Margin Requirements, Bond Futures Prices and Volatilities

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Abstract

Using a partial equilibrium model where the margin level is defined to be equivalent to the settlement interval, we show that the effects of the margin requirement on both prices and price volatilities of bond futures depend on the autoregressive covariance structure of the interest rate process and on the margin level. While such effects are predictable, they may reverse themselves during the life of the futures contract. This result may explain the prolonged and unresolved debate in the literature about the empirical relationship between margin levels and volatility. This result confirms in part and generalizes the theoretical results of Hartzmark (1986) and Kupiec and Sharpe (1991). We also show by using simulations that margin requirements, as proxied by the settlement interval, have a positive effect on volatility and futures prices, and that they induce a leverage effect. Other results are that the futures price decreases with volatility, and that futures on longer term bonds have higher volatility.
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1. Introduction

The purpose of initial margins in futures markets is to ensure contract performance. In contrast to stock margins which are sometimes imposed with the objective of reducing speculation, futures margins are set by exchanges to yield reasonable protection against default by accounting for recent volatility and other relevant factors. However, raising margin requirements may affect futures market factors other than default risk, such as the volume of transactions, price levels and volatility. These factors may in turn increase default risk, and thus prevent the fulfillment of the prudential function of margin requirements in futures markets. Since legislators and regulators believe that margins are related to volatility (Kupiec 1993), margins are used increasingly since the October 1987 stock market crash by the security exchanges when increased volatility threatens contract performance. To illustrate, when Sumitomo Bank announced on June 13, 1996, that it lost $1.8 billion in its London copper trading, the New York Mercantile Exchange raised its margin requirements twice over a three-day period.

The reported evidence is mixed on whether an unambiguous relationship exists between margins and volatility. While not specifically examining this relationship, Brennan (1986) concludes that "a daily settlement rule greatly reduces the dispersion of the liability under a contract, and makes it possible to attain a given level of credit risk with a much lower level of margin ..." He demonstrates that price limits act as a partial substitute for margin requirements. When margin costs differ for various types of market participants, Hartzmark (1986) shows that changes in margin requirements result in changes in the market composition, and that futures price volatility must change, although in an unpredictable manner. Hardouvelis (1988) finds an economically and statistically significant negative relationship between initial margin requirements and stock market excess (and total) volatility. Kupiec and Sharpe (1991) show that binding initial margin
requirements may either increase or decrease stock price volatility, depending upon the structure of risk aversion among market participants, when the source of heterogeneity is the beliefs of noise traders. Although Kupiec (1993) finds a positive relationship between margins and volatility, he can not reject the Excessive Volatility Hypothesis, which states that volatility is controlled because increasing margins increases speculation costs. In contrast to these studies, Becketti and Roberts (1990) find that the imposition of circuit breakers and higher margins on stock index futures since 1987 are unlikely to reduce stock market volatility, whether such volatility is measured by the frequency or the size of large swings in stock prices. Moser (1991) argues that raising margins does not reduce subsequent volatilities. Goldberg and Hachey (1992) find no consistent relationship between margin changes and price volatility in the foreign exchange futures markets.

Thus, in this study, we derive a partial equilibrium model to re-examine the unresolved issue of whether margin levels affect the volatility of futures prices. Unlike past studies, our approach allows for the investigation of the effects of margin levels on price volatility without invoking the assumption of heterogeneity or invoking specific assumptions about cost, information and market structures.

Our intuition is simple: If the marking-to-market of positions for bond futures is continuous and price changes are smooth, there is no need for initial or maintenance margins. Therefore, when marking-to-market is discrete, increasing the time interval between two consecutive settlement dates, all other things being equal (especially, constant margin levels) implies a lowering of the effective margin requirements. Increasing the settlement period a priori might mean higher default risk, encouraging speculation and increasing volatility.\(^1\) Thus, the time interval between two

\[^1\] If \(\dot{o}\) is the instantaneous standard deviation of the interest rate stochastic process and \(L\) is the settlement interval, then the volatility of that process over that time period is \(\dot{o} \sqrt{L}\). However, note that we investigate the effect of increasing \(L\) on bond futures prices and their volatilities, and not on the volatility of the interest rate process.
consecutive settlements can be used as a proxy for the margin level.² When both the margin and settlement intervals are explicitly modeled, an increase in the length of the settlement interval of a futures contract should be matched by a proportional increase in the margin level in order to provide the same protection to traders. Thus, in our model, we can assume that an increase in the length of the settlement interval is an implicit decrease of the margin. Therefore the results that we obtain are based on the validity of the joint hypothesis that our model of bond futures prices is correct and that margin requirements can be proxied by the time interval between two settlement dates.³ This relationship implies that the partial derivatives of the prices of bond futures with respect to the settlement intervals can be used to uncover the effects of margin variations on bond futures prices and their volatility.

We first develop a model of bond prices using the Jamshidian (1989) methodology, and a bond futures price model using the Duffie and Stanton (1992) algorithm, where futures are continuously marked-to-market. Then the model is discretized, and comparative statics are used to examine the effects of increasing the settlement period on both the bond futures price and its volatility. Our findings show that the effect of margin on the price volatility of futures contracts depends on the autocorrelation structure of the interest rate process and on the margin level, when the latter is implicitly modeled by the length of the settlement period. This result is compatible with those of Hartzmark (1986) and Kupiec and Sharpe (1991). However, while both of these studies show that the direction of the relationship between margins and volatility depends on market conditions.

² Brennan (1986) similarly shows that price limits are proxies for margin requirements.

³ More formally, our intuition is more easily understood if a model of continuous time settlement is taken as a benchmark. Since continuous time settlement protects traders against significant price variations leading to default, no need for margin exists. However, if the settlement is allowed to become discrete, a margin proportional to the length of time between two settlements is needed to maintain the same level of protection. Therefore, an increase of the settlement interval without a matching margin increase is an implicit decline in the effective margin.
settings which change because of changes in margin requirements, our research shows that the
direction of the relationship is rooted in the parameters of the interest rate process itself. For
example, for bond futures prices derived using the Vasicek framework, an increase in the margin
level for a futures contract leads to a decrease in price volatility. The effect on the bond futures
price itself is predictable, although it may reverse itself during the life of the futures contract. In the
general case of a multivariate Itô process, the directional effects of increasing margin levels on bond
futures price and its volatility depend on the time covariance structure of the processes. The
directional effects are predictable, although they depend upon the functional form of the volatility
term of the process. Our results explain why the debate on this subject has been both prolonged
and contradictory.

The remainder of our paper is organized as follows. In section 2, a closed-form solution
of bond futures prices under continuous-time marking-to-market is derived under the Vasicek
framework. In section 3, the initial model of bond futures is extended to obtain an expression of
futures prices under discrete-time marking-to-market. In section 4, the equation derived in section
3 is used to determine the effect of margin variations on the volatility of bond futures prices. In
section 5, we present a generalization of our results to multidimensional Itô processes. Section 6
provides some concluding remarks.

2. Futures prices under continuous marking-to-market

In this section, in order to introduce the required notation, the Jamshidian (1989)
methodology is followed to derive the price of a discount bond, and the Duffie and Stanton (1992)
algorithm is used to obtain the futures price of the discount bond. While these results are fairly well
known, they are required subsequently to derive the discrete marking-to-market variant of the
continuously resettled claim. Assume that the term structure of interest rates is completely
determined by the instantaneous interest rate \( r(t) \) which follows a mean-reverting Gaussian process [as in Vasicek (1977)]:

\[
\frac{dr(t)}{dt} = \mu (\mu - r(t)) dt + \sigma dw(t)
\]  

(1)

where  
\( \mu \) is the instantaneous mean of the interest rate process, \( r_0 > 0 \);  
\( \sigma \) is the instantaneous standard deviation of the interest rate process, \( \sigma > 0 \);  
\( a \) is the speed of reversion towards the mean by the interest rate, \( a > 0 \); and  
\( w(t) \) is a standard Wiener process.

Because of its Gaussian volatility, the choice of the Vasicek model facilitates analytical tractability. However, like all traditional models which assume a Gaussian volatility structure [e.g., Brace and Musiela (1994), Jamshidian (1989, 1991), Hull and White (1990)], negative interest rates are possible [see Rogers (1996)]. Alternative models which assume lognormal volatility, such as the models proposed by Heath, Jarrow and Morton (1992), Black, Derman and Toy (1990) and Black and Karasinski (1991), have a nonzero probability of negative interest rates, but can have a positive probability of infinite interest rates if the instantaneous short rate is allowed to compound. Unlike the square root process of Cox, Ingersoll and Ross (1985), the assumption of a lognormal term structure of interest rates is the most natural way to exclude the possibility of negative spot rates. Fortunately, Sandmann and Sondermann (1997) show that the infinite rate problem disappears, if the rates with a finite accrual period (e.g. 3-month Libor rates) instead of the instantaneous rate with an infinitesimal accrual period are assumed to have a lognormal volatility. While the Vasicek model is not lognormal, we assume in this paper that the coupon rate \( h(r,t) \) is continuously compounded, while the rate \( r(t) \) itself follows an Ornstein-Uhlenbeck process.
Leblanc, Renault and Scaillet (2000) provide an argument for the simpler and more tractable Gaussian models. They show that the first passage time of an Ornstein-Uhlenbeck process to a boundary can be computed.\(^4\) Using annual values for the parameters that are representative of recent real economic situations, we calculate and report the probabilities of negative interest rates in Table 1. We find that the probability of a negative interest rate is extremely small for future time intervals of at least one hour. To illustrate, the probability is less than .02 for a time interval of one hour, less than 10\(^{-4}\) for a time interval of one day, and less than 10\(^{-11}\) for a time interval of one week. For the next future extremely short time interval (e.g., the next minute),\(^5\) we find that the interest rate almost surely becomes negative. The reason for these apparently anomalous results is found in the design of the Ornstein-Uhlenbeck process itself. Although the interest rate reverts to its mean, the probability of first hitting a zero boundary at a given time \(t\) does not depend on the value of the mean.\(^6\) When the interest rate is both close to zero and has a high volatility (i.e., \(\mu=.06, \sigma=.06\) and \(r_0=.05\)), there is a high probability that interest rates could become negative in the next small time interval. However, as time passes, the mean-reverting nature of interest rates substantially increases the probability that they will be moved in a positive direction away from zero. Hypothetically, although time brings interest rates anywhere,

\(^4\) Their equation considers a positive boundary and a starting point for the process that is within that boundary. Because the Ornstein-Uhlenbeck process is Gaussian and thus locally symmetric, the equation has been easily adapted to our problem of a positive rate process hitting a lower boundary at zero. The adapted equation and the results are given in Table 1.

\(^5\) Computations at small values such as 1/(60x24x365) can give indefinite and meaningless values, e.g. greater than one.

\(^6\) The equation in Table 1 indicates that the mean has a net effect on the probability. However, this effect must be very small relatively to other variables such the mean reversion speed and the initial value. See the next footnote for greater details.
their ending value will be beyond a fixed point (or level). Indeed, our results suggest that if interest rates are not negative in the next second, then they almost never will be in the future. Because the Ornstein-Uhlenbeck process has no memory (a Markovian property), these results are consistent internally.

Based on Table 1, other results are that the value of $\mu$, the instantaneous long term mean of the process, does not have any measurable effect on the probability of the process hitting the zero boundary at any given time $t$, which is beyond a small future time interval. This is true even if $\mu$ is set equal to zero. The effects of the instantaneous standard deviation, $\sigma$, and of the speed of reversion to the mean, $a$, are extreme in relative terms, again for dates that capture a small future time interval. Surprisingly, the initial value of the process, $r_0$, appears to have a somewhat linear and adverse effect on the probability of the process hitting the zero boundary. The closer the initial value is to zero, the smaller is the probability of the process hitting the zero boundary.

A further rationale for the use of the Ornstein-Uhlenbeck process is provided by Carrière (1999) who demonstrates that if yield rates follow a continuous multivariate Ho-Lee model, then the no-arbitrage condition implies that these rates can be expressed as a generalized multivariate Ornstein-Uhlenbeck process. Ho and Lee (1986) develop a one-factor model in discrete time which Duffie and Kan (1996) generalize to a multivariate continuous time affine model with stochastic volatilities. Carrière suggests that if his model is well calibrated, the chances of attaining a negative rate is very small. Carrière also finds that the Gaussian assumption (Vasicek model or Ornstein-Uhlenbeck process) is very well fitted to yield data for stripped bonds.

Prices of bonds and their derivative securities depend on $r(t)$; that is, on only one state variable. Using Itô's Lemma, the price of a security, such as a bond, which pays a rate $h(r,t)$

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7 Numerous values for the mean (including zero) and for the mean reversion speed (a values as high as 1) find that the higher the value of $a$, the smaller the probability, all else being equal. In contrast, changes in the mean value always have no effect, all else being equal. See the previous footnote for more details.
continuously and yields a terminal payoff \(g[r(T)]\) at time \(T\), is given as the solution of the following two-equation system:\(^8\)

\[
U_t + \frac{1}{2} \sigma^2 U_{rr} \partial^2(\tilde{r} \& r) U_{t} \& r(t) U(r, t) \partial h(r, t)
\]

\[U(r, T) = g[r(T)]\] \hspace{1cm} (2)

where \(U(r, t)\) is the price of a futures contract (or a forward contract) to deliver at maturity date \(T\) the bond with terminal payoff \(g[r(T)]\);\(^9\)

\[
U_t = \frac{\mathcal{M} U(r,t)}{\mathcal{M}};
U_r = \frac{\mathcal{M} U(r,t)}{\mathcal{M}};
U_{rr} = \frac{\mathcal{M} U(r,t)}{\mathcal{M}^2};
\tilde{r} = \mu \frac{\delta}{\sigma}; \text{ and }
\]

\(\delta\), the market price of interest rate risk, is the expected instantaneous excess return above the riskless rate divided by the instantaneous standard deviation of return on the bond.

According to Jamshidian (1989), the solution to the above equation system is:\(^{10}\)

\[
U(r, t) = E_{t, \tilde{r}} \{ g[\tilde{r}(T)] e^{\tilde{Y}(t, T)} \} \% F r, t \{ \int_{t}^{T} h[\tilde{r}(s)] e^{\tilde{Y}(t, s)} ds \}
\] \hspace{1cm} (4)

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\(^8\) Note that since \(h(r,t)\) is time-dependent, it can be a floating rate.

\(^9\) The maturity date \(T\) applies to both the bond and the futures contract. Also, at this point in time, it is neither useful nor necessary to distinguish between the maturity dates of the futures contract and the bond. Later, \(T_f\) will refer to the maturity date of the futures contract, \(T_b\) will refer to the maturity date of the bond, \(U(r, t, T_f, T_b)\) will refer to the futures price, and \(P(r, t, T_b)\) will refer to the bond price.

\(^{10}\) The solution is based on Friedman (1975), Theorem 5.3, chapter 6.
where \( \tilde{r} \) is the "risk neutral interest rate process" as defined by: \(^{11}\)

\[
d\tilde{r} = a(\tilde{r} \& \tilde{r}) \, dt + \sigma \, dw(t)
\]

and

\[
Y(t, s) = \int_s^t \tilde{r}(v) \, dv
\]

The first term on the right-hand side of equation (4) is simply the expected value of the terminal pay-off at maturity, discounted at the appropriate zero-coupon rate \( Y(t,T) \). The second term represents the sum of the values of the continuous coupons \( h(r,t) \), discounted at the appropriate rate \( Y(t,s) \). Suppose that \( U(r,t,T_F) \) is the futures price of a discount bond whose current price is \( P(r,t,T_B) \), where \( T_F \) is the maturity date of the futures contract and \( T_B \) is the maturity date of the discount bond. Then, using Itô's lemma, the futures price for delivery of a discount bond at time \( T_F \) is given as the solution of:

\[
U_t - \frac{1}{2} \sigma^2 U_{rr} \%a(\tilde{r} \& r(t)) U_r \%b(r,t)
\]

(5)

Duffie and Stanton (1992) show that the resettlement price of a continuously resettled claim that is marked at the terminal date \( T_F \) to \( P[r(T_F),T_F,T_B] \) is:\(^{12}\)

\[
U(r,t,T_F) \cdot E_{r,t} \{ P[r(T_F),T_F,T_B] \}
\]

(6)

\(^{11}\) Arnold (1974, section 8.3) shows that \( E_{r,t} [\tilde{r}(s)] \cdot m(r,t,s) \cdot e^{\delta a(s \& t)} \%[1 \& e^{\delta a(s \& t)}] \tilde{r} \), and \( \text{var}_{r,t} [\tilde{r}(s)] \) \( \frac{\sigma^2 [1 \& e^{\delta a(s \& t)}]}{2a} \), for all \( s > t \).

\(^{12}\) A continuously resettled contingent claim is completely described by its maturity date, dividend rate and underlying process. A futures on a bond is a continuously resettled claim, with the bond price process as the underlying process, and a zero dividend rate. A continuously resettled claim is continuously marked-to-market.
Thus, equation (6) is the general solution to equation (5). $U(r,t,T_F)$ in (5) and (6) more specifically and exclusively designates the futures contract, and $P(r,t,T_B)$ in (7) below more specifically and exclusively designates the bond price as a solution to (4). The following closed-form solution to the bond futures price at any time $t$ can be obtained from (6):

$$U(r,t,T_F,T_B) = \frac{P(r,t,T_B)}{P(r,t,T_F)} \exp \left\{ \frac{\sigma^2}{2} \frac{1}{2a} \left[ \frac{1}{1 + \frac{a \delta T}{2}} \right]^{2} \right\}$$

(7)

Equation (7) is a specific solution for the assumed Vasicek process.\textsuperscript{13} In (7), the ratio $P(r,t,T_B)$ to $P(r,t,T_F)$ is simply the forward price at time $T_F$ of the bond maturing at $T_B$. Therefore, the futures price of the discount bond is equal to its forward price multiplied by an exponential function which

\small
\begin{align*}
\text{To obtain this solution, first recall that if } r \text{ and } P \text{ are two random variables such that } r = \ln(P) \text{, and if } r \text{ is normal with mean } \mu \text{ and variance } \sigma^2 \text{, then } \mu = \exp(\mu - \frac{1}{2} \sigma^2) \text{ and } P \text{ is lognormal. Note that } P([\bar{r}(T_F), T_F, T_B]) \text{ is the price at time } T_F \text{ of a bond with maturity date } T_B. \text{ Such a price follows a lognormal distribution when it is considered at time } t, \text{ with } r = \ln(P) \text{, a locally normal variable with } \mu_r = E[\ln(P)] \text{ and } \sigma^2 = \text{var}[\ln(P)]. \text{ Furthermore:}

\text{var}_r \left\{ \ln P \left[ \bar{r}(T_F), T_F, T_B \right] \right\} = \sigma^2 \frac{1}{2a} \left[ \frac{1}{1 + \frac{a \delta T}{2}} \right]^{2}

\text{Under these conditions, we have:}

E_{r,t} \left\{ P \left[ \bar{r}(T_F), T_F, T_B \right] \right\} = \exp \left\{ E_{r,t} \left\{ \ln \left[ P \left[ \bar{r}(T_F), T_F, T_B \right] \right] \right\} + \frac{\sigma^2}{2} \frac{1}{2a} \left[ \frac{1}{1 + \frac{a \delta T}{2}} \right]^{2} \right\}

\text{Consequently, (7) is obtained.}
\end{align*}

\textsuperscript{13}
is the outcome of the continuous marking-to-market. Using the definition of $Y(.,.)$ and of the covariance operator, the argument of this exponential function can be decomposed into the product of the conditional covariance of $Y(T_F,T_B)$ and $\tilde{r}(T_B)$, with the conditional variance of $\tilde{r}(T_F)$.

Therefore, the futures price at time $t$ becomes:

$$U(r,t,T_F,T_B) \cdot \frac{P[r,t,T_B]}{P[r,t,T_F]}$$

$$\exp \left\{ \frac{1}{2} \text{cov}_{r,T_F} [\tilde{r}(T_B), \int_{T_F}^{T_B} \tilde{r}(v) \, dv] \text{var}_{r,t}[\tilde{r}(T_F)] \right\}$$

This expression for the futures price is used in the next section to derive a pricing formula for a discount bond futures under discrete-time marking-to-market.

3. Futures prices under discrete marking-to-market

In this section, a closed-form continuous-time pricing equation is derived for a futures contract maturing at time $T_F$, which is written on a discount bond that matures at time $T_B$ and is discretely marked-to-market at an equi-spaced sampling interval $L$. Consider equations (7) and (8). From the previous section, it is known that the only part of the discount bond futures price equation which results from the continuous marking-to-market is:

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14 Chen (1992) derives a similar result. It is easily shown that our equation (7) is equivalent to Chen's equation (6). While Chen solves a differential equation, the general approach of Duffie and Stanton (1992) is used herein.
\[
\exp\left\{ \frac{1}{\sigma^2} \text{cov}_{r,T_F} \left[ \tilde{r}(T_B^f), \tilde{m}^{T_B^f}_{T_F^f} \var_{r,T}\{\tilde{r}(T_F^f)\} \right] \right\}
\]

The discrete marking-to-market equivalent of the above expression for an equi-spaced sampling interval of length \(L\) is:

\[
\exp\left\{ \frac{\sigma^2}{2} \left[ \sum_{i=1}^{n_1} \frac{e^{\delta \alpha (n_1 \% n_2 \% L)}}{e^{\delta \alpha L}} \right] \right\}
\]

where \(n_1\) and \(n_2\) are the number of equi-spaced sampling intervals of length \(L\) between \(t\) and \(T_F^f\), and \(T_F^f\) and \(T_F^B\), respectively (see proof in the appendix). Therefore, \(U^{D}\), which is the discount bond futures price at time \(t\) under discrete marking-to-market, is given by:

\[
U^{D}(r,t,T_F^f,T_B^f) = \exp\left\{ \frac{\sigma^2}{2} \left[ \sum_{i=1}^{n_1} \frac{e^{\delta \alpha (n_1 \% n_2 \% L)}}{e^{\delta \alpha L}} \right] \right\}
\]

This equation is more tractable and compact than the alternative expression developed by Chen (1992) for the same interest rate process by applying a backward dynamic programming methodology. The analytical differences between the outcomes for discrete and continuous marking-to-market are shown clearly by equation (9), unlike the case for Chen's expression. Furthermore, the representation of the futures price under discrete marking-to-market in equation (9) is similar to that of Flesaker (1993) for the interest rate model developed by Heath, Jarrow and Morton (1992).\(^{15}\)

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\(^{15}\) Flesaker (1993) shows, as we do in equation (7), that a discount bond futures price under discrete marking-to-market is equal to the product of the forward price of the bond and an exponential term.
4. Effects of margin variations on bond futures price volatility

The discrete sampling interval of length L defined in the previous section can be used to study the reaction of bond futures prices to margin variations. This is possible because an increase in L, all other things being equal, is an implicit decrease of the margin level when the purpose of the margin is to ensure contract performance. Although bankruptcy costs are not explicitly assumed in our model, they are implicitly considered in the formulation of the problem which links margin requirements to contract performance. The effect of bankruptcy costs on futures prices and volatility can be predicted by using the relationship between the margin level and the leverage of the futures position. High bankruptcy costs reduce leverage and should increase the margin level, all other things being equal. Therefore, the prices of bond futures and their volatilities should be reduced in the presence of bankruptcy costs, all other things being equal.

Equation (9) from the previous section can be rewritten as:

\[
U^D(r,t,T_F,T_B) = \left[ \frac{P[r,t,T_B]}{P[r,t,T_F]} \right] \exp \left\{ \sigma^2 \left[ \frac{\delta a(T_F,\delta t) \sigma e}{\frac{\delta a(T_B,\delta t)}{\delta L}} \right] \right\} 
\]  

(10)

The partial derivative of the bond futures price in equation (10) with respect to the sampling interval L is:  

Because L varies simultaneously with both \( n_1 \) and \( n_2 \), the partial derivative of \( U^D \) with respect to L is obtained under the following chain rule form:

\[
\frac{M_1^D}{M_1}, \frac{M_2^D}{M_2}, \frac{\%}{\%}, \frac{M_1^D}{M_1}, \frac{M_2^D}{M_2}
\]

The same applies to the total differential of the volatility of \( U^D \) with respect to L which is examined next. Note that both the partial derivative of \( U^D \) with respect to L and the total differential of the volatility of \( U^D \) with respect to L are presented using the following equivalences:

\[ n_1 L' (T_F \& t) ; \quad n_2 L' (T_B \& T_F) ; \quad (n_1 \% n_2) L' (T_B \& t) \]
\[
\frac{M^U_D}{M_L} \cdot \left[ \hat{\delta}_b \cdot e^{\delta_b \delta a_L} \& e^{\delta_b} \cdot e^{\delta_b L} \right] \frac{\delta^2}{1 \& e^{\delta a_L}} \cdot U^D
\]

where \( \delta_b \) and \( \delta_f \) are measures of time, adjusted by the speed of reversion of the interest rate to its mean, \( a \). This condition is obtained using the equivalences presented in footnote 12.

From equation (11):

\[
\hat{\delta}_b \cdot e^{\delta_b \delta a_L} \& e^{\delta_b} \cdot e^{\delta_b L} > 0 \quad \text{or} \quad \frac{n_1 \cdot \% n_2}{n_1} e^{\delta a n_2 L \& 1} > 0
\]

The above condition can also be rewritten as:

\[
e^{\delta a n_2 L} > \frac{n_1}{n_1 \cdot \% n_2} \quad \text{or} \quad e^{\delta (\delta_b \& \delta_f \cdot \% a L)} > \frac{\hat{\delta}_b}{\hat{\delta}_b},
\]

which clearly shows that sign of \( \frac{M^U_D}{M_L} \) depends upon the speed of adjustment toward the mean, \( a \), the settlement interval, \( L \), and the maturities of the bond and of the futures, \( n_1 \) and \( n_2 \), respectively. Since \( n_1 \) appears only on the right-hand-side of the third of the four inequalities depicting the same condition for positiveness of \( \frac{M^U_D}{M_L} \), then for fixed values of \( a, n_2 \) and \( L \), arbitrary values of \( n_1 \) can be chosen so as to ensure whether or not \( \frac{M^U_D}{M_L} \) is positive. Hence, over the course of the time to maturity of the futures, the effect of \( L \) on the bond futures price may be constant or may reverse itself. Moreover, the effect of the settlement interval \( L \) on the bond futures price \( U(r,t,T,F) \), although predictable from the above inequality condition, is not known a priori. Furthermore, since the inequality is more likely to hold for smaller \( n_1 \), it can be said that \( \frac{M^U_D}{M_L} \)

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17 Note that \( \delta_b \) and \( \delta_f \) are measures of time, which are adjusted by the speed of reversion of the interest rate to its mean, \( a \).

18 This condition is obtained using the equivalences presented in footnote 12.
is positive for futures contracts with a shorter time to maturity, everything else being equal. The reverse also is true. If the above condition holds, a positive sign of the derivative of the bond futures price with respect to the settlement interval \( L \) means that an increase (decrease) in the settlement interval, which is interpreted as a decrease (increase) in the margin level, leads to an increase (decrease) of the bond futures price. In other words, futures contracts with a shorter time to maturity are more likely to have decreasing prices with increasing margin levels. Futures contracts with a longer time to maturity are more likely to have increasing prices with increasing margin levels, and they are also likely to have such effects reversed as they approached maturity.

The volatility of the bond futures price under discrete settlement, \( \delta_U^D \), is:

\[
\delta_U^D \equiv \frac{\partial}{\partial a} \left\{ \left[ 1 + \epsilon_{a(T_F \Delta t)} \right] \left[ 1 + \epsilon_{a(T_B \Delta t)} \right] \right\} U^D(r, t, T_F, T_B) \tag{12}
\]

The total differential of \( \delta_U^D \) with respect to the settlement interval \( L \) is:

\[
\frac{d\delta_U^D}{dL} = \left\{ n \left[ \left( e^{\delta_L} \epsilon_{e^{\delta_L}} \right)^2 \epsilon_{\delta_L} \epsilon_{\delta_L} \right] \right\} \left[ \left( e^{\delta_L} \epsilon_{e^{\delta_L}} \right)^2 \epsilon_{\delta_L} \epsilon_{\delta_L} \right] \left[ \left( e^{2\delta_L} \epsilon_{e^{2\delta_L}} \epsilon_{\delta_L} \epsilon_{\delta_L} \right) \right]
\]

\[
\frac{\delta}{1 + \epsilon_{a \delta L}} U^D \tag{13}
\]

Since \( 1 + \epsilon_{a \delta L} > 0 \) and \( T_B > T_F \), we have \( \delta_B > \delta_F \). Thus, the sign of this total differential is always positive. This means that an increase (decrease) in \( L \) implies a decrease (increase) in the margin level, and leads to an increase (decrease) in the instantaneous volatility of the bond futures price. The sign of the partial derivative of the bond futures price with respect to the settlement
interval L is determined by:

$$\left( \frac{n_1 \% n_2}{n_1} e^{\frac{\delta \sigma L}{\delta \sigma L}} \& 1 \right) \frac{\sigma^2}{1 \& e^{\frac{\delta \sigma L}{\delta \sigma L}}} ,$$

where $R_p, p = 0, 1, ..., $ is a first-order linear normal autoregressive process resulting from sampling at uniform interval $L$ of the short-term continuous-time interest rate process. Equation (14) implies that the time-covariance structure of the interest rate process determines the directional effect of margin variations on the bond futures price. For an extensive discussion of a related issue, see Cox, Ingersoll and Ross (1981). More generally, on inspection of the expressions of the bond futures price under discrete marking-to-market in (10) and of its volatility in (12), it becomes apparent that the time-covariance structure, $\sum_{k=1}^{n_2} \text{cov} (R_{n_2}, R_{n_2 \& k})$, is of prime importance in determining the relationship between $L$ and $U^D$ and $\sigma_{U^D}^2$.

To have a better sense of the comparative statics results obtained so far, a simulation is

---

19 To obtain this condition, the discretization results presented in the appendix are used.

20 In our model, the futures price under continuous settlement depends on

$$\text{cov}_{T_B, T_F} \left[ \bar{f}(T_B), \int_{T_B}^{T_F} \bar{f}(v) d\nu \right]$$

which is equivalent to

$$\sum_{k=1}^{n_2} \text{cov} (R_{n_2}, R_{n_2 \& k}) \left[ \frac{1 \& e^{\frac{\delta \sigma L (n_2 \% L)}}{1 \& e^{\frac{\delta \sigma L}{\delta \sigma L}}} \sigma^2 \right],$$

under discrete settlement. Both sums of covariances are positive, which implies that interest rates in each version of the model are negatively correlated to the terminal cash flow of the underlying discount bond so that futures prices decrease with the covariances. Increasing the sampling interval decreases the sum of the covariances, which results in an increase in the bond futures price.
carried out with three futures contracts: A 60-month futures contract on a 90-month short term bond, a one-year futures contract on a 15-year medium term bond, and a one-year futures contract on a 30-year long term bond. Also, three levels of $\sigma$, the instantaneous standard deviation of the interest rate process are used: .03, .06, and .09. Various settlement intervals are considered from the close to instantaneous (one minute) to the life of the futures contract (either 60 days or one year). The long term instantaneous mean of the interest rate process, $\mu$, is set at .06, the risk neutral rate, $r_0$, is equal to .07, and the mean reversion speed, $a$, is set at .05. Other values for the couple $(\mu, r_0)$ (namely, (.03,.04), (.12,.13), and (.03, .15)) are used with no change in the direction of the simulation results.

The results are presented in Table 2, and are consistent with the analytical results obtained from the comparative statics. The volatility of the price of bond futures increases with the settlement interval for all three futures contracts and for all three levels of volatility. This means that an increase (decrease) in the (proxied) margin level leads to a decrease (increase) in the instantaneous volatility. However, the increase in volatility levels off quickly with the increase in the settlement interval, especially for the futures on the short bond, and for the smaller volatility level for all three futures contracts.

While the effect of the settlement interval on the futures price theoretically depends on the mean reversion speed, and the times to maturity of both the futures and the underlying bond, and thus is predictable but unknown a priori, the simulation results uniformly show that the bond futures price increases with the settlement interval, and decreases with the volatility of the interest rate process itself. Therefore, the (potential) reversal of the price trend during the life of the futures contract which has been found through comparative statics is not observed in our simulation results. Furthermore, the bond futures price decreases from that on the short bond to that on the medium

---

17 Our equations (10) and (12) expectedly give indefinite and meaningless results with values of $L$ are close to zero.
bond to that on the long bond. This is mostly due to the time-to-maturity of these essentially
discount bonds, which greatly affects their prices and therefore the prices of the futures written on
them. Also, the volatility relative to the futures increases from the futures on the short bond to that
on the medium bond to that on the long bond. Moreover, for the futures on the long bond and for
the highest volatility level only, the volatilities are higher in absolute terms than the futures prices
themselves. One can surmise that these results are due to the higher “compounded” uncertainty
associated with the longer term bonds.

When the settlement interval is small (e.g. one minute for the futures on the short bond, up
to one hour for the futures on the medium bond, and up to one day for the futures on the long
bond), the futures price is next to zero. The explanation might be found in the leverage effect of the
margin, as proxied here by the settlement interval. The margin level determines the leverage of a
trader's position in futures contracts. This leverage has an effect on futures prices because the value
of a futures contract depends on financing conventions in the presence of interest rate risk. When
leverage is non existent (close to continuous marked-to-market, or very short settlement interval,
or 100% margin requirements), the futures value is close to zero.

Finally, it should be observed that while continuous marking-to-market simulation
(represented by L=0 in the simulations) produces indefinite (zero) values for both futures prices and
its volatility, marking-to-market only at the maturity of the futures contract (L=60 days or 1 year
depending upon the futures contract) yields finite values for both the futures price and its volatility
which are not much higher than those under marking-to-market at shorter time intervals. These
values are useful benchmarks on which alternative marking-to-market and margin requirement
models may be evaluated. They are alternatives to the actual daily marking-to-market and margin
requirements of around 2% of a futures contract value.

The reported simulation results closely mimic the observable values in real markets. They
help us to better understand the effects of the settlement interval, or proxied margin requirements, on the bond futures price and its volatility.

5. Generalization to multidimensional Itô processes

To this point, our results show that the effects of increasing margin on bond futures price and its volatility depend upon the time-covariance structure of the interest rate process when the Vasicek framework is used. In this section, we extend these results to the more general case of the multivariate Itô processes.

Duffie and Stanton (1992), following Cox, Ingersoll and Ross (1981), show that:

\[
F_t \cdot E \left[ \delta(\hat{X}_T, T) \right]
\]

\[
L_t \cdot \frac{E_t \left\{ \delta(\hat{X}_T, T) \exp \left[ \int_t^T \mu(\hat{X}_s, \sigma) \, ds \right] \right\}}{E_t \left\{ \exp \left[ \int_t^T \mu(\hat{X}_s, \sigma) \, ds \right] \right\}}
\]

(15)

where \( \delta(X_t, \tau) \) is the underlying process for the continuously marked-to-market asset;

\( X_t \) is a k-dimensional state vector satisfying the stochastic differential equation: \( dX_t \cdot \dot{i}(X_t, t) \, dt + \dot{\epsilon}(X_t, t) \, dB_t \), where B is a k-dimensional standard Brownian motion, and \( \dot{i} \) and \( \dot{\epsilon} \) are known functions;

\( \{\hat{X}_s : 0 \leq s \leq T\} \) is a no arbitrage process defined by \( \hat{X}_t \cdot X_t \) and \( d\hat{X}_s \cdot \mu(\hat{X}_s, \sigma) \, d\sigma + \%_\epsilon(\hat{X}_s, \sigma) \, dB_s \).
\[ r(\hat{X}_t, \hat{\delta}) \text{ is the instantaneous interest rate at time } \hat{\delta}; \]

\[ F_t \text{ is the futures price at any time } t \text{ for delivery of the asset at time } T; \]

\[ L_t \text{ is the forward price at any time } t \text{ for delivery of the asset at time } T; \]

\[ E_t \text{ is the conditional expectation operator.} \]

From the definition of the covariance between two random variables, and using (15), one has:

\[
\text{cov}
\left( \hat{X}_t, T \right)
, \exp
\left[ \int_{t}^{T} r(\hat{X}_t, \hat{\delta}) \, d\hat{\delta} \right]
\]

\[
F_t \quad \& \quad L_t
\]

\[
E_t
\left\{ \exp
\left[ \int_{t}^{T} r(\hat{X}_t, \hat{\delta}) \, d\hat{\delta} \right] \right\}
\]

Based on (16), the futures price is the difference between the forward price and an expression which represents the continuous marking-to-market involved in futures pricing. This expression is the covariance between the underlying asset price and the discount factor, adjusted by the expected value of this discount factor. Therefore, when moving from continuous marking-to-market to discrete marking-to-market, only that expression must be taken into consideration to obtain:
where \( r_{t,i \% L} (\hat{X}_{t\% L}, t \% L) \) is the interest rate sampled at the end of interval \( i \) after date \( t \), and \( T-t = n \ L \). From (17), regardless of whether marking-to-market is continuous or discrete, the bond futures price always involves the time-covariance structure of the interest rate process, and this covariance structure affects both the bond futures price and its volatility. While the relationship between volatility and the length of the settlement period (the proxy for the margin level) can be deduced from the time-covariance structure, this structure is itself dependent on the real underlying interest rate process, and on the maturities of both the bond and the futures. A priori, one can not determine the direction of the relationship between margin level and volatility. This result can be generalized to any type of process, including jump processes which are often used to model interest rate processes.

6. Concluding Remarks

The direction and the magnitude of the reaction of prices and volatility to margin changes depend on the time-correlation structure of the underlying interest rate process and the margin level, where the latter is implicitly modeled by the length of the settlement period. This result confirms those of Hartzmark (1986) and Kupiec and Sharpe (1991) for the relationship between margins
and volatility under different market settings. While the direction of the relationship changes because of changes in market structures induced by changes in margin requirements in these studies, our research shows that the ambiguous relationship between margins and volatility is determined by the nature of the autoregressive structure of interest rates for the general case of multidimensional Itô processes. Our findings explain why the debate on the relationship between margins and volatility has not been resolved previously.

For the special case of the Vasicek process, an increase in the margin level leads to a decrease in volatility. This result explains the empirical finding of Hardouvelis that an economically and statistically significant relationship exists between margins and volatility. The effect of increasing the margin level on bond futures prices is predictable, but depends on the parameters of the interest rate process and on the maturities of the bond and the bond futures, and can not be known a priori. Furthermore, this effect may reverse itself during the life of the futures contract. For futures contracts with a shorter time to maturity, increasing margin levels may imply decreasing prices. For futures contracts with a longer life to maturity, increasing margin levels may lead to increasing prices. The simulation results based on our model show that both futures prices and volatility increase with the settlement interval, or decrease with the proxied margin requirements. The potential reversion of the trend in futures price is not observed. Furthermore, the leverage effect of margin requirements is very strong close to the continuous marking-to-market (or 100% margin requirements). Finally, we show that the futures price on the longer term bonds has higher volatility, while futures price decrease with volatility of the interest rate process.

Our initial results lay the foundation for more sophisticated investigations which account for a differential cost structure due to informational asymmetry. Our results also illustrate the importance of settlement conventions for asset pricing.
Table 1

Probability $Q$ of first hitting at time $t$ a lower barrier of zero for an Ornstein-Uhlenbeck process

$Q$ is computed from the following equation adapted from Leblanc, Renault and Scaillet (2000):

$$
Q = \frac{r_0}{\sqrt{2\delta}} \exp \left( \frac{a}{2} \left[ (\mu \& & r_0)^2 \& & \mu^2 \& & \delta \& & r_0^2 \& & \coth (a^1t) \right] \right) \left[ \frac{a}{\sinh (a^1t)} \right]^{3/2}
$$

where $r_0$ is the starting value; $a$ is the mean reversion speed; $\mu$ is the instantaneous long term mean; $\sigma$ is the instantaneous standard deviation, and $\delta = \sigma^2$. To obtain $Q$ from Leblanc, Renault and Scaillet (2000), write $r = -R_0$, where $R_0$ is the process studied by those authors. This implies $r_0 = -R_0$, and $\delta(r) = -\delta(R_0)$.

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<th>$\sigma$</th>
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<th>$t=1/(24\times365)$</th>
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Table 2
Simulated Prices and Price Volatilities for Bond Futures

$L$ is the settlement interval, $T_F$ is the time-to-maturity of the futures, $T_B$ is the time-to-maturity of the bond, $s$ is the instantaneous standard deviation of the spot rate, $U^D$ is futures price, and $s_U^D$ is its standard deviation. The mean reversion speed $\alpha$ is set at .05, the risk neutral rate $r$ is set at .07, and the instantaneous mean rate $r_0$ is set at .06. All values are annualized. Each panel provides prices and price volatilities for bond futures for various settlement lengths $L$ for fixed values of $T_F$ and $T_B$. The price and price volatilities for bond futures, respectively, are given by:

$$U^D(r,t,T_F,T_B) = \exp \left\{ \frac{\alpha e^{\sigma a(T_F,\delta t)} - e^{\sigma a(T_B,\delta t)}}{1 + e^{\sigma a L}} \right\}$$

$$\delta U^D \approx \frac{\delta}{\delta a} \left\{ e^{\sigma a(T_F,\delta t)} \right\} \left[ 1 + e^{\sigma a(T_B,\delta t)} \right] U^D(r,t,T_F,T_B)$$

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Simulated Prices and Price Volatilities of Bond Futures

### Panel B

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Appendix

Derivation of the discount bond futures price expression
under discrete marking-to-market

Consider the argument of the exponential function on the right-hand-side of equation (8). This expression is obtained from the mean-reverting Gaussian process of the short-term interest rate (1). When the continuous-time interest rate process is at a uniform sampling interval (say, \(L\)), the resulting discrete-time process, \(R_p, p = 0, 1, 2, \ldots\), follows a first-order linear normal autoregressive process of the form [Phadke and Wu (1974) and Vasicek (1977)]:

\[
S \quad R_p = R_0 (1 \& \alpha \& L) \% \alpha \& L \cdot R_{p-1} \% Q
\]

Suppose that \(n\) equi-spaced samplings exist between \(t\) and \(T_e\), so that the contract is marked-to-market \(n\) times. The discrete time equivalent of

\[
\text{var}_{r,t} \left( \tilde{r}(T_F) \right)
\]

is:

\[
\delta^2 \cdot \alpha \cdot n \cdot L
\]

Since \(n\) equi-spaced samplings between \(t\) and \(T_e\) implies \(n\) equi-spaced samplings between \(T_F\) and \(T_B\),

\[
\text{cov}_{r,T_p} \left[ \tilde{r}(T_B), \gamma \tilde{r}(v) dv \right]_{T_F}^{T_B}
\]

becomes:

\[
\sum_{k=1}^{n} \text{cov} \left( R_{n_1}, R_{n_2} \& k \right) \cdot \left( 1 \& \alpha \& L \right) \% \alpha \& L \cdot \delta^2
\]

[Nelson (1972), pp. 11 and 12]. By replacing the variance and covariance expressions with their discrete time equivalents, the expression for bond futures prices under discrete marking-to-market is obtained.
References


