrightarrow} N\left(0, e^{-2rT}\mathrm{Var}[g(S_T)]\right)$$

where $\Longrightarrow$ stands for "convergence in distribution." Since $\mathrm{Var}[g(S_T)]$ can be consistently estimated by the sample variance, one simulation sample suffices in characterizing the distribution of the standard Monte Carlo price estimator. The distribution of the price estimator serves an important function. It can be used to gauge the magnitude of simulation error, and thus helps us determine whether the quality of a price estimate is acceptable.

The standard Monte Carlo simulation method is known to be inefficient, however. A variety of error reduction schemes have been proposed over the years. One specific scheme being considered here is the empirical martingale simulation.

2.2. Empirical Martingale Simulation (EMS)

The EMS method was motivated by a simple observation, which is

$$\frac{1}{n}\sum_{i=1}^{n} e^{-rT} S_{i,T} \neq S_0. \tag{4}$$

In words, the simulated asset price does not satisfy the martingale property prescribed by the pricing theory. To ensure the martingale property is respected in simulation, Duan and Simonato (1998) proposed a simple adjustment to the simulated sample before computing the option value. First, define $\overline{S}_{n,t} = n^{-1}\sum_{i=1}^{n} S_{i,t}$. The EMS adjustment is defined as

$$S_{i,T}^{*} \equiv \frac{S_0 e^{rT}}{\overline{S}_{n,T}} S_{i,T}, \tag{5}$$

where $\mathrm{E}[S_{i,T}] = S_0 e^{rT}$, because of the martingale property discussed earlier. Duan and Simonato (1998) defined the EMS adjustment in a recursive manner to reflect the typical multiperiod simulation situation. Solving their recursive system up to time T can actually yield a direct expression as in Equation (5). It is easy to verify that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} e^{-rT} S_{i,T}^{*} &= \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\frac{S_0 e^{rT}}{\overline{S}_{n,T}} S_{i,T}\\ &= \frac{S_0}{\overline{S}_{n,T}}\frac{1}{n}\sum_{i=1}^{n} S_{i,T} = S_0.\end{aligned} \tag{6}$$

Thus, the EMS-adjusted price (after discounting by the risk-free interest rate) behaves like a martingale in the simulated sample. Even though the EMS-adjusted contract payoffs are still identically distributed, independence is destroyed by the adjustment.

Duan and Simonato (1998) have shown, using simulations, that the estimator $C_{EMS}^{(n)}$, defined by

$$C_{EMS}^{(n)} = e^{-rT}\frac{1}{n}\sum_{i=1}^{n} g(S_{i,T}^{*}), \tag{7}$$

has a smaller variance than $C_{MCS}^{(n)}$. They have also proved that $C_{EMS}^{(n)}$ converges to the theoretical value for any derivative contract with a well-behaved payoff function. The lack of independence, however, prevents them from getting a simulation error estimate based on one simulation sample. As argued earlier, the knowledge of the magnitude of the simulation error can be very useful in gauging the quality of a price estimate. The next section is thus devoted to the development of a scheme that is capable of accurately assessing the EMS-pricing error.

3. Asymptotic Properties of the EMS Price Estimator

We begin by the definition of the piecewise linear continuous function $f\colon \mathcal{R} \to \mathcal{R}$ as

$$f(x) = \sum_{j=1}^{m} (a_j x + b_j)\delta_{A_j}(x) \tag{8}$$

where $\forall\, j \in \{1, \ldots, m\}$, $a_j, b_j \in \mathcal{R}$, the collection of intervals $\{A_1, \ldots, A_m\} \subseteq \mathcal{B}(\mathcal{R})$ form a partition of $\mathcal{R}$ ($A_1 \equiv (-\infty, k_1)$ and $A_j = [k_{j-1}, k_j)$ for $j \in \{2, 3, \ldots, m\}$, $k_m = \infty$) and $\delta(\cdot)$ is the indicator function

$$\delta_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{9}$$

Continuity requires that, for any $j \in \{1, 2, \ldots, m-1\}$,

$$a_j k_j + b_j = a_{j+1} k_j + b_{j+1} \Leftrightarrow b_{j+1} = (a_j - a_{j+1})k_j + b_j. \tag{10}$$

The EMS price estimate for a contingent claim with the payoff function $f(S_T)$ can be written as

$$\begin{aligned} C_{EMS}^{(n)} &= e^{-rT} \frac{1}{n} \sum_{i=1}^{n} f\big(S_{i,T}^{*}\big) \\ &= e^{-rT} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \big(a_j S_{i,T}^{*} + b_j\big)\delta_{A_j}\big(S_{i,T}^{*}\big). \end{aligned} \tag{11}$$

The asymptotic properties of the EMS estimator are completely characterized in the following two-part theorem.

Theorem 1. *If the asset price S_T is a positive random variable with a continuous distribution and a finite second moment, then, for any piecewise linear and continuous payoff function f as in* (8),

(i)

$$\begin{aligned} C_{EMS}^{(n)} = C_{MCS}^{(n)} &- \big(e^{-rT}\overline{S}_{n,T} - S_0\big) \\ &\times \mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0 e^{rT}}\right] + o^{Q}(n^{-1/2}), \end{aligned} \tag{12}$$

where $o^Q(n^{-k})$ denotes some random variable H_n that satisfies $\lim_{n\to\infty} n^k H_n = 0$ in Q probability, and $\varphi'(x) \equiv \sum_{j=1}^{m} a_j \delta_{A_j}(x)$.

(ii)

$$\sqrt{n}\Big(C_{EMS}^{(n)} - C\Big) \overset{n\to\infty}{\Longrightarrow} N(0, V) \tag{13}$$

where

$$\begin{aligned} V = {} & e^{-2rT}\mathrm{Var}[f(S_T)] + e^{-2rT}\mathrm{Var}[S_T]\Phi^2 \\ & - 2e^{-2rT}\Phi\,\mathrm{Cov}[f(S_T), S_T], \end{aligned} \tag{14}$$

and $\Phi = \mathrm{E}[\varphi'(S_T)S_T/(S_0 e^{rT})]$,

Remark. $f(x)$ is nondifferentiable only at $\{k_j\colon j = 1, 2, \ldots, m-1\}$. In general, $\varphi'(x) = f'(x)$ for $x \notin \{k_j\colon j = 1, 2, \ldots, m-1\}$. At the nondifferentiable points, $\varphi'(x)$ is taken as the right-hand derivative of $f(x)$, i.e., $\varphi'(k_j) = f'(k_j^{+})$ for $j = 1, 2, \ldots, m-1$.

Proof. See Appendix.

Part (i) of the above theorem implies that the EMS method is asymptotically equivalent to a linear control-variate simulation scheme with the asset price serving as the control. This result is in a way similar to Glynn and Whitt's (1989) characterization of nonlinear control-variate schemes in a different context. They reached a conclusion that moving from a linear control-variate scheme to a nonlinear one (with the same fixed control variable) cannot improve asymptotic efficiency, which is the same conclusion reached earlier by Cheng and Feast (1980). Later, we will show by examples that the EMS method has a performance comparable to the optimal linear control-variate method. The optimal linear control-variate method is, however, computationally more demanding because of the optimization step.

Part (ii) of the theorem completely characterizes the asymptotic distribution of the EMS estimator. Indeed, Part (ii) implies that $C_{EMS}^{(n)}$ converges weakly to the constant C. Consequently, $C_{EMS}^{(n)}$ converges in probability to C. The consistency of the EMS price estimator was first established in Duan and Simonato (1998) via a different approach for payoff functions that are more general than piecewise linear. Obtaining consistency is thus not our intention here. The reason for choosing a more restricted set-up is to go beyond consistency and provide a complete characterization of the distributional properties of this estimator. The sum of the last two terms on the right-hand side of the variance formula in (14) is expected to be negative. Although we cannot prove this assertion, all specific cases that we have examined so far turn out to be consistent with this prediction. The sum being negative makes the simulation error of the EMS estimator smaller than that of the standard Monte Carlo estimator, which is the first term of the right-hand side of the variance formula.

The variance formula can be used to obtain a consistent variance estimate using just one simulation sample. Since the EMS adjustment is made to the standard Monte Carlo simulated prices, the unadjusted simulated prices $\{S_{i,T}, i=1,\ldots,n\}$ can be used for the consistent estimation of $\mathrm{Var}[S_T]$, $\mathrm{Var}[f(S_T)]$, $\mathrm{E}[\varphi'(S_T)S_T/(S_0e^{rT})]$, and $\mathrm{Cov}[f(S_T), S_T]$.

4. Examples of European Call Options

In this section we show how the asymptotic results given in the preceding section can be applied to European call options. We also examine how close is the actual distribution of the EMS estimator to the asymptotic distribution prescribed by our theoretical result.

A European call option with maturity T and strike price K has the payoff function $\max(S_T-K, 0)$. The EMS price estimate at time 0 using a sample size n is

$$C_{EMS}^{(n)} = \frac{e^{-rT}}{n}\sum_{i=1}^{n}\left(S_{i,T}^{*}-K\right)\delta_{[K,\infty)}\left(S_{i,T}^{*}\right). \tag{15}$$

In other words, the relevant piecewise linear function has the following parameter values: $m=2$, $a_1=0$, $b_1=0$, $A_1=(-\infty, K)$, $a_2=1$, $b_2=-K$, and $A_2=[K,\infty)$.

Theorem 1 can be used to construct a confidence interval around the EMS price estimate. More specifically, the $1-\alpha$ confidence interval is

$$\left[C_{EMS}^{(n)}-z_{\frac{\alpha}{2}}\sqrt{\mathrm{Var}\left(C_{EMS}^{(n)}\right)};\, C_{EMS}^{(n)}+z_{\frac{\alpha}{2}}\sqrt{\mathrm{Var}\left(C_{EMS}^{(n)}\right)}\right] \tag{16}$$

where $z_{\alpha/2}$ is such that $\Pr[Z>z_{\alpha/2}]=\alpha/2$ and Z stands for a standard normal random variable.

4.1. Black-Scholes Model

The variance formula for the EMS price estimator given in Theorem 1 is only an approximate relationship for any finite sample size. It is therefore important to find out whether it is a good approximation for different sample sizes commonly used in practice. The same concern applies to the asymptotic normal distribution of the EMS price estimator. We use European call options in the Black Scholes option pricing framework to demonstrate the performance of the variance estimator and the distributional result. Three maturities (30, 90, and 270 days) and three asset-to-strike price ratios (1.1, 1, and 0.9) are used in the study. In our numerical analyses, one year is assumed to have 365 days. According to Black and Scholes (1973), the asset price after risk-neutralization follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = r\,dt + v\,dW_t. \tag{17}$$

In accordance with the above model, the future asset price has a lognormal distribution. The parameter values used in our analysis are $S_0=100$, $r=0.1$ (annualized), and $v=0.2$ (annualized). In Table 1, the first and second panels report the results obtained with n equal to 500 and 10,000 sample paths, respectively.

In each panel, we first report the standard deviation estimates obtained from one Monte Carlo simulation for $C_{MCS}^{(n)}$ and $C_{EMS}^{(n)}$. We also report the standard errors of two other Monte Carlo price estimates based on the linear control-variate scheme with the terminal stock price as the control variate. The two linear control-variate estimates use different multipliers. Specifically,

$$C_{CV1}^{(n)} = C_{MCS}^{(n)} - \left(e^{-rT}\frac{1}{n}\sum_{i=1}^{n}S_{i,T}-S_0\right), \quad \text{and} \tag{18}$$

$$C_{CV2}^{(n)} = C_{MCS}^{(n)} - \beta_n\left(e^{-rT}\frac{1}{n}\sum_{i=1}^{n}S_{i,T}-S_0\right), \tag{19}$$

where β_n is the sample regression coefficient from regressing $f(S_{i,T})$ on $(e^{-rT}S_{i,T}-S_0)$. It is clear that $C_{CV2}^{(n)}$ must by design dominate, in terms of simulation error, all linear control-variate schemes including $C_{CV1}^{(n)}$. By Theorem 1, $C_{EMS}^{(n)}$ can also be viewed as a linear control-variate scheme, and thus its simulation error cannot be less than that of $C_{CV2}^{(n)}$. It is worth noting that $C_{CV2}^{(n)}$ may not be the most efficient scheme when computing time is taken into account because more calculations are required. We want to examine how well EMS performs in relation to these two linear control-variate schemes. By comparing $C_{EMS}^{(n)}$ to

Table 1 European Calls Based on the Black-Scholes Option Pricing Model

	$n = 500$ sample paths								
	Maturity = 30 days			Maturity = 90 days			Maturity = 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1)									
MCS	0.2545	0.1689	0.0317	0.4008	0.3001	0.1397	0.7109	0.6213	0.4764
EMS	0.0227	0.0782	0.0286	0.0550	0.1183	0.0976	0.1614	0.2139	0.2352
CV1	0.0248	0.1382	0.2513	0.0642	0.1967	0.3467	0.2051	0.3246	0.4761
CV2	0.0229	0.0787	0.0288	0.0554	0.1212	0.1000	0.1684	0.2289	0.2524
STD(1,000)									
MCS	0.2428	0.1619	0.0311	0.3883	0.2987	0.1537	0.7221	0.6254	0.4645
EMS	0.0213	0.0745	0.0287	0.0735	0.1274	0.1123	0.1427	0.2061	0.2371
CV1	0.0229	0.1297	0.2406	0.0835	0.1975	0.3345	0.1725	0.2967	0.4686
CV2	0.0214	0.0748	0.0287	0.0732	0.1273	0.1114	0.1429	0.2059	0.2369
Cov. rates									
25% cov. rate ems	0.2370	0.2670	0.2360	0.2520	0.2450	0.2500	0.2280	0.2240	0.2220
50% cov. rate ems	0.4970	0.5300	0.4850	0.4780	0.4710	0.4860	0.4610	0.4590	0.4600
75% cov. rate ems	0.7450	0.7520	0.7450	0.7030	0.7470	0.7290	0.7190	0.6990	0.7020
95% cov. rate ems	0.9120	0.9510	0.9180	0.9160	0.9410	0.9510	0.9340	0.9370	0.9250
	$n = 10{,}000$ sample paths								
	Maturity = 30 days			Maturity = 90 days			Maturity = 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1)									
MCS	0.0557	0.0373	0.0069	0.0915	0.0695	0.0346	0.1541	0.1330	0.1001
EMS	0.0047	0.0168	0.0063	0.0153	0.0280	0.0245	0.0298	0.0421	0.0487
CV1	0.0050	0.0293	0.0548	0.0183	0.0463	0.0799	0.0378	0.0656	0.1019
CV2	0.0047	0.0169	0.0064	0.0157	0.0287	0.0251	0.0311	0.0452	0.0525
STD(1,000)									
MCS	0.0538	0.0359	0.0075	0.0940	0.0711	0.0345	0.1541	0.1321	0.0997
EMS	0.0048	0.0170	0.0070	0.0158	0.0277	0.0247	0.0318	0.0451	0.0520
CV1	0.0052	0.0294	0.0532	0.0180	0.0453	0.0812	0.0394	0.0673	0.1029
CV2	0.0048	0.0171	0.0070	0.0157	0.0276	0.0247	0.0318	0.0450	0.0518
Cov. rates									
25% cov. rate ems	0.2650	0.2340	0.2400	0.2370	0.2410	0.2380	0.2540	0.2240	0.2490
50% cov. rate ems	0.5210	0.4930	0.4680	0.5040	0.4910	0.5130	0.4780	0.4640	0.4720
75% cov. rate ems	0.7370	0.7420	0.7270	0.7300	0.7500	0.7550	0.7310	0.7020	0.7080
95% cov. rate ems	0.9550	0.9480	0.9300	0.9400	0.9540	0.9400	0.9340	0.9380	0.9300

Note. STD(1) are the sample standard deviations obtained from one Monte Carlo simulation sample for MCS, EMS, CV1, and CV2. STD(1,000) are the standard deviations obtained from 1,000 independent Monte Carlo price estimates, each generated with n sample paths. The coverage rate is the percentage of the 1,000 EMS prices for which the theoretical price is contained in the α% confidence interval implied by the asymptotic distribution. Basic parameter values: $S_0 = 100$, $r = 0.10$ (annualized), and $v = 0.2$ (annualized).

$C^{(n)}_{CV1}$ and $C^{(n)}_{CV2}$, we can determine how well EMS performs in relation to the most simple and optimal linear control-variate schemes, respectively.

The second set of results appearing in both panels reports the simulated standard errors obtained directly from calculating the standard errors of the price estimates using $C^{(n)}_{MCS}$, $C^{(n)}_{EMS}$, $C^{(n)}_{CV1}$, and $C^{(n)}_{CV2}$. These standard errors are computed using 1,000 independent simulation samples corresponding to each of two sample sizes—500 and 10,000. The results show that theoretical and simulated standard errors are very similar in every case, which in turn suggests that the variance formulas based on the asymptotic distribution are fairly reliable even when one deals with a simulation sample size as small as 500. The EMS estimate, $C_{EMS}^{(n)}$, dominates by a wide margin the first linear control-variate scheme and has a performance nearly identical to the optimal linear control-variate scheme. The benefit brought about by the EMS adjustment is substantial. The simulation error reduction delivered by EMS is particularly large for shorter-maturity in-the-money options and/or longer-maturity options. As noted in Duan and Simonato (1998), the EMS estimator ensures that the typical option pricing bounds are satisfied. Since the simulation error of $C_{EMS}^{(n)}$ is comparable to that of $C_{CV2}^{(n)}$, this unique feature of the EMS estimator is really an added advantage.

To see whether asymptotic normality is a reasonable description for the EMS price estimator, we compute various coverage rates constructed from the asymptotic normal distribution. For every EMS price, there corresponds a sample of n simulated paths. The distribution of each EMS price estimate can be directly inferred from this simulation sample by using our theoretical result. Such a distribution can be used to construct the α% confidence band. We then use it to check whether the band covers the theoretical price, which is in this case the Black-Scholes formula price. We repeat this comparison 1,000 times to obtain an empirical estimate of the coverage rate. If the theoretical distribution is a good approximation, we should expect to have a coverage rate close to α%. Each panel of Table 1 reports the results for four coverage rates, 25%, 50%, 75%, and 95%. Our results indicate that the actual coverage rates are very close to their theoretical values.

4.2. GARCH Model

We now perform a similar study for the GARCH option pricing model of Duan (1995). We use a linear GARCH(1, 1)-in mean model for this purpose. In this valuation framework, the price dynamic of the underlying asset under the locally risk-neutralized probability measure Q is

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1}, \tag{20}$$

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t(\varepsilon_t - \lambda)^2, \tag{21}$$

$$\varepsilon_{t+1} | \mathcal{F}_t \overset{Q}{\sim} N(0,1), \tag{22}$$

where $\varepsilon_{t+1} | \mathcal{F}_t \overset{Q}{\sim} N(0,1)$ means that the conditional distribution of ε_{t+1} under the probability Q is standard normal, $\beta_0, \beta_1, \beta_2$ are the linear GARCH(1, 1) parameters, and λ is the unit risk-premium parameter (per unit of conditional standard deviation) that defines the conditional mean equation of the GARCH(1, 1)-(in mean) dynamic under the data generating probability measure. In the following numerical study, we set the initial conditional variance to its stationary level under the data generating probability measure, i.e., $h_1 = \beta_0(1-\beta_1-\beta_2)^{-1}$. Our results in Table 2 are based on $S_0 = 100$, $r = 0.1$ (annual), $\beta_0 = 0.00001$, $\beta_1 = 0.7$, $\beta_2 = 0.2$, and $\lambda = 0.01$. Again the number of sample paths are varied and the calculation is repeated 1,000 times to generate the numbers in the table. Since there is no analytical formula for the GARCH option pricing model, the theoretical prices required for the coverage rate calculation are approximate values, which are averages of 500 MCS prices with each price being calculated using 10,000 sample paths with the Black-Scholes price as the control variable. More precisely, the benchmark is obtained using

$$C_{CV3}^{(n)} = C_{MCS}^{(n)} - \left(e^{-rT} \frac{1}{n} \sum_{i=1}^{n} \max\left(\widehat{S}_{i,T} - K; 0\right) - C_{BS} \right) \tag{23}$$

where $\widehat{S}_{i,T}$ is simulated according to a geometric Brownian motion with the same gaussian shocks that we used in generating the GARCH path and C_{BS} is the Black-Scholes formula price with h_1 as volatility parameter. The results in Table 2, as it was the case for the Black-Scholes framework, show coverage rates that are very close to their theoretical values. In short, the asymptotic distribution based on our theory does provide a good approximation to the finite-sample distribution of the EMS price estimator under a different theoretical option pricing framework.

Table 2 European Calls Based on the GARCH Option Pricing Model

	$n = 500$ sample paths								
	Maturity = 30 days			Maturity = 90 days			Maturity = 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1)									
MCS	0.2459	0.1722	0.0470	0.3938	0.3026	0.1441	0.6289	0.5280	0.3789
EMS	0.0156	0.0739	0.0421	0.0496	0.1093	0.1013	0.1359	0.1825	0.2020
CV1	0.0163	0.1202	0.2307	0.0558	0.1745	0.3310	0.1766	0.2991	0.4599
CV2	0.0156	0.0741	0.0410	0.0502	0.1120	0.1034	0.1432	0.1964	0.2173
STD(1,000)									
MCS	0.2378	0.1595	0.0406	0.3906	0.3005	0.1537	0.6483	0.5589	0.4169
EMS	0.0282	0.0753	0.0378	0.0731	0.1274	0.1152	0.1360	0.1983	0.2337
CV1	0.0308	0.1325	0.2367	0.0831	0.1973	0.3399	0.1624	0.2815	0.4427
CV2	0.0275	0.0764	0.0373	0.0735	0.1279	0.1139	0.1357	0.1987	0.2335
Cov. rates									
25% cov. rate ems	0.2180	0.2360	0.2280	0.2340	0.2520	0.2360	0.2500	0.2390	0.2300
50% cov. rate ems	0.4720	0.4990	0.4810	0.4960	0.4950	0.4790	0.4650	0.4770	0.4660
75% cov. rate ems	0.7180	0.7640	0.7120	0.7300	0.7450	0.7270	0.7140	0.7100	0.7130
95% cov. rate ems	0.8880	0.9480	0.8950	0.9210	0.9430	0.9330	0.9400	0.9260	0.9290
	$n = 10{,}000$ sample paths								
	Maturity = 30 days			Maturity = 90 days			Maturity = 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1)									
MCS	0.0536	0.0368	0.0096	0.0879	0.0675	0.0341	0.1473	0.1272	0.0950
EMS	0.0065	0.0172	0.0088	0.0165	0.0277	0.0249	0.0280	0.0400	0.0469
CV1	0.0069	0.0289	0.0524	0.0190	0.0445	0.0771	0.0343	0.0609	0.0968
CV2	0.0065	0.0173	0.0088	0.0166	0.0284	0.0254	0.0288	0.0428	0.0505
STD(1,000)									
MCS	0.0517	0.0362	0.0095	0.0891	0.0682	0.0339	0.1483	0.1286	0.0985
EMS	0.0063	0.0176	0.0089	0.0162	0.0277	0.0245	0.0293	0.0441	0.0537
CV1	0.0066	0.0281	0.0505	0.0183	0.0436	0.0761	0.0358	0.0630	0.0987
CV2	0.0062	0.0176	0.0087	0.0162	0.0276	0.0241	0.0293	0.0439	0.0532
Cov. rates									
25% cov. rate ems	0.2680	0.2390	0.2450	0.2580	0.2380	0.2400	0.2570	0.2370	0.2250
50% cov. rate ems	0.5170	0.4900	0.4830	0.4970	0.4900	0.5080	0.5240	0.4840	0.4600
75% cov. rate ems	0.7460	0.7330	0.7200	0.7370	0.7590	0.7540	0.7460	0.7150	0.6990
95% cov. rate ems	0.9550	0.9410	0.9380	0.9490	0.9470	0.9540	0.9440	0.9290	0.9170

Note. STD(1) are the sample standard deviations obtained from one Monte Carlo simulation sample for MCS, EMS, CV1, and CV2. STD(1,000) are the standard deviations obtained from 1,000 independent Monte Carlo price estimates, each generated with n sample paths. The coverage rate is the percentage of the 1,000 EMS prices for which the theoretical price is contained in the α% confidence interval implied by the asymptotic distribution. Basic parameter values: $S_0 = 100$, $r = 0.10$ (annualized), $\beta_0 = 0.00001$, $\beta_1 = 0.7$, $\beta_2 = 0.2$, and $\lambda = 0.01$.

5. Conclusion

In this paper, we have offered a complete characterization of the asymptotic distribution of the EMS price estimator. We have shown that the theoretically derived asymptotic distribution is a good approximation to the actual distribution obtained from a simulation study. This result solves one major problem facing the EMS method developed by Duan and Simonato (1998). With these results, one can now compute the simulation error using just one sample. In short, we are now in the position to take advantage of the efficiency gain offered by the EMS method while retaining the convenience of the standard Monte Carlo simulation method in terms of assessing simulation errors.

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Appendix: Proof of the Main Result

Proof of part (i). The proof of Theorem 1 is based on a sequence of smooth approximations to the piecewise linear continuous function. Recall that the piecewise linear continuous function defined in Equation (8) is

$$f(x) = \sum_{j=1}^{m} (a_j x + b_j)\delta_{A_j}(x),$$

where, for any $j \in \{1, 2, \ldots, m-1\}$,

$$b_{j+1} = (a_j - a_{j+1})k_j + b_j.$$

Consider a sequence $\{\varphi_n: n \in \mathbb{N}\}$ of differentiable functions:

$$\varphi_n(x) \equiv \begin{cases} a_1 x + b_1 & \text{if } x < k_1 - \varepsilon_n, \\ a_j x + b_j & \text{if } k_{j-1} + \varepsilon_n < x < k_j - \varepsilon_n,\ j \in \{2, 3, \ldots, m-1\}, \\ a_m x + b_m & \text{if } x > k_{m-1} + \varepsilon_n, \\ p_n^{(j)}(x) & \text{if } k_j - \varepsilon_n \le x \le k_j + \varepsilon_n,\ j \in \{1, 2, \ldots, m-1\}, \end{cases}$$

where ε_n is a positive decreasing function of n such that $\lim_{n\to\infty} \sqrt{n}\varepsilon_n = 0$ (we may think of $\varepsilon_n = 1/n$, $\varepsilon_n = 1/n^2, \ldots$) and, for any $j \in \{1, 2, 3, \ldots, m-1\}$,

$$p_n^{(j)}(x) \equiv b_j + \left(\frac{a_{j+1} - a_j}{4\varepsilon_n}\right)(k_j - \varepsilon_n)^2 + \left(a_j - \frac{a_{j+1} - a_j}{2\varepsilon_n}(k_j - \varepsilon_n)\right)x + \frac{a_{j+1} - a_j}{4\varepsilon_n}x^2.$$

(Note that using a quadratic function over $[k_j - \varepsilon_n, k_j + \varepsilon_n]$ gives rise to a smooth (differentiable) pasting of two adjacent linear functions. A quadratic function suffices in the current context because of the continuity of the target function.)

Recall that $\varphi'(x) \equiv \sum_{j=1}^{m} a_j \delta_{A_j}(x)$. Since

$$\varphi_n'(x) \equiv \begin{cases} a_1 & \text{if } x < k_1 - \varepsilon_n, \\ a_j & \text{if } k_{j-1} + \varepsilon_n < x < k_j - \varepsilon_n, \\ & \qquad j \in \{2, 3, \ldots, m-1\}, \\ a_m & \text{if } x > k_{m-1} + \varepsilon_n, \\ a_j + \frac{a_{j+1} - a_j}{2\varepsilon_n}(x - k_j + \varepsilon_n) & \text{if } k_j - \varepsilon_n \le x \le k_j + \varepsilon_n, \\ & \qquad j \in \{1, 2, \ldots, m-1\}, \end{cases}$$

φ_n is continuous and has a continuous first derivative φ_n'. Furthermore,

$$\begin{aligned} \left|\varphi_n'(x) - \varphi'(x)\right| &= \left|\sum_{j=1}^{m-1} \frac{a_{j+1} - a_j}{2\varepsilon_n}(x - k_j + \varepsilon_n)\delta_{(k_j - \varepsilon_n, k_j)}(x) \right. \\ &\qquad \left. + \frac{a_j - a_{j+1}}{2\varepsilon_n}(k_j + \varepsilon_n - x)\delta_{[k_j, k_j + \varepsilon_n)}(x)\right| \\ &\le \sum_{j=1}^{m-1} \frac{|a_{j+1}| + |a_j|}{2\varepsilon_n}\Big[(x - k_j + \varepsilon_n)\delta_{(k_j - \varepsilon_n, k_j)}(x) \\ &\qquad + (k_j + \varepsilon_n - x)\delta_{[k_j, k_j + \varepsilon_n)}(x)\Big] \\ &\le \sum_{j=1}^{m-1} \frac{|a_{j+1}| + |a_j|}{2\varepsilon_n}\Big[\varepsilon_n \delta_{(k_j - \varepsilon_n, k_j)}(x) + \varepsilon_n \delta_{[k_j, k_j + \varepsilon_n)}(x)\Big] \\ &\le \sum_{j=1}^{m} |a_j|. \end{aligned}$$

Thus, there is a constant κ, which does not depend on n, such that

$$\|\varphi' - \varphi_n'\|_\infty = \sup_{x \in \mathbb{R}} |\varphi'(x) - \varphi_n'(x)| \le \kappa.$$

We also have

$$\|f - \varphi_n\|_\infty \equiv \sup_{x \in \mathbb{R}} |f(x) - \varphi_n(x)| \le \kappa\varepsilon_n.$$

Using the nth approximation and recalling $\sqrt{n}\varepsilon_n \xrightarrow{n\to\infty} 0$, we have

$$\begin{aligned} e^{rT}\sqrt{n}C_{EMS}^{(n)} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f\big(S_{i,T}^*\big) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \varphi_n\big(S_{i,T}^*\big) + O(\sqrt{n}\varepsilon_n) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(S_{i,T}) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} (\varphi_n(S_{i,T}) - f(S_{i,T})) \\ &\quad + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} (\varphi_n(S_{i,T}^*) - \varphi_n(S_{i,T})) + O(\sqrt{n}\varepsilon_n) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(S_{i,T}) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} (\varphi_n\big(S_{i,T}^*\big) - \varphi_n(S_{i,T})) + O(\sqrt{n}\varepsilon_n) \\ &\qquad \Big(\text{because } \|f - \varphi_n\|_\infty \equiv \sup_{x \in \mathbb{R}} |f(x) - \varphi_n(x)| \le \kappa\varepsilon_n\Big) \\ &= e^{rT}\sqrt{n}C_{MCS}^{(n)} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \big(\varphi_n\big(S_{i,T}^*\big) - \varphi_n(S_{i,T})\big) + O(\sqrt{n}\varepsilon_n), \end{aligned}$$

where $O(c_n)$ represents a function $h(n)$ such that $\sup_n h(n)/c_n < \infty$.

Note that for any fixed $s > 0$, the function $v \to \varphi_n(s/v)$ is continuous and differentiable on the interval $(0, \infty)$. Since the asset price is a positive random variable, we can use the mean value theorem to expand $\varphi_n(s/v)$. That is, there exists a random variable $V_{i,n}^*$ which depends on $S_{i,T}$ and $\overline{S}_{n,T}$ and satisfies

$$\begin{aligned}\varphi_n(S^*_{i,T}) &= \varphi_n\left(\frac{S_{i,T}S_0e^{rT}}{\overline{S}_{n,T}}\right)\\ &= \varphi_n\left(\frac{S_{i,T}S_0e^{rT}}{S_0e^{rT}}\right) - \varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{(V^*_{i,n})^2}(\overline{S}_{n,T}-S_0e^{rT})\\ &= \varphi_n(S_{i,T}) - \varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2}(\overline{S}_{n,T}-S_0e^{rT}),\end{aligned}$$

and $\min(S_0e^{rT}, \overline{S}_{n,T}) \le V^*_{i,n} \le \max(S_0e^{rT}, \overline{S}_{n,T})$. Note that $V^*_{1,n}, \ldots, V^*_{n,n}$ are identically distributed but not necessarily independent. It follows that

$$\begin{aligned}&e^{rT}\sqrt{n}C^{(n)}_{EMS}\\ &= e^{rT}\sqrt{n}C^{(n)}_{MCS} - \sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})\\ &\quad\times\frac{1}{n}\sum_{i=1}^{n}\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2} + O(\sqrt{n}\varepsilon_n)\\ &= e^{rT}\sqrt{n}C^{(n)}_{MCS} - \sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})\mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] + \sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})\\ &\quad\times\left(\mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2}\right) + O(\sqrt{n}\varepsilon_n).\end{aligned}$$

Since $\sup_n \sup_s |\varphi_n'(s)| \le w$ for some finite w and $\varphi_n'(S_T)$ converges Q-almost surely to $\varphi'(S_T)$ (because $\Pr[S_{i,T} \in \{k_1, \ldots, k_{m-1}\}] = 0$), the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] = \mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0e^{rT}}\right]. \tag{A1}$$

Note that by the central limit theorem, $\sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})$ converges weakly to a proper normal random variable. Recall that $o^Q(n^{-k})$ denotes some random variable H_n that satisfies $\lim_{n\to\infty} n^k H_n = 0$ in Q-probability. Therefore, (A1) can be used to obtain

$$\sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})\left\{\mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0e^{rT}}\right]\right\} = o^Q(n^0).$$

We thus have

$$\begin{aligned}&e^{rT}\sqrt{n}C^{(n)}_{EMS}\\ &= e^{rT}\sqrt{n}C^{(n)}_{MCS} - \sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})\mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0e^{rT}}\right] + \sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})\\ &\quad\times\left(\mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2}\right)\\ &\quad + o^Q(n^0).\end{aligned} \tag{A2}$$

It remains to show that the third item on the right-hand side of (A2) is also a term of $o^Q(n^0)$. Again by the central limit theorem, $\sqrt{n}(\overline{S}_{n,T}-S_0e^{rT})$ converges weakly to a proper normal random variable. It suffices to show that the second part of the third item goes to zero in Q-probability. Consider

$$\begin{aligned}&\mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2}\\ &= \mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'(S_{i,T})\frac{S_{i,T}}{S_0e^{rT}}\\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\varphi_n'(S_{i,T})\frac{S_{i,T}}{S_0e^{rT}} - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2}.\end{aligned} \tag{A3}$$

The first two items on the right-hand side of (A3) tend together to zero in Q-probability because

$$\begin{aligned}&\mathrm{E}\left[\varphi_n'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'(S_{i,T})\frac{S_{i,T}}{S_0e^{rT}}\\ &= \mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'(S_{i,T})\frac{S_{i,T}}{S_0e^{rT}} + o(n^0) \quad \text{by (A1)}\\ &= \mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0e^{rT}}\right] - \frac{1}{n}\sum_{i=1}^{n}\varphi'(S_{i,T})\frac{S_{i,T}}{S_0e^{rT}}\\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\left[\varphi'(S_{i,T}) - \varphi_n'(S_{i,T})\right]\frac{S_{i,T}}{S_0e^{rT}} + o(n^0).\end{aligned}$$

Clearly, the first two items together converge to zero in Q-probability by the law of large numbers since the integrability of S_T and the boundedness of φ' implies that $\mathrm{E}[|\varphi'(S_T)S_T|] < \infty$. The third item approaching zero in Q-probability can be shown as follows:

$$\begin{aligned}&\mathrm{E}\left[\left|\frac{1}{n}\sum_{i=1}^{n}\left[\varphi'(S_{i,T}) - \varphi_n'(S_{i,T})\right]\frac{S_{i,T}}{S_0e^{rT}}\right|\right]\\ &\le \mathrm{E}\left[\left|\varphi'(S_{1,T}) - \varphi_n'(S_{1,T})\right|\frac{S_{1,T}}{S_0e^{rT}}\right]\\ &\le \sqrt{\mathrm{E}\left[\left|\varphi'(S_{1,T}) - \varphi_n'(S_{1,T})\right|^2\right]}\sqrt{\mathrm{E}\left[\left(\frac{S_{1,T}}{S_0e^{rT}}\right)^2\right]}.\end{aligned}$$

The sequence $\varphi_n'(\bullet)$ converges to $\varphi'(\bullet)$ on the set $\mathbb{R}\backslash\{k_1, \ldots, k_m\}$. It in turn implies that $|\varphi'(S_{1,T}) - \varphi_n'(S_{1,T})|^2$ converges Q-almost surely to zero, because $\Pr[S_{1,T} \in \{k_1, \ldots, k_{m-1}\}] = 0$. Since $\sup_n \sup_s |\varphi_n'(s)| \le w$ for some finite w, the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\mathrm{E}\left[\left|\varphi'(S_{1,T}) - \varphi_n'(S_{1,T})\right|^2\right] = 0.$$

Combining this result with the assumption that $\mathrm{E}[(S_{1,T}/(S_0e^{rT}))^2] < \infty$ gives rise to the L^1 convergence to zero, which in turn implies convergence to zero in Q-probability as required.

For the remaining two items on the right-hand side of (A3), we perform the following calculation:

$$\begin{aligned}&\left|\frac{1}{n}\sum_{i=1}^{n}\varphi_n'(S_{i,T})\frac{S_{i,T}}{S_0e^{rT}} - \frac{1}{n}\sum_{i=1}^{n}\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2}\right|\\ &\le \frac{1}{n}\sum_{i=1}^{n}\left|\varphi_n'(S_{i,T})\frac{S_{i,T}}{S_0e^{rT}} - \varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(S_0e^{rT}\right)^2}\right|\\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\left|\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(S_0e^{rT}\right)^2} - \varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V^*_{i,n}}\right)\frac{S_{i,T}S_0e^{rT}}{\left(V^*_{i,n}\right)^2}\right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\left|\varphi_n'(S_{i,T}) - \varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V_{i,n}^*}\right)\right|\frac{S_{i,T}}{S_0e^{rT}} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\left|\varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V_{i,n}^*}\right)\right|\left|\frac{S_{i,T}}{S_0e^{rT}} - \frac{S_{i,T}S_0e^{rT}}{\left(V_{i,n}^*\right)^2}\right| \\
&\le \frac{1}{n}\sum_{i=1}^{n}\left|\varphi_n'(S_{i,T}) - \varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V_{i,n}^*}\right)\right|\frac{S_{i,T}}{S_0e^{rT}} \\
&\quad + \|\varphi_n'\|_\infty \frac{1}{n}\sum_{i=1}^{n}\left|\frac{S_{i,T}}{S_0e^{rT}} - \frac{S_{i,T}S_0e^{rT}}{\left(V_{i,n}^*\right)^2}\right|. \qquad \text{(A4)}
\end{aligned}
$$

The fact that the first item of the last inequality in (A4) converges to zero in L^1-norm can be shown as follows:

$$
\begin{aligned}
&\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left|\varphi_n'(S_{i,T}) - \varphi_n'\left(\frac{S_{i,T}S_0e^{rT}}{V_{i,n}^*}\right)\right|\frac{S_{i,T}}{S_0e^{rT}}\right] \\
&\quad = \mathrm{E}\left[\left|\varphi_n'(S_{1,T}) - \varphi_n'\left(\frac{S_{1,T}S_0e^{rT}}{V_{1,n}^*}\right)\right|\frac{S_{1,T}}{S_0e^{rT}}\right] \\
&\quad \le \sqrt{\mathrm{E}\left[\left|\varphi_n'(S_{1,T}) - \varphi_n'\left(\frac{S_{1,T}S_0e^{rT}}{V_{1,n}^*}\right)\right|^2\right]}\sqrt{\mathrm{E}\left[\left(\frac{S_{1,T}}{S_0e^{rT}}\right)^2\right]}.
\end{aligned}
$$

We know that $V_{1,n}^*$ converges Q-almost surely to S_0e^{rT} as $n \to \infty$ by recalling that

$$
\min\left(S_0e^{rT}, \overline{S}_{n,T}\right) \le V_{1,n}^* \le \max\left(S_0e^{rT}, \overline{S}_{n,T}\right).
$$

The sequence $\varphi_n'(\bullet)$ converges to $\varphi'(\bullet)$ on the set $\mathbb{R}\backslash\{k_1, \ldots, k_m\}$. It in turn implies that $|\varphi_n'(S_{1,T}) - \varphi_n'(S_{1,T}S_0e^{rT}/V_{1,n}^*)|^2$ converges Q-almost surely to 0, since $\Pr[S_{1,T} \in \{k_1, \ldots, k_{m-1}\}] = 0$. Since $\sup_n \sup_s |\varphi_n'(s)| \le w$ for some finite w, the Lebesgue dominated convergence theorem implies that

$$
\lim_{n\to\infty}\mathrm{E}\left[\left|\varphi_n'(S_{1,T}) - \varphi_n'\left(\frac{S_{1,T}S_0e^{rT}}{V_{1,n}^*}\right)\right|^2\right] = 0.
$$

By the assumption in the statement of the theorem, $\mathrm{E}[(S_{1,T}/S_0e^{rT})^2] < \infty$, we thus have L^1 convergence to zero for the first item of the last inequality in (A4), which in turn implies convergence to zero in Q-probability.

The second item of the last inequality in (A4) can also be shown to converge to zero. First, we define

$$
q_{n,T} \equiv \sup_{x\in[\min(S_0e^{rT}, \overline{S}_{n,T}), \max(S_0e^{rT}, \overline{S}_{n,T})]}\left|\frac{1}{\left(S_0e^{rT}\right)^2} - \frac{1}{x^2}\right|.
$$

Note that $q_{n,T}$ converges to zero Q-almost surely as n goes to infinity. Thus,

$$
\begin{aligned}
&\|\varphi_n'\|_\infty \frac{1}{n}\sum_{i=1}^{n}\left|\frac{S_{i,T}}{S_0e^{rT}} - \frac{S_{i,T}S_0e^{rT}}{\left(V_{i,n}^*\right)^2}\right| \\
&\quad = \|\varphi_n'\|_\infty \frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\left(S_0e^{rT}\right)^2} - \frac{1}{\left(V_{i,n}^*\right)^2}\right| S_{i,T}S_0e^{rT} \\
&\quad \le \|\varphi_n'\|_\infty q_{n,T}\frac{1}{n}\sum_{i=1}^{n}S_{i,T}S_0e^{rT} \\
&\quad = \|\varphi_n'\|_\infty q_{n,T}\overline{S}_{n,T}S_0e^{rT} \\
&\quad \longrightarrow 0 \quad Q\text{-almost surely.}
\end{aligned}
$$

In summary, we have proved that

$$
e^{rT}\sqrt{n}C_{EMS}^{(n)} = e^{rT}\sqrt{n}C_{MCS}^{(n)} - \sqrt{n}\left(\overline{S}_{n,T} - S_0e^{rT}\right)\mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0e^{rT}}\right] + o^Q(n^0),
$$

or

$$
C_{EMS}^{(n)} = C_{MCS}^{(n)} - \left(e^{-rT}\overline{S}_{n,T} - S_0\right)\mathrm{E}\left[\varphi'(S_T)\frac{S_T}{S_0e^{rT}}\right] + o^Q\left(n^{-\frac{1}{2}}\right).
$$

Proof of part (ii). Let $\Phi = \mathrm{E}[\varphi'(S_T)S_T/(S_0e^{rT})]$. Use the result in Part (i) to write

$$
\begin{aligned}
\sqrt{n}\left(C_{EMS}^{(n)} - C\right) &= \sqrt{n}\left(C_{MCS}^{(n)} - C\right) - \sqrt{n}\left(e^{-rT}\overline{S}_{n,T} - S_0\right)\Phi + o^Q(n^0) \\
&= e^{-rT}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(f(S_{i,T}) - S_{i,T}\Phi - \mathrm{E}[f(S_T)] \\
&\quad + \mathrm{E}[S_T]\Phi) + o^Q(n^0).
\end{aligned}
$$

Since the first item of the second equality converges, by the central limit theorem, to a normal random variable with mean 0, one can invoke the converging-together theorem (Theorem 4.1, Billingsley, 1968) to conclude that

$$
\sqrt{n}\left(C_{EMS}^{(n)} - C\right) \overset{n\to\infty}{\Longrightarrow} N(0, V)
$$

where

$$
V = e^{-2rT}\mathrm{Var}[f(S_T)] + e^{-2rT}\mathrm{Var}[S_T]\Phi^2 - 2e^{-2rT}\Phi\mathrm{Cov}[f(S_T), S_T]. \quad \square
$$

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