Pricing Discretely Monitored Barrier Options by a Markov Chain

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The method here to price discretely monitored barrier options in both constant and time-varying volatility valuation frameworks uses a time-homogeneous Markov chain to approximate the underlying asset price process. It provides a natural framework for this pricing process because the discrete time step of the Markov chain can be easily matched with the monitoring frequency of the barrier. The underlying asset price can also be partitioned so as to place the barrier suitably. The method can efficiently handle difficult cases where the barrier is close to the initial asset price.

Examples include both knock-in and knock-out barrier options. Different types of barriers such as single, double, and moving barriers are also analyzed.

Barrier options have become almost as popular as their plain vanilla counterparts. They are desirable risk management tools because hedging costs are reduced by surrendering (via a knock-out or knock-in provision) a portion of the option's payoff that is deemed non-essential from a risk management or trading perspective. The typical analytical pricing formulas were derived assuming continuous monitoring of the barrier, but real-life barrier options are monitored at times that are discretely spaced over the life of the option contract, and by now it is well known that the frequency of monitoring has an important effect on an option's price.

Several numerical schemes have been proposed to address the pricing of barrier options in a discrete monitoring framework. A common technique is the trinomial tree scheme. Ritchken [1995] evaluates barrier options with continuous monitoring by adjusting a stretch parameter in the trinomial tree so that a row of nodes coincides with the barrier. Ahn, Figlewski, and Gao [1999] modify the trinomial tree scheme by introducing an adaptive mesh designed to improve resolution in specific critical areas of the tree.

Cheuk and Vorst [1996] price discretely monitored barrier options by adjusting the geometry of the trinomial tree so that the barrier always lies exactly halfway between two nodes at each monitoring time. With this method, a minimum of 50 trinomial steps between two consecutive barrier monitoring points are needed to achieve a reasonable level of pricing accuracy. If daily monitoring is required, the Cheuk and Vorst method can become computationally intensive.

Boyle and Tian [1998] and Zvan, Vetzal, and Forsyth [1998] use a finite-difference (or finite-element) scheme to price both discretely and continuously monitored barrier options. Their methods are flexible, but that flexibility comes at a price. The finite-difference (finite-element) method is computationally demanding, as one must partition the price and time dimensions into a reasonably fine grid to obtain pricing accuracy.

An alternative methodology for pricing barrier options is provided by Reimer and Sandmann [1995]. They employ backward reduction and quadratic interpolation in a binomial tree framework to obtain barrier option prices. Heynen and Kat [1995] find closed-form solutions for discretely monitored European style barrier options by applying the Girsanov theorem to express the barrier option price as a function of an *n*-dimensional integral. The dimension of the integral must grow with the number of monitoring time points, and the increase in dimension as required by more frequent monitoring quickly renders this method numerically inoperable.

Wei [1998] improves upon the Heynen and Kat [1995] method by using a combination of integral reduction technique and linear interpolation. He also evaluates exponentially decreasing and increasing barrier options.

Broadie, Glasserman, and Kou [1997] provide a computationally efficient adjustment to Rubinstein and Reiner's [1991] closed-form solution for continuously monitored European style barrier options. Their adjustment works well for up-and-out European puts and down-and-out European calls. The method deteriorates noticeably for other types of barrier options, though, and has difficulty dealing with a barrier that is close to the underlying stock price. Finally, Boyle, Broadie, and Glasserman [1997] propose a conditional Monte Carlo approach for the valuation of discretely monitored barrier options. Their method retains the typical features of the Monte Carlo scheme, which is robust to different contract specifications but slower in computation and less able to deal with American style options.

The numerical schemes for barrier options are typically developed for the constant-volatility Black-Scholes [1973] option pricing framework. Whether these numerical methods can be easily generalized to a more general time-varying volatility setting remains to be seen. (Monte Carlo simulation is an obvious method that can be generalized, with the exception of the conditional Monte Carlo method.)

We devise a numerical framework that is capable of dealing with both European- and American-style discretely monitored barrier options in either the constant or time-varying volatility framework. Our method uses the Markov chain approximation design proposed by Duan and Simonato [2001] for dealing with constant and time-varying volatility option pricing problems.

Unlike the traditional lattice scheme, the Markov chain approach unties the link between the number of asset prices and the number of time steps used in the approximation. This independence of price and time dimensions allows us to adjust the time step of the Markov chain to exactly fit the barrier monitoring frequency without sacrificing the fineness of the asset price approximation. In comparison to the finite-difference (finite-element) approach, the method avoids the computational burden associated with the unnecessary refinement of time due to the numerical approximation of the partial differential equation.

The Markov chain method also allows us to place the barrier suitably in relation to the discretized asset prices of the Markov chain. As discussed in Boyle and Lau [1994] and Boyle and Tian [1998], the placement of the barrier is important because it determines the performance of an algorithm in handling the difficult case where the barrier is located near the initial asset price.

The numerical performance of the Markov chain method is assessed under the constant-volatility (Black-Scholes) and time-varying volatility (GARCH) pricing models. Our results show that the Markov chain method performs accurately in various situations. An analysis of the pricing accuracy/computing time trade-off also suggests that the Markov chain method is superior to the finite difference method of Boyle and Tian [1998].

I. CONSTANT-VOLATILITY OPTION PRICING FRAMEWORK

We first review the method for an underlying asset that has a constant volatility.

The Markov Chain Method for Plain Vanilla Options

Let the asset price at time t be S_t. The Black-Scholes [1973] constant-volatility option pricing framework assumes that the asset price follows a geometric Brownian motion process under the data-generating probability measure $\mathbb{P}$:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

By the Black-Scholes option pricing theory, option valuation can be implemented simply as a discounted expected value of the contingent payoff associated with a derivative contract under the risk-neutralized asset price dynamic. This risk-neutralized asset price dynamic is also a geometric Brownian motion process with a change in drift. That is:

$$dS_t = rS_t dt + \sigma S_t dW_t^* \tag{2}$$

where r is a constant risk-free rate of interest, and W_t^* is the standard Brownian motion with respect to the risk-neutralized probability measure $\mathbb{Q}$. For option pricing, one needs to be concerned only with the system in Equation (2). To approximate this stochastic process, we use a Markov chain $X = \{X_t : t \in \{0, 1, \ldots\}\}$ with state space $p_1, p_2, \ldots, p_m\}$ and transition probability matrix Q as an approximation for $\{\ln(S_t): t \geq 0\}$, where m is an odd integer and $p_{(m+1)/2} = \ln(S_0)$. As shown in Duan and Simonato [2001], one can construct a time-homogeneous Markov chain in such a way that, as $m \to \infty$, the chain converges (weakly) to the target stochastic process over the time index set $\{t = 0, 1, \ldots\}$, and option prices computed with this chain converge to the theoretical option values.

In constructing the approximating Markov chain, two decisions need to be made. First, one must choose the set of discrete prices, i.e., $\{p_1, p_2, \ldots, p_m\}$. The second decision is the length of a time step. For a given set of prices, a different length of the time step simply produces a transition matrix with a different set of entries. We let $\vec{p} = [p_1, p_2, \ldots, p_m]'$, and its associated transition probability matrix be

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & \ddots & \vdots \\ q_{m1} & \cdots & q_{mm} \end{bmatrix} \tag{3}$$

The specific procedures for setting $\vec{p}$ and computing Q are described in Duan and Simonato [2001]. We provide a brief account of the procedures in Appendix A. It should be noted that all entries of Q are computed analytically.

The price of an American option with maturity T and strike price K can be computed by the recursive system:

$$V(\vec{p}, t) = \max\left[g(\vec{p}, K), e^{-r} Q V(\vec{p}, t+1)\right]$$

$$t \in \{0, 1, ..., T-1\} \tag{4}$$

with

$$V(\vec{p}, T) = g(\vec{p}, K)$$

where $V(\vec{p}, t)$ is the time t option price vector corresponding to the vector $\vec{p}$; max[·, ·] is a vector-valued function returning the maximum value on an element-by-element basis; $g(\vec{p}, K)$ is the option's payoff function upon exercise. Note that $g(\vec{p}, K) = \max\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\}$ where $\vec{0}$ and $\vec{1}$ denote vectors of zeros and ones, respectively, and w indicates a call ($w = 1$) or a put ($w = -1$).[1] The time 0 option price is the ($[m+1]/2$)-th element of $V(\vec{p}, 0)$.

For European options, where early exercise is not permitted, the recursive system can be simplified to:

$$V(\vec{p}, 0) = e^{-rT} Q^T \max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\} \tag{5}$$

Two points are worth noting. First, the transition probability matrix associated with this Markov chain is usually sparse. In other words, many elements of this matrix are numerically negligible. This property is important, because it drastically reduces storage and computation costs. This in turn ensures a better numerical result because a higher-dimensional Markov chain can actually be implemented.

Second, compared to the typical lattice approach, the Markov chain method offers one more degree of freedom in design, which is convenient for dealing with discrete monitoring. In a standard trinomial tree scheme, for example, the number of possible prices directly depends on the number of time steps chosen. In fact, the formula is $2n + 1$ where n is the number of time steps.[2]

In the Markov chain framework, these two decisions can be made independently for a European option. Indeed, once the step size is fixed, the Markov chain reproduces at discrete times the probabilistic behavior of the asset price process as the number of states goes to infinity (see Duan and Simonato [2001]). In other words, the entire conditional distribution of the stock price, at an arbitrary time, can be well approximated by the Markov chain.

Exhibit 1 illustrates this independence property between the step size and the number of states by examining the convergence of the Markov chain prices for a plain vanilla European call option with 75 days to maturity. Four different choices of time steps—1, 10, 50, and 75—are used to produce this table. As the results indi-

EXHIBIT 1
European Call Options in Black-Scholes Framework

	Markov Chain			
Time steps	75	50	10	1
$m = 101$	2.5978	2.5673	2.5612	2.5601
$m = 201$	2.5703	2.5620	2.5604	2.5601
$m = 301$	2.5648	2.5610	2.5603	2.5601
$m = 401$	2.5628	2.5606	2.5602	2.5601
$m = 501$	2.5619	2.5604	2.5602	2.5601
$m = 1001$	2.5606	2.5602	2.5602	2.5601
$m = 2001$	2.5602	2.5602	2.5602	2.5602

Time steps is the number of transition steps used by the Markov chain to reach maturity. Parameters: $S_0 = 50$, $r = 0.05$, $K = 50$, $T = 75$ days with a year defined as 250 days, and $\sigma = 0.20$. Black-Scholes formula price is 2.5602.

cate, convergence to the theoretical Black-Scholes price is achieved regardless of the number of time steps used.

Valuing Barrier Options

To value barrier options, it is convenient to augment the system by an auxiliary variable a_t. This auxiliary variable takes on two possible values: $a_t = 1$ if the barrier condition is triggered before or at time t, and $a_t = 0$ otherwise. Discrete monitoring need not take place in every period. If, for example, monitoring takes place every other period, say, 1, 3, 5, …, such a case can be easily handled as moving barriers. In our setup, monitoring is assumed to take place at $\{0, 1, \ldots, T\}$.

An American-style barrier option can be valued by the recursive bivariate system below, for $t \in \{0, 1, \ldots, T-1\}$:

$$v(p_i, t; a_t = 0) = \max \left[\begin{array}{c} g(p_i, K, a_t = 0), \\ e^{-r} \sum_{j=1}^{m} \mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 0\} \times \\ v(p_j, t+1; a_{t+1} = 0) \\ + e^{-r} \sum_{j=1}^{m} \mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 1 | X_t = p_i, a_t = 0\} \times \\ v(p_j, t+1; a_{t+1} = 1) \end{array} \right] \quad (6)$$

$$v(p_i, t; a_t = 1) = \max \left[\begin{array}{c} g(p_i, K, a_t = 1), \\ e^{-r} \sum_{j=1}^{m} \mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 1\} \times \\ v(p_j, t+1; a_{t+1} = 0) \\ + e^{-r} \sum_{j=1}^{m} \mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 1 | X_t = p_i, a_t = 1\} \times \\ v(p_j, t+1; a_{t+1} = 1) \end{array} \right] \quad (7)$$

where $v(p_i, t; a_t)$ captures the relevant barrier option value at time t corresponding to the underlying asset price p_i and the auxiliary condition $a_t = 1$ or 0; and $g(p_i, K, a_t)$ is the immediate exercise value at time t corresponding to the underlying asset price p_i and the barrier condition a_t. The values of $g(p_i, K, a_t)$ and the terminal conditions for $v(p_i, T; a_T = 0)$ and $v(p_i, T; a_T = 1)$, of course, depend on the nature of the barrier option under consideration.

It is important to note that the appropriate barrier option value at time t must be suitably chosen from the two alternative values: $v(p_i, t; a_t = 0)$ and $v(p_i, t; a_t = 1)$. Since a particular p_i may not be compatible with $a_0 = 0$, some adjustment may be needed. This becomes clear when we consider an example of the knock-out option.

If p_i is in the knock-out region, then it is consistent only with $a_t = 1$. The knock-out option value at time t thus equals $v(p_i, t; a_t = 1)$. If, on the other hand, p_i is not in the knock-out region, there are two possibilities. First, the prices at the previous times have already knocked out the option so that the knock-out option value equals $v(p_i, t; a_t = 1)$. Second, the option has not yet been knocked out so that the correct value equals $v(p_i, t; a_t = 0)$.

The recursive valuation system in Equations (6) and (7) is, however, unaffected by this complication if we make sure that the transition probability corresponding to any null set equals zero. Since all incompatible combinations of p_i and a_t constitute a null set, we can ignore such a complication entirely until we are already at the time of option valuation. In other words, assigning an arbitrary value to an incompatible combination does not affect the integrity of the valuation system. (The way to choose the appropriate value between $v(p_i, t; a_t = 0)$ and $v(p_i, t; a_t = 1)$ is discussed in specific cases later.)

For European-style barrier options, one simply sets $g(p_i, K, a_t) = 0$ for $t < T$. The maximum function can also be ignored because the discounted one-period average value will always be non-negative. For some European- style barrier options, the recursive system can be simplified in a way that is similar to the plain vanilla contract described in Equation (5), but this is not true for all barrier options.

Knock-Out Barrier Options. A knock-out barrier option differs from the standard option in that the option is knocked out and becomes worthless whenever the underlying asset price touches or crosses a constant barrier H at any monitoring time. For a double barrier option, there are two barriers: the lower barrier H and the upper barrier H^*, and the underlying asset price must remain between these barriers at the monitoring times, or the option will be knocked out. The auxiliary variable $a_t =$

1 if the barrier option is knocked out at or prior to time t, and $a_t = 0$ otherwise.

It is obvious that the knock-out option's value at time t equals zero when $a_t = 1$, regardless of the prevailing underlying asset price. In other words, $v(p_i, t; a_t = 1) = 0$. This simplifies the recursive bivariate system in (6) and (7) to a recursive univariate system:

$$v(p_i, t; a_t = 0) = \max\left[\begin{array}{c} g(p_i, K, a_t = 0), \\ e^{-r}\sum_{j=1}^{m}\mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 0\} \times \\ v(p_j, t+1; a_{t+1} = 0) \end{array}\right] \quad (8)$$

where the terminal condition is $v(p_i, T; a_T = 0) = \max\{w[\exp(p_i) - K], 0\}$; and $g(p_i, K, a_t = 0)$ equals $\max\{w[\exp(p_i) - K], 0\}$ or 0, depending on whether it is an American or European knock-out option.[3]

To compute the transition probability, it is convenient to first define the set of the states for which the option is knocked out (in):

$$\mathcal{S} = \begin{cases} \{i \in \{1, ..., m\} : \exp(p_i) \leq H\} & \text{for a down-and-out (in) option} \\ \{i \in \{1, ..., m\} : \exp(p_i) \geq H^*\} & \text{for an up-and-out (in) option} \\ \{i \in \{1, ..., m\} : \exp(p_i) \leq H \text{ or } \exp(p_i) \geq H^*\} & \\ \quad \text{for a double-barrier-out (in) option.} & \end{cases} \quad (9)$$

S denotes the region where a barrier has been crossed. We will also use the symbol S^c to denote the complement of S, i.e., the set of prices on the same side of the barrier as the initial price. Note that the definition of S is also used later for knock-in options, which is the reason we include "(in)" in the definitions.

Since the option's value equals zero if the barrier is crossed, we focus on the transition probabilities π_{ij} of passing from state p_i to state p_j without crossing the barrier:

$$\pi_{ij} = \mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 0\} = \begin{cases} q_{ij} & \text{if } i \notin S \text{ and } j \notin S \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

where q_{ij} is taken from Equation (3).

It is obvious that the conditional probability equals q_{ij} when going from one price that is not in the knock-out region to another that is also not in the knock-out region, provided that the auxiliary variables are in agreement with such a transition. If the transition is to a price that is in the knock-out region but the auxiliary variable states otherwise, we are evaluating the probability of a null set, which clearly has a zero conditional probability.

If the current price is in the knock-out region but the auxiliary variable states otherwise, we also have a null set. The probability conditional on a null set such as $\{\exp(p_i) \leq H, a_t = 0\}$ is technically undefined. Since the probability of reaching such a null set is zero, we can conveniently set such a conditional probability to zero without affecting the integrity of the recursive system.

In order to put the recursive system in a vector-matrix form as in Equation (4), we define three quasi-transition probability matrices for the down-and-out, up-and-out, and double barrier-out options, respectively:

$$\Pi_{DO} = \begin{bmatrix} \mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,m-k+1} \\ \mathbf{0}_{m-k+1,k-1} & Q(k,m;k,m) \end{bmatrix} \quad (11)$$

$$\Pi_{UO} = \begin{bmatrix} Q(1,l;1,l) & \mathbf{0}_{l,m-l} \\ \mathbf{0}_{m-l,l} & \mathbf{0}_{m-l,m-l} \end{bmatrix} \quad (12)$$

$$\Pi_{DBO} = \begin{bmatrix} \mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,l-k+1} & \mathbf{0}_{k-1,m-l} \\ \mathbf{0}_{l-k+1,k-1} & Q(k,l;k,l) & \mathbf{0}_{l-k+1,m-l} \\ \mathbf{0}_{m-l,k-1} & \mathbf{0}_{m-l,l-k+1} & \mathbf{0}_{m-l,m-l} \end{bmatrix} \quad (13)$$

where k is the index number of the price located immediately above the lower barrier H, and l is the index number of the price located immediately below the upper barrier H^*; $\mathbf{0}_{i,j}$ is an $i \times j$ matrix of zeros; and $Q(i, j: k, l)$ is the submatrix of Q taken from rows i to j and columns k to l inclusively.

In the vector matrix form, a knock-out option's price with maturity T and strike price K can be computed by the recursive system:

$$V(\overrightarrow{p}, t; a_t = 0) = \max\left[g(\overrightarrow{p}, K, a_t = 0), e^{-r}\Pi V(\overrightarrow{p}, t+1; a_{t+1} = 0)\right], \quad t \in \{0, 1, ..., T-1\} \quad (14)$$

$$V(\overrightarrow{p}, t; a_t = 1) = \overrightarrow{0}, \quad t \in \{0, 1, ..., T\} \quad (15)$$

with

$$g(\overrightarrow{p}, K, a_t = 0) = \begin{cases} \max\left\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\right\} & \text{if it is of American style} \\ \overrightarrow{0} & \text{if it is of European style} \end{cases}$$

$$V(\overrightarrow{p},T;a_T = 0) = \max\left\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\right\}$$

where $V(\overrightarrow{p}, t; a_t = 0)$ is the vector form of $v(p_i, t; a_t = 0)$; $g(\overrightarrow{p}, K; a_t = 0)$ is the vector form of $g(p_i, K, a_t = 0)$; and Π is either Π_{DO}, Π_{UO}, or Π_{DBO}, depending on the nature of the knock-out option. Recall that w indicates a call ($w = 1$) or a put ($w = -1$).

Suitably combining Equations (14) and (15), we have a final valuation system for the knock-out option as follows:

$$V(\overrightarrow{p},0) = \begin{cases} BV(\overrightarrow{p},0;a_0=0) & \text{if } a_0 = 0 \\ \overrightarrow{0} & \text{if } a_0 = 1 \end{cases} \tag{16}$$

where $B = [\beta_{ij}]$ is an $m \times m$ matrix satisfying the condition $\beta_{ij} = 1$ if $i = j \in S^c$, the complement of S, and 0 otherwise.

If $a_0 = 1$, the result is obvious. If $a_0 = 0$, the value of the knock-out option depends on p_i. If it is in the knock-out region, the option value should be zero. If it is not in the knock-out region, then the value must equal $v(p_i, 0; a_0 = 0)$. The expression $BV(\overrightarrow{p}, 0; 0)$ performs exactly this operation. Again, the time 0 option price is the $[(m + 1)/2]$-th element of $V(\overrightarrow{p}, 0)$.

For European options, the recursive valuation system can be simplified to:

$$V(\overrightarrow{p},0) = \begin{cases} e^{-rT}\Pi^T \max\left\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\right\} & \text{if } a_0 = 0 \\ \overrightarrow{0} & \text{if } a_0 = 1 \end{cases} \tag{17}$$

As pointed out by Boyle and Lau [1994] and Boyle and Tian [1998], the position of discrete prices in relation to the barrier is important in obtaining accurate barrier option prices. It is particularly sensitive when the barrier is located near the initial stock price. The Markov chain valuation setting is flexible enough to allow for such an adjustment. In Duan and Simonato [2001], the Markov chain is constructed so that each p_i is at the center of a cell with which the transition probability is computed. The cells can be easily constructed so that the barrier corresponds exactly to one particular cell's border.

Such a construction ensures that the probability of being below or above the barrier is precisely equal to the value prescribed by the theory. Specifically, if $\ln(H)$ is contained between p_{i-1} and p_i, then we set the lower boundary of the cell for p_i to $\ln(H)$. Similarly, if $\ln(H^*)$ is contained between p_{j-1} and p_j, then we set the upper boundary of the cell for p_{j-1} to $\ln(H^*)$.

Knock-In Barrier Options. A knock-in barrier option differs from the standard option in that the option is activated only when the underlying asset price has touched or crossed a constant barrier H at least once at monitoring times. Specifically, the auxiliary variable $a_t = 1$ if the barrier option gets knocked in at or prior to time t, and $a_t = 0$ if otherwise.

It is obvious that the knock-in option becomes the standard option once $a_t = 1$. It differs significantly from the knock-out option, because knock-in does not simplify the valuation problem as much as does knock-out. The relevant transition probability can be stated as

$$\begin{aligned}\mathbb{Q}\{X_{t+1} &= p_j, a_{t+1} = 0 | X_t = p_i, a_t = 0\} \\ &= \begin{cases} q_{ij} & \text{if } i \notin S \text{ and } j \notin S \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{18}$$

$$\begin{aligned}\mathbb{Q}\{X_{t+1} &= p_j, a_{t+1} = 1 | X_t = p_i, a_t = 0\} \\ &= \begin{cases} q_{ij} & \text{if } i \notin S \text{ and } j \in S \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{19}$$

$$\mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 1\} = 0 \tag{20}$$

$$\mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 1 | X_t = p_i, a_t = 1\} = q_{ij} \tag{21}$$

where q_{ij} is taken from Equation (3).

Equations (18) and (19) are true for a reason similar to that for Equation (10). Equation (20) is true because once the knock-in option is activated, it cannot be deactivated. Since the transition probability from p_i to p_j remains unaffected as long as the option remains activated, we have Equation (21).

The recursive bivariate system in Equations (6) and (7) can be partially simplified but only to:

$$\begin{aligned}v(p_i, t; a_t = 0) &= e^{-r}\sum_{j=1}^{m} \mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 0\} \times \\ &\quad v(p_j, t+1; a_{t+1} = 0)\end{aligned}$$

$$+e^{-r}\sum_{j=1}^{m}\mathbb{Q}\{X_{t+1}=p_j,a_{t+1}=1|X_t=p_i,a_t=0\}\ \times$$
$$v(p_j,t+1;a_{t+1}=1) \tag{22}$$

$$v(p_i,t;a_t=1) \;=\; \max\left[g\left(p_i,K,a_t=1\right),\right.$$
$$\left. e^{-r}\sum_{j=1}^{m} q_{ij}v(p_j,t+1;a_{t+1}=1)\right] \tag{23}$$

because $g(p_i, K, a_t = 0)$ equals 0 whether the option is American or European, and $g(p_i, K, a_t = 1)$ equals $\max\{w[\exp(p_i) - K], 0\}$ or 0, depending on whether it is an American or European knock-in option.[4]

The terminal conditions are

$$v(p_i,T;a_T \;=\; 0)=0 \tag{24}$$

$$v(p_i,T;a_T \;=\; 1)=\max\{w[\exp(p_i)-K],0\} \tag{25}$$

To put the system in a vector matrix form, we let $V(\vec{p}, t; a_t = 0)$ be the vector form of $v(p_i, t; a_t = 0)$. Similarly, we let $V(\vec{p}, t; a_t = 1)$ be the vector form of $v(p_i, t; a_t = 1)$. We let

$$\Pi_{DI}=\Pi_{DO}=\begin{bmatrix}\mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,m-k+1}\\ \mathbf{0}_{m-k+1,k-1} & Q(k,m;k,m)\end{bmatrix}$$

$$\Gamma_{DI}=\begin{bmatrix}\mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,m-k+1}\\ Q(k,m;1,k-1) & \mathbf{0}_{m-k+1,m-k+1}\end{bmatrix}$$

$$\Pi_{UI}=\Pi_{UO}=\begin{bmatrix}Q(1,l;1,l) & \mathbf{0}_{l,m-l}\\ \mathbf{0}_{m-l,l} & \mathbf{0}_{m-l,m-l}\end{bmatrix}$$

$$\Gamma_{UI}=\begin{bmatrix}\mathbf{0}_{l,l} & Q(1,l;l+1,m)\\ \mathbf{0}_{m-l,l} & \mathbf{0}_{m-l,m-l}\end{bmatrix}$$

$$\Pi_{DBI}=\Pi_{DBO}$$

$$\Gamma_{DBI}=\begin{bmatrix}\mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,l-k+1} & \mathbf{0}_{k-1,m-l}\\ Q(k,l;1,k-1) & \mathbf{0}_{l-k+1,l-k+1} & Q(k,l;l+1,m)\\ \mathbf{0}_{m-l,k-1} & \mathbf{0}_{m-l,l-k+1} & \mathbf{0}_{m-l,m-l}\end{bmatrix}$$

The vector matrix form of the recursive valuation system becomes

$$V(\vec{p},t;a_t \;=\; 0)=e^{-r}\Pi V(\vec{p},t+1;a_{t+1}=0)+$$
$$e^{-r}\Gamma V(\vec{p},t+1;a_{t+1}=1) \tag{26}$$

$$V(\vec{p},t;a_t \;=\; 1)=\max\left[g(\vec{p},K,a_t=1),\right.$$
$$\left.e^{-r}QV(\vec{p},t+1;a_{t+1}=1)\right] \tag{27}$$

with

$$g(\vec{p},K,a_t \;=\; 1)=$$
$$\begin{cases}\max\left\{w[\exp(\vec{p})-K\vec{1}],\vec{0}\right\} & \text{if it is of American style}\\ \vec{0} & \text{if it is of European style}\end{cases}$$

$$V(\vec{p},T;a_T \;=\; 0)=\vec{0}$$

$$V(\vec{p},T;a_T \;=\; 1)=\max\left\{w[\exp(\vec{p})-K\vec{1}],\vec{0}\right\}$$

Note that appropriate Π and Γ can be plugged into the recursive system, depending on the nature of the knock-in option. If the knock-in barrier option has been activated initially, it is actually a standard option. According to this recursive system, its valuation does not depend on $V(\vec{p}, t; a_t = 0)$, which is hardly a surprise.

Suitably combining Equations (26) and (27), we have a final valuation system for the knock-in option as follows:

$$V(\vec{p},0)=$$

$$\begin{cases}BV(\vec{p},0;a_0=0)+AV(\vec{p},0;a_0=1) & \text{if } a_0=0\\ V(\vec{p},0;a_0=1) & \text{if } a_0=1\end{cases} \tag{28}$$

where $A = [\alpha_{ij}]$ and $B = [\beta_{ij}]$ are two $m \times m$ matrices satisfying the condition $\alpha_{ij} = 1$ if $i = j \in S$ and 0 otherwise, and $\beta_{ij} = 1$ if $i = j \in S^c$ and 0 otherwise. Note that $A + B = \mathbf{I}$, the identity matrix. If $a_0 = 1$, the result is obvious. If $a_0 = 0$, the value of the knock-in option becomes $BV(\vec{p}, 0; a_0 = 0) + AV(\vec{p}, 0; a_0 = 1)$. We need to treat the two cases separately because the valuation result of $BV(\vec{p}, 0; a_0 = 0) + AV(\vec{p}, 0; a_0 = 1)$ may not apply to the case that $a_0 = 1$.

Consider a particular asset price at time 0, say, p_i, outside the knock-in region. This situation does not preclude the option from having been knocked in previously, however. (Note that time 0 denotes the time of option valuation, and the option may have been in effect for some time.) In other words, the event that p_i, is outside the knock-in region and $a_0 = 1$ need not be a null set. In such cases, the valuation equation of $BV(\vec{p}, 0; a_0 = 0) + AV(\vec{p}, 0; a_0 = 1)$ will assign values to the elements of this set according to $V(\vec{p}, 0; a_0 = 0)$, which should have been assigned according to $V(\vec{p}, 0; a_0 = 1)$ instead. Again, the time 0 option price is the $[(m + 1)/2]$-th element of $V(\vec{p}, 0)$.

For European-style knock-in options, we simply set $g(\vec{p}, 0; a_t = 1) = \vec{0}$ for $t < T$, and the recursive valuation system becomes

$$V(\vec{p}, t; a_t = 0) = e^{-r}\Pi V(\vec{p}, t+1; a_{t+1} = 0) + e^{-r}\Gamma V(\vec{p}, t+1; a_{t+1} = 1) \tag{29}$$

$$V(\vec{p}, t; a_t = 1) = e^{-r} Q V(\vec{p}, t+1; a_{t+1} = 1) \tag{30}$$

Combining this with the terminal conditions, we can solve the recursive system to yield (see Appendix B for details):

$$V(\vec{p}, 0; a_0 = 0) = e^{-rT}\left(\sum_{i=1}^{T} \Pi^{T-i}\Gamma Q^{i-1}\right) \times \max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\} \tag{31}$$

$$V(\vec{p}, 0; a_0 = 1) = e^{-rT} Q^T \max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\} \tag{32}$$

For European-style knock-in and knock-out options, an in-out parity can also be used to price one type of option by the other. The in-out parity comes from the fact that simultaneously holding a knock-in and a knock-out option is equivalent to holding a standard option. Combining this idea with the earlier results in Equations (5) and (17) leads to an expression for the value of a knock-in option:

$$V(\vec{p}, 0) = \begin{cases} e^{-rT}\left(Q^T - \Pi^T\right)\max\left(w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right) & \text{if } a_0 = 0 \\ e^{-rT} Q^T \max\left(w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right) & \text{if } a_0 = 1 \end{cases} \tag{33}$$

This is true because the T-step transition probabilities from the i-th state to the j-th state with at least one crossing during the life of the option are given by $(Q^T - \prod^T)$ if the option has not been knocked in previously. If the option has already been knocked in, it is effectively a standard option.

We prove in Appendix C that using Equations (31), (32), and (28) to value European-style knock-in options is equivalent to using Equation (33) directly.

Moving Barrier Options. Suppose the barrier is set according to $\{H_t: t \in \{0, 1, 2, \ldots, T\}\}$ where H_t changes over time deterministically. Examples are step-barrier, partial-barrier, exponential-barrier, and intermittent-barrier options. For valuation purposes, we need to modify the procedure presented earlier for the fixed barrier option. First we change S to S_t to reflect the moving barrier:

$$\mathcal{S}_t = \begin{cases} \{i \in \{1, ..., m\} : \exp(p_i) \leq H_t\} \text{ for a down-and-out (in) option} \\ \{i \in \{1, ..., m\} : \exp(p_i) \geq H_t^*\} \text{for an up-and-out (in) option} \\ \{i \in \{1, ..., m\} : \exp(p_i) \leq H_t \text{ or } \exp(p_i) \geq H_t^*\} \\ \quad \text{for a double-barrier-out (in) option} \end{cases} \tag{34}$$

For knock-out options:

$$\mathbb{Q}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 0\} = \begin{cases} q_{ij} & \text{if } i \notin S_t \text{ and } j \notin S_{t+1} \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

Thus, all three quasi-transition probability matrices must be indexed by time, i.e., $\Pi_{DO,t}$; $\Pi_{UO,t}$; and $\Pi_{DBO,t}$. The recursive valuation system in Equations (14) and (15) can be modified slightly to reflect the fact that the relevant quasi-transition probability matrix is time-varying:

$$V(\vec{p}, t; a_t = 0) = \max\left[g(\vec{p}, K, a_t = 0), e^{-r}\Pi_t V(\vec{p}, t+1; a_{t+1} = 0)\right], \quad t \in \{0, 1, ..., T-1\} \tag{36}$$

$$V(\vec{p}, t; a_t = 1) = \vec{0}, \quad t \in \{0, 1, ..., T\} \tag{37}$$

with

$$g(\vec{p}, K, a_t = 0) = \begin{cases} \max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\} & \text{if it is of American style} \\ \vec{0} & \text{if it is of European style} \end{cases}$$

$$V(\overrightarrow{p},T;a_T = 0) = \max\left\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\right\}$$

Similar to Equation (16), the final valuation system after suitably combining cases becomes

$$V(\overrightarrow{p},0) = \begin{cases} BV(\overrightarrow{p},0;a_0=0) & \text{if } a_0 = 0 \\ \overrightarrow{0} & \text{if } a_0 = 1 \end{cases} \tag{38}$$

where $B = [\beta_{ij}]$ is an $m \times m$ matrix satisfying the condition $\beta_{ij} = 1$ if $i = j \in S_0^c$ and 0 otherwise. For European knock-out moving barrier options, the recursive system can be simplified to:

$$V(\overrightarrow{p},0) = \begin{cases} e^{-rT}\Pi_0\Pi_1\ldots\Pi_{T-1}\max\left\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\right\} & \text{if } a_0 = 0 \\ \overrightarrow{0} & \text{if } a_0 = 1 \end{cases} \tag{39}$$

Similarly, we can deal with knock-in options by indexing all quasi-transition probability matrices.

II. TIME-VARYING VOLATILITY OPTION PRICING FRAMEWORK

One advantage of the Markov chain framework is its ability to handle time-varying volatility.

Markov Chain Method Under GARCH

We show how the Markov chain method is used to approximate a time-homogeneous bivariate Markov process that incorporates as a particular case the GARCH(1, 1) model. Specifically, we consider the non-linear asymmetric GARCH(1, 1) process, NGARCH(1, 1) for short, proposed in Engle and Ng [1993]. The NGARCH(1, 1) process is also the model Duan and Simonato [2001] use in demonstrating their Markov chain approximation method.

Assume the asset price dynamic as follows under the data-generating probability measure $\mathbb{P}$:

$$\ln\frac{S_{t+1}}{S_t} = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1} \tag{40}$$

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t(\varepsilon_t - \theta)^2 \tag{41}$$

$$\varepsilon_{t+1}|\mathcal{F}_t \overset{\mathbb{P}}{\sim} N(0,1) \tag{42}$$

where $\mathcal{F}_t$ is the σ-field generated by $\{S_0, h_0, \varepsilon_\tau; \tau = 0, 1, 2, \ldots, t\}$; r is the one-period, continuously compounded risk-free rate of interest; λ is the constant unit risk premium; h_{t+1} is the conditional variance of the asset return; and θ determines the leverage effect. The conditional variance follows the NGARCH(1, 1) process with the typical parameter restrictions: $\beta_0 > 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, and $\beta_1 + \beta_2(1 + \theta^2) < 1$.

According to the valuation theory developed by Duan [1995], the derivative contracts contingent on S_t can be valued using a locally risk-neutralized price dynamic. Specifically, the asset price dynamic under the locally risk-neutralized probability measure $\mathbb{Q}$ is

$$\ln\frac{S_{t+1}}{S_t} = r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\epsilon_{t+1} \tag{43}$$

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t(\epsilon_t - \theta - \lambda)^2 \tag{44}$$

$$\epsilon_{t+1}|\mathcal{F}_t \overset{\mathbb{Q}}{\sim} N(0,1) \tag{45}$$

Note that this system is Markovian when expressed in a vector form of $\{(S_t, h_{t+1}): t \in \{0, 1, 2, \ldots,\}\}$.

To construct a Markov chain approximation, we have to discretize the state space of the process. As described in Duan and Simonato [2001], one can use $\{(p_i, u_j): i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}\}$ and an $mn \times mn$ transition probability matrix Q to approximate $\{(\ln S_t, \ln h_{t+1})': t \in \{0, 1, 2, \ldots\}\}$. The transition probability matrix looks like

$$Q = \begin{bmatrix} q(1,1;1,1) & \cdots & q(1,1;m,1) & q(1,1;1,2) & \cdots & q(1,1;m,n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q(m,1;1,1) & \cdots & q(m,1;m,1) & q(m,1;1,2) & \cdots & q(m,1;m,n) \\ q(1,2;1,1) & \cdots & q(1,2;m,1) & q(1,2;1,2) & \cdots & q(1,2;m,n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q(m,n;1,1) & \cdots & q(m,n;m,1) & q(m,n;1,2) & \cdots & q(m,n;m,n) \end{bmatrix} \tag{46}$$

In vectoring the system for option valuation, the stock price vector including m discretized logarithmic asset prices must be repeated n times to correspond to n different values of the conditional volatility:[5]

$$\overrightarrow{p} = [p_1, p_2, \cdots, p_m, \cdots, p_1, p_2, \cdots, p_m]' \tag{47}$$

With values for Q and $\overrightarrow{p}$, Equation (4) can be used to price standard call and put options. See Duan and Simonato [2001] for details as to how to assign values to Q and $\overrightarrow{p}$.

Moving Barrier Options

Since the constant-barrier option is a special case of the moving barrier option, we directly analyze the valuation of moving barrier options under time-varying volatilities. Let S_t be the set of indexes corresponding to the states where p_i is on the other side of the barriers at time t, as defined in (34). The transition probability $\pi_{t,t+1}(i, j; k, l)$ from state (p_i, u_j) to state (p_k, u_l) without crossing the barriers is:

$$\pi_{t,t+1}(i, j; k, l) = \begin{cases} q(i, j; k, l) & \text{if } i \in \mathcal{S}_t^c \text{ and } k \in \mathcal{S}_{t+1}^c \\ 0 & \text{if } i \in \mathcal{S}_t \text{ or } k \in \mathcal{S}_{t+1} \end{cases} \tag{48}$$

We store these probabilities in an $mn \times mn$ matrix Π_t. Using this matrix and the asset price vector in (47), the recursive valuation system developed earlier for knock-out options becomes

$$V(\overrightarrow{p}, t; a_t = 0) = \max[g(\overrightarrow{p}, K, a_t = 0), \\ e^{-r}\Pi_t V(\overrightarrow{p}, t+1; a_{t+1} = 0)], \quad t \in \{0, 1, ..., T-1\} \tag{49}$$

$$V(\overrightarrow{p}, t; a_t = 1) = \overrightarrow{0}, \quad t \in \{0, 1, ..., T-1\} \tag{50}$$

with

$$g(\overrightarrow{p}, K, a_t = 0) = \begin{cases} \max\left\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\right\} & \text{if it is of American style} \\ \overrightarrow{0} & \text{if it is of European style} \end{cases}$$

$$V(\overrightarrow{p}, T; a_T = 0) = \max\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\}$$

Similar to the case of constant volatility, the final valuation system after suitably combining different cases becomes

$$V(\overrightarrow{p}, 0) = \begin{cases} BV(\overrightarrow{p}, 0; a_0 = 0) & \text{if } a_0 = 0 \\ \overrightarrow{0} & \text{if } a_0 = 1 \end{cases} \tag{51}$$

where $B = [\beta_{ij}]$ is an $mn \times mn$ matrix satisfying the condition $\beta_{ij} = 1$ if $i = j \in S_0^c$ and 0 otherwise. For European knock-out moving barrier options, the recursive system can be simplified to:

$$V(\overrightarrow{p}, 0) = \begin{cases} e^{-rT}\Pi_0\Pi_1 \ldots \Pi_{T-1} \max\left\{w[\exp(\overrightarrow{p}) - K\overrightarrow{1}], \overrightarrow{0}\right\} & \text{if } a_0 = 0 \\ \overrightarrow{0} & \text{if } a_0 = 1 \end{cases} \tag{52}$$

The value of the option is the entry of $V(\overrightarrow{p}, 0)$ corresponding to the current asset price and conditional volatility. In accordance with Duan and Simonato's [2001] design, the final option value is determined by the formula:

$$C(S_0, h_1) = \frac{d(j+1) - \ln(h_1)}{d(j+1) - d(j)} v(j) + \frac{\ln(h_1) - d(j)}{d(j+1) - d(j)} v(j+1) \tag{53}$$

where j is the index number satisfying $d(j)$ ″ $\ln(h_1)$ ″ $d(j + 1)$, and $v(j)$ is the $[(j-1)m + (m+1)/2]$-th element of $V(\overrightarrow{p}, 0)$. A linear interpolation is performed because h_1 may not correspond exactly to any of the discretized volatilities (see Duan and Simonato [2001, Equation (35)].

It is clear that we can deal with knock-in options by indexing all quasi-transition probability matrices similar to those in the constant-volatility framework. The in-out parity relationship is again applicable to European-style options. This property can then be used to speed up the calculation of the European knock-in option price by using the analytical approximation formula developed by Duan, Gauthier, and Simonato [1999] for computing the plain vanilla option value under GARCH.

III. NUMERICAL RESULTS

We first provide results showing the performance of this approach for different cases in the Black-Scholes and GARCH contexts. Then, a cost-benefit analysis is conducted to compare the Markov chain approach with the explicit finite-difference approach of Boyle and Tian [1998]. Specifically, a large test pool of options is used to assess the trade-off between the accuracy of a method and the computing time required.

The pricing of a down-and-out option in the Black-Scholes framework is analyzed for three different barriers monitored daily and weekly. The results are summarized in Exhibit 2. The parameter values employed in the analysis are provided below the table. The benchmark values at the top of the table are based on the conditional Monte Carlo simulation method of Boyle, Broadie, and Glasserman [1997].

We also include cases where the barrier is close to the initial price of the underlying asset. These are known to be difficult cases for obtaining accurate barrier option prices. We assume in these calculations 5 and 250 days for a week and a year, respectively.

The convergence of the Markov chain approximation appears to be fairly fast. In most cases, using 701 discrete asset prices in the approximation, i.e., $m = 701$, is enough to obtain penny accuracy. As in the Cheuk and Vorst [1996] method, the Markov chain approximation method can tackle the difficult cases where the barrier is close to the initial price of the underlying asset. Yet it requires far fewer discrete asset prices to achieve the same level of accuracy.

Also reported in Exhibit 2 are computation times for a C-code implementation of the Markov chain with $m = 501$ and $m = 1{,}001$ states performed on a desktop computer (Pentium III 933 MHz). These numbers show that the Markov chain method can accurately compute option prices in fractions of seconds, even for as many states as 1,001.

Exhibit 3 shows prices for down-and-out and up-and-out options in the Black-Scholes framework. The benchmark option prices are again computed by the conditional Monte Carlo simulation method of Boyle, Broadie, and Glasserman [1997]. We compare the Markov chain approximation with the analytical approximation of Broadie, Glasserman, and Kou [1997] as well as those based on an 80,000-step trinomial tree that they report. Their analytical approximation performs more poorly when the barrier is closer to the initial price of the underlying asset. Such cases can be handled without difficulty using our Markov chain approximation.

Exhibit 4 presents the results for down-and-in options in the Black-Scholes framework. The results for double knock-out barrier options are presented in Exhibit 5. The choice of parameter values follows that of Cheuk and Vorst [1996]. The Markov chain method obtains penny accuracy with discrete asset prices as few as $m = 501$.

Our analysis of three types of moving barrier options is summarized in Exhibit 6. The first category is step-barrier options. We consider two scenarios for these options. In the first scenario, the barrier is moved from 94 to 92 after three months, and in the second, the barrier is shifted from 99.9 to 95.0 after three months. The speeds of convergence for the Markov chain option prices are similar

EXHIBIT 2

European Down-and-Out Call Options in Black-Scholes Framework

	Daily			Weekly		
Barrier	95	99.5	99.9	95	99.5	99.9
			Monte Carlo			
Price	6.1662	1.9580	1.5104	6.6370	3.3494	3.0118
Std.	0.0045	0.0051	0.0047	0.0040	0.0058	0.0058
			Markov Chain			
$m = 51$	6.8367	1.8935	1.3928	6.8145	3.3494	2.9713
$m = 101$	6.1887	2.0468	1.4804	6.6347	3.3825	2.9980
$m = 201$	6.2304	2.0906	1.5046	6.6496	3.3922	3.0059
$m = 301$	6.1779	1.9663	1.5096	6.6334	3.3563	3.0077
$m = 401$	6.1688	1.9853	1.5115	6.6304	3.3622	3.0084
$m = 501$	6.1735	1.9610	1.5124	6.6323	3.3547	3.0088
$m = 601$	6.1683	1.9686	1.5129	6.6306	3.3572	3.0091
$m = 701$	6.1709	1.9602	1.5133	6.6316	3.3545	3.0092
$m = 801$	6.1680	1.9636	1.5135	6.6306	3.3557	3.0094
$m = 901$	6.1693	1.9602	1.5137	6.6312	3.3546	3.0095
$m = 1001$	6.1678	1.9617	1.5138	6.6307	3.3552	3.0095
time for $m = 501$	0.0037	0.0034	0.0034	0.0016	0.0012	0.0012
time for $m = 1001$	0.0297	0.0265	0.0266	0.0109	0.0094	0.0110

Daily and weekly denote discretely monitored barrier options with daily and weekly frequency based on the assumption that 1 day = 1/250 years and 1 week = 1/50 years. Monte Carlo prices are computed using the conditional Monte Carlo simulation method described in Boyle, Broadie, and Glasserman [1997] using 200,000 sample paths. Std. are standard deviations of the Monte Carlo prices. Markov chain denotes prices computed using the Markov chain method. Time is computing time in seconds. Parameters: $S_0 = 100$, $r = 0.10$, $K = 100$, $T = 0.5$ (annualized), and $\sigma = 0.20$ (annualized).

EXHIBIT 3

European Knock-Out Call Options in Black-Scholes Framework

	Down-and-out			Up-and-out		
Barrier	85	93	99	115	135	155
			Monte Carlo			
Price	10.5054	7.5694	3.4812	0.8085	8.9618	12.8940
Std.	0.0019	0.0040	0.0043	0.0035	0.0171	0.0081
			Markov Chain			
$m = 51$	10.9259	7.6955	3.5241	0.7705	8.6792	13.0242
$m = 101$	10.6303	7.8166	3.5710	0.8140	8.9100	12.9052
$m = 201$	10.5351	7.5808	3.4724	0.8013	8.9452	12.8998
$m = 301$	10.5159	7.5911	3.4843	0.8049	8.9486	12.8961
$m = 401$	10.5166	7.5661	3.4991	0.8064	8.9545	12.8959
$m = 501$	10.5093	7.5695	3.4770	0.8071	8.9570	12.8954
$m = 601$	10.5068	7.5636	3.4839	0.8074	8.9577	12.8949
$m = 701$	10.5069	7.5643	3.4751	0.8076	8.9578	12.8945
$m = 801$	10.5056	7.5660	3.4789	0.8076	8.9577	12.8943
$m = 901$	10.5057	7.5631	3.4745	0.8067	8.9577	12.8943
$m = 1001$	10.5050	7.5635	3.4767	0.8068	8.9577	12.8944
BGK	10.505	7.566	3.414	0.819	8.994	12.905
Trinomial	10.505	7.563	3.475	0.807	8.959	12.894

Down-and-out and up-and-out denote discretely monitored barrier options with daily frequency (1 day = 1/250 years). Monte Carlo prices are computed using the conditional Monte Carlo simulation method described in Boyle, Broadie,and Glasserman [1997] using 500,000 sample paths. Std. are standard deviations of the Monte Carlo prices. Markov chain denotes prices computed using the Markov chain method. BGK are prices obtained from Broadie, Glasserman, and Kou [1997] using their analytical approximation. Trinomial are trinomial tree prices (80,000 steps) taken from Broadie, Glasserman, and Kou [1997]. Parameter values for down-and-out options: S_0 = 100, r = 0.10, K = 100, T = 0.2, and σ = 0.60; parameter values for up-and-out options: S_0 = 110, r = 0.10, K = 100, T = 0.20,σ = 0.30.

EXHIBIT 4

European Down-and-In Call Options in Black-Scholes Framework

		Daily			Weekly	
Barrier	**95**	**99.5**	**99.9**	**95**	**99.5**	**99.9**
			Monte Carlo			
Price	2.1116	6.3198	6.7674	1.6408	4.9284	5.2661
Std.	0.0045	0.0051	0.0047	0.0040	0.0058	0.0058
			Markov Chain			
$m = 51$	1.9997	6.9429	7.4436	1.5675	5.0326	5.4107
$m = 101$	2.2462	6.3882	6.9546	1.6697	4.9219	5.3064
$m = 201$	2.0886	6.2283	6.8144	1.6339	4.8913	5.2776
$m = 301$	2.1180	6.3296	6.7864	1.6463	4.9234	5.2719
$m = 401$	2.1189	6.3023	6.7762	1.6479	4.9162	5.2699
$m = 501$	2.1103	6.3228	6.7714	1.6455	4.9231	5.2689
$m = 601$	2.1134	6.3131	6.7688	1.6469	4.9203	5.2684
$m = 701$	2.1096	6.3203	6.7672	1.6458	4.9228	5.2681
$m = 801$	2.1116	6.3160	6.7661	1.6466	4.9216	5.2679
$m = 901$	2.1097	6.3189	6.7654	1.6461	4.9226	5.2678
$m = 1001$	2.1108	6.3169	6.7649	1.6465	4.9220	5.2677

Daily and weekly denote discretely monitored barrier options with daily and weekly frequency based on the assumption that 1 day = 1/250 years and 1 week = 1/50 years. Monte Carlo prices are computed using the conditional Monte Carlo simulation method described in Boyle, Broadie, and Glasserman [1997] using 200,000 sample paths. Std. are standard deviations of the Monte Carlo prices. Markov chain denotes prices computed using the Markov chain method. Parameters: S_0 = 100, r = 0.10, K = 100, T = 0.5 (annualized), and σ = 0.20 (annualized).

EXHIBIT 5

European Double Knock-Out Call Options in Black-Scholes Framework

	Daily			Weekly		
Upper barrier	110	125	150	110	125	150
Lower barrier	95	95	95	95	95	95
			Monte Carlo			
Price	0.0752	2.4822	5.7919	0.1633	3.0160	6.2757
Std	0.0007	0.0059	0.0108	0.0011	0.0064	0.0110
			Markov Chain			
$m = 51$	0.0560	2.5027	6.1703	0.1582	3.0567	6.4411
$m = 101$	0.0656	2.3287	5.7611	0.1602	2.9759	6.2959
$m = 201$	0.0751	2.4715	5.8438	0.1636	3.0093	6.3161
$m = 301$	0.0742	2.4667	5.8044	0.1626	3.0040	6.3018
$m = 401$	0.0749	2.4702	5.7964	0.1628	3.0038	6.2987
$m = 501$	0.0752	2.4772	5.8025	0.1629	3.0057	6.3006
$m = 601$	0.0753	2.4783	5.7981	0.1629	3.0056	6.2990
$m = 701$	0.0757	2.4797	5.8013	0.1631	3.0060	6.3000
$m = 801$	0.0754	2.4800	5.7986	0.1629	3.0059	6.2990
$m = 901$	0.0756	2.4803	5.8001	0.1630	3.0059	6.2995
$m = 1001$	0.0756	2.4802	5.7988	0.1630	3.0058	6.2990

Daily and weekly denote discretely monitored barrier options with daily and weekly frequency based on the assumption that 1 day = 1/250 years and 1 week = 1/50 years. Monte Carlo prices are computed with a crude Monte Carlo simulation using 200,000 sample paths. Std. are standard deviations of the Monte Carlo prices. Markov chain denotes prices computed using the Markov chain method. Parameters: $S_0 = 100$, $r = 0.10$, $K = 100$, $T = 0.5$ (annualized), and $\sigma = 0.20$ (annualized).

EXHIBIT 6

European Down-and-Out Call Options with Moving Barriers in Black-Scholes Framework

	Step		Partial		Exponential	
Barrier	94 ▷ 92	99.9 ▷ 95	95	99.9	95exp(at)	99.9exp(at)
			Monte Carlo			
Price	6.7713	1.5726	7.6576	6.5311	6.0033	1.4217
Std.	0.0039	0.0050	0.0019	0.0040	0.0045	0.0045
			Markov Chain			
$m = 51$	6.8655	1.4089	8.0996	7.1136	6.2245	2.1269
$m = 101$	6.7973	1.5231	7.8076	6.7036	6.0513	1.7634
$m = 201$	6.8440	1.5612	7.6895	6.5871	6.0048	1.5694
$m = 301$	6.7872	1.5686	7.6753	6.5442	6.0231	1.5027
$m = 401$	6.7755	1.5713	7.6659	6.5419	5.9976	1.4691
$m = 501$	6.7802	1.5727	7.6673	6.5366	6.0110	1.4486
$m = 601$	6.7756	1.5734	7.6632	6.5349	5.9999	1.4343
$m = 701$	6.7772	1.5738	7.6606	6.5357	6.0096	1.4246
$m = 801$	6.7794	1.5741	7.6598	6.5371	6.0034	1.4171
$m = 901$	6.7760	1.5743	7.6614	6.5376	6.0081	1.4114
$m = 1001$	6.7771	1.5744	7.6600	6.5382	6.0069	1.4058

Step denotes discretely monitored barrier options with daily frequency (1 day = 1/250 years), and the barrier goes from 94 to 92 at the third month. Partial denotes discretely monitored barrier options with daily frequency, and the barrier starts at the 63th day. Exponential denotes discretely monitored barrier options with daily frequency and an exponential barrier. Monte Carlo prices are computed with the conditional Monte Carlo simulation method described in Boyle, Broadie, and Glasserman [1997] using 200,000 sample paths. Std. are standard deviations of the Monte Carlo prices. Markov chain denotes prices computed with the Markov chain method. Parameters: $S_0 = 100$, $r = 0.10$, $K = 100$, $T = 0.5$ (annualized), $\sigma = 0.20$ (annualized), and $a = (r - 0.5\sigma^2)/\sigma$.

under these two scenarios. One needs approximately $m = 501$ to obtain penny accuracy.

The second type of option is the partial barrier option, where the barrier starts at the 63rd day at 95.0 in the first case and at 99.9 in the second. Penny accuracy is again obtained with $m = 501$.

The last two columns in Exhibit 6 describe the convergence pattern of the exponential-barrier option. In the first scenario, the barrier starts at 95 and then increases exponentially at a rate computed by the formula in Ritchken [1995]; that is, $(r - 0.5\sigma^2)/\sigma$. The convergence speed of the Markov chain price to its Monte Carlo benchmark value in this barrier scenario is comparable to the other cases in this table.

EXHIBIT 7
European Down-and-Out Call Options in NGARCH Framework

	Daily			Weekly		
Barrier	95	99.5	99.9	95	99.5	99.9
			Monte Carlo			
Price	6.1614	1.9406	1.3906	6.5784	3.3368	2.9245
Std.	0.0064	0.0112	0.0116	0.0057	0.0010	0.0103
			Markov Chain			
$n = 25$ $m = 25$	5.8890	0.8355	0.6071	6.2758	0.9449	0.6639
$n = 31$ $m = 31$	8.5691	1.2925	0.9198	9.4729	1.5341	1.0449
$n = 35$ $m = 35$	6.2506	1.4944	1.0558	6.3921	1.8347	1.2338
$n = 41$ $m = 41$	6.4074	1.6712	1.1714	6.8282	2.1629	1.4390
$n = 45$ $m = 45$	6.7323	1.7550	1.2252	7.4531	2.3506	1.5532
$n = 51$ $m = 51$	7.5332	1.8384	1.2765	8.5268	2.5816	1.6971
$n = 25$ $m = 75$	6.2454	1.9871	1.3617	6.6131	3.2009	2.1485
$n = 31$ $m = 93$	6.2200	2.0316	1.3828	6.6027	3.4419	2.3594
$n = 35$ $m = 105$	6.3577	2.0525	1.3917	6.9296	3.5334	2.4458
$n = 41$ $m = 123$	6.2809	2.0703	1.3979	6.8024	3.6223	2.5626
$n = 45$ $m = 135$	6.1987	2.0802	1.4012	6.6360	3.6547	2.6124
$n = 51$ $m = 153$	6.1845	2.0914	1.4066	6.6125	3.6895	2.6821
$n = 25$ $m = 125$	6.3003	2.0729	1.4048	6.8369	3.6279	2.5764
$n = 31$ $m = 155$	6.1925	2.0935	1.4119	6.6374	3.6954	2.6979
$n = 35$ $m = 175$	6.1834	2.1000	1.4139	6.6282	3.7072	2.7507
$n = 41$ $m = 205$	6.1594	1.9205	1.4159	6.5624	3.1628	2.8043
$n = 45$ $m = 225$	6.1599	1.9222	1.4161	6.5694	3.2128	2.8216
$n = 51$ $m = 255$	6.1674	1.9308	1.4183	6.6079	3.2768	2.8497
$n = 25$ $m = 175$	6.1805	2.0984	1.4165	6.6238	3.7042	2.7519
$n = 31$ $m = 217$	6.1808	1.9257	1.4201	6.6336	3.2038	2.8268
$n = 35$ $m = 245$	6.1568	1.9288	1.4207	6.5709	3.2639	2.8490
$n = 41$ $m = 287$	6.1564	1.9391	1.4205	6.5810	3.3290	2.8752
$n = 45$ $m = 315$	6.1614	1.9474	1.4209	6.5994	3.3577	2.8850
$n = 51$ $m = 357$	6.1629	1.9585	1.4207	6.6032	3.3917	2.8996

Daily and weekly denote discretely monitored barrier options with daily and weekly frequency based on the assumption that 1 day = 1/250 years and 1 week = 1/50 years. Monte Carlo prices are computed with Monte Carlo simulation using 500,000 sample paths and the Black-Scholes price as the control variate. Std. are standard deviations of the Monte Carlo prices. Markov chain prices are computed with the Markov chain method; parameters: $S_0 = 100$, $r = 0.10$, $K = 100$, $T = 0.5$ *(annualized),* $\beta_0 = 0.00001$, $\beta_1 = 0.80$, $\beta_2 = 0.10$, $\theta = 0.3$, $\lambda = 0.2$, *and* $\sqrt{h_1} = 0.010483$.

Convergence appears to be slower in the second exponential barrier scenario, where the barrier increases exponentially at the same rate but starts at 99.9 instead. This is likely due to the same numerical difficulty as when the barrier is close to the current price of the underlying asset. Although the Markov chain method can successfully tackle such cases if the barrier is constant, it is a different matter when the barrier is exponential. An exponential barrier, in fact, prevents the discretized asset prices from matching up with the continuously increasing barrier.

Analysis of the Markov chain method for pricing barrier options in the NGARCH(1, 1) pricing framework appears in Exhibits 7, 8, and 9. The GARCH methodology, due to time-varying volatilities, requires setting an initial conditional volatility. Here, we consider an average situation; that is, we assume its initial conditional volatility is equal to the stationary volatility under the data-generating probability measure $\mathbb{P}$. Specifically, the formula is $h_1 = \beta_0[1 - \beta_1 - \beta_2(1 + \theta^2)]^{-1}$. The specific parameter values used in the analysis are the same as in Duan and Simonato [2001], and are shown in the exhibits.

All price estimates in Exhibit 7 converge to their respective Monte Carlo benchmark values as the number of states increases. For a given precision level, the numerical scheme for the NGARCH model requires more computing time, because the transition probability matrix under the GARCH model is much larger than that under the Black-Scholes model.

Unlike the constant-volatility case, it is possible to match the time step of the Markov chain with the monitoring frequency only if the underlying GARCH model is defined exactly over the monitoring frequency. That

EXHIBIT 8
European Knock-Out Call Options in NGARCH Framework

	Down-and-out			Up-and-out		
Barrier	85	93	99	115	135	155
			Monte Carlo			
Price	4.2099	4.1053	1.9694	2.4021	12.1035	12.3620
Std.	0.0014	0.0017	0.0054	0.0057	0.0047	0.0019
			Markov Chain			
$n = 25$ $m = 25$	4.8610	4.6439	2.2572	1.6082	11.8833	12.6777
$n = 31$ $m = 31$	4.7395	4.5399	2.4239	2.0752	12.0393	12.6191
$n = 35$ $m = 35$	4.6409	4.5058	2.4803	3.0620	11.8839	12.5695
$n = 41$ $m = 41$	4.5267	4.3811	2.5449	2.0167	11.9697	12.5076
$n = 45$ $m = 45$	4.4866	4.3936	2.5742	2.2925	12.0576	12.4851
$n = 51$ $m = 51$	4.4272	4.3082	2.6040	2.0967	12.0024	12.4563
$n = 25$ $m = 75$	4.3252	4.2142	1.9815	2.4863	12.0289	12.3986
$n = 31$ $m = 93$	4.2910	4.1846	2.0307	2.2864	12.0639	12.3875
$n = 35$ $m = 105$	4.2618	4.1539	2.0642	2.3197	12.0751	12.3740
$n = 41$ $m = 123$	4.2539	4.1470	1.9653	2.3511	12.0772	12.3709
$n = 45$ $m = 135$	4.2501	4.1466	1.9712	2.3456	12.0776	12.3684
$n = 51$ $m = 153$	4.2343	4.1296	1.9875	2.3857	12.0806	12.3617
$n = 25$ $m = 125$	4.2753	4.1693	1.9692	2.3292	12.0508	12.3715
$n = 31$ $m = 155$	4.2447	4.1381	1.9923	2.3870	12.0743	12.3658
$n = 35$ $m = 175$	4.2306	4.1279	1.9642	2.3661	12.0758	12.3583
$n = 41$ $m = 205$	4.2219	4.1196	1.9712	2.3814	12.0769	12.3540
$n = 45$ $m = 225$	4.2215	4.1197	1.9830	2.3989	12.0773	12.3534
$n = 51$ $m = 255$	4.2219	4.1189	1.9662	2.3777	12.0795	12.3552
$n = 25$ $m = 175$	4.2455	4.1412	1.9662	2.3448	12.0683	12.3620
$n = 31$ $m = 217$	4.2351	4.1308	1.9794	2.3621	12.0706	12.3594
$n = 35$ $m = 245$	4.2172	4.1161	1.9651	2.3848	12.0809	12.3525
$n = 41$ $m = 287$	4.2167	4.1150	1.9759	2.3865	12.0794	12.3525
$n = 45$ $m = 315$	4.2164	4.1148	1.9649	2.3851	12.0778	12.3509
$n = 51$ $m = 357$	4.2077	4.1074	1.9742	2.3973	12.0824	12.3482
time for $n = 51$ $m = 255$	6.6	6.5	6.5	6.9	6.9	6.9
time for $n = 51$ $m = 357$	12.8	12.6	12.6	13.4	13.4	13.4

Down-and-out and up-and-out denote discretely monitored barrier options with daily frequency (1 day = 1/250 years). MonteCarlo prices are computed with Monte Carlo simulation using 500,000 sample paths and the Black-Scholes price as the control variate. Std. are standard deviations of the Monte Carlo prices. Markov chain denotes prices computed with the Markov chain method. Time is the computing time in seconds. Parameter values for the down-and-out options: $S_0 = 100$, $r = 0.10$, $K = 100$, and $T = 0.2$; parameter values for the up-and-out options: $S_0 = 110$, $r = 0.10$, $K = 100$, and $T = 0.20$; GARCH parameter values: $\beta_0 = 0.00001$, $\beta_1 = 0.80$, $\beta_2 = 0.10$, $\theta = 0.3$, $\lambda = 0.2$, and $\sqrt{h_1} = 0.010483$.

is, if the GARCH model is defined on a daily basis but monitoring takes place only weekly, then weekly monitoring of a constant barrier is equivalent to monitoring an intermittent barrier on a daily basis. Alternatively, one can first obtain the relevant transition probability matrix over one week by raising the daily transition probability matrix to an appropriate power so that option valuation can be conducted on a weekly basis.

Exhibit 8 examines barrier options identical to those in Exhibit 3 except in the GARCH pricing framework. The convergence speed in Exhibit 8 improves over that in Exhibit 7, which can be attributed to shorter maturity of the options in Exhibit 8 (from $T = 0.5$ down to $T = 0.2$). A shorter maturity requires fewer states to obtain an equally good approximation.

In Exhibit 9, we examine the same double barrier options as in Exhibit 5. The results suggest that monitoring frequency does not affect the precision of the price estimates. By comparison with Exhibit 7, we can also conclude that the presence of two barriers does not adversely affect the performance of the algorithm. In short, the precision is mostly a function of the option's maturity.

We study the convergence pattern of American options and report the results in Exhibits 10, 11, and 12. American down-and-out and down-and-in option results under Black-Scholes are summarized in Exhibits 10 and 11. For down-and-out options, we use the recursive system described in Equations (14), (15), and (16); for

EXHIBIT 9

European Double Knock-Out Call Options in NGARCH Framework

	Daily			Weekly		
Upper barrier	110	125	155	110	125	155
Lower barrier	95	95	95	95	95	95
			Monte Carlo			
Price	0.1985	3.6035	6.1006	0.3430	4.1386	6.5501
Std.	0.0015	0.0086	0.0128	0.0020	0.0090	0.0129
			Markov Chain			
$n = 25\ m = 25$	1.3868	3.5555	5.8219	1.6567	4.0127	6.2247
$n = 31\ m = 31$	0.3072	3.5733	8.1659	0.4583	4.1633	9.1445
$n = 35\ m = 35$	0.5160	2.0538	5.7329	0.7371	2.1472	5.9162
$n = 41\ m = 41$	0.0708	2.4147	5.9379	0.1267	2.8556	6.3603
$n = 45\ m = 45$	0.3743	2.6560	6.4740	0.7779	3.2987	7.2564
$n = 51\ m = 51$	0.1883	4.1806	7.2122	0.3892	5.5037	8.2298
$n = 25\ m = 75$	0.2071	3.0914	6.0947	0.4270	3.5936	6.4896
$n = 31\ m = 93$	0.1600	3.2927	6.0845	0.3010	3.8647	6.4817
$n = 35\ m = 105$	0.2077	3.4217	6.2419	0.4275	4.0849	6.8269
$n = 41\ m = 123$	0.1852	3.4909	6.1790	0.3504	4.1373	6.7082
$n = 45\ m = 135$	0.1928	3.4437	6.1038	0.3567	3.9749	6.5488
$n = 51\ m = 153$	0.1835	3.4887	6.0976	0.3220	4.0261	6.5360
$n = 25\ m = 125$	0.2070	3.4698	6.2055	0.4042	4.0410	6.7564
$n = 31\ m = 155$	0.1876	3.4506	6.1029	0.3419	3.9446	6.5575
$n = 35\ m = 175$	0.1888	3.5025	6.0990	0.3379	4.0519	6.5541
$n = 41\ m = 205$	0.1907	3.5295	6.0810	0.3266	4.0269	6.4915
$n = 45\ m = 225$	0.1903	3.5596	6.0825	0.3245	4.0907	6.5001
$n = 51\ m = 255$	0.1956	3.5832	6.0938	0.3348	4.1253	6.5426
$n = 25\ m = 175$	0.1885	3.4894	6.0933	0.3376	4.0378	6.5474
$n = 31\ m = 217$	0.1947	3.5364	6.1028	0.3458	4.0589	6.5657
$n = 35\ m = 245$	0.1897	3.5379	6.0794	0.3225	4.0462	6.5021
$n = 41\ m = 287$	0.1918	3.5616	6.0810	0.3276	4.0934	6.5142
$n = 45\ m = 315$	0.1960	3.5828	6.0893	0.3311	4.1017	6.5359
$n = 51\ m = 357$	0.1964	3.5875	6.0910	0.3327	4.1189	6.5397

Daily and weekly denote discretely monitored barrier options with daily and weekly frequency based on the assumption that 1 day = 1/250 years and 1 week = 1/50 years. Monte Carlo prices are computed with Monte Carlo simulation using 500,000 sample paths and the Black-Scholes price as the control variate. Std. are standard deviations of the Monte Carlo prices. Markov chain prices are computed with the Markov chain method; parameters: S_0 = 100, r = 0.10, K = 100, T = 0.5 (annualized), β_0 = 0.00001, β_1 = 0.80, β_2 = 0.10, θ = 0.3, λ = 0.2, and $\sqrt{h_1}$ = 0.010483.

EXHIBIT 10

American Down-and-Out Put Options in Black-Scholes Framework

	European			American		
Barrier	85	93	99	85	93	99
			Markov Chain			
$m = 51$	2.0297	0.3698	0.0000	2.9301	2.6708	0.0000
$m = 101$	2.0299	0.3952	0.0009	2.8472	2.6349	0.2844
$m = 201$	2.0221	0.3975	0.0011	2.8240	2.6159	0.2902
$m = 301$	2.0224	0.3992	0.0011	2.8194	2.6124	0.2858
$m = 401$	2.0218	0.3999	0.0011	2.8177	2.6112	0.2823
$m = 501$	2.0213	0.4001	0.0011	2.8169	2.6107	0.2855
$m = 601$	2.0210	0.3995	0.0011	2.8164	2.6105	0.2854
$m = 701$	2.0213	0.3995	0.0011	2.8162	2.6102	0.2867
$m = 801$	2.0210	0.3996	0.0011	2.8160	2.6101	0.2859
$m = 901$	2.0211	0.3996	0.0011	2.8159	2.6100	0.2855
$m = 1001$	2.0210	0.3996	0.0011	2.8158	2.6099	0.2862

European and American denote two styles of discretely monitored barrier options with daily frequency (1 day = 1/250 years). For American options, early exercise is permitted on a daily basis. Parameters: S_0 = 100, r = 0.10, K = 100, T = 0.2 (annualized), and σ = 0.20 (annualized).

EXHIBIT 11

American Down-and-In Put Options in Black-Scholes Framework

Barrier	European 85	European 93	European 99	American 85	American 93	American 99
Markov Chain						
$m = 51$	2.6337	3.7082	3.8014	2.9857	4.2468	4.3423
$m = 101$	2.2455	3.3750	3.5175	2.5366	3.8944	4.0416
$m = 201$	2.1686	3.3172	3.4367	2.4505	3.8332	3.9562
$m = 301$	2.1381	3.2875	3.4200	2.4145	3.8020	3.9386
$m = 401$	2.1626	3.2732	3.4140	2.4457	3.7868	3.9322
$m = 501$	2.1373	3.2771	3.4111	2.4145	3.7910	3.9291
$m = 601$	2.1484	3.2796	3.4094	2.4285	3.7936	3.9274
$m = 701$	2.1546	3.2724	3.4085	2.4363	3.7859	3.9264
$m = 801$	2.1579	3.2746	3.4078	2.4405	3.7883	3.9257
$m = 901$	2.1385	3.2763	3.4073	2.4165	3.7900	3.9252
$m = 1001$	2.1410	3.2711	3.4070	2.4195	3.7846	3.9249

European and American denote two styles of discretely monitored barrier options with daily frequency (1 day = 1/250 years). For American options, early exercise is permitted on a daily basis. Parameters: S_0 = 100, r = 0.10, K = 100, T = 0.2 (annualized), and σ = 0.20 (annualized).

EXHIBIT 12

American Down-and-Out Put Options in NGARCH Framework

Barrier	European 85	European 93	European 99	American 85	American 93	American 99
Markov Chain						
$m = 25$ $n = 25$	1.9847	0.1110	0.0000	4.1086	3.4670	0.0000
$m = 31$ $n = 31$	1.5766	0.1452	0.0000	4.6099	3.4736	0.0000
$m = 35$ $n = 35$	1.0486	0.2817	0.0000	4.5704	3.2675	0.0000
$m = 41$ $n = 41$	1.0887	0.0931	0.0000	4.3984	3.4076	0.0000
$m = 45$ $n = 45$	0.9798	0.1296	0.0000	4.3169	3.3579	0.0000
$m = 51$ $n = 51$	1.0204	0.0960	0.0000	4.1571	3.2511	0.0000
$m = 25$ $n = 75$	1.1066	0.1130	0.0000	3.7887	3.0914	0.0000
$m = 31$ $n = 93$	1.1074	0.1188	0.0002	3.7013	3.0428	0.4256
$m = 35$ $n = 105$	1.1076	0.1198	0.0002	3.6421	3.0247	0.4079
$m = 41$ $n = 123$	1.1169	0.1272	0.0002	3.5721	2.9876	0.3783
$m = 45$ $n = 135$	1.1190	0.1271	0.0002	3.5362	2.9674	0.3584
$m = 51$ $n = 153$	1.1175	0.1319	0.0002	3.5116	2.9496	0.3299
$m = 25$ $n = 125$	1.1176	0.1324	0.0002	3.5464	2.9770	0.3720
$m = 31$ $n = 155$	1.1105	0.1251	0.0002	3.5246	2.9630	0.3251
$m = 35$ $n = 175$	1.1184	0.1284	0.0002	3.4957	2.9494	0.2984
$m = 41$ $n = 205$	1.1186	0.1305	0.0003	3.4645	2.9313	0.3511
$m = 45$ $n = 225$	1.1203	0.1287	0.0003	3.4595	2.9293	0.3410
$m = 51$ $n = 255$	1.1217	0.1301	0.0003	3.4380	2.9189	0.3239
$m = 25$ $n = 175$	1.1265	0.1294	0.0002	3.4903	2.9505	0.2973
$m = 31$ $n = 217$	1.1172	0.1292	0.0003	3.4715	2.9340	0.3442
$m = 35$ $n = 245$	1.1269	0.1303	0.0003	3.4550	2.9275	0.3285
$m = 41$ $n = 287$	1.1239	0.1305	0.0003	3.4470	2.9228	0.3343
$m = 45$ $n = 315$	1.1244	0.1314	0.0003	3.4281	2.9139	0.3337
$m = 51$ $n = 357$	1.1248	0.1310	0.0003	3.4303	2.9137	0.3279

European and American denote two styles of discretely monitored barrier options with daily frequency (1 day = 1/250 years). For American options, early exercise is permitted on a daily basis. Parameters : S_0 = 100, r = 0.10, K = 100, T = 0.5 (annualized), β_0 = 0.00001, β_1 = 0.80, β_2 = 0.10, θ = 0.3, λ = 0.2, and $\sqrt{h_1}$ = 0.010483.

EXHIBIT 13

Delta and Gamma for European Down-and-Out Call Options in Black-Scholes Framework

	Daily			Weekly		
Barrier	95	99.5	99.9	95	99.5	99.9
			Explicit Finite Difference Delta			
	0.9895	1.2761	1.1674	0.9291	1.0714	1.0378
			Markov Chain Delta			
$m = 51$	1.0308	1.2846	1.2184	0.9743	1.1262	1.0950
$m = 101$	1.1143	1.3836	1.2758	0.9434	1.0840	1.0475
$m = 201$	1.0100	1.3226	1.1882	0.9285	1.0756	1.0384
$m = 301$	0.9980	1.3112	1.1731	0.9290	1.0748	1.0375
$m = 401$	0.9937	1.3082	1.1692	0.9290	1.0747	1.0374
$m = 501$	0.9910	1.2767	1.1678	0.9287	1.0709	1.0374
$m = 601$	0.9907	1.2767	1.1673	0.9289	1.0711	1.0374
$m = 701$	0.9899	1.2745	1.1670	0.9287	1.0709	1.0374
$m = 801$	0.9899	1.2747	1.1669	0.9289	1.0709	1.0374
$m = 901$	0.9896	1.2739	1.1668	0.9288	1.0709	1.0374
$m = 1001$	0.9897	1.2740	1.1668	0.9289	1.0709	1.0374
			Explicit Finite Difference Gamma			
	-0.0208	0.2621	0.3944	-0.0129	0.1229	0.1484
			Markov Chain Gamma			
$m = 51$	0.0060	0.7993	0.9435	-0.0172	0.1425	0.1703
$m = 101$	-0.0250	0.3354	0.5049	-0.0143	0.1211	0.1494
$m = 201$	-0.0215	0.2291	0.4006	-0.0131	0.1196	0.1480
$m = 301$	-0.0213	0.2204	0.3933	-0.0130	0.1196	0.1480
$m = 401$	-0.0211	0.2186	0.3921	-0.0129	0.1196	0.1480
$m = 501$	-0.0210	0.2594	0.3917	-0.0129	0.1226	0.1480
$m = 601$	-0.0210	0.2576	0.3916	-0.0129	0.1224	0.1480
$m = 701$	-0.0209	0.2603	0.3916	-0.0129	0.1226	0.1480
$m = 801$	-0.0209	0.2594	0.3916	-0.0129	0.1226	0.1481
$m = 901$	-0.0209	0.2605	0.3916	-0.0129	0.1226	0.1481
$m = 1001$	-0.0209	0.2601	0.3916	-0.0129	0.1226	0.1481

down-and-in options, we use the recursive system defined by Equations (26), (27), and (28). We also implement the Markov chain method for American down-and-out options using the GARCH pricing framework, and report these results in Exhibit 12. The convergence patterns in all these exhibits suggest that the Markov chain method works well in pricing American barrier options.

Exhibit 13 examines the hedge parameters obtained with the Markov chain method for the down-and-out case. Similar to the finite-difference approach, the Markov chain method can produce a vector of option prices, i.e., one option price corresponds to each discrete value of the stock price. Using these, delta and gamma values can be computed from the option prices adjacent to $p[(m + 1)/2]$; that is, the option price corresponding to the stock price located at the center of the price partition.

In order to obtain higher-quality Greeks via the finite-difference calculation, it is advisable to have adjacent prices very close to the initial stock price. This can be easily accomplished by adding $p[(m + 1)/2] - \varepsilon$ and $p[(m + 1)/2] + \varepsilon$ to the states defining the Markov chain, where ε is some small number.

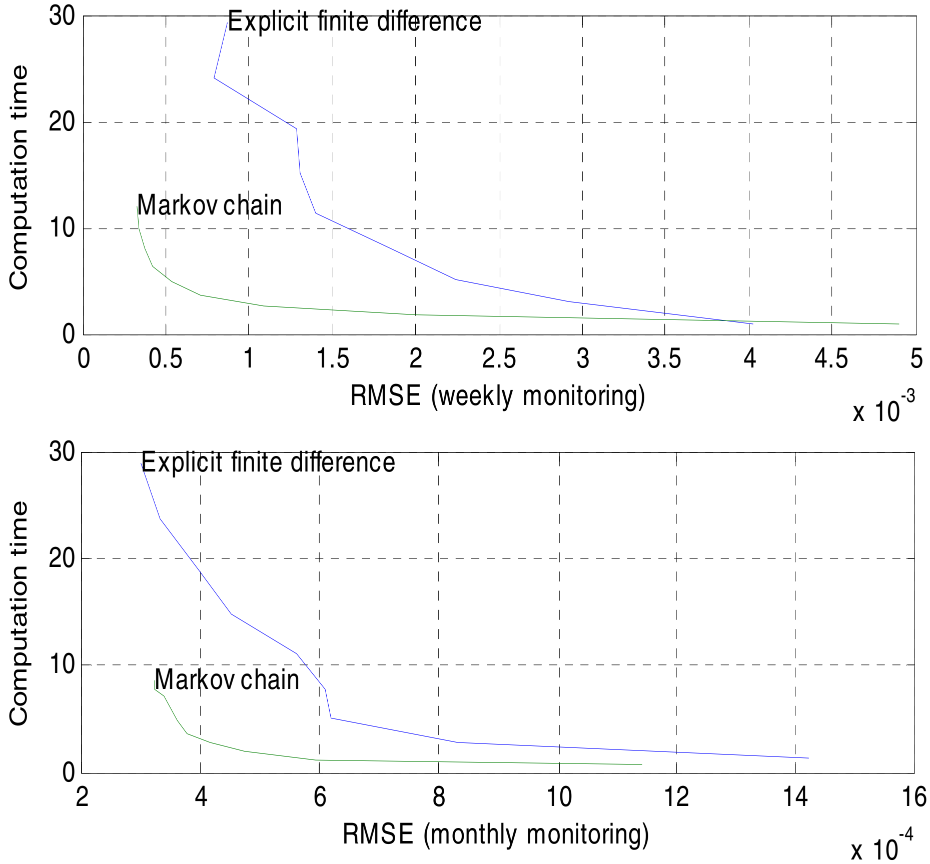
Since the price partition is constructed on the logarithmic stock price, we use expressions as follows to compute the delta and gamma values:

$$\Delta = \frac{\partial V}{\partial \ln S_0}\frac{1}{S_0} \approx \frac{V\left(p_{\frac{m+1}{2}} + \varepsilon\right) - V\left(p_{\frac{m+1}{2}} - \varepsilon\right)}{2\varepsilon}\frac{1}{S_0}$$

$$\Gamma = \frac{\partial}{\partial S_0}\left(\frac{\partial V}{\partial \ln S_0}\frac{1}{S_0}\right)$$

$$\approx \left(\frac{V\left(p_{\frac{m+1}{2}} + \varepsilon\right) + V\left(p_{\frac{m+1}{2}} - \varepsilon\right) - 2V\left(p_{\frac{m+1}{2}}\right)}{\varepsilon^2} - \frac{V\left(p_{\frac{m+1}{2}} + \varepsilon\right) - V\left(p_{\frac{m+1}{2}} - \varepsilon\right)}{2\varepsilon}\right)\frac{1}{S_0^2}$$

EXHIBIT 14
Cost-Benefit Analysis for Pool of European Down-and-Out Call Options



The numbers obtained in Exhibit 13 are calculated using $\varepsilon = 1 \times 10^{-6}$. As these results show, the delta and gamma values obtained with a reasonable number of states are very close to those obtained by the explicit finite-difference approach of Boyle and Tian [1998] with 5000 time steps.

For a cost-benefit analysis, we use a random sample of options to assess the trade-off between computing time and pricing accuracy (as in Broadie, Glasserman, and Kou [1999]).

The option contract parameters are drawn from distributions as follows: $S_0 = 100$, σ is uniform on [0.1, 0.6], r is uniform on [0.0, 0.1], T is uniform on [0.1, 1.0], H is uniform on [80.00, 99.9], and K is uniform on [$1.1 \times H$, 130]. Two different monitoring frequencies are considered, weekly and monthly. We measure the aggregate relative pricing error for the test pool of options by a root mean square error (RMSE), defined as:

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\frac{C_i(MC \text{ or } FD) - C_i}{C_i}\right)^2} \tag{54}$$

where $C_i(MC$ or $FD)$ is the i-th Markov chain (or explicit finite-difference) price, C_i is the i-th benchmark price obtained from a high-precision Monte Carlo simulation using 6 million sample paths, and N is the number of option prices in the test pool. Although our sampling procedure results in the pricing of 500 options, we restrict the analysis to a subset of options whose Monte Carlo prices are greater than or equal to 0.50. This restriction avoids a large pricing error due to a small divider.

The results for both weekly and monthly monitoring are shown in Exhibit 14. Two curves are presented in each graph. The horizontal axis marks the RMSE, and the vertical axis indicates the total computing time (in seconds) for the test pool of options. A curve closer to the origin implies better computational efficiency, because for a given level of RMSE, it takes less time to complete the calculations.

It is clear from Exhibit 14 that the Markov chain method dominates the explicit finite-difference approach.

IV. CONCLUSION

Barrier options are popular financial derivatives for which we should find faster and more reliable numerical methods. These methods must be able to accommodate at least one important real-life feature of these options: discrete monitoring of the barrier.

We have proposed a valuation method for discretely monitored barrier options that is derived from the general Markov chain approach put forth by Duan and Simonato [2001]. The method is fast and flexible in handling various barrier scenarios. It can easily deal with both European- and American-style barrier options.

Besides pricing barrier options in the constant-volatility option valuation framework, our method also works in a time-varying volatility option valuation framework such as the GARCH model. This added benefit can be immensely valuable, given our increasing understanding of the advantages of time-varying volatility option valuation theory.

APPENDIX A

Approximating Markov Chain for Black-Scholes Model

A Markov chain that will, in the limit, share the same probabilistic properties as the asset price process S can be constructed in two steps. First, we find an interval that extends for δ standard deviations in each direction from S_0, the starting point of the target process. This interval can then be partitioned into equal-size cells. The middle point of these cells is used as the state values of the Markov chain. In the second step, these state values in conjunction with the cell boundaries allow us to compute the entries of the transition probability matrix. Our description pertains specifically to geometric Brownian motion under the risk-neutral measure.

The overall interval covering the set of representative logarithmic asset prices is written as $[\ln S_0 - I_p, \ln S_0 + I_p]$. The quantity I_p is computed using the conditional standard deviation of the asset return over the life of the option contract multiplied by a scaling factor. Formally:

$$I_p = \delta(m)\sigma_a\sqrt{T \times \Delta t}$$

where Δt denotes the length in years between two discrete times, and σ_a denotes the volatility parameter expressed on an annual basis. The quantity, $\sigma_a\sqrt{T \times \Delta t}$, is the standard deviation of the asset return at time T. The scaling factor $\delta(m)$ is an increasing function of m, satisfying the partition conditions: 1) $\delta(m) \to \infty$ as $m \to \infty$, and 2) $[\delta(m)]/m \to 0$ as $m \to \infty$. We opt for $\delta(m) = 2 + \ln[\ln(m)]$, which has been shown in Duan and Simonato [2001] to be a good choice.

Once the overall interval for the asset price is determined, the interval is divided equally into $m - 1$ segments to yield m discrete values:

$$p_i = \ln S_0 + \frac{2i - m - 1}{m - 1} I_p \quad \text{for} \quad i \in \{1, \cdots, m\}$$

for the logarithmic asset price. Note that $p_1 = \ln S_0 - I_p$ and $p_m = \ln S_0 + I_p$ are the minimum and maximum values, respectively. In order to have $\ln S_0$ among the state prices, m needs to be an odd integer, in which case, $p_{(m+1)/2} = \ln S_0$. The m cells are constructed as:

$$C_1 = (c_1, c_2), \quad C_i = [c_i, c_{i+1}) \quad \text{for } i = \{2, \cdots, m\}$$

where $c_1 = -\infty$, $c_i = (p_i + p_{i-1})/2$ for $i = 2$ to m, and $c_{m+1} = +\infty$.

Using these cells and states, the transition probabilities can now be computed using

$$q_{ij} = \Phi\left(\frac{c_{j+1} - p_i - \left(r_a - 0.5\sigma_a^2\right)\Delta t}{\sigma_a\sqrt{\Delta t}}\right) - \Phi\left(\frac{c_j - p_i - \left(r_a - 0.5\sigma_a^2\right)\Delta t}{\sigma_a\sqrt{\Delta t}}\right)$$

where $\Phi(\cdot)$ stands for the cumulative standard normal distribution function, and r_a is the annualized continuously compounded risk-free interest rate. Using these transition probabilities and the discretized asset prices, an option price can be computed using Equations (4) or (5).

APPENDIX B

Derivation of European-Style Knock-In Option Valuation Equation

The recursive system in Equations (29) and (30):

$$V(\overrightarrow{p}, t; a_t = 0) = e^{-r}\Pi V(\overrightarrow{p}, t+1; a_{t+1} = 0) + e^{-r}\Gamma V(\overrightarrow{p}, t+1; a_{t+1} = 1)$$

$$V(\overrightarrow{p}, t; a_t = 1) = e^{-r} Q V(\overrightarrow{p}, t+1; a_{t+1} = 1)$$

can be used to prove by induction that for any $t \in \{0, 1, \ldots, T-1\}$:

$$V(\vec{p}, t; a_t = 0) = e^{-(T-t)r} \left(\sum_{i=1}^{T-t} \Pi^{T-t-i} \Gamma Q^{i-1} \right) V(\vec{p}, T; a_T = 1) \tag{B-1}$$

$$V(\vec{p}, t; a_t = 1) = e^{-(T-t)r} Q^{T-t} V(\vec{p}, T; a_T = 1) \tag{B-2}$$

Indeed, if $t = T - 1$, we have

$$\begin{aligned} & V(\vec{p}, T-1; a_{T-1} = 0) \\ = \ & e^{-r} \Pi V(\vec{p}, T; a_T = 0) + \\ & e^{-r} \Gamma V(\vec{p}, T; a_T = 1) \\ = \ & e^{-r} \Gamma V(\vec{p}, T; a_T = 1) \end{aligned}$$

$$V(\vec{p}, T-1; a_{T-1} = 1) = e^{-r} Q V(\vec{p}, T; a_T = 1)$$

because the terminal conditions are

$$\begin{aligned} V(\vec{p}, T; a_T &= 0) = \vec{0} \\ V(\vec{p}, T; a_T &= 1) = \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \end{aligned}$$

Assuming that Equations (B-1) and (B-2) hold for some $t \in \{0, 1, \ldots, T-1\}$, we compute

$$\begin{aligned} & V(\vec{p}, t-1; a_{t-1} = 0) \\ = \ & e^{-r} \Pi V(\vec{p}, t; a_t = 0) + e^{-r} \Gamma V(\vec{p}, t; a_t = 1) \\ = \ & e^{-r} \Pi e^{-(T-t)r} \left(\sum_{i=1}^{T-t} \Pi^{T-t-i} \Gamma Q^{i-1} \right) V(\vec{p}, T; a_T = 1) + \\ & e^{-r} \Gamma e^{-(T-t)r} Q^{T-t} V(\vec{p}, T; a_T = 1) \\ = \ & e^{-(T-(t-1))r} \left(\sum_{i=1}^{T-(t-1)} \Pi^{T-(t-1)-i} \Gamma Q^{i-1} \right) V(\vec{p}, T; a_T = 1) \\ & V(\vec{p}, t-1; a_{t-1} = 1) \\ = \ & e^{-r} Q e^{-(T-t)r} Q^{T-t} V(\vec{p}, T; a_T = 1) \\ = \ & e^{-(T-(t-1))r} Q^{T-(t-1)} V(\vec{p}, T; a_T = 1) \end{aligned}$$

which completes the induction. If $t = 0$, we obtain

$$\begin{aligned} & V(\vec{p}, 0; a_0 = 0) \\ & = e^{-rT} \left(\sum_{i=1}^{T} \Pi^{T-i} \Gamma Q^{i-1} \right) \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \\ & V(\vec{p}, 0; a_0 = 1) = \\ & \quad e^{-rT} Q^T \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \end{aligned}$$

APPENDIX C

In-Out Parity

If $a_0 = 1$, the result is obvious. If $a_0 = 0$, the value of the knock-in option, according to Equation (28), can be expressed as

$$V^*(\vec{p}, 0) = BV(\vec{p}, 0; a_0 = 0) + AV(\vec{p}, 0; a_0 = 1)$$

We intend to prove by induction that

$$V^*(\vec{p}, 0) = e^{-rT} \left(Q^T - \Pi^T \right) \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \tag{C}$$

Indeed, if $T = 1$:

$$\begin{aligned} & V^*(\vec{p}, 0) - e^{-r} (Q - \Pi) \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \\ = \ & BV(\vec{p}, 0; a_0 = 0) + AV(\vec{p}, 0; a_0 = 1) - \\ & e^{-r} (Q - \Pi) \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \\ = \ & e^{-r} \left(B\Gamma + \underbrace{AQ - Q}_{=-BQ} + \underbrace{\Pi}_{=A\Pi + B\Pi} \right) \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \end{aligned}$$

[by Equations (31) and (32)]

$$= e^{-r} \left(\underbrace{A\Pi}_{=\mathbf{0}} + \underbrace{B(\Gamma + \Pi - Q)}_{=\mathbf{0}} \right) \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\}$$

Now assume that Equation (C) holds for some maturity date $T - 1 \in \{2, 3, 4, \ldots\}$. If the maturity becomes T, then

$$\begin{aligned} & V^*(\vec{p}, 0) - e^{-rT} \left(Q^T - \Pi^T \right) \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \\ = \ & BV(\vec{p}, 0; a_0 = 0) + AV(\vec{p}, 0; a_0 = 1) - e^{-rT} \left(Q^T - \Pi^T \right) \\ & \max\left\{ w[\exp(\vec{p}) - K\vec{1}], \vec{0} \right\} \end{aligned}$$

$$= e^{-Tr}\left(B\sum_{i=1}^{T}\Pi^{T-i}\Gamma Q^{i-1} + AQ^{T} - \left(Q^{T}-\Pi^{T}\right)\right)$$

$$\max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\}$$

[by Equations (31) and (32)]

$$= e^{-Tr}\left(B\Pi\sum_{i=1}^{T-1}\Pi^{T-1-i}\Gamma Q^{i-1} + B\Gamma Q^{T-1} + \underbrace{AQ^{T} - Q^{T}}_{=-BQ^{T}} + \Pi^{T}\right)$$

$$\max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\}$$

$$= e^{-Tr}\left(\Pi\left(B\sum_{i=1}^{T-1}\Pi^{T-1-i}\Gamma Q^{i-1} + AQ^{T-1}\right) - \underbrace{\Pi AQ^{T-1}}_{=\Pi Q^{T-1}-\Pi BQ^{T-1}} + \underbrace{B\Gamma Q^{T-1} - BQ^{T}}_{=B(\Gamma-Q)Q^{T-1}} + \Pi^{T}\right) \times$$

$$\max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\}$$

(because $B\Pi = \Pi B$)

$$= e^{-Tr}\left(\Pi\left(Q^{T-1} - \Pi^{T-1}\right) - \Pi Q^{T-1} + \Pi BQ^{T-1} + B\left(\Gamma - Q\right)Q^{T-1} + \Pi^{T}\right)\max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\}$$

(from the induction hypothesis)

$$= e^{-Tr}\underbrace{B\left(\Gamma + \Pi - Q\right)}_{=\mathbf{0}}Q^{T-1}\max\left\{w[\exp(\vec{p}) - K\vec{1}], \vec{0}\right\} = \vec{0}$$

(again because $B\Pi = \Pi B$)

which completes the induction.

ENDNOTES

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[1]Our Markov chain approximation differs slightly from that of Duan and Simonato [2001]. We partition $\ln(S_t)$ instead of $\ln(e^{-(r-\sigma^2/2)t}S_t)$ as in Duan and Simonato [2001]. As a result, we do not need to include the argument t in the function $g(\cdot, \cdot)$. Duan and Simonato [2001] wanted to remove the drift in the asset price before partitioning so that the price evolution is properly centered. Our modification is designed specifically for barrier options so that we can control the placement of the barrier in relation to the discrete asset prices.

[2]The trinomial tree scheme can be viewed as a special Markov chain. For example, a two-step standard trinomial tree (five possible prices from low to high) with the probability of going down (p) and up (q) has the transition probability matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & 1-p-q & q & 0 & 0 \\ 0 & p & 1-p-q & q & 0 \\ 0 & 0 & p & 1-p-q & q \\ 0 & 0 & p & 0 & 1 \end{bmatrix}$$

[3]The combination that p_i at time t is in the knock-out region and $a_t = 0$ constitutes a null set. In this null set, we still let $g(p_i, K, a_t = 0) = \max\{w[\exp(p_i) - K], 0\}$ if it is of American style, but its value should actually equal zero. As discussed earlier, however, we can assign any value to the null set without affecting the integrity of the recursive valuation system. Setting $g(p_i, K, a_t = 0)$ the way we do nevertheless simplifies the valuation formula.

[4]If a particular p_i fails to trigger a knock-in, the immediate exercise value for the American option clearly equals zero. If p_i does trigger a knock-in, the event of having such a p_i and $a_t = 0$ constitutes a null set. As discussed earlier, we can assign an arbitrary value to this null set. Thus, we can have $g(p_i, K, a_t = 0)$ for all p_i.

[5]Although the option payoff vector is typically defined only in terms of the underlying asset price, repetition is necessary because using a Markovian representation of the GARCH process enlarges the relevant dimension of the system.

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