Incomplete Information, Heterogeneity and Asset Pricing

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Abstract

We consider a pure exchange economy where the drift of aggregate consumption is unobservable. Agents with heterogeneous beliefs and preferences act competitively on a financial and good markets. We discuss how equilibrium market prices of risk differ across agents, and in particular we discuss the properties of the market price of risk under the physical (objective) probability measure. We provide a number of specification of risk aversions and beliefs where the market price of risk is much higher, and the riskless rate of return lower, than in the equivalent full information economy (homogeneous and heterogeneous preferences) and thus could provide an(other) answer to the equity premium and risk free rate puzzles. We also provide a representation of the equilibrium volatility and numerically assess the role of heterogeneity in beliefs. We show that high level of stock volatility can be obtained with low level of aggregate consumption volatility when beliefs are heterogeneous. Finally we discuss how incomplete information may explain the apparent predictability in stock return, and show that in-sample predictability can not be exploited by the agents, as it is in fact a result of their learning processes.

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1 Introduction

The determination of the equilibrium excess return of stock prices has been a central question in the asset pricing literature. The traditional approach, due to Breeden (1979), links the variations of the stock price to the variations of aggregate consumption in an elegantly simple manner. However, the consumption CAPM lacks empirical support. Observed stock returns are on average too high relative to the riskless rate. As Mehra and Prescott (1985) pointed out, in order to calibrate the model to consumption and stock price data one needs to assume a very high level of risk aversion for the representative agent. This in turn would yield unreasonably high riskless rate, as discussed in Weil (1989). Furthermore, the second moment of the stock return distribution is too high relative to the volatility of dividend and aggregate consumption, as initially mentioned in Grossman and Shiller (1981). In this paper, we show how incomplete information and heterogeneity in beliefs and preferences can help understanding the behavior of the stock price return and riskless rate and the failure of the consumption CAPM.

Several authors have studied the impact of incomplete information on portfolio consumption decision (Detemple (1986), Genotte (1986), Karatzas and Xue (1991), Lakner (1995, 1998), Merton (1973), Rogers (2002)) and equilibrium (Brennan and Xia (2001), Cechetti, Lam and Mark (2001), Detemple and Murthy (1994, 1997), Jouini and Napp (2003), Riedel (2001), Veronesi (2000)). They have shown that problems under incomplete information can be transformed into problems under complete information, where unknown parameters and state variables are projected on the information set of agents. One can show that under this reformulation it is possible to write a representative agent model where the consumption CAPM holds. Most of this literature has focused on the equilibrium behavior of stock prices as it is perceived by the agents in the economy, and has not discussed the properties of the empirically measured returns and volatilities. In an incomplete information environment, random events are perceived differently by agents with heterogeneous beliefs. What appears as a negative surprise to an agent can be interpreted as a positive surprise by another agent. The econometrician who estimates the mean return and volatility does not observe the beliefs of the agents, and his measurements are affected by the true (objective) underlying evolution of the stochastic processes. Therefore, in order to understand the statistical properties of asset prices, one needs to identify the corresponding evolution of the stock prices in the objective probability measure. This is what we propose to do in this paper.

We consider a pure exchange economy with incomplete information and heterogeneous agents. In our model the expected growth rate of consumption is unobservable and follows a mean reverting process. We depart from existing literature by assuming that the long term mean of the unobserved process is unknown and must be filtered as well. Agents have heterogeneous beliefs and are endowed with iso-elastic utility with different level of relative risk aversion. We develop the model by focusing on three problems.

Our first contribution is to consider how incomplete information and heterogeneity affect the market price of risk (MPR) and the riskless rate. Brennan and Xia (2001) and Cechetti, Lam and Mark (2000) also study the return dynamics under the objective probability measure but in an homogeneous agent economy. Jouini and Napp (2003) discuss the properties of the MPR and riskless rate with heterogeneous agent but do not model the learning process specifically. One important finding is that in an homogenous economy high excess return are obtained by assuming a pessimistic representative agent. In our setting, perceived MPR differ across agents, and incorporate the differences in beliefs and preferences. In the objective probability measure the market price of risk is equal to its complete information equivalent plus a consumption weighted average of the errors made by the agents. We show that a number of configuration of risk aversions and beliefs yield high market price of risk and low riskless rate. For instance when initial beliefs are Gaussian and symmetric in mean around the true value of the unobserved parameter (no average pessimism or optimism), but the pessimistic agents have lower risk aversion and conditional variances, we obtain market prices of risk and riskless rate of comparable magnitude to the empirical values. When multiple agents are considered, the representative agent may be pessimistic without having to assume that the agents themselves are pessimistic in majority.

Our second contribution is to provide an analytical formula for the volatility of the stock price. It is shown that it incorporates the immediate effect of the heterogeneity in beliefs and the expected impact of the future evolution of the beliefs in addition to the volatility of the aggregate consumption. Furthermore, the volatility is perceived identically by all agents and does not differ under the objective probability measure. The effects of incomplete information and heterogeneity in beliefs can be either positive or negative. Therefore, the volatility of the stock price can be either higher or lower than the volatility of aggregate consumption. The idea that incomplete information can help understand the high volatility of stock prices was first expressed in Bulkey and Tonks (1989), for the case of homogeneous beliefs. Timmermann (1993, 1996) formalized the argument in a discrete time model with a partial equilibrium approach. Lewellen and Shanken (2000) extended the approach to a discrete time equilibrium model, with short lived agents endowed with exponential utility function. Zapatero (1998) considered a setting close to ours but with logarithmic agents, and focused on the volatility of interest rate induced by the heterogeneity in beliefs. Brennan and Xia (2001) and Veronesi (2000) provide an analysis of the volatility when agents have homogeneous beliefs. In a concurrent paper, Gallmeyer (2000) obtains an analytical representation of the equilibrium volatility when beliefs are heterogeneous. His approach and the one developed in this paper rely on the same mathematical tools and differs essentially in the choice of the underlying unobservable process. Gallmeyer (2000) does not discuss the impact of heterogeneous beliefs and preferences on the MPR and riskless rate in the objective measure, which is our central result.

Our last contribution relates to the predictability of asset returns. We show how past dividend variations affect future stock returns. Timmermann (1993, 1996) and Lewellen and Shanken (2001) have also studied the impact of incomplete information on the predictability of asset return and we validate their results in a more general framework. We simulate the economy and regress the stock returns against past values of dividend yield and dividend growth and we obtain statistically significant coefficients. We explain this result by looking at the long term equilibrium value of the excess return. A term related to the average error made by the agents in the economy is shown to be related to past dividends. However that term is by definition unpredictable conditional on the information available to the agents. They rationally update their beliefs by taking into account the observable variations of the dividend, and this creates an appearance of predictability.

The paper is organized as follows. Section 2 describes the economy, along with the optimal consumption and equilibrium. Section 3 discusses the connection between subjective and objective measures and section 4 provides the numerical results for the MPR, the volatility and the predictability. We conclude in section 5.

2 Economy

We consider a pure exchange economy with finite horizon T, where the uncertainty is described by probability space $(\Omega, F, \mathcal{F}, \mathbb{P}^o)$ on which two independent Brownian motions, W_{1t} and W_{2t} are defined. We let $\mathcal{F} = \mathcal{F}_t^{W_1,W_2}$ and $F = \mathcal{F}_T$. Ω is the canonical state space $\mathcal{C}^0$ and $\mathbb{P}^0$ is the Wiener measure. There is a single perishable good which serves as the numeraire. The financial market consists of two securities, a locally riskless money market account in zero net supply which pays an interest rate of r_t and a risky stock with price S_t representing a claim to an exogenous stream of dividend payment D_t. The dividend process is of the form

$$D_t = D_0 \exp\left[\int_0^t \gamma_s - \frac{1}{2}\lambda_s^2 ds + \int_0^t \lambda_s dW_{1t}\right] \tag{1}$$

where the volatility λ_t is a square integrable process adapted to the natural filtration of W_{1t} and the drift process γ_t is a square integrable process adapted to $\mathcal{F}$. The initial value of the money market account is normalized to one, so in equilibrium the price process B_t is given by

$$B_t = \exp \int_0^t r_s ds. \tag{2}$$

There are 2 agents in the economy endowed with initial number of shares $\pi_{i0} > 0$. Units of money market account are denoted π_{it}^0. The flow of information available to the agents is restricted to the filtration generated by the dividend process, $\mathcal{F}^D$. We assume that agents have heterogeneous gaussian initial beliefs about the drift process γ_t, with mean m_{i0} and variance ε_{i0}. Subjective probability measure are represented by $\mathbb{P}^i$. Agents preference over consumption are described by the expected utility functional

$$U_i(c) := E^i \left[\int_0^T \rho_t \frac{c_{it}^{1-R_i}}{1-R_i} dt \right] \tag{3}$$

where R_i refers to the individual specific relative risk aversion, $\rho_t := e^{-\psi t}$ is the common discount rate and ψ is a positive constant. The dividend process is represented in the subjective probability space as a function of the innovation process $\chi_{it} := \int_0^t \lambda_s^{-1} \left(\frac{dD_s}{D_s} - m_{is} ds \right)$, namely we have

$$D_t = D_0 \exp \left[\int_0^t m_{is} - \frac{1}{2} \lambda_s^2 ds + \int_0^t \lambda_s d\chi_{is} \right]. \tag{4}$$

The cum dividend stock price process in the subjective measure is given by

$$S_t + \int_0^t D_s ds = S_0 + \int_0^t S_s \mu_{is} ds + \int_0^t S_s \sigma_s d\chi_{it} \tag{5}$$

The quantities $(\mu_{it}, \sigma_t, r_t)$ are obtained endogenously in equilibrium.

Trading takes place continuously and there are no frictions. The wealth process X_{it} associated to a trading strategy, π_{it}, takes the form

$$X_{it} = \pi_{it} S_t + \pi_{it}^0 B_t. \tag{6}$$

A trading strategy is admissible if X_{it} is uniformly bounded from below by a constant (this guarantees the absence of arbitrage, see Dybvig and Huang (1988)). A consumption plan (c_i) is said to be feasible if there exists an admissible trading strategy which solves the agent's dynamic budget constraint

$$dX_{it} = \pi_{it}^0 dB_t + \pi_{it} \left[dS_t + D_t dt \right] - c_{it} dt. \tag{7}$$

2.1 Individual Choices

Note that the economy is initially incomplete as the number of risky securities is inferior to the number of Brownian motions. However, every contingent claim adapted to $\mathcal{F}_t^D$ is attainable, and therefore when information is incomplete and restricted to $\mathcal{F}_t^D$, the market can be treated as complete for the purpose of individual consumption and investment decision. We therefore use the standard result of Cox and Huang (1991) to solve the static problem equivalent to the dynamic consumption investment decision. In order to present optimal consumption we define the individual specific state price density process

$$\xi_{it} := \exp \left[- \int_0^t r_s ds - \int_0^t \theta_{is} d\chi_{is} - \frac{1}{2} \int_0^t \|\theta_{is}\|^2 ds \right] \tag{8}$$

where $\theta_{it} := \frac{\mu_{it} - r_t}{\sigma_t}$ is the subjective market price of risk (MPR). Individual state price densities are linked by the expression $\xi_{2t} = \eta_{2t} \xi_{1t}$, where η_{2T} satisfies

$$\eta_{2t} = \eta_{20} \exp \left[\frac{1}{2} \int_0^t \left(\frac{\Delta_{2t}}{\lambda} \right)^2 dt + \int_0^t \frac{\Delta_{2t}}{\lambda} d\chi_{1t} \right] \tag{9}$$

and $\Delta_{2t} = m_{1t} - m_{2t}$.The static budget set is defined as

$$B(x_{i0}) = \left\{ c_i : E^i \int_0^T \xi_{it} c_{it} dt \leq \pi_{i0} S_0 \right\}. \tag{10}$$

The static optimization problem is given by

$$\begin{aligned} &\max_{c_i} U(c_i) \\ c_i \quad \in \quad &B(\pi_{i0} S_0). \end{aligned} \tag{11}$$

which yields the following optimal individual policies

$$c_{it} = \left(\frac{y_i \xi_{it}}{\rho_t} \right)^{-\frac{1}{R_i}} \tag{12}$$

where y_i is a Lagrange multiplier which solves agent i's static budget constraint

$$x_{i0} = E^i \int_0^T \xi_{it} \left(\frac{y \xi_{it}}{\rho_t} \right)^{-\frac{1}{R_i}} dt. \tag{13}$$

2.2 Equilibrium

An equilibrium is a combination $\{(c_{it}, \pi_{it}); (r_t, S_t)\}$, where (c_{it}, π_{it}) is an optimal admissible strategy for all i taking (r_t, S_t) as given and all markets clear $\sum_i c_{it} = D_t$, $\sum_i \pi_{it} = 1$, $\sum_i \pi_{it}^0 = 0$. In the goods market, the market clearing condition implies that all dividends are consumed. In the financial market, the market clearing conditions imply that the stock is held and the bond is in zero net supply. In order to derive the equilibrium, we choose a reference agent, which is arbitrarily called agent 1. This enables us to compute all expectations under the probability measure $\mathbb{P}^1$. The equilibrium is expressed in terms of the reference agent subjective measure.

Proposition 1 *When it exists, the competitive equilibrium for the economy described in section 2 is given by*

$$\theta_{1t} = R_{at} \lambda_t + \frac{c_{2t}}{C_t} \frac{R_{at}}{R_2} \lambda_t^{-1} \Delta_{2t} \tag{14}$$

$$\theta_{2t} = \theta_{1t} - \lambda_t^{-1} \Delta_{2t} \tag{15}$$

$$r_t = \psi + R_{at} m_{1t} - \frac{c_{2t}}{C_t} \frac{R_{at}}{R_2} \lambda_t^{-1} \Delta_{2t} \theta_{2t} - \sum_i \frac{c_{it}}{C_t} \frac{R_{at}(1 + R_i)}{R_i^2} \theta_{it}^2 \tag{16}$$

$$S_t = E^1 \left[\int_t^T \rho_{t,v} \frac{u_1'(c_{1v})}{u_1'(c_{1t})} D_v dv \mid \mathcal{F}_t^D \right] \tag{17}$$

where $R_{at} = \left[\sum_i \frac{c_{it}}{C_t} \frac{1}{R_i} \right]^{-1}$ *and* $C_t = c_{1t} + c_{2t}$.*(Proof see Detemple and Murthy (1997))*

Existence and uniqueness of this equilibrium can be established as in Riedel (2001) provided that the volatility of the stock remains invertible and other coefficient satisfy integrability conditions, we will assume this is the case.

The perceived MPRs, θ_{it}, differ across agents, and the difference is due to the heterogeneity in beliefs. The stock price is equal to the expected discounted sum of future dividends, and the discount factor is the intertemporal marginal rate of substitution. The equilibrium is stated relative to agent 1, but there are no disagreements about the price of the risky security, i.e. $E_t^1 \int_t^T \rho_{t,v} \frac{u_1'(c_{1v})}{u_1'(c_{1t})} D_v dv = E_t^i \int_t^T \rho_{t,v} \frac{u_i'(c_{iv})}{u_i'(c_{it})} D_v dv$ for all $i = 2....n$. The full information value of the market price of risk is given here as a benchmark. In the full information case, $m_{it} = \gamma_t$ for both agents, and therefore $\Delta_{2t} = 0$, which yields

$$\theta_t = R_{at} \lambda_t. \tag{18}$$

This result leads to the well known consumption CAPM.

3 Subjective and Objective Measures

Under the objective measure the stock price may be written as

$$\begin{aligned} \frac{dS_t + D_t dt}{S_t} &= \mu_{it} dt + \sigma_{it} \left(dW_{2t} + \lambda_t^{-1} \left(\gamma_t - m_{it} \right) dt \right) && (19) \\ &: \quad = \mu_t^o dt + \sigma_t^o dW_{2t}. && (20) \end{aligned}$$

μ_t^o and σ_t^o denotes the instantaneous stock return and volatility under the objective measure. An econometrician considering the time series of stock prices will get an estimate of μ_t^o, and not μ_{it}. Extending our definition of the subjective MPR, we define $\theta_t^o \equiv \frac{\mu_t^o - r_t}{\sigma_t}$ the MPR in the objective probability measure.

3.1 Market Price of Risk

The perceived MPR for any agent differs from the market price of risk under the objective probability measure. The latter is given by the formula

$$\theta_t^o = R_{at} \lambda_t + \lambda_t^{-1} \left[\gamma_t - \sum_i \frac{c_{it}}{C_t} \frac{R_{at}}{R_i} m_{it} \right]. \tag{21}$$

Note that this construction is equivalent to the consensus characteristic construction as exposed in Jouini and Napp (2003). As they point out the difference between the homogeneous belief MPR and the heterogeneous beliefs MPR for the case of power utility is given by a weigthed average of the individual subjective beliefs. Let us define the aggregate belief $m_{at} = \frac{\sum_i \frac{c_{it}}{C_t} \frac{1}{R_i} m_{it}}{\sum_i \frac{c_{it}}{C_t} \frac{1}{R_i}}$, the MPR is then given by

$$\theta_t^o = R_{at} \lambda_t + \lambda_t^{-1} \left[\gamma_t - m_{at} \right]. \tag{22}$$

We recover here a well known result that a representative agent must be pessimistic, i.e. $m_{at} < \gamma_t$, in order to obtain values of θ_t^o, which are above the full information result (see Cechetti, Lam and Mark (2000) and Abel (2002)). It is however difficult to justify, in the single agent framework why the beliefs, if formed rationally, should systematically remain below the true value. Heterogeneity seems to provide a justification as the process m_{at} does not represent a Bayesian updating, but rather a weighted sum of beliefs. The restriction to consider in order to observe high values for the market price of risk becomes

$$m_{at} < \gamma_t.$$

Apparent pessimism at the aggregate level, m_{at}, can thus be driven by any combination of three effects (as shown also in Jouini and Napp (2003)) (i) wealth effect, if the agent with lowest beliefs have larger wealth (ii) risk aversion effect, when agents with lowest beliefs also have the lowest risk aversion (iii) beliefs effect, when risk aversion and wealth are identical, aggregate pessimism is obtained by assuming that the dispersion of beliefs is no longer symmetric on average. This is obtained by assuming for example symmetric initial beliefs and lowest conditional variance for the pessimistic agents.

3.2 Volatility

In this section we discuss the equilibrium volatility and provide an analytical representation which emphasizes the role of incomplete information and heterogeneity in beliefs.

Proposition 2 *In the economy described in section 2, the volatility of the stock price is identical under the subjective and objective probability measures. For the case of $\lambda_t = \lambda$, a constant, it is given by the formula*

$$\sigma_t = \lambda + \Omega_t + \Pi_t + \Xi_t \tag{23}$$

where Ω_t is the immediate impact of heterogeneity in beliefs and preferences, Π_t is the expected impact of the future evolution of the beliefs of agent 1, and Ξ_t is the correction for heterogeneity in beliefs and preferences on the future impact of incomplete information. Ω_t, Π_t and Ξ_t are adapted to the filtration $\mathcal{F}^D$ and are derived in the Appendix.

The volatility is obtained by considering the equilibrium stock price[1].

$$S_t = E_t^1 \int_t^T \xi_{t,v}^1 D_v dv \tag{24}$$

which can also be written as

$$\xi_t S_t = E_t^1 \int_0^T \xi_v^1 D_v dv - \int_0^t \xi_v^1 D_v dv. \tag{25}$$

[1] The term $\xi_{t,v}$ is to be understood as $\frac{\xi_v}{\xi_t}$.

We use the fact that $E_t^1 \int_0^T \xi_v^1 D_v dv$ is a $\left(F_t^D, P^1\right)$ −martingale and admits a representation as an integral of the innovation process.

$$E_t^1 \int_0^T \xi_v^1 D_v dv = E_0^1 \int_0^T \xi_v^1 D_v dv + \int_0^t \phi_s d\chi_{1s}. \tag{26}$$

As χ_1 is an $\left(F_t^D, P^1\right)$ − Brownian motion, Clark-Ocone formula (Clark (1970)) applies, and the integrand ϕ_s can be expressed as a conditional expectation of the Malliavin derivative[2] of $\int_0^T \xi_v^1 D_v dv$. The integrand ϕ_s plays an important role in the volatility formula presented in Proposition 2. Malliavin derivatives and the generalized version of Clark's theorem have been used in Karatzas and Ocone (1991) and recently in Detemple, Garcia and Rindisbacher (2003) for the representation of optimal portfolios.

4 Numerical Analysis

4.1 Market Price of Risk and Riskless Rate

Equation (21) provides a closed form solution for the market price of risk under the objective probability measure which we have denoted θ_t^o. An econometrician observing the time series of stock returns will obtain an estimate of this quantity. Ideally we would like to obtain an expression for the average of the market price of risk over a given time interval, i.e. $E_t\left[\frac{1}{T-t}\int_t^T \theta_t^o dt\right]$. Due to the complex path-dependency of the quantities appearing in θ_t^o, it is not possible to obtain a closed form solution to the latter expression. To overcome this difficulty, we will simulate the economy and approximate this expectation. Furthermore we restrict the preferences to the case $R_2 = pR_1$. In this case, the state price density is the root of a polynomial of order n which admits a unique real positive solution for $p \in \{2, 3, 4, 5\}$[3]. We proceed by describing the evolution of the underlying stochastic processes, then we discuss the filtering procedure, the simulation methodology and we present and explain the numerical results.

4.1.1 Filtering

We assume that the unobservable dividend growth rate γ_t follows a mean reverting process of the form

$$d\gamma_t = \alpha(\beta - \gamma_t) dt + \delta_1 dW_{1t} + \delta_2 dW_{2t}. \tag{27}$$

The speed of convergence α, and the volatility coefficients δ_1 and δ_2 are common knowledge. Unlike previous studies (Brennan and Xia (2001)) we do not assume that the long term mean of the dividend growth rate is known. The agents must construct their beliefs based on the observations of the dividend D_t. 2

[2] Notice here that the Malliavin operator acts on the innovation process and not on the brownian motion.

[3] This result was first pointed out in Wang (1996).

conditional means and a conditional covariance matrix must be constructed. Let us define the individual specific covariance matrix and conditional mean vectors

$$
\begin{aligned}
\Sigma_t^i &= \begin{bmatrix} \varepsilon_{t,11}^i & \varepsilon_{t,12}^i \\ \varepsilon_{t,12}^i & \varepsilon_{t,22}^i \end{bmatrix} \\
M_{it} &= \begin{bmatrix} m_{it} \\ l_{it} \end{bmatrix}
\end{aligned} \tag{28}
$$

where $l_{it} = E^i\left[\beta \mid \mathcal{F}_t^D\right]$ and $m_{it} = E^i\left[\gamma_t \mid \mathcal{F}_t^D\right]$. The filtering equations are obtained using theorem 12.6 in Lipster and Shiryayev (1978). The conditional expectations evolve according to

$$
\begin{aligned}
dm_{it} &= \alpha(l_{it} - m_{it})dt + \frac{\delta_2\lambda + \varepsilon_{t,11}^i}{\lambda^2}\left[\frac{dD_t}{D_t} - m_{it}dt\right] \\
dB_{it} &= \frac{\varepsilon_{t,12}^i}{\lambda^2}\left[\frac{dD_t}{D_t} - m_{it}dt\right]
\end{aligned} \tag{29}
$$

while the covariance matrix evolution is given by

$$
\begin{aligned}
\frac{d\varepsilon_{t,11}^i}{dt} &= 2\alpha\left(\varepsilon_{t,12}^i - \varepsilon_{t,11}^i\right) + \delta_1^2 + \delta_2^2 - \left[\frac{\delta_2\lambda + \varepsilon_{t,11}^i}{\lambda}\right]^2 \\
\frac{d\varepsilon_{t,12}^i}{dt} &= \alpha\left(\varepsilon_{t,22}^i - \varepsilon_{t,11}^i\right) - \frac{\delta_2\lambda + \varepsilon_{t,11}^i\varepsilon_{t,12}^i}{\lambda^2} \\
\frac{d\varepsilon_{t,22}^i}{dt} &= -\left(\frac{\varepsilon_{t,12}^i}{\lambda}\right)^2.
\end{aligned} \tag{30}
$$

Under the agents' subjective probability measures the conditional mean m_{it} follows a mean reverting process with speed of convergence equal to the unobservable dividend growth rate and long term mean equal to the conditional expectation of the true long term mean.

4.1.2 Simulation Methodology

We follow a two step procedure. First, for a given value of the dividend at time 0, we calibrate the Lagrange multiplier to obtain symmetric wealth distribution. Then, we simulate the evolution of the economy using the filtering equations and the optimal consumption demands. Using (12) along with the market clearing we identify the state price density process as the solution to the polynomial expression

$$
\left(\frac{y_1\xi_{1t}}{\rho_t}\right)^{-\frac{1}{R_1}} + \left(\frac{y_2\xi_{1t}\eta_{2t}}{\rho_t}\right)^{-\frac{1}{R_2}} - D_t = 0 \tag{31}
$$

The only real positive root can be obtained analytically when $R_2 = pR_1$ and $p \in \{2, 3, 4, 5\}$. The Radon-Nykodim derivative η_{2t} is simulated using (9) and the divergence process, Δ_{2t}, is simply obtained from the filtering equations. All expectations given in the following results are given relative to the objective probability measure $\mathbb{P}^0$.

4.1.3 Results

In the simulation exercise the horizon is set at 20 years and it is assumed that the subjective discount rate $\psi = 0.02$. Parameters value for the consumption process are obtained from the Shiller dataset (*www.econ.yale.edu/~shiller/data.htm*), which provides per capita consumption in the US over the period 1871-2003. Parameters are listed in table 1, the salient feature is the low volatility of consumption, which forces high level of risk aversion to fit the full information CCAPM to the stock return data.

We want to assess the level of heterogeneity necessary to obtain high level of MPR (0.3) and low level of interest rate (0.01), and more importantly whether the two issues can be addressed simultaneously. For our set of parameters and a risk aversion level of 4.5, the full information, and homogeneous preference, level for the MPR and riskless rate are respectively 0.16 and 0.0749. We compute average values of the MPR and riskless rate when varying the average divergence, defined as $\frac{1}{T}\int_0^T m_{1t} - m_{2t}dt$, from 0.5 % to 2 %. The computation is repeated with $R_2 = pR_1$ and $p \in \{2, 3, 4, 5\}$. In all computation we assign the lowest initial conditional standard deviation to the pessimistic agent (1 % for pessimistic and 2.25 % for optimistic) and set initial beliefs symmetric around the true long term mean. Results are displayed in figure (1). As expected the MPR increases, and the riskless rate decreases, as a function of the divergence in beliefs. When preferences are close ($p = 2$) the decrease in interest rate is fast and a level of MPR of 0.3 is obtained only by allowing the interest rate to drop to -3%. In order to maintain an acceptable level of interest rate we must consider important differences in risk aversion, indeed, when $p = 5$, a level of MPR of 0.3 is obtained with an interest rate of 1.22 %. This is achieved with an average divergence of 1.68 %, which is less than half of a standard deviation of the growth rate of aggregate consumption, and therefore perfectly plausible. Figure (2) and (3) displays values obtained for the MPR and riskless rate by varying the level of disagreement.

4.2 Volatility

It is computationally not feasible to simulate dynamic properties of the stock price volatility, as each point in time requires extensive simulation to obtain the value of conditional expectation of stochastic integrals. However we may consider comparative statics at a given point in time. In this section we simplify the filtering procedure and consider the case where the long term mean is known and agents differ in anticipations but not in preferences. The volatility equation can be then be expressed as[4]

$$
\begin{aligned}
\sigma_t &= \lambda \\
&+ \left(\frac{c_{2t}}{C_t} + \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \eta_{2v} dv \right) \lambda_t^{-1} \Delta_{2t} \\
&(1-R) \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \xi_{1v} D_v \int_t^s \mathcal{D}_t m_{1s} ds dv \\
&+ \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \eta_{2v} \left[\int_t^v \lambda^{-1} \Delta_{2s} \lambda^{-1} \mathcal{D}_t \Delta_{2s} ds + \int_t^v \lambda^{-1} \mathcal{D}_t \Delta_{2s} d\chi_{1s} \right] dv
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
\frac{\partial \xi_{1v}}{\partial \eta_{2v}} &= -\frac{\left(\frac{D_v}{\left(\frac{y_1}{\rho_v}\right)^{-\frac{1}{R}} + \left(y_2 \frac{\eta_{2v}}{\rho_v}\right)^{-\frac{1}{R}}} \right)^{-R}}{\left(\frac{y_1}{\rho_v}\right)^{-\frac{1}{R}} + \left(y_2 \frac{\eta_{2v}}{\rho_v}\right)^{-\frac{1}{R}}} \frac{\left(y_2 \frac{\eta_{2v}}{\rho}\right)^{-\frac{1}{R}}}{\eta_{2v}} \\
\mathcal{D}_t m_{1s} &= \frac{\delta_2 \lambda + \varepsilon_{1t}}{\lambda} \exp\left[\alpha (t-s)\right] \\
\mathcal{D}_t \Delta_{2v} &= \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda} \exp\left[-\int_t^v \frac{\delta_2 \lambda + \varepsilon_{2s}}{\lambda^2} + \alpha ds \right].
\end{aligned}
$$

The stock volatility is equal to the full information volatility λ, i.e. the volatility of the dividend, plus three extra terms. The first extra term is due to the immediate effect of the heterogeneity in beliefs on the perceived market prices of risk. The second extra term is the effect of incomplete information only, and the last term is the long run effect of heterogeneity. It is important to note that the last term vanishes when $\varepsilon_{1t} = \varepsilon_{2t}$. Heterogeneity has a long term effect only through the divergence in precision. When the conditional variances are equal, the evolution of the divergence in beliefs, Δ_{2t}, is deterministic. The evolution of Δ_{2t} is given by

$$
d\Delta_{2t} = \left(-\frac{\delta_2 \lambda + \varepsilon_{2t}}{\lambda^2} - \alpha\right) \Delta_{2t} dt + \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda} d\chi_{1t} \tag{33}
$$

and has therefore a volatility equal to $\frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda}$.

Figure (4) displays the volatility as a function of risk aversion and divergence in beliefs. Parameters values are identical to the previous section and conditional standard deviations for the pessimistic and optimistic agents are set equal to 2% and 2.5% respectively. The model is able to generate high level of volatility (15 - 20 %) with relatively low level of risk aversion (4-5) and low dividend volatility (3.6 %) when beliefs are not identical. Notice also, that for some parameter values the volatility can be lower that the dividend volatility.

[4] see Appendix

4.3 Predictability of Asset Returns

Empirical studies such as Fama and Schwert (1977) and Kothari and Shanken (1997), to name a few, have found that variables such as interest rates, dividend yields and default premium have the ability to forecast stock returns. This troubling fact might raise doubts about either market efficiency or investors rationality, since past dividend yields and interest rates are available to the investors any informational content should be included in the price and the variations in returns should not be predictable based on these informations. In this section, we look at the implication of incomplete information on the issue of stock returns predictability. Our model displays no predictability when considering the perspective of the investor (and therefore his subjective probability measure), but when we look at the statistical properties of the excess return we indeed find that there is a link between past dividends and future expected stock returns.

We will first consider the case of homogeneous beliefs which simplifies the exposition and yet convey the general idea. When beliefs are homogeneous, the excess return in the objective probability measure is given by

$$\mu_t^o - r_t = R\lambda\sigma_t + \frac{\sigma_t}{\lambda_t}(\gamma_t - m_t) \tag{34}$$

which is a particular case of equation 21 when $m_{it} = m_t$ for all i. Considering the mean reverting assumption for the drift of the dividend process and assuming that α and β are known, the current belief m_t is given by

$$m_t = m_0 + \int_0^t \alpha \left[\beta - m_s\right] ds + \int_0^t \frac{\delta_2\lambda_t + \varepsilon_s}{\lambda_t^2} \left[\frac{dD_s}{D_s} - m_s ds\right] \tag{35}$$

which we can alternatively write for any $v \in [0, T]$

$$m_t = m_v + \int_v^t \alpha \left[\beta - m_s\right] ds + \int_v^t \frac{\delta_2\lambda_t + \varepsilon_s}{\lambda_t^2} \left[\frac{dD_s}{D_s} - m_s ds\right]. \tag{36}$$

Let us consider an interval $[v, t]$ where $\int_v^t \frac{\delta_2\lambda + \varepsilon_s}{\lambda^2} \left[\frac{dD_s}{D_s} - m_s ds\right] >> 0$, implying realized dividend growth far exceeds the one anticipated by the agent. If the time interval is sufficiently short (e.g. 1 month) the mean reversion effect based on empirical values of the coefficient α is very small. Therefore, the main impact on variation of m_t is due to the innovation effect. So, in this case high dividend growth period implies an increase in m_t, but what matters for the predictability is how m_t compares to γ_t. If we take $m_v = \gamma_v$, then we have a negative relation between dividend growth and stock return ; high dividend growth will be followed by lower expected excess return. But depending on the relation between m_v and γ_v, it may well be the case that high dividend growth periods are followed by higher expected excess return.

Since in the long run m_t will fluctuate around γ_t the negative relation should prevail. Note that in the perspective of the investor the expected excess return is always equal to $R\lambda\sigma_t$ since under his probability measure $E\left[\frac{\sigma_t}{\lambda}(\gamma_t - m_t)\right] = 0$ by definition. Predictability in this model appears only to the researcher studying a sufficiently long sample to observe the true distributional properties of the returns, and it is therefore possible to have perfectly rational agents operating in an efficient market and still observe predictability.

The discussion for heterogeneous beliefs goes along the same lines. The expected excess return is given under the original probability measure by

$$\mu_t^o - r_t = R\lambda\sigma_t + \frac{\sigma_t}{\lambda_t}\left[\gamma_t - \left(\frac{c_{1t}}{C_t}m_{1t} + \frac{c_{2t}}{C_t}m_{2t}\right)\right]. \tag{37}$$

Once the steady state is reached, i.e. when the conditional variance of the 2 agents are identical, the behavior of $\mu_t^o - r_t$ is similar to the homogeneous case. On average, high past dividend growth implies low future returns. The heterogeneity in beliefs makes the relationship less stable, and before the steady state is reached various patterns can be observed. Using the equilibrium stock price we can write the dividend yield, which is defined as the ratio of the dividend to the stock price as

$$Dyield_t = \frac{D_t}{E_t^1 \int_t^T \xi_{1t,v} D_v dv} \tag{38}$$

which simplifies to

$$Dyield_t = \frac{1}{E_t^1 \int_t^T \xi_{1t,v} \exp\left(\int_t^v m_{1s} - \frac{1}{2}\lambda^2 ds + \int_t^v \lambda d\chi_{1s}\right) dv}. \tag{39}$$

According to equation (39), the dividend yield is a decreasing function of the current belief m_{1t}. High dividend yield should be observed when the conditional expectation of γ_t is low, recall that in this case the excess return in the objective probability measure is high since it is an increasing function of $\gamma_t - m_{1t}$.

To numerically assess the implication of incomplete information for the predictability of excess return, we perform the following simulations. Using the homogeneous information setting[5], we simulate the evolution of the stock price based on the present value formula. From that simulation, we construct the following variables (observed at discrete time intervals)

$$\begin{aligned} DivYield_t &= \frac{D_t}{S_t} \\ DivGrowth_t &= \frac{D_t - D_{t-h}}{D_{t-h}} \\ Excess_t &= \frac{S_t - S_{t-h} + D_t h}{S_{t-h}} \end{aligned}$$

[5] We use the homogeneous information setting to simplify the procedure and limit the number of simulation required. For the question of predictability the homogeneous and heterogeneous setting are similar, the effect should just be stronger for the homogeneous economy. The heterogenety in belief remains a crucial assumption when considering deviation from the CCAPM, as explain in the previous sections.

we then estimate the following linear regression.

$$Excess_t = \alpha + \beta_1 Divyield_{t-h} + \beta_2 DivGrowth_{t-h} + \varepsilon_t. \tag{40}$$

Based on the previous theoretical analysis, we expect β_2 to be negative and β_1 to be positive. A summary of the estimation results is presented in table 2. The theoretical results is obtained strongly for the dividend yield with approximately 2/3 of the estimations yielding a significant positive coefficient. The result for the growth rate of dividend are in general not significant. Notice, again, that predictability is apparent only to the outside observer, since the deviations from the full information benchmark, are a function of the *ex post* error made by the agents. Our simulation results confirm the results of Timmermann (1996) and Lewellen and Shanken (2000), which both found, in a discrete time setting, that predictability could be induced by estimation risk. We show here, that the results still hold in a continuous time pure exchange economy setting, when agents display risk aversion and have intermediate consumption. Recently Menzly, Santos and Veronesi (2004) have shown that return predictability based on dividend yield in a model with habit formation is compatible with equilibrium and efficient markets. An equilibrium with heterogeneous beliefs is isomorphic to an equilibrium with homogeneous beliefs and state dependent preferences as shown in Riedel (2001), therefore we might conjecture that the simulation results obtained in this section for homogeneous beliefs would be strengthened with the introduction of heterogeneity.

5 Conclusion

We have shown in this paper how incomplete information and heterogeneity in beliefs affect the statistical properties of asset prices, making an important distinction between objective and subjective probability measures. We have seen that it is possible, under particular beliefs and preference configuration and information structure, to observe large deviations from the full information market price of risk and riskless rate. Empirical values were matched with low level of aggregate risk aversion (4.5) and low level of aggregate consumption volatility (3.6 %). This was obtained with average divergence below 2 %, in a 2 agent economy, where the pessimistic agent was endowed with lower initial conditional variance and risk aversion. We have also shown how the volatility of stock prices is affected by incomplete information and heterogeneous beliefs and have demonstrated that high level of volatility could be obtained with low level of dividend volatility. Finally we have discussed how predictability of return based on dividend yield could be motivated by incomplete information.

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Appendix A: Proof of proposition 2

From the equilibrium stock price and first order condition, we write the stock price as

$$S_t = E^1 \left[\int_t^T \frac{\xi_v}{\xi_t} D_v dv \mid \mathcal{F}_t^D \right]. \tag{41}$$

The innovation process $\chi_{1t} := \int_0^t \lambda_s^{-1} \left(\frac{dD_s}{D_s} - m_{1s} ds \right)$ is a $\left(P^1, \mathcal{F}_t^D\right)$ - Brownian motion. Let M_t be the following $\left(P^1, \mathcal{F}_t^D\right)$ - martingale

$$M_t := \xi_t S_t = E^1 \left[\int_0^T \xi_v D_v dv \mid \mathcal{F}_t^D \right] \tag{42}$$

it has a representation in term of χ_1, namely

$$M_t = M_0 + \int_0^t \phi_s d\chi_{1s}. \tag{43}$$

for some unique square integrable process ϕ. Applying Itô's lemma to the product $\xi_t S_t$, we identify its diffusion term and obtain

$$\xi_{1t} S_t [-\theta_{1t} + \sigma_t] = \phi_t \tag{44}$$

and therefore the volatility is given by

$$\sigma_t = \xi_{1t}^{-1} S_t^{-1} \phi_t + \theta_{1t}. \tag{45}$$

To identify the process ϕ we need the following lemma.

Lemma 3 *(Proposition 1.3.5 (Clark-Ocone) - Nualart (1995))*
Let $F \in D^{1,2}$, where $D^{1,2}$ is the closure of the class of smooth random variable S with respect to the norm

$$\|F\|_{1,2} = \left[E\left(|F|^2 \right) + E\left(\|\mathcal{D}F\|^2_{L^2(T)} \right) \right]^{1/2}. \tag{46}$$

Suppose that W is a one-dimensional Brownian motion. Then

$$F = E(F) + \int_0^T E\left(\mathcal{D}_s F \mid \mathcal{F}_s \right) dW(s), \tag{47}$$

taking conditional expectations

$$E\left(F \mid \mathcal{F}_t\right) = E(F) + \int_0^t E\left(\mathcal{D}_s F \mid \mathcal{F}_s \right) dW(s). \tag{48}$$

Proof see Nualart (1995) page 42.

The operator $\mathcal{D}$ in our setting is the Malliavin derivative with respect to the one-dimensional Brownian motion χ_1, and S is the class of smooth functionals of χ_1. The process ϕ can be identified from Lemma 3

$$\phi_t = E_t^1\left[\int_t^T \mathcal{D}_t\left(\xi_{1v}D_v\right)dv\right] \tag{49}$$

which we rewrite using the chain rule of Malliavin calculus as

$$\phi_s = E_t^1\int_t^T D_v\mathcal{D}_t\xi_{1v}dv + E_t^1\int_t^T \xi_{1v}\mathcal{D}_tD_vdv. \tag{50}$$

Using the market clearing condition, $c_{1t}+c_{2t}=D_t$, and the optimal consumptions $c_{1t}=\left(\frac{y_1\xi_{1t}}{\rho_t}\right)^{-\frac{1}{R_1}}$ and $c_{2t}=\left(\frac{y_2\eta_{2t}\xi_{1t}}{\rho_t}\right)^{-\frac{1}{R_2}}$, to identify the state price density process, we obtain

$$\begin{aligned}
\phi_t \quad = \quad & E_t^1\int_t^T D_v\left[\frac{\xi_{1v}D_v\left[\lambda+\int_t^v\mathcal{D}_tm_{1s}ds\right]}{\sum_i -\frac{1}{R_i}\left(y_i\eta_{iv}\xi_{1v}\right)^{-\frac{1}{R_i}}}\right]dv \\
& +E_t^1\int_t^T\frac{\partial\xi_1v}{\partial\eta_{2v}}\left(\eta_{2v}\left[\begin{array}{c}\int_t^v\lambda^{-1}\Delta_{2s}\lambda^{-1}\mathcal{D}_t\Delta_{2s}ds\\ +\int_t^v\lambda^{-1}\mathcal{D}_t\Delta_{2s}d\chi_{1s}+\lambda^{-1}\Delta_{2t}\end{array}\right]\right)dv \\
& +E_t^1\int_t^T\xi_{1v}D_v\left[\lambda+\int_t^v\mathcal{D}_tm_{1s}dv\right]dv
\end{aligned} \tag{51}$$

Finally we use the equilibrium market price of risk $\theta_1 = R_{at}\lambda_t+\frac{c_{2t}}{C_t}\frac{R_{at}}{R_2}\lambda_t^{-1}\Delta_{2t}$, to obtain the result in proposition 2

$$\sigma_t = \lambda+\Omega_t+\Pi_t+\Xi_t \tag{52}$$

where

$$\begin{aligned}
\Omega_t \quad : & = \left(\frac{c_{2t}}{C_t}\frac{R_{at}}{R_2}+\xi_{1t}^{-1}S_t^{-1}E_t^1\int_t^T\frac{\partial\xi_{1v}}{\partial\eta_{2v}}\eta_{2v}dv\right)\lambda_t^{-1}\Delta_{2t} \\
\Pi_t \quad : & = R_{at}\lambda+\xi_{1t}^{-1}S_t^{-1}E_t^1\int_t^T D_v\left[\frac{\xi_{1v}D_v\left[\lambda+\int_t^s\mathcal{D}_tm_{1s}ds\right]}{-\frac{1}{R_1}\left(y_1\xi_{1v}\right)^{-\frac{1}{R1}}-\frac{1}{R_2}\left(y_2\eta_{2v}\xi_{1v}\right)^{-\frac{1}{R_2}}}\right]dv \\
& \quad +\xi_{1t}^{-1}S_t^{-1}E_t^1\int_t^T\xi_{1v}D_v\int_t^v\mathcal{D}_tm_{1s}dsdv \\
\Xi_t \quad : & = \xi_{1t}^{-1}S_t^{-1}E_t^1\int_t^T\frac{\partial\xi_{1v}}{\partial\eta_{2v}}\eta_{2v}\left[\begin{array}{c}\int_t^v\lambda^{-1}\Delta_{2s}\lambda^{-1}\mathcal{D}_t\Delta_{2s}ds\\ +\int_t^v\lambda^{-1}\mathcal{D}_t\Delta_{2s}d\chi_{1s}\end{array}\right]dv.
\end{aligned}$$

When preferences are homogeneous the volatility simplifies to

$$
\begin{aligned}
\sigma_t &= \lambda \\
&+ \left(\frac{c_{2t}}{C_t} + \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \eta_{2v} dv \right) \lambda_t^{-1} \Delta_{2t} \\
&(1-R) \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \xi_{1v} D_v \int_t^s \mathcal{D}_t m_{1s} ds dv \\
&+ \xi_{1t}^{-1} S_t^{-1} E_t^1 \int_t^T \frac{\partial \xi_{1v}}{\partial \eta_{2v}} \eta_{2v} \left[\int_t^v \lambda^{-1} \Delta_{2s} \lambda^{-1} \mathcal{D}_t \Delta_{2s} ds + \int_t^v \lambda^{-1} \mathcal{D}_t \Delta_{2s} d\chi_{1s} \right] dv
\end{aligned} \tag{53}
$$

The Malliavin derivative of m_1 and Δ_2 are given by the following expressions when the long term mean is known

$$
\mathcal{D}_t \left(m_{1s} \right) = \frac{\delta_2 \lambda + \varepsilon_{1t}}{\lambda} \exp \left[\alpha \left(t - s \right) \right] \tag{54}
$$

$$
\mathcal{D}_t \left(\Delta_{2v} \right) = \frac{\varepsilon_{1t} - \varepsilon_{2t}}{\lambda} \exp \left[- \int_t^v \frac{\delta_2 \lambda + \varepsilon_{2s}}{\lambda^2} + \alpha ds \right] \tag{55}
$$

Using (54) and (55) with (53) gives equation (32) in the text.

Appendix B: Tables

	Consumption Growth Rate Coefficients
α	1.16
δ_1	0.011
δ_2	0.011
λ	0.036
β	0.018
	Prior Conditional Covariance (Pessimistic)
$\varepsilon^1_{11,0}$	0.001
$\varepsilon^1_{12,0}$	-0.0003
$\varepsilon^1_{22,0}$	0.0005
	Prior Conditional Covariance (Optimistic)
$\varepsilon^2_{11,o}$	0.005
$\varepsilon^2_{12,o}$	0.0001
$\varepsilon^2_{22,o}$	0.001

Table 1: Parameters used in the simulation for the computation of average MPR and Riskless rate. Consumption coefficients are obtained from per capita consumption over the period 1871-2003 in the United-States.

		confidence	level		
	5%	10%	15%	positive	negative
β_1	20%	40%	60%	96%	4%
β_2	12%	24%	24%	56%	44%

Table 2: Estimation results of the linear regressions. The dependent variable is the excess return (cum dividend), the independent variables are 1-period lagged dividend yield and dividend growth

Appendix C: Figures

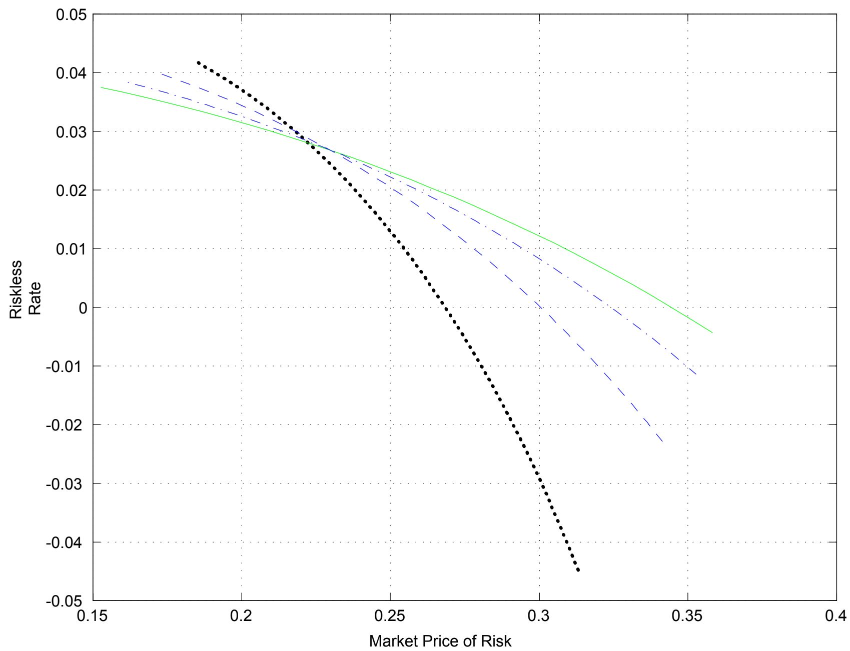

Figure 1: Pairs of MPR and riskless rate for level of average divergence ranging from 0.55 % to 2.14 %. $R_2 = pR_1$, dotted line $p = 2$, dashed line $p = 3$ dashed-dotted line $p = 4$ solid line $p = 5$.

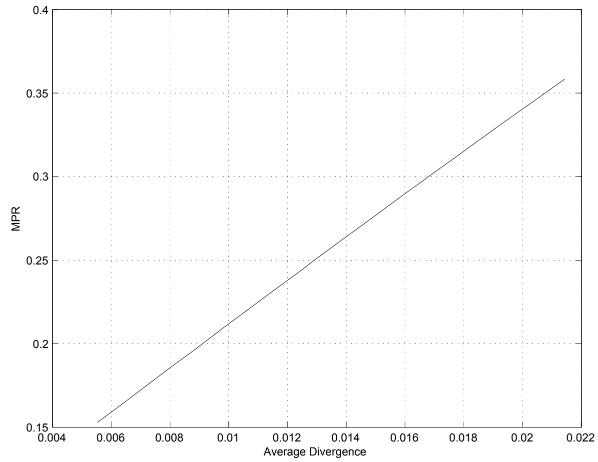

Figure 2: MPR as a function of average divergence in beliefs, when $R_2 = 7.5$ and $R_1 = 1.5$.

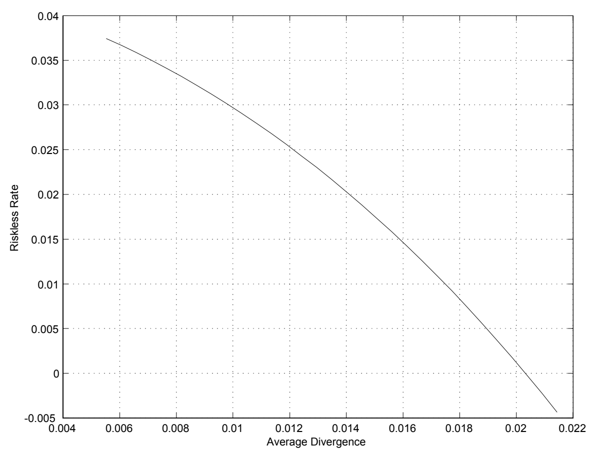

Figure 3: Riskless rate as a function of average divergence in beliefs, when $R_2 = 7.5$ and $R_1 = 1.5$.

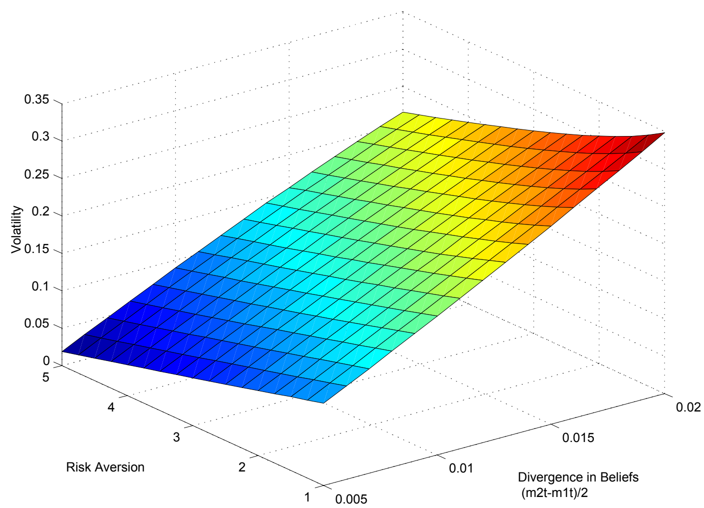

Figure 4: Equilibrium volatility as a function of risk aversion and divergence in beliefs. Beliefs are symmetric around the true unobserved parameter. $T = 20$, $\psi = 0.02$, $(\varepsilon_{1t})^{1/2} = 2\%$, $(\varepsilon_{2t})^{1/2} = 2.5\%$, $\delta_1 = 0.011$, $\delta_2 = 0.011$, $\beta = 0.018$, $\lambda = 0.036$.