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FUTURES MARKET EQUILIBRIUM WITH HETEROGENEITY AND A SPOT MARKET AT HARVEST

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This paper studies equilibrium in the futures market for a commodity in a single good economy, which is populated by heterogeneous producers and speculators. The commodity is traded only in the spot market at harvest whereas futures contracts written on the commodity are traded continuously. The model illustrates the role of heterogeneity and non-tradeness in a futures market equilibrium. The results show that the futures price is driven by aggregate wealth, rather than the spot price as in other models and that the futures price process is a simple one which depends on the relative risk process.

Keywords: Futures price dynamics, volatility of futures prices, demand for futures contracts, heterogeneity of producers, non-tradeness of commodity.

JEL Classification Codes: C61, G13.

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1. Introduction

This paper studies hedging and (partial) equilibrium in the futures market for a commodity which is populated by heterogeneous producers and speculators. The commodity is traded in a spot market only at harvest time. Each producer is endowed with a non-traded private technology and trades in futures contracts in order to reduce her quantity and price risks. Speculators invest their initial wealth in bonds and take positions in futures contracts written on the commodity.

The model is motivated by the observation that a spot market is open only at harvest time for some commodities.[1] In such a setting, the producer faces both a price risk and a quantity risk. However, most of the optimal hedging literature deals only with price risk. Hirshleifer (1990, 1991) discusses the effects of both types of risk on the hedging decision but does not model them simultaneously. A simple way to represent the quantity and price risks of each producer is to model them as a private cash flow risk, where the cash position of a hedger is the present value of her terminal cash flows. The difficulty in solving the equilibrium under these conditions is that both the futures price and the cash position of a

[1] This observation is especially true for seasonal and not storable commodities such as fresh strawberries. Since these characteristics imply a discontinuous cash market, which is open only at harvest time, a food processing firm or a restaurant operator using fresh strawberries as input has a strong incentive to trade in strawberry futures contracts during the interim period in order to hedge against price and quantity risks. Likewise, a farmer producing strawberries might want to hedge against price and quantity risks of her crop.

hedger are endogenous. In more traditional models, the cash position, which is the fixed quantity of the commodity held by the producer, is independent of the futures price.[2]

The optimal demand for a futures contract depends on tastes, on the resolution of uncertainty, and on the formulation of the hedging problem. Early contributions in the optimal hedging literature usually assumed that hedgers were monoperiodic expected utility maximizers (see for instance Stein (1961), Johnson (1960), Anderson and Danthine (1980) and Losq (1982)). A common result from most of these papers is that the optimal hedge ratio consists of a pure hedge component, and a mean variance component.

More recent models are derived in a continuous-time framework, whereby the hedger maximizes the expected utility of intertemporal consumption subject to a wealth-budget or cash-budget dynamic constraint. The cash-budget formulation has been used by Ho (1984) in a model in which a farmer, subject to both output and price risk during the production period, hedges a non-traded position. The optimal demand for futures contracts depends on the exogenous cash position (the output), and includes both a mean-variance efficiency component and a Merton-Breeden dynamic hedging component. The wealth-budget formulation, initiated by Stulz (1984), is used by Adler and Detemple (1988) to solve a problem similar to that of Ho. The resulting demand for futures contracts depends on the output, but also includes a mean-variance efficiency term, a dynamic hedging component, and a minimum-variance component. For myopic investors, both formulations yield tractable optimal demands, similar to those obtained in the single period models. Therefore, investors are assumed to have logarithmic utility functions in the model developed herein.

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[2] See, for example, Anderson and Danthine (1983).

The demand for futures contracts depends on the uncertainty structure of the spot and the futures market. A producer that hedges against price risk to protect a non-traded position has a demand dependent on the output. Without trade constraints, and in the absence of frictions, the demand for futures contracts is independent of the output [see Briys, Crouhy and Schlesinger (1990)]. In the model developed herein, a producer is concerned with both quantity and price risks, and thus hedges against the risk of individual cash flows.

Although equilibrium futures price and portfolio policies are determined simultaneously, the determination herein proceeds in two stages for mathematical convenience. First, the demand of each agent is derived assuming that the parameters of the futures price are known. Then, individual demands are aggregated to obtain the equilibrium futures price at any point in time.

The optimal demand for futures contracts by each investor is derived by assuming temporarily that the parameters of the futures price are known, and by applying the equivalent martingale method developed by Karatzas, Lehoczky and Shreve (1987). Diversity in cash flow risk leads to different hedging strategies. The demand for futures contracts by a producer depends on her wealth, the present value of her terminal production, and the relative risk process of the futures contract. This structure of demand is motivated by a desire to diversify, and to insure the actual value of the terminal production. Producers are long or short in the futures contract if there is heterogeneity in cash flow risk. A similar result is obtained by Hirshleifer (1991) in a discrete-time model where heterogeneity in the resolution of uncertainty leads to different hedging strategies. In the model developed herein, the differential in cash flow risk can also be interpreted as a differential in the resolution of uncertainty.

Once the optimal demand for futures contracts is derived, the aggregation follows easily. A continuous-time equilibrium exists in the futures market when producers account for both quantity and price risks by hedging against their individual and terminal cash flow risks. Stochastic cash positions and equilibrium futures price are endogenous, unlike previous equilibrium models which ignore quantity risk. The futures price depends only on the volatility of the aggregate wealth. The volatility of the futures price is stochastic and increases, on average, as the time to maturity decreases. This maturity effect in the model is consistent with Samuelson (1965). The stochastic volatility obtained in the model is also consistent with the recent empirical evidence on the behaviour of daily futures prices.[3] The positive mean of the instantaneous growth rate of futures prices indicates normal backwardation in the model. The relative risk process of futures prices is a weighted sum of the present values of all the terminal productions of hedgers.

Aggregation is made easier by assuming a logarithmic utility function. The relative risk process of futures prices in the model of heterogeneous agents developed herein has a weighted-average structure. Detemple and Murty (1994), who appear to be the first to deal with heterogeneity in an intertemporal production economy, obtain a similar result for the equilibrium interest rate.[4]

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[3] See, for example, Milonas (1986) and Anderson (1985).

[4] In the model of Detemple and Murty (1994), heterogeneity is implied by different beliefs about the rates of return of the production technologies. In our paper, heterogeneity proceeds from the parameters of the production processes which differ for different producers.

Our results show that the futures price is driven by aggregate wealth, rather than the spot price as in other models and that the futures price process is a simple one which depends on the relative risk process itself. Contrary to the contango hypothesis (a futures market dominated by risk averse commodity producers) or the normal backwardation (a futures market dominated by commodity users), our model shows that producers could be long or short on the futures. Another interesting result which is consistent with that of Anderson and Danthine (1983) and the Samuelson hypothesis, and the empirical observations of Milonas (1986), is that volatility in futures prices increases as time to maturity shortens. Finally, our results are obtained while allowing for the simultaneous resolution of both price and quantity risks.

The remainder of the paper is organized as follows. In section 2, the economy is described. In section 3, optimal demands for futures contracts by each type of investor are derived. In section 4, the equilibrium futures price is derived; and in section 5, some concluding remarks are offered.

2. The Economy

Consider a continuous-time production economy with a Brownian uncertainty structure that is populated with two classes of agents. The first class consists of producers indexed by $i = 1, 2, ..., n$, and the second class consists of speculators indexed by $k=1,2...,m$. Consider a complete probability space $(\Omega, \Im, P)$ and a finite time horizon $[0,T]$. On $(\Omega, \Im, P)$, define a Brownian motion z with values in $\Re$. Also assume the following:

H1. All producers grow the same commodity, and each producer is endowed with their own specific production technology.[5] The value of production in process S satisfies:

$$dS_t^i = S_t^i(m^i dt + s^i dz_t) ,$$

where the constant parameter μ^i represents the expected instantaneous change in the rate of production growth, and σ^i is the constant local variance of the process. *S* is the production at harvest time, expressed as a cash flow in units of the numeraire.[6] The only source of risk, z_t, which denotes a standard Wiener process, equally affects all production processes.[7] The production in process S_T^i is not tradeable, and $S_t^i \rangle o \forall i$.

H2. Producers and speculators have free and unlimited access to a financial market where a futures contract written on the specified commodity is traded. The settlement price of this contract, F_t, satisfies:

$$dF_t = a\left(F_t, t\right)dt + d\left(F_t, t\right)dz_t \ .$$

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[5] The aggregate production at maturity time *T* is homogenous. However, the differences in the production technology lead to differences in the individual quantities of production at maturity time *T*.

[6] The numeraire obviously is not the commodity which exists only at harvest time. As the model is a partial equilibrium model, such a differentiation does not pose any problems.

[7] The unique source of technology could be, for example, meteorological conditions which condition the strawberry crop.

The production growth rate and the futures contract have the same uncertainty structure, and α and δ are determined in equilibrium later [see equation (4.3)].[8]

H3. An instantaneously riskless bond is available and is held by both types of agents. The roll-over value of an investment I in the riskless rate solves:

$$dI_t = I_t \, r_t \, dt , \qquad I_o = 1 ,$$

where r_t is deterministic and non-stationary.[9]

H4. A producer i chooses a feasible trading strategy $\{N_t^i\}$, that is, the number of futures contracts which can be bought or sold. This strategy is adapted to her information set so as to maximize the expected utility of her terminal consumption $C_T^i = S_T^i + X_T^i$. The terminal gain from trading in the futures contract X_T^i can be written as:

$$X_T^i = \int_o^T \exp\left[\int_s^T r_u \, du\right] N_s^i \, dF_s ,$$

where X_T^i is the terminal value of a margin account. The position N_s^i is credited (debited) with any gain (loss) caused by price changes. The margin account is also credited with interest at the continuously compounded

8 The production in process S is not tradeable before the harvest time T because a market for the immature commodity does not exist. However, commodity futures contracts are traded continuously before harvest time. This ensures the completeness of the market. The inexistence of a cash market for the production in process precludes both the cash and carry and the reverse cash and carry arbitrage for the pricing of the futures contract. More specifically, although short sales are feasible in the spot market, its price is not observable (it does not exist) before harvest time T. Therefore, any replication of the stock using the futures is not feasible.

9 r_t is exogeneous in our partial equilibrium model.

interest rate r_t. Borrowing is possible at the same interest rate if the investor incurs losses that cause the value of the account to become negative. Transaction costs are ignored.

H5. A producer i solves:

(2.1)
$$\underset{\{N_t^i\}}{\text{Max}} \; E\left[e^{-bT} Ln(C_T^i)\right]$$

(2.2)
$$s.t. \quad E_Q\left\{\exp[-\int_o^T r_u \, du]\, C_T^i\right\} \leq x^i + E_Q\left\{\exp[-\int_o^T r_u \, du]\, S_T^i\right\}$$

where x^i is the initial cash-position of producer i, and ß is a subjective discount rate which is identical across agents. E_Q is the expectation under the risk-neutral probability measure Q which is equivalent to P.[10] Given one source of risk in the economy and one traded asset (the futures), there always exists a unique martingale risk-neutral measure Q which is equivalent to P.[11] Equations (2.1) and (2.2) are the static equivalents of the maximization of terminal wealth under the wealth-budget dynamic constraint [see Karatzas, Lehoczky, and Shreve (1987)].

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[10] E_Q is the expectation operator associated with a particular probability measure Q on $(\Omega, \Im)$. This probability measure Q is equivalent to probability measure P, meaning that both measures have the same null set, i.e. $Q(A)=0$ if and only if $P(A)=0$.

[11] See Bensoussan (1984) or Harrison and Pliska (1981).

H6. A speculator trades in the futures contract. She chooses a feasible strategy $\{N_t^k\}$, that is, the number of futures constracts which can be bought or sold, so as to maximize his terminal wealth X_T^k, which satisfies:

$$X_T^k = \int_o^T \exp[\int_s^T r_u \, du] N_s^k \, dF_s$$

H7. The utility function of speculator k is logarithmic. Thus the speculator solves:

(2.3) $$\underset{\{N_t^k\}}{\text{Max}} \quad E[e^{-bT} Ln(W_T^k)]$$

(2.4) $$s.t. \quad E_Q \left\{ \exp[-\int_o^T r_u \, du] W_T^k \right\} \le x^k$$

where x^k is the initial wealth of speculator k, ß is the subjective discount rate used by producers, and W is the terminal wealth.[12] Equations (2.3) and (2.4) are also equivalent to the static problem of terminal wealth maximization under a dynamic wealth-budget constraint. E_Q is the expectation under a probability measure Q which is equivalent to probability measure P.

[12] The logarithmic utility function is used for both producers and speculators because it allows for aggregation of wealth in a heterogeneous setting. See Dothan (1978) for more details.

3. Optimal Demands for the Futures Contract

In this section, the optimization problems for producers and speculators are solved when a futures contract is available and the interest rate is exogenous. Before deriving the demands for investment, the present value of terminal production S_T^i is determined.

Lemma 1

Assume for the time being that a bounded equilibrium relative risk process $\left\{ q_t = \frac{a_t}{d_t} \right\}$ exists for the futures price at time t. The present value of the terminal production S_t^i is:

$$V_t^i = E_Q \left\{ \exp\left[-\int_t^T r_u \, du \right] S_T^i \,\middle|\, \Im_t \right\}$$

where E_Q is the expectation under the equivalent martingale measure Q.

Comment: The relative risk process $\{\alpha_t / \delta_t\}$ is a function of the parameters of the futures contract (i.e. the only traded asset). The interest rate, r_t, does not appear in the expression of the relative risk process defined in Lemma 1 because no need exists for cash (borrowing) to buy or sell a futures contract. Even if a margin must be posted to trade a futures contract, it could be done using Treasury bills, and that results in no cost for the trader of the futures contract. If the commodity was a traded asset, then cash (borrowing) would be needed and the relative risk process would be $\{(\alpha_t - r_t)/\delta_t\}$.

Proof: See the Appendix.

The futures contract is a continuously traded asset in an economy with only one source of uncertainty. Therefore, the market is complete, and the present value, V, of each terminal production, *S*, is unique. The volatility of *V* is of interest because of its relevance to the hedging decision of the producer. Since the exact form of the relative risk process is not yet known, the volatility of the present value of terminal production is:[13]

$$(3.1) \qquad s^{i}_{vt} V^{i}_{t} = s^{i} V^{i}_{t} - E_{Q}\left\{ \exp\left[-\int_{t}^{T} r_{u}\, du\right] S_{T} \int_{t}^{T} D_{t} q_{s}\, d\tilde{z}_{s} \,\middle|\, \Im_{t} \right\}$$

where D_t is the Malliavin derivative operator at time *t*, and $\tilde{z}_t = \int_o^t [q_u\, du + dz_u]$ is the Brownian motion under the equivalent martingale measure *Q*, and is thus translated from the Brownian motion z_t under probability measure P.[14] The second expression on the right hand side (RHS) of equation (3.1) is justified by the uncertainty about the future evolution of the relative risk process at time *t*. If the relative risk process is deterministic, the Malliavin derivatives are equal to zero.

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13 Note that V_t^i can be rewritten as:

$$h_t^{-1} E\left\{ \exp\left[-\int_t^T r_u\, du\right] h_T \ S_o \exp\left\{ \left[u - \frac{1}{2} (s^{i})^2\right] T + s^{i} z_T \right\} \middle|\, \Im_t \right\}$$

where:

$$h_t = \exp - \left[\int_o^t \frac{1}{2} q_u^2\, du + q_u\, dz_u \right]$$

For more details, please see the Appendix.

14 The Malliavin derivative $D_t\, \theta_s$ of the functional θ_s represents the effect on θ_s of a perturbation in the Brownian motion z_t at time t. Such a perturbation affects the whole path of z_t from t on, and of the functional θ_s accordingly. The Malliavin derivative arises in our model since the present value of the terminal production depends on the path of the futures relative risk function θ_s. For a detailed introduction to Malliavin calculus, see Ocone (1988). For previous financial applications of Malliavin calculus, see Ocone and Karatzas (1991) for the optimal portfolio problem, and Detemple and Zapatero (1991) for the asset pricing problem.

The allocation problem of the individual agent given that the relative risk process of futures price is known is now solved. The demands for the futures contract by producers and by speculators are given below in propositions 1 and 2, respectively.

Proposition 1

Consider the problem described by equations (2.1) and (2.2). The optimal demand for the futures contract by producer *i* at time *t* is:

$$N_t^i = \frac{W_t^i q_t}{d_t} - \frac{V_t^i s_{vt}^i}{d_t} \tag{3.2}$$

where $W_t^i = X_t^i + V_t^i$ (wealth) is the absolute risk tolerance of investor *i*'s utility function, and V is the present value of the cash flows generated by the position in the commodity at time T.[15]

Proof: See the Appendix.

The first and second terms on the RHS of equation (3.2) are the demand for mean variance efficiency and the hedge ratio, respectively. The present value of the non-traded position in the commodity is insured. The producer uses futures contracts to offset variations in the present value V_t^i of the cash flow S_T^i received from her non-traded position at maturity. The diversity in initial endowments and technologies implies that some producers might be short in the futures contract at any time *t*, whereas others are long in the futures contract. The net position of a producer in futures contracts is determined by the

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[15] Detemple and Murty (1994) obtain similar results in a model with heterogeneous beliefs about the rates of return of production technologies. In our paper, the source of heterogeneity arises from the parameters of the specific production processes.

difference between the volatility of her wealth and the volatility of the present value of her terminal production. Producers with highly (un)certain terminal cash flows smooth their terminal consumptions by buying (selling) futures contracts. The fact that some producers are long when others are short in the futures contract is a consequence of heterogeneity. Hirshleifer (1991) obtains a similar result for a discrete time model in which heterogeneity concerns the resolution of uncertainty for producers. In contrast, heterogeneity is part of the equilibrium in the present model, and the hedging position of the producer is determined by the relative volatility of her production process. In the Hirshleifer model, arrival of information is independent of the equilibrium futures price, and the assumption of additively separable preferences, complete markets, and a non-random endowment of the numeraire are required to obtain a martingale property for the futures price. This martingale property of the futures price is instrumental in deriving the demand for futures contracts. No restriction on the futures or the spot price of the commodity are needed to solve for the equilibrium in the next section of the paper.

Before doing so, the optimal demand for investment by speculators is derived. The difference between a speculator and a producer herein is only in the position in the non-traded technology.

Proposition 2

For the problem described by equations (2.3) and (2.4) the optimal demand for futures contracts by speculator k is:

(3.3) $$N_t^k \;=\; W_t^k \;\; \frac{q_t}{d_t} \;,$$

where W_t^k (wealth) is the absolute risk tolerance of the speculator's utility function. Demands for futures contracts by speculators depend on the mean and the variance of the instantaneous changes in the futures price.

Proof: See the Appendix.

4. Equilibrium with Heterogenous Agents

The equilibrium value, θ_t, of the relative risk process of the futures price at any time *t* is derived in this section. The market clearing solution is straightforward if producers hedge only against the price risk. In the model developed herein, producers hedge against fluctuations in the present values of their respective terminal cash flows, which depend on the equilibrium value of the relative risk process for futures price. This equilibrium value depends on the present values of the terminal cash flows of all producers. Thus, the present values of the terminal cash flows and the relative risk process are endogenous to the model developed herein.

Proposition 3

For the economy described in section 2, a unique equilibrium relative risk process of the futures price exists at each time which satisfies:

$$q_t = \sum_i b_t^i \, s_{vt}^i \tag{4.1}$$

where:

$$b_t^i = \frac{V_t^i}{W_{Gt}} \qquad\qquad W_{Gt} = \sum_{i=1}^{n} V_t^i + \sum_{i=1}^{n} X_t^i + \sum_{k=n+1}^{m+n} W_t^k \tag{4.2}$$

$$dW_t^i = [W_t^i \, (r_t + q_t^2) - V_t^i s_{vt}^i q_t^i] \, dt + W_t^i q_t \, d\tilde{z}_t, \quad i = 1, 2, \ldots, n,$$

$$dW_t^k = W_t^k (r_t + q_t^2) \, dt + W_t^k q_t \, d\tilde{z}_t, \; k = n+1, n+2, \ldots, \; n+m,$$

$$dW_{Gt} = W_{Gt} \, r_t \, dt + W_{Gt} q_t \, d\tilde{z}_t$$

$$db_t^i = b_t^i \, q_t \, (q_t - s_{vt}^i) \, dt + b_t^i \, (s_{vt}^i - q_t) \, d\tilde{z}_t, \quad i = 1, \, 2, \, \ldots, \, n.$$

Proof: See the Appendix.

b_t^i is the contribution of producer i to the aggregate wealth of the economy, W_{Gt}, expressed as the current value of his production V_t^i. θ_t is the relative risk process expressed in an alternative form which more directly relates to the volatilities of the production values, s_t^i. Equation (4.1) characterizes the relative risk process in equilibrium, and is obtained by observing that a zero net supply of the futures contract exists in equilibrium. The equilibrium value of the relative risk process depends on the ratios b^i's, and the volatilities s_{vt}^i which are also determined endogenously. This weighted-average characterization follows from a structural property of linear technologies with logarithmic utility [for another example, see Detemple and Murty (1994)].

Proposition 4

The equilibrium futures price process for the delivery of a unit of the commodity at time T is described by:

$$dF_t = q_t^2 \, dt + q_t \, dz_t \tag{4.3}$$

Proof: See the Appendix.

The futures price process is relatively simple since it is strictly dependent on the relative risk process. The drift term is equal to the square of the relative risk process. The futures price is an increasing (decreasing) function of time, if the relative risk is greater (smaller) than one. The volatility term of the futures price is the relative risk process itself, which implies that volatility depends on time. Equation (4.3) characterizes an arbitrage free futures price which depends only on the volatility structure of aggregate wealth. A similar result is obtained by Heath, Jarrow and Morton (1992) who constrain the drifts of forward prices to get arbitrage-free bond prices. Their constraint leads to a unique equivalent martingale probability measure, as is the case for our model. Proposition 4 simply states that the futures price is driven by the volatility of aggregate wealth, and not by spot prices as in other models. The premium paid by the short side to the long side of the contract diminishes with increasing aggregate risk tolerance (that is, increasing aggregate wealth), which is consistent with logarithmic utilities. The higher the aggregate wealth the less producers are willing to pay for insurance. This wealth effect on insurance confirms a well-known result due to Mossin (1968) under decreasing absolute risk aversion. Price changes per unit of time are stochastic, and have conditional variances that increase with time as the present values of terminal production increase. An alternative expression for the instantaneous volatility of the futures price change at time t is:

$$(4.4) \qquad \frac{\sum_i V_t^i \, s_{vt}^{\,i}}{W_{Gt}} = \frac{\sum_i V_t^i \, s^{\,i} - \sum_i E_Q \left\{ \exp\left[-\int_t^T r_u \, du\right] S_T^i \int_t^T D_t q_s \, d\tilde{z}_s \,\middle|\, \Im_t \right\}}{W_{Gt}}$$

The instantaneous conditional volatility at time t of the futures price change at time $t+j$, dF_{t+j}, for all $j > 0$, is denoted by:

$$(4.5) \qquad E_Q \left\{ \frac{\sum_i V_{t+j}^i \, s_{v_{t+j}}^{\,i} \,\middle|\, \Im_t}{W_{Gt}} \right\}$$

This conditional volatility increases with the time index j.[16] This behaviour is consistent with the Samuelson hypothesis, which states that changes per unit of time in the volatility of futures prices increase as the time to maturity decreases. The non-tradeness of production technologies also contributes to the increase of futures price volatility. To see this, observe that:

$$dF_t = q_t^2 \, dt + \left\{ \frac{\sum V_t^i \, s^{\,i}}{W_{Gt}} - \frac{E\left[\exp\left(-\int_t^T r_u \, du\right) S_T^i \int_t^T D_t \, q_s \, d\tilde{z}_s \,\middle|\, \Im_t \,\right]}{W_{Gt}} \right\} dz_t$$

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16 It can be noted that b is a P-martingale and that, consequently, the first term of the RHS of (4.4), $\frac{\sum_i V_t^i \, s^{\,i}}{W_{Gt}} = \sum_i b_t^i \, s_{\,i}$ has no drift term. On the other hand, $\int_t^T D_t \, q_s \, dz_{\bar{s}}$ decreases with j, and the second term on the RHS of (4.4) thus increases with j. The final result is an increase of the conditional volatility in (4.5) with j.

The first part of the volatility of dF_t is stationary, and the second part of the volatility tends to zero as t tends to T. Therefore, the volatility of dF_t is negatively correlated to time to maturity. This increasing volatility property is explained by the fact that a non-traded position held by a producer cannot be changed before time T. So, at any time t, the time remaining before the next trade in the production technology is equal to the time to maturity of the futures contract. As time to maturity decreases, so does the time until the next trade in the production technology. This increases the resolution rate of production uncertainty, and is consistent with results by Anderson and Danthine (1983), who establish an explicit link between the time pattern of the volatility of futures prices and the resolution of production uncertainty.

The futures price dynamics described by proposition 4 also appear to be consistent with the empirical evidence. Although the behaviour of futures prices is still an open question, several studies have uncovered a maturity effect in the volatility of futures price series. Anderson (1985) reports a maturity effect in the daily prices for commodities. Milonas (1986) concludes that a strong maturity effect exists in 10 out of 11 futures price series examined (including agricultural, financial and metal series).

5 Concluding Remarks

Hedging and equilibrium are studied herein for a setting in which: (1) heterogeneous producers hedge to reduce their respective terminal cash flow risks, which emanate from non-traded production technologies, by trading in futures contracts written on the commodity they grow, and (2) speculators invest their initial wealth in bonds and trade in futures contracts. The distinctive feature of the model is that cash flow (and not price) risk drives trades in futures contracts. Consequently, the futures price and the cash position are endogenous. Furthermore, equilibrium prices are derived when both quantity and price risks can be hedged simultaneously.

This theoretical framework yields some interesting insights. First, the net position in futures contracts depends on the relative volatility of the producers' cash flows. Producers with cash flow uncertainty, which are resolved less rapidly than the aggregate, will tend to go short in order to smooth their terminal consumption, or alternatively, will hold futures contracts. This result differs from that obtained in traditional models in which producers sell futures contracts to hedge their non-traded positions against spot price fluctuations. This result is consistent with the casual observation that the distinction between hedgers and speculators is ambiguous, and thus producers can be both on the long and the short sides.

Second, changes in the equilibrium futures price depend on wealth and the relative risk process. The futures price dynamics resulting from this model are consistent with empirical evidence (Milonas, 1986) for commodity futures price, where trading is absent. Specifically, the model predicts an inverse relation between volatility of futures price increments and maturity (the Samuelson hypothesis): Volatility increases as time to maturity shortens.

An interesting avenue for future research would be to study a similar model in an incomplete market setting. This might account better for the ARCH behaviour reported in studies of futures prices.[17]

[17] Yang and Brorsen (1993) report ARCH/GARCH behaviour of commodity futures prices.

Appendix

Proof of lemma 1

Assume that an equilibrium bounded relative risk process θ_t exists. The terminal production S obtained by a producer i from her non-traded technology can be attained by using bond and futures contracts. For the admissible strategy N_t, the following must be verified:

(A1) $$dV_t^i = N_t^i \, dF_t + V_t^i \, r_t \, dt \ ,$$

Equation (A1) means that the instantaneous return on a production process can be achieved using the strategy $\{N_t^i\}$ which involves trading the futures contract and the riskless bond. Integrating (A1), one obtains:

$$V_T^i = V_o^i + \int_o^t N_s^i \, a_s \, ds + N_s^i \, d_s \, dz_s + r_s \, V_s^i \, ds \ .$$

Or, recalling that $q_s = \dfrac{a_s}{d_s}$, and using the transformation of the Brownian motion under the P-martingale measure, z_t, into its associated equivalent Q-martingale measure, $\tilde{z}_t$, by way of the relation $\tilde{z}_t \equiv z_t + \int_o^t q_s \, ds$, one can write:

$$V_T^i = S_T^i = V_o^i + \int_o^T (N_s^i \, d_s \, d\tilde{z}_s + r_s \, V_s^i \, ds) \ .$$

Re-arranging terms, one obtains:

(A2) $$\exp[-\int_o^T r_u \, du \,] S_T^i = V_o^i + \int_o^T N_s^i \, d_s \, d\tilde{z}_s \ .$$

Taking expectations on both sides of (A2) under Q gives:

(A3) $$E_Q \left\{ \exp\left[-\int_o^T r_u \, du\right] S_T^i \right\} = V_o^i \ .$$

Therefore the time t value of the production process i is:

$$V_t^i = E_Q \left\{ \exp\left[-\int_o^t r_u \, du\right] S_T^i \,\middle|\, \Im_t \right\} \ .$$

Proof of proposition 1

Consider the optimal consumption of a producer i at time T, C_T^{i*}. Since the futures price has the same uncertainty structure as the non-traded technology, it can be used to obtain the terminal consumption of producer i. It is known that:

(A4) $$C_T^{i*} = X_T^i + S_T^i \ ,$$

where X represents the terminal payoff of the investment in futures contracts by the producer i. Define:

$$h_T = \exp\left[\int_o^T -q_u \, dz_u - \frac{1}{2} q_u^2 \, du\right]$$

and

$$x_T = e^{bT} \exp\left[-\int_o^T r_u \, du\right] h_T \ ,$$

where $q_t = \dfrac{u_t}{d_t}$ for t in [0,T] denotes the relative risk process of the traded technology correlated to the non-traded position of producer i. According to Karatzas, Lehoczky and

Shreve (1987), the optimal terminal consumption can be written as $C_T^{i*} = I(y^* x_T)$, where $I(.)$ is a strictly decreasing function and y^* is the Lagrange multiplier. From equation (A4),

(A5) $$X_t^i = E_Q \left\{ \exp\left[-\int_o^T r_u \, du \right] [I(y^* x_T) - S_T^i] \,|\, \Im_t] \right\} \exp[\int_o^t r_u \, du] \ ,$$

where E_Q is the expectation under the measure Q. From equation (A5),

(A6) $$X_t^i = E_Q \left\{ \exp[-\int_o^T r_u \, du] \, I(y^* x_T) | \, \Im_t \right\} \exp[\int_o^t r_u \, du \,] - E_Q \left\{ \exp[-\int_o^T r_u \, du] S_T^i \,|\, \Im_t \right\} \exp[\int_o^t r_u \, du]$$

The first part on the right-hand side (RHS) of (A6) is the present value of the optimal wealth W_t^{i*} at time t, which is also the optimal consumption. In this model, consumption occurs at the terminal date, and therefore $C_T = W_T$, and at any date t before T, $C_t = W_t$, and C_t is the present value of C_T. The second part of the RHS of equation (A6) is the present value, V_t^i , of the final production, S_t^i , at time t. This value is derived in lemma 1. The instantaneous volatility of V_t^i is $s_{vt}^i \; V_t^i$. To witness this, write:

$$V_t^i = E_Q \left\{ \exp[-\int_t^T r_u \, du] S_T^i \,|\, \Im_t \right\}$$

and transform this into:

$$V_t^i = h_t^{-1} M_t d_t^{-1}$$

where

$$h_t = \exp -\left[\int_o^t \frac{1}{2} q_u^2 \, du + q_u \, dz_u\right]$$

$$d_t = \exp\left[-\int_o^t r_u \, du\right]$$

$$M_t = E\left[e^{-\int_o^T r_u \, du} h_T \, S_o \, e^{\left[m - \frac{1}{2}(s^i)^2\right]T + s^i Z_T} \mid \Im_t\right]$$

The transformation $V_t^i = h_t^{-1} M_t \, d_t^{-1}$ allows us to do the derivation in the original probability space P. Since we want to obtain the stochastic part of V_t^i, we apply Itô's rule for a product of stochastic functions, while noting that d_t is deterministic. Doing such:

$$V_t^i = d_t^{-1}\left[-M_t h_t^{-2}(-h_t q_t)\, dz_t + h_t^{-1} g_t dz_t\right]$$

where $g_t = S_o \, h_t$ E { $e^{-\int_o r_u du}$

$$\exp[(m - ½(s^i)^2)T + s^i \tilde{z}_T - \int_o^T ½ q_u^2 \, du + q_u dz_u]$$

$$[-\int_t D_t q_u d\tilde{z}_u - q_t + s^i] \mid \Im_t \}$$

can be reduced to:

$$g_t = h_t E_Q \left\{ e^{-\int_o^T r_u\, du} S_T^i \left[-\int_t^T D_t q_u d\tilde{z}_u - q_t + s^i \right] | \Im_t \right\}$$

Hence, the volatility of V_t^i is:

$$= V_t^i s^i - E_Q \left\{ \exp \left[-\int_t^t S_T^i \left(\int_t^t D_t q_u\, d\tilde{z}_u \right) \right] | \Im_t \right\}$$

$$= V_t^i s_{vt}^i$$

To derive the volatility of W_t^{i*} for an investor with a logarithmic utility function, first note that:

$$C_t^{i*} = \frac{1}{h_t y^{i*} e^{-bT}} \exp \left[\int_o r_u\, du \right]$$

where y^{i*} is the Lagrange multiplier. Thus, the random part of dW_t^{i**} is:

$$-\left\{ y^{i*} \exp\left[-\int_o^t r_v\, dv \right] e^{-bT} \right\} h_t^{-2} (-h_t q_t) dz_t +$$

$$h_t^{-1} \left\{ \exp\left[-\int_o^t r_v\, dv \right] e^{-bT} \right\}^{-1} r_t^i\, dz_t = W_t^{i*} q_t\, dz_t \ ,$$

given that the volatility of y^{i*} is $r_t^i = 0$, since y^{i*} is the non-stochastic Lagrange multiplier. Therefore the instantaneous volatility of W^* is $W^* \theta_t$.

Given that:

(A7) $$X_t^i = \int_o^t N_s^i \, [a_s \; ds + d_s \; dz_s \;] \; ,$$

the optimal demand for futures contracts by producer *i* is obtained by equating the instantaneous volatility of the RHS of equation (A7) with the instantaneous volatility of the RHS of equation (A6) to get:

$$N_t^i \, d_t \;\; = \;\; W_t^i \, q_t \;\; - \;\; V_t^i \, s_{vt}^{\,i} \;\; .$$

Therefore, the optimal demand for the futures contract by producer *i* is:

(A8) $$N_t^{i*} = \frac{q_t}{d_t} W_t^i \; - \; \frac{V_t^i \; s_{vt}^{\,i}}{d_t} \; .$$

Proof of proposition 2

The demand for investment by a speculator is obtained by setting V to zero in equation (A8) and then solving.

Proof of proposition 3

The relative risk process of the futures price is obtained by setting the aggregate demand for futures contracts by producers and speculators to zero, because the futures contracts are in zero net supply. The existence and the uniqueness of this process is shown by first assuming that the conditions of proposition 3 are respected. Then, the aggregate production at time t can be written as a function of the relative risk process θ_t. Doing this yields:

(A9) $$A_t \, (q_t) = \sum_i V_t^i = \; E_Q \left\{ \exp\,[-\int_t^T r_s \; ds\,]\, A_T \,\middle|\, \Im_t \right\} \;\; ,$$

where:

(A10) $$A_T = \sum_i S_T^i \ .$$

Equation (A10) means simply that at the harvest (terminal) time T, the aggregate production process is equal to the marketable production. The instantaneous variation of aggregate production is:

(A11) $$\sum_i V_t^i r_t \, dt + \sum_i V_t^i s_{vt}^{\,i} \, d\tilde{z}_t \ .$$

Using the market clearing condition $\sum_{j=1}^{n+m} N_t^j = 0$, and propositions 1 and 2, we have $q_t = \sum_i b_t^i s_{vt}^{\,i}$ and setting $\sum_{i=1}^{n} X_t^i + \sum_{k=n+1}^{n+m} W_t^k = l_t$ the following process is obtained for $\tilde{A}_t$ (the value of A_t under equilibrium condition):

(A12) $$d\tilde{A}_t = \tilde{A}_t r_t dt + (\tilde{A}_t + l_t) q_t \, d\tilde{z}_t \ .$$

This is equivalent to:

(A13) $$d\tilde{A}_t = \left[\tilde{A}_t \left(r_t + q_t^{\,2}\right) + l_t q_t^{\,2}\right] dt + (\tilde{A}_t + l_t) q_t \, d\, z_t \ .$$

Remark: Since

$$dl_t = l_t r_t \ dt + \sum_{j=1}^{m+n} N_t^j \ (a_t \ dt + d_t \ dz_t) \ ,$$

$$dl_t = l_t r_t dt, \text{ with } \ l_o = \sum_j x^j \ .$$

l_t depends only on the interest rate and the initial cash position of agents, both of which are exogeneous. The system of equations (A9)-(A13) is consistent if and only if:

(A14) $$\tilde{A}_T = A_T \ .$$

Thus, at each point in time t, $\tilde{A}_t$ and θ_t should be such that $\tilde{A}_T = A_T$. The above problem can be viewed as a design problem in which $\tilde{A}_t$ and θ_t are chosen at each time t such that the terminal condition is met. This is equivalent to solving a backward stochastic differential equation of the type studied by Pardoux and Peng (1990), namely:

(A15) $$x(t) + \int_t^1 f[s, x(s), y(s)]ds + \int_t^1 g[s, x(s), y(s)]dz_s = X$$

where *x(t)* denotes an Itô process;

y(t) is a state variable;

f[t, x(t), y(t)] and *g[t, x(t), y(t)]* are some known functions;

z_s is a standard Wiener process; and

X is the terminal condition.

The conditions for the existence of a unique solution to equation (A15) are:

a) *f(., 0, 0)* and *g(., 0, 0)* belong to $M^2(0, 1, R)$, the set of integrable and progressively measurable functions relative to the information set.

b) $c > 0$ exists such that:

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| + |g(t, x_1, y_1) - g(t, x_2, y_2)| \leq c\left(|x_1 - x_2| + |y_1 - y_2|\right)$$

for all x_1, x_2, y_1, y_2 ε R, and for all couples (t, W) a.e.

c) There exists a constant a such that:

$$\left|g\left(t,x,y_1\right)-g\left(t,x,y_2\right)\right| \geq a\left|y_1-y_2\right|$$

for all x, y_1, y_2 e R and for all couples (t, W) a.e.

In the case of $\tilde{A}_t$:

$$f\left[t,x(t),y(t)\right]=\tilde{A}_t\left(r_t+q_t^2\right)+l_t\left(q_t^2\right), \qquad q_t=\sum_i b_t^i s_{vt}^i$$

$$g[t,x(t),y(t)] = (\tilde{A}_t+l_t)q_t .$$

Condition a) obviously holds for the differential equation of $\tilde{A}_t$. Condition b) is the Lipschitz condition. Since θ_t is bounded, the Lipschitz condition holds for the differential equation of $\tilde{A}_t$. Condition c) also holds for the differential equation of $\tilde{A}_t$. This is apparent after setting $a=\min\left\{\frac{l_t}{t}\in\left[0,T\right]\right\}$, and knowing that l_t is always positive. When these three conditions are respected, a unique couple ($\tilde{A}_t$,θ_t) which solves the differential equation (A11) with the terminal condition (A4) exists at each time t. The solution to the system (A11)-(A14) is equivalent to an optimal allocation made by a central planner.

That the solution is compatible with equilibrium in the initial decentralised economy is now shown. If a unique couple ($\tilde{A}_t$,θ_t) solution exists to the system of (A11)-(A14) at each time t, then:

$$\tilde{A}_t^*=E_Q^*\left\{\exp\left[-\int_t^T r_u du\right]A_T \mid \Im_t\right\}$$

$$\tilde{A}_t = \sum_i E_Q^* \left\{ \exp\left[-\int_t^T r_u \, du \right] S_T^i \mid \Im_t \right\}$$

The volatility of $\tilde{A}_t^*$ is then: $\sum_i V_t^{i*} s_{vt}^{i*}$.

However, from equation (A12), the volatility of $\tilde{A}_t^*$ is: $\left(\tilde{A}_t^* + l_t\right) q_t^*$.

Equating the two volatilities leads to the equilibrium condition of the decentralised economy:

$$\sum_i b_t^{i*} s_{vt}^{i*} = q_t^* \quad .$$

Proof of proposition 4

The futures price, F_{Gt}, for the delivery of aggregate production is obtained by setting the instantaneous volatility of dF_{Gt} at time t, d_t, to $\sum_i V_t^i s_{vt}^i$, and then, the relation $\alpha_t/\delta_t = \theta_t$ is used to determine the optimal α_t . The futures price for delivery of one unit of production follows.

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