Exact tests for possibly fractionally integrated stochastic volatility models *

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ABSTRACT

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Mots clé:

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1. Introduction

A growing body of empirical evidence supports the view that important economic data series exhibit persistent statistical dependence. Indeed, Granger (1966) notes that economic time series display typical spectral shape of long-range dependence which is distinct from unit-root processes. As a result, an alternative representation to both short-range and unit-root processes is proposed by Mandelbrot and Van Ness (1968), Granger and Joyeux (1980), Hosking (1981), through the class of *fractionally differenced* processes or *ARFIMA* models. This new class is particularly appealing for embedding both ARMA and ARIMA models.

Mandelbrot (1971) was the first to seize the important implications of *long memory* features in asset returns, and more generally for modern financial economics. More specifically, the presence of long-term memory may critically affect the pricing of derivative securities. Thus Comte, Coutin and Renault (2003), Ohanissian, Russell and Tsay (2003) judiciously incorporate a long memory component in continuous time stochastic volatility models to price options. On the other hand, traditional tests of the capital asset pricing model and the arbitrage theory are no longer valid since the usual forms of statistical inference do not apply to time series exhibiting such persistence. Likewise, test of *efficient markets* hypothesis also hang precariously on the presence or absence of long-term memory[1]

Several empirical studies have supported Mandelbrot's findings, among which Greene and Fielitz (1977), have found long-range dependance in the daily returns of many securities listed on the NYSE. Fama and French (1988), Jegadeesh (1990, 1991),Poterba and Summers (1988) have further noted anomalous behavior in long horizon stock returns.

Thus, several methods to test for long memory can be found in the literature which can be classified in three types: 1) the rescaled range statistic, 2) the periodogram regression test and 3) the KPSS test.

First, Lo (1991) propose a formal test for long-run memory based on the *rescaled range* statistic or R/S statistic, named after the English hydrologist Harold Edwin Hurst (1951).

Mandelbrot (1972, 1975), Mandelbrot and Taqqu (1979), Mandelbrot and Wallis (1968) introduce some refinements to the R/S statistic without distinguishing between short-range and long-range dependence[2]. By incorporating this crucial distinction in the R/S statistic, Lo robustify the R/S statistic to short-range dependence called the *modified R/S* statistic. This one is then applied to daily and monthly stock return indexes and contrary to previous findings, Lo (1991) finds no evidence of long-range dependence once short-range dependence is taken into account.

Second, the other kind of test consists in regressing the logarithm of the periodogram at low frequencies on a function of the frequencies; the expected slope is dependent on the long-memory parameter d. This method was introduced by Geweke and Porter-Hudak (1983) and developed by Robinson (1995). However this test involves the truncation of the ordinates of the periodogram and arises some severe bias issues. No clear rules exists to provide some guidance about such a truncation.

[1] See Mandelbrot (1989) and Merton (1987) for excellent surveys of this literature.

[2] One of the most widely used concepts of short-range dependence is the notion of *strong mixing* due to Rosenblatt(1956), a measure of the decline in statistical dependence between events separated by successively longer spans of time.

Third, Lee and Schmidt (1996) also test for short memory by using the KPSS test of the null hypothesis of stationarity. Described as a test of the trend stationarity hypothesis, the KPSS test is less general than Lo's modified rescaled range test. They show nevertheless that the power of the KPSS test in finite sample is comparable to that of Lo's modified rescaled range test.

Among these three techniques, the R/S statistic appears to be the easiest and the more straightforward to apply. However, the applications of R/S analysis by Mandelbrot and Wallis (1969) and others do suffer from common shortcomings: (i) They provide no sampling theory with which to judge the statistical inference of their empirical results; (ii) they use the classical non-robust short-range R/S statistic; (iii) they do not focus on the R/S statistic itself, but rather on the regression of its logarithm on samples sizes. In particular, Davies and Harte (1987) show such regression tests to be significantly biased toward rejection even for a weakly dependent stationary AR(1) process (with an autoregressive parameter of 0.3). By contrast, Lo (1991) provides a relevany asymptotic sampling theory showing that the modified rescaled range statistic converges weakly under the null of weak dependence toward the range of a Brownian Bridge[3]. He also characterizes the distributional properties under the alternative of strong dependence. However, there is no guarantee anymore for these distributional properties to hold in a stochastic volatility framework.

Indeed, Breidt, Crato and Lima (1998) introduce a long memory component in the conditional variance representation. A long memory stochastic volatility model (LMSV) is constructed by incorporating an ARFIMA process in the standard stochastic volatility scheme [see Harvey, Ruiz and Shephard (1994), Jacquier, Polson and Rossi (1994), Danielsson (1994), Gallant, Hsieh and Tauchen (1997), Tauchen (1997)...]. The authors propose a spectral likelihood estimator for the LMSV parameter vector for which they provide a strong consistency characterization. However no asymptotic distributional theory is available for this spectral estimator, making thus difficult any significant statistical assessment.

By contrast, we propose a test procedure which remains valid in the absence of asymptotic distributional theory in the context of the LMSV model.

In this paper, we focus on hypothesis testing in parametric SV models with short and long memory. Our main objective is to develop both exact tests as well as asymptotically justified procedures that are markedly more reliable than those based on usual large-sample approximations, especially in the presence of non-standard asymptotic distributions. The proposed procedures are also designed to be computationally manageable.

Exploiting the fact that short-range and long-range SV models are parametric models involving only a finite number of unknown parameters, our basic outlook is to develop finite-sample simulation-based procedures as opposed to procedures based on establishing asymptotic distributions. For that purpose, we rely on extensions of the basic idea of Monte Carlo (MC) tests originally proposed by Dwass (1957) and Barnard (1963). When the distribution of a test statistic does not depend on (unknown) nuisance parameters, the technique of MC tests yields an exact test provided one can generate a few i.i.d. (or exchangeable) replications of the test statistic under the null hypothesis; for example, 19 replications are sufficient to get a test with level 0.05; see Dufour and Khalaf (2001). This technique can be extended to test statistics which depend on nuisance parameters

[3]The distribution for the range of the Brownian Bridge is implicitly contained in Feller (1951), and given explicitly by Kennedy (1976) and Siddiqui (1976) as $F_V(v) = 1+2\sum_{k=1}^{\infty}(1-4k^2v^2)\exp^{-2(kv)^2}$. $E(V) = \sqrt{\pi/2}$, $E(V^2) = \pi^2/6$.

by considering maximized Monte Carlo (MMC) tests; see Dufour (2005). MMC tests yield exact tests whenever the distribution of the test statistic can be simulated as a function of the nuisance parameters: no additional assumption on its distribution is needed. Further, computationally more tractable versions of this procedure, such as MMC tests on consistent set estimators of model nuisance parameters, provide asymptotically valid tests irrespective of the presence of non-regularities and non-standard asymptotic distributions. Parametric bootstrap tests may also be interpreted as degenerate MMC tests, where the simulated p-value function is evaluated at a single nuisance-parameter point estimate. However, the asymptotic validity of the parametric bootstrap method requires stronger assumptions than the MMC procedure and it may fail to control the level of the test even asymptotically, especially in non-regular problems (where the MMC procedure remains valid).

To be more specific, the contributions of the paper can be summarized as follows.

The paper is organized as follows.

2. Framework

2.1. Autoregressive SV model

The basic form of the stochastic volatility model we study here comes from Gallant et al. (1997). Let us denote by y_t the variable of interest. For example, y_t can denote the first difference over a short time interval, a day for instance, of the log-price of a financial asset traded on security markets.

Assumption 2.1 *The process $\{y_t : t \in \mathbb{N}\}$ follows a stochastic volatility model of the type:*

$$y_t - \mu_y = \sum_{i=1}^{L_y} c_i (y_{t-i} - \mu_y) + u_t \ , \tag{2.1}$$

$$u_t = \exp(w_t/2) r_y z_t \ , \tag{2.2}$$

$$w_t - \mu_w = \sum_{j=1}^{L_w} a_{wj} (w_{t-j} - \mu_w) + r_w v_t \ , \tag{2.3}$$

where μ_y, $\{c_j\}_{j=1}^{L_y}$, r_y, μ_w, $\{a_{wj}\}_{j=1}^{L_w}$ and r_w are unknown parameters and $s_t = (y_t,\, w_t)'$ is initialized from its stationary distribution.

In the above model, (2.1) is the mean equation, while (2.3) is the volatility equation. We shall call the model represented by (2.1)-(2.3) the stochastic volatility model of order L_w with autoregressive mean of order L_y [ARSV$(L_y,\, L_w)$ for short]. The lag lengths of the autoregressive specifications used in the literature are typically short. Usual configurations are $(L_y,\, L_w) = (0,\, 1)$, $(1,\, 1)$ or $(2,\, 2)$; see Andersen and Sørensen (1996), Gallant et al. (1997) and Andersen, Chung and Sørensen (1999). An important special case of (2.1) - (2.3) consists in setting $\mu_w = 0$, $c_j = a_{wj} = 0, \forall j \geq 2$, and $\delta = (c,\, \theta')'$ with $\theta = \theta_1$, where $\theta_1 = (a_w,\, r_y,\, r_w)'$. We then have:

$$y_t - \mu_y = c(y_{t-1} - \mu_y) + u_t \ , \quad |c| < 1 \ , \tag{2.4}$$

$$u_t = [r_y \exp(w_t/2)] z_t \,, \tag{2.5}$$
$$w_t = a_w w_{t-1} + r_w v_t \,, \quad |a_w| < 1 \,. \tag{2.6}$$

Let $\mathcal{M}_1$ denote the model defined by equations (2.4) to (2.6).

Assumption 2.2 *The vectors $(z_t,\, v_t)'$, $t \in \mathbb{N}$ are i.i.d. according to a $N(0,\, I_2)$ distribution.*

Assumption 2.3 *The process $s_t = (y_t,\, w_t)'$ is strictly stationary.*

The ARSV$(L_y,\, L_w)$ process is Markovian of order $L_s = \max(L_y,\, L_w)$. Let us denote by

$$\delta = (\mu_y,\, c_1,\, \ldots\,,\, c_{L_y},\, r_y,\, \mu_w,\, a_{w1},\, \ldots\,,\, a_{wL_w},\, r_w)' \tag{2.7}$$

the parameter vector of the model. Here $\{y_t\}$ is observed, while $\{w_t\}$ is a latent variable. Accordingly, the joint density of the observation vector $y_{(T)} = (y_1,\, \ldots\,,\, y_T)$ is not available in closed form, for it requires evaluating an integral with dimension equal to the whole path of the latent volatilities. Let

$$F(y_1,\, \ldots\,,\, y_T) = \mathsf{P}[Y_1 \le y_1,\, \ldots\,,\, Y_T \le y_T|\delta] \equiv F_0(y_{(T)}|\delta)$$

denote its unknown distribution function.

We shall now focus on the ARSV$(1,\, 1)$ model. To estimate it, we consider a two-step method whose first step consists in applying ordinary least squares (OLS) to the mean equation which yields a consistent estimate of the autoregressive parameter c and of the mean parameter μ_y, denoted by $\hat{c}$, $\hat{\mu}_y$ and the residuals $\hat{u}_t \equiv u_t(\hat{c}) = y_t - \hat{\mu}_y - \hat{c}(y_{t-1} - \hat{\mu}_y)$. Then, we apply in a second step a method of moments to the residuals $\hat{u}_t$ to get the estimate of the parameter $\theta_1 = (a_w,\, r_y,\, r_w)'$ of the mean and volatility equations. Unlike the other estimators proposed in the financial literature for estimating SV models, this two-step moment estimator is easy to implement and available in closed form, an appealing feature for complicated latent variable models. Besides, its simplicity allows for simulation-based inference and will be further exploited to obtain simulated testing procedures. In the sequel we will focus on the particular case where $\mu_y = 0$ but all the results still hold in the general case.

Under the assumptions **2.1** to **2.3**, with $\mu_y = \mu_w = 0$ and $c_i = a_{wi} = 0$, $\forall i \ge 2$, the perturbation term u_t has the following moments for positive even values of j and k:

$$\mu_k(\theta_1) \equiv \mathsf{E}(u_t^k) = r_y^k \frac{k!}{2^{(k/2)}(k/2)!} \exp\left[\frac{k^2}{8} r_w^2/(1-a_w^2)\right] , \tag{2.8}$$

$$\begin{aligned} \mu_{j,\,k}(l|\theta_1) &\equiv \mathsf{E}(u_t^j u_{t+l}^k) \\ &= r_y^{j+k} \frac{j!}{2^{(j/2)}(j/2)!} \frac{k!}{2^{(k/2)}(k/2)!} \exp\left[\frac{r_w^2}{8(1-a_w^2)}(j^2 + k^2 + 2jk a_w^{|l|})\right] . \end{aligned} \tag{2.9}$$

The odd moments are equal to zero. In particular, for $j = 2$, $j = 4$ and $j = k = 2$ and $l = 1$, we have:

$$\mu_2(\theta_1) = \mathsf{E}(u_t^2) = r_y^2 \exp[(1/2)r_w^2/(1 - a_w^2)]\,, \tag{2.10}$$

$$\mu_4(\theta_1) = \mathsf{E}(u_t^4) = 3r_y^4 \exp[2r_w^2/(1 - a_w^2)]\,, \tag{2.11}$$

$$\mu_{2,\,2}(1|\theta_1) = \mathsf{E}[u_t^2 u_{t-1}^2] = r_y^4 \exp[r_w^2/(1 - a_w)]\,; \tag{2.12}$$

see Dufour and Valéry (2005). Let

$$\kappa = \frac{\mu_4(\theta_1)}{\mu_2^2(\theta_1)} \tag{2.13}$$

be the kurtosis coefficient of the process. It is easy to see that $\kappa \geq 3,$ with $\kappa > 3$ as soon as $r_w \neq 0$ (*i.e.*, when the volatility is constant). Solving the above moment equations corresponding to $j = 2$, $j = 4$ and $l = 1$ yields the following expressions: provided $\kappa > 3$,

$$a_w = \frac{\log[\mu_{2,\,2}(1|\theta_1)] + \log\left[\kappa/3\mu_2^2(\theta_1)\right]}{\log(\kappa/3)} - 1\,, \tag{2.14}$$

hence

$$r_y = \frac{3^{1/4}\mu_2(\theta_1)}{\mu_4(\theta_1)^{1/4}} = \left(\frac{3\mu_2^2(\theta_1)}{\kappa}\right)^{1/4}, \quad r_w = \left[(1 - a_w^2)\log\left(\kappa/3\right)\right]^{1/2}, \quad \text{if } \kappa > 3. \tag{2.15}$$

If $\kappa \leq 3,$ the volatility is constant and it is natural to set

$$a_w = r_w = 0 \;\text{ and }\; r_y = \sqrt{\mu_2(\theta_1)} \;\text{ if }\; \kappa \leq 3\,. \tag{2.16}$$

Given the latter definitions, it is easy to compute a method-of-moment estimator for $\theta_1 = (a_w,\, r_y,\, r_w)'$ replacing the theoretical moments by sample counterparts based on the residuals $\hat{u}_t$. Let $\hat{\theta}_T$ denote the method-of-moments estimator of θ_1. Typically, $\mathsf{E}(u_t^2)$, $\mathsf{E}(u_t^4)$ and $\mathsf{E}(u_t^2 u_{t-1}^2)$ are approximated by:

$$\hat{\mu}_2 = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2 \quad \hat{\mu}_4 = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^4\;, \quad \hat{\mu}_2(1) = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2\hat{u}_{t-1}^2$$

respectively. This yields the following estimators of the stochastic volatility coefficients:

$$\hat{a}_w = \begin{cases} \varDelta & \text{if } \tilde{a}_w > \varDelta\,, \\ \tilde{a}_w & \text{if } |\tilde{a}_w| \leq \varDelta\,, \\ -\varDelta & \text{if } \tilde{a}_w < -\varDelta\,, \end{cases} \tag{2.17}$$

$$\begin{aligned} \hat{r}_y &= \left(3\hat{\mu}_2^2/\hat{\kappa}\right)^{1/4} && \text{if } \hat{\kappa} > 3, \\ &= \hat{\mu}_2^{1/2} && \text{if }\ \hat{\kappa} \leq 3, \end{aligned} \tag{2.18}$$

$$
\begin{aligned}
\hat{r}_w &= \left[(1-\hat{a}_w^2)\log\left(\hat{\kappa}/3\right)\right]^{1/2} && \text{if } \hat{\kappa} > 3, \\
&= 0 && \text{if } \hat{\kappa} \leq 3,
\end{aligned}
\tag{2.19}
$$

where $\hat{\kappa} = \hat{\mu}_4/\hat{\mu}_2^2$ and

$$
\begin{aligned}
\tilde{a}_w &= \left[\log[\hat{\mu}_2(1)] + \log\left(\hat{\kappa}/3\hat{\mu}_2^2\right)\right]/\log\left(\hat{\kappa}/3\right) && \text{if } \hat{\kappa} > 3, \\
&= 0 && \text{if } \hat{\kappa} \leq 3.
\end{aligned}
\tag{2.20}
$$

In (2.17), $\varDelta$ is a number close to one which is used to bound the estimator away from the stationary boundary. This is important to avoid numerical instability. In the simulations and application below, we used $\varDelta = 0.99,$ but a value closer to one could be considered. Under the assumptions of the model, the restriction $\hat{\kappa} \geq 3$ must hold with probability converging to one. Provided $|a_w| < \varDelta,$ the estimator $\hat{\theta}_T = [\hat{a}_w,\, \hat{r}_y,\, \hat{r}_w]'$ is consistent and asymptotically normally distributed; see Dufour and Valéry (2005) for a detailed presentation of its asymptotic properties.

2.2. The fractionally integrated SV model

Let $\mathcal{M}_2$ denote the ARFIMA(0,d,0)-SV model defined as follows:

$$
y_t - \mu_y = c(y_{t-1} - \mu_y) + v_t(\theta_2)\ , \quad |c| < 1\ , \tag{2.21}
$$

$$
v_t(\theta_2) = [r_y \exp(w_t/2)] z_t\ , \tag{2.22}
$$

$$
(1-B)^d w_t = \eta_t \quad , \quad \eta_t \overset{i.i.d.}{\sim} N(0, \sigma_\eta^2)\ , \tag{2.23}
$$

where $d \in (-0.5, 0.5)$ and $\theta_2 = (d, r_y, \sigma_\eta^2)'$. When d is restricted to this domain, w_t is stationary and invertible [see Hosking (1981)]. We briefly review the first two moments of $v_t(\theta_2)$ obtained from the properties of the log-normal distribution as it is stated in Breidt, Crato and de Lima (1998):

$$
\mu_2(\theta_2) = E(v_t(\theta_2)^2) = r_y^2 \exp[\gamma(0)/2]\ , \tag{2.24}
$$

$$
\mu_4(\theta_2) = E(v_t(\theta_2)^4) = 3r_y^4 \exp[2\gamma(0)]\ , \tag{2.25}
$$

and

$$
\mu_{2,2}(h|\theta_2) = E[v_t(\theta_2)^2 v_{t-h}(\theta_2)^2] = r_y^4 \exp[\gamma(0)(1+\rho(h))]\ , \tag{2.26}
$$

where the auto-covariance and autocorrelation functions for the long-memory process $\{w_t\}$, denoted by $\gamma(\cdot)$ and $\rho(\cdot)$ are given by:

$$
\gamma(0) = \sigma_\eta^2 \frac{\varGamma(1-2d)}{\varGamma^2(1-d)}\ , \tag{2.27}
$$

$$
\rho(h) = \frac{\varGamma(h+d)\varGamma(1-d)}{\varGamma(h-d+1)\varGamma(d)}\ , \quad , h = 1, 2, \ldots, \tag{2.28}
$$

[see Brockwell and Davis, (1991), p.522]. When d is restricted to $(-0.5, 0.5)$, the process exhibits a unique kind of dependence that is positive or negative depending on whether d is positive or negative, *i.e.* the autocorrelation coefficients of w_t are of the same sign as d. So slowly do the autocorrelations decay that when d is positive their sum diverges to infinity, and collapses to zero when d is negative. Mandelbrot and others have called the $d < 0$ case *antipersistence*, reserving the term *long-range dependence* for the $d > 0$ case.

The moment conditions (2.24) to (2.26) will define a just-identified GMM estimator of $\theta_2 = (d, r_y, \sigma_\eta^2)'$ for drawing inference on the fractional integration parameter d.

3. Testing problems and test statistics

In this section, we set the framework for testing general hypotheses as $H_0(\psi_0) : F \in \mathcal{H}_0(\psi_0)$ where $\mathcal{H}_0(\psi_0)$ is a subset of all possible distributions for the ARSV(1,1) model (2.4)-(2.6) [or for the ARFIMA(0,d,0)-SV model (2.21)-(2.23)], that is,

$$\mathcal{H}_0(\psi_0) \equiv \{F(\cdot) : F(y_{(T)}) = F_0[y_{(T)}|\delta] \text{ with } \psi(\theta) = \psi_0\} \tag{3.1}$$

where $\delta = (c, \theta^{'})^{'}$, $\psi(\theta)$ is a $p \times 1$ continuously differentiable function of θ and ψ_0 is the hypothesized value of $\psi(\theta)$, such as $\psi_0 = 0$. $H_0(\psi_0)$ is usually abbreviated as:

$$H_0(\psi_0) : \psi(\theta) = \psi_0\,.$$

Let $\hat{\theta}$ be the unrestricted estimator and $\hat{\theta}^0$ the constrained estimator obtained by minimizing the following criterion

$$M_T^*(\theta) \equiv [\bar{g}_T(\hat{U}_T) - \mu(\theta)]'\hat{\Omega}_*^{-1}[\bar{g}_T(\hat{U}_T) - \mu(\theta)] \tag{3.2}$$

where $\bar{g}_T(\hat{U}_T)$ denotes the vector of empirical moments based on the residual vector $\hat{U}_T$ corresponding to $\mu(\theta)$. $\hat{\Omega}_*$ denotes a consistent estimator of Ω_*,

$$\Omega_* = \lim_{T\to\infty} \mathsf{E}\{T\left[\bar{g}_T(U_T) - \mu(\theta_0)\right]\left[\bar{g}_T(U_T) - \mu(\theta_0)\right]'\}\,, \tag{3.3}$$

with θ_0 denoting the true value of θ, [$\theta = \theta_1$ for the ARSV(1,1) $\mathcal{M}_1$ model and $\theta = \theta_2$ for the ARFIMA(0,d,0)-SV $\mathcal{M}_2$ model. Such an estimator $\hat{\Omega}_*$ can easily be obtained [see Newey and West (1987*b*)] using a Bartlett kernel:

$$\hat{\Omega}_* = \hat{\Gamma}_0 + \sum_{k=1}^{K(T)} \Big(1 - \frac{k}{K(T)+1}\Big)(\hat{\Gamma}_k + \hat{\Gamma}_k') \tag{3.4}$$

where

$$\hat{\Gamma}_k = \frac{1}{T}\sum_{t=k+1}^{T} [g_{t-k}(\hat{U}_T) - \mu(\tilde{\theta})][g_t(\hat{U}_T) - \mu(\tilde{\theta})]' \tag{3.5}$$

where $\tilde{\theta}$ is a consistent estimator of θ, $g_t(\hat{U}_T) = [\hat{u}_t^2,\ \hat{u}_t^4,\ \hat{u}_t^2\hat{u}_{t-1}^2]'$ for the ARSV(1,1) $\mathcal{M}_1$ model (2.4) - (2.6), and $g_t(\hat{U}_T) = [\hat{v}_t(\theta_2)^2,\ \hat{v}_t(\theta_2)^4,\ \hat{v}_t(\theta_2)^2\hat{v}_{t-1}(\theta_2)^2]'$ for the ARFIMA(0,d,0)-SV $\mathcal{M}_2$ model (2.21) - (2.23), respectively. In a just-identified framework, the choice of weight metric $\hat{\Omega}_*^{-1}$

is irrelevant.

3.1. Specification test

In this section, we propose a specification test to test the null hypothesis that the true model, denoted by $\mathcal{M}_2$ corresponds to the ARFIMA(0,d,0)-SV model defined in equations (2.21)-(2.23) against the alternative ARSV(1,1) model [*i.e.* the $\mathcal{M}_1$ model defined in equations (2.4)-(2.6)]. Then, the likelihood-ratio-type test statistic for comparative fit that is investigated here is given by:

$$\tilde{\xi}_T^C = T[M_T^*(\hat{\theta}_{2T}|\mathcal{M}_2) - M_T^*(\hat{\theta}_{1T}|\mathcal{M}_1)] \tag{3.6}$$

where

$$M_T^*(\theta|\mathcal{M}_i) \equiv [\bar{g}_T(\hat{U}) - \mu(\theta|\mathcal{M}_i)]'\hat{\Omega}^{*-1}[\bar{g}_T(\hat{U}) - \mu(\theta|\mathcal{M}_i)], \quad i = 1, 2\ . \tag{3.7}$$

3.2. Hypothesis tests

In the simulations, we will focus on parametric functions of the form

$$\psi(\theta) = (1,\ 0)\begin{pmatrix}\theta_{s1}\\ \theta_{s2}\end{pmatrix} = \theta_{s1}\ ,$$

in which case the null hypothesis $H_0(\psi_0)\ :\ \psi(\theta) = \psi_0$ simplifies to $H_0(\psi_0)\ :\ \theta_{s1} = \theta_{s1}^0$. For example, we may have $\theta_{s1} \equiv d$.

We will test two null hypotheses of interest; one corresponds to test no fractional integration by testing $d = 0$ (against fractional integration) while the other one, namely $d = 0.4$ corresponds to test a parameter value close to the boundaries of the stationarity domain. In particular, in the $d = 0.4$ case, the standard asymptotic theory might not behave properly anymore as we get closer to the boundaries of the domain.

Inference on the parameter of interest will be drawn by means of a LR-type statistic. Thus, the difference between the restricted and unrestricted optimal values of the objective function yields the following LR-type statistic:

$$\xi_T^C = T[M_T^*(\hat{\theta}^0) - M_T^*(\hat{\theta})]\ . \tag{3.8}$$

Provided

$$T[M_T^*(\theta) - M_T(\theta)] \underset{T\to\infty}{\longrightarrow} 0 \tag{3.9}$$

where

$$M_T(\theta) \equiv [\bar{g}_T(\hat{U}_T) - \mu(\theta)]'\Omega_*^{-1}[\bar{g}_T(\hat{U}_T) - \mu(\theta)]\ , \tag{3.10}$$

the LR-type statistic ξ_T^C is well-defined.

However, the asymptotic distribution of both LR-type statistics aforementioned for testing model specification and parameter hypotheses has not been stated analytically. Therefore, using the standard χ^2 critical points may be strongly misleading. Hence, an easy and straightforward solution to circumvent a defaulting asymptotic distributional theory is to resort to the technique of Monte Carlo tests which is *provably valid* under general regularity conditions [see Dufour (2005)].

4. Monte Carlo tests

The technique of Monte Carlo tests was originally proposed by Dwass (1957) for implementing permutation tests and did not involve nuisance parameters. This technique was also independently proposed by Barnard (1963) and Birnbaum (1974); for a review, see Dufour and Khalaf (2001). It has the great attraction of providing *exact* (randomized) tests based on any statistic whose finite sample distribution may be intractable but can be simulated. We briefly review the methodology of Monte Carlo tests covering both cases, first without nuisance parameters and then with nuisance parameters. The technique of Monte Carlo tests provides a simple method allowing one to replace the unknown or intractable theoretical distribution $F(y|\delta)$, where $\delta = (c,\, \theta')'$, by its sample analogue based on the statistics $S_1(\delta),\, \ldots\,,\, S_N(\delta)$ simulated under the null hypothesis. We shall now describe how MC tests can be performed in practice.

For the sake of clarity, let us first consider the case where no nuisance parameters are present.

1. Using the observed sample, we calculate the relevant statistic S_0.

2. Using draws under H_0, we generate N simulated samples $S_1,\, \ldots\,,\, S_N$.

3. Then we consider the following simulated survival function

$$\hat{G}_N[y;\, S(N)] = \frac{1}{N}\sum_{i=1}^{N} s(S_i - y)$$

   and the associated p-value function

$$\hat{p}_N(y) = \frac{N\hat{G}_N(y) + 1}{N + 1}$$

   where $s(x) = 1$ if $x \geq 0$, and $s(x) = 0$ if $x < 0$. If the distribution of S is continuous and N is chosen so that $\alpha(N + 1)$ is an integer, then

$$\mathsf{P}[\hat{p}_N(S_0) \leq \alpha] = \alpha\ ,\ \text{under } H_0,$$

   yielding an exact test.

In most econometric models, the relevant case is the one where the distribution of the test statistic depends on nuisance parameters. To deal with this complication, the MC test procedure can be modified as follows, where $\bar{\delta}$ represents the true parameter vector.

1. To test the null hypothesis

$$H_0 : \bar{\delta} \in \Omega_0\ ,$$

   we use first the observed sample to calculate the relevant statistic denoted by S_0.

2. For each $\delta \in \Omega_0$, we generate N replications of S: $S_1(\delta),\, \ldots\,,\, S_N(\delta)$.

3. Using these simulations we compute the corresponding simulated p-value function:

$$\hat{p}_N[y|\delta] = \frac{N\hat{G}_N[y|\delta] + 1}{N+1} \; .$$

4. The p-value function $\hat{p}_N[S_0|\delta]$ as a function of δ is maximized over the parameter values compatible with the null hypothesis (Ω_0), and H_0 is rejected if

$$\sup\{\hat{p}_N(S_0|\delta) : \delta \in \Omega_0\} \leq \alpha \, . \tag{4.1}$$

   If the number of simulated statistics N is chosen so that $\alpha(N+1)$ is an integer, then we have under H_0:

$$\mathsf{P}[\sup\{\hat{p}_N(S_0|\delta) : \delta \in \Omega_0\} \leq \alpha] \leq \alpha \, , \tag{4.2}$$

   that is we control for the level of the test [for a proof, see Dufour (2005)].

Because of the maximization, the critical region in (4.1) is called a *maximized Monte Carlo* (MMC) test. MMC tests provide valid inference under general regularity conditions such as almost-unidentified models or time series processes involving unit roots. In particular, even though the moment conditions defining the estimator are derived under the stationarity assumption, this does not question in any way the validity of *maximized* MC tests, unlike the parametric bootstrap whose distributional theory is based on strong regularity conditions. Only the power of MMC tests may be affected.

A simplified approximate version of the MMC procedure can alleviate the computational load of the MMC procedure whenever a consistent point or set estimate of δ is available. To do this, we shall need to reformulate the setup in order to allow for an increasing sample size.

1. To test the null hypothesis

$$H_0 : \bar{\delta} \in \Omega_0 \; , \quad \text{with} \;\; \Omega_0 \in \Omega, \quad \Omega_0 \neq \emptyset \, ,$$

   we use first the observed sample to calculate the relevant statistic denoted by S_{T0}.

2. We consider C_T, $T \geq I_0$ a sequence of (possibly random) subsets of Ω instead of Ω_0, such that

$$\lim_{T\to\infty} \mathsf{P}[\bar{\delta} \in C_T] = 1 \text{ under } H_0. \tag{4.3}$$

3. For each $\delta \in C_T$, we generate N replications of S: $S_{T1}(\delta), \ldots, S_{TN}(\delta)$, with $T \geq I_0$.

4. Using these simulations we compute the corresponding simulated p-value function:

$$\hat{p}_{TN}[y|\delta] = \frac{N\hat{G}_{TN}[y|\delta] + 1}{N+1} \; .$$

5. The p-value function $\hat{p}_{TN}[S_{T0}|\delta]$ is maximized with respect to δ in $C_T,$ and H_0 is rejected if

$$\sup\{\hat{p}_{TN}(S_{T0}|\delta) : \delta \in C_T\} \leq \alpha\,. \tag{4.4}$$

If the number of simulated statistics N being chosen so that $\alpha(N+1)$ is an integer, we have under H_0:

$$\lim_{T\to\infty} \mathsf{P}[\sup\{\hat{p}_{TN}(S_{T0}|\delta) : \delta \in C_T\} \leq \alpha] \leq \alpha\ , \tag{4.5}$$

i.e., we control for the level asymptotically.

In practice, it is easy to find a consistent set estimate of $\bar{\delta}$, whenever a consistent point estimate $\hat{\delta}_T$ of $\bar{\delta}$ is available. For instance, any set of the form

$$C_T = \{\delta \in \Omega\ : \|\hat{\delta}_T - \delta\| < d\} \tag{4.6}$$

with d a fixed positive constant independent of T, satisfies (4.3). It is worth noting that there is no need to maximize the p-value function with respect to possibly unidentified parameters under the null hypothesis. Thus, parameters which are unidentified under the null hypothesis can be set to any fixed value and the maximization be performed only over the remaining identified nuisance parameters. When there are several nuisance parameters, one can use simulated annealing [see Goffe, Ferrier and Rogers (1994)], an optimization algorithm which does not require differentiability. Indeed $\hat{G}_N[S_0|\delta]$ is a step-type function which typically has zero derivatives almost everywhere, except on isolated points where it is not differentiable. For an example, where this is done using in a VAR model involving a large number of nuisance parameters, see **?**.

Finally, if the set C_T in (4.4) is reduced to a single point estimate $\hat{\delta}_T,$ *i.e.* $C_T = \{\hat{\delta}_T\},$ we get a *local MC* (LMC) test

$$\hat{p}_{TN}(S_{T0}|\hat{\delta}_T) \leq \alpha\,, \tag{4.7}$$

which can be interpreted as a *parametric bootstrap* test. Even if $\hat{\delta}_T$ is a consistent estimate of δ (under the null hypothesis), the condition (4.3) is not usually satisfied in this case, so that additional assumptions are needed to show that the parametric bootstrap procedure yields an asymptotically valid test. It is computationally less costly but clearly less robust to violations of regularity conditions than the MMC procedure; for further discussion, see Dufour (2005).

5. Simulation results

In this section, we present some simulation evidence on the finite-sample properties of the procedures described in the previous sections. In particular, we provide results on the actual level of LR-type tests for the three main hypotheses discussed: (1) the hypothesis of autoregressive volatility (against fractionally integrated volatility) by means of a specification test; (2) the hypothesis of no fractional integration in the volatility ($d = 0$) (against fractional integration $d \neq 0$); (3) the hypothesis of fractional integration in the volatility for specific value of the parameter (*i.e.* $d = 0.4$). Three ways of implementing the tests are considered: asymptotic critical values, parametric bootstrap, and MMC. We also present results on power for the three types of hypotheses described above.

The LR-type test statistic $LR(\hat{\Omega}) \equiv \xi_T^C$ corresponds to the difference between the restricted and the unrestricted optimal values of the objective function, with $\hat{\Omega} \equiv \Omega(\hat{\theta})$. The weighting matrix $\hat{\Omega}$ is estimated by a kernel estimator with fixed-bandwith Bartlett kernel, where the lag truncation is set at $K = 2$ [see Newey and West (1987*a*)].

Let S denote the test statistic which takes the form of the LR-type statistic, and S_0 the statistic computed from the "pseudo-true" data obtained by simulation under the true data generating process. The critical regions have the following forms:

$$\mathcal{R}_a = \{S_0 > \chi^2_\alpha(\nu)\}$$

for the standard asymptotic tests, where $\mathsf{P}[\chi^2(\nu) \geq \chi^2_\alpha(\nu)] = \alpha$ and ν is the number of constraints tested,

$$\mathcal{R}_B = \{\hat{p}_N[S_0|\hat{\delta}^0] \leq \alpha\}$$

for the bootstrap test, and

$$\mathcal{R}_{MMC} = \left\{ \sup\{\hat{p}_{TN}(S_{T0}|\delta) : \delta \in C_T\} \leq \alpha \right\},$$

where

$$\begin{aligned} \hat{p}_N[x|\delta] &= \frac{N\hat{G}_N[x|\delta] + 1}{N+1}, \\ \hat{G}_N[x;\, S(N,\, \delta)] &= \frac{1}{N}\sum_{i=1}^{N} s\big(S_i(\delta) - x\big), \end{aligned}$$

$\hat{\delta}^0$ is a consistent point restricted estimate $\delta = (c,\, \theta')'$, θ is the vector of the SV parameters [*e.g.*, $\theta = \theta_1 = (a_w,\, r_y,\, r_w)'$ for the ARSV model, $\theta = \theta_2 = (d,\, r_y,\, \sigma_\eta)'$ for the ARFIMA(0,d,0)-SV model], and C_T is a restricted consistent set estimator of δ.

For MMC tests of the specification hypothesis (ARFIMA(0,d,0)-SV model against ARSV(1,1) model), the set C_T over which we maximize the simulated $p - value$ is:

$$C_T^{(1)} = \{(\, c,\, r_y\,) : |c - \hat{c}| \leq 0.20,\ |c| \leq 0.99,\ |r_y - \hat{r}_y| \leq 0.20,\ , \tag{5.1}$$

and for MMC tests of the no-fractional-integration hypothesis $(d = 0)$ [or of fractional-integration hypothesis $(d = 0.4)$ respectively], the set C_T over which we maximize the simulated $p - value$ is:

$$C_T^{(2)} = \{(\, c,\, r_y,\, \sigma_\eta) : |c - \hat{c}| \leq 0.20,\ |c| \leq 0.99,\ |r_y - \hat{r}_y^{(1)}| \leq 0.20,\ |\sigma_\eta - \hat{\sigma}_\eta^{(1)}| \leq 0.20\}, \tag{5.2}$$

where $\hat{c}$ is the least squares estimate of c [based on fitting the AR(1) model (2.1) with no drift], $\hat{r}_y$ the unrestricted GMM estimate and $(\hat{r}_y^{(1)},\, \hat{\sigma}_\eta^{(1)})$ is the restricted GMM estimate of $(r_y,\, \sigma_\eta)$ in the ARFIMA(0,d,0)-SV model [based on minimizing $M_T^*(\theta)$ subject to the restriction $d = 0$ (or $d = 0.4$ respectively)]. Since the number of nuisance parameters is relatively small, maximization was achieved through a grid search (with points separated by a distance of 0.05 for each coefficient).

Note that many other restricted consistent estimates of the relevant nuisance parameters could be used to build the sets C_T.

The nominal level is $\alpha = 0.05$. The number of replications used for Monte Carlo tests is $N = 99$, while the rejection frequencies are estimated with $M = 1000$. T is the sample size of the series y_t whose data generating process is assumed to be specified as in equations (2.4)-(2.6) for the ARSV model $\mathcal{M}_1$ and as in equations (2.21)-(2.23) for the ARFIMA(0,d,0)-SV model. Calculation were performed with the GAUSS software (version 3.2.37). The autoregressive parameters c, a_w (and the fractional integration parameter d, respectively) are restricted to an interval inside $(-1, 1)$ [$(-0.5, 0.5)$ respectively] to guarantee stationarity and invertibility requirements.

In the power study (Section 5.2), the asymptotic critical points are *locally level-corrected, i.e.* the critical points are modified to ensure that the rejection frequency under the null hypothesis (for the specific nuisance parameter values considered) is equal to 0.05; the corrected critical value is obtained by simulating the test statistic under the null hypothesis with a large number of replications.

As for simulating the ARFIMA(0,d,0)-SV model, we resort to the infinite-order MA process, namely

$$(1 - B)^d w_t = \eta_t$$

or equivalently,

$$w_t = (1 - B)^{-d}\eta_t = B(L)\eta_t = \sum_{k=0}^{\infty} B_k \eta_{t-k} \ , \quad with \quad B_k = \frac{\Gamma(k + d)}{\Gamma(d)\Gamma(k + 1)} \ .$$

In practice, the infinite-order MA representation has been truncated to $k = 60$.

5.1. Level

5.2. Power

6. Empirical application

7. Concluding remarks

Table 1. Empirical levels of asymptotic, bootstrap and MMC tests.

(A) Specification test									
$H_0 : \theta = \theta_2 = (d, r_y, \sigma_\eta)'$ (ARFIMA(0,d,0)-SV)									
ARFIMA(0,d,0)-SV: $d = 0.3,\ r_y = \sigma_\eta = 0.5,$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	0	3.5	2.3	0	4.4	3.5	0.1	4.7	3.9
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	0.5	6.7	4.2	1.8	7.6	4.8	5.3	8.8	6

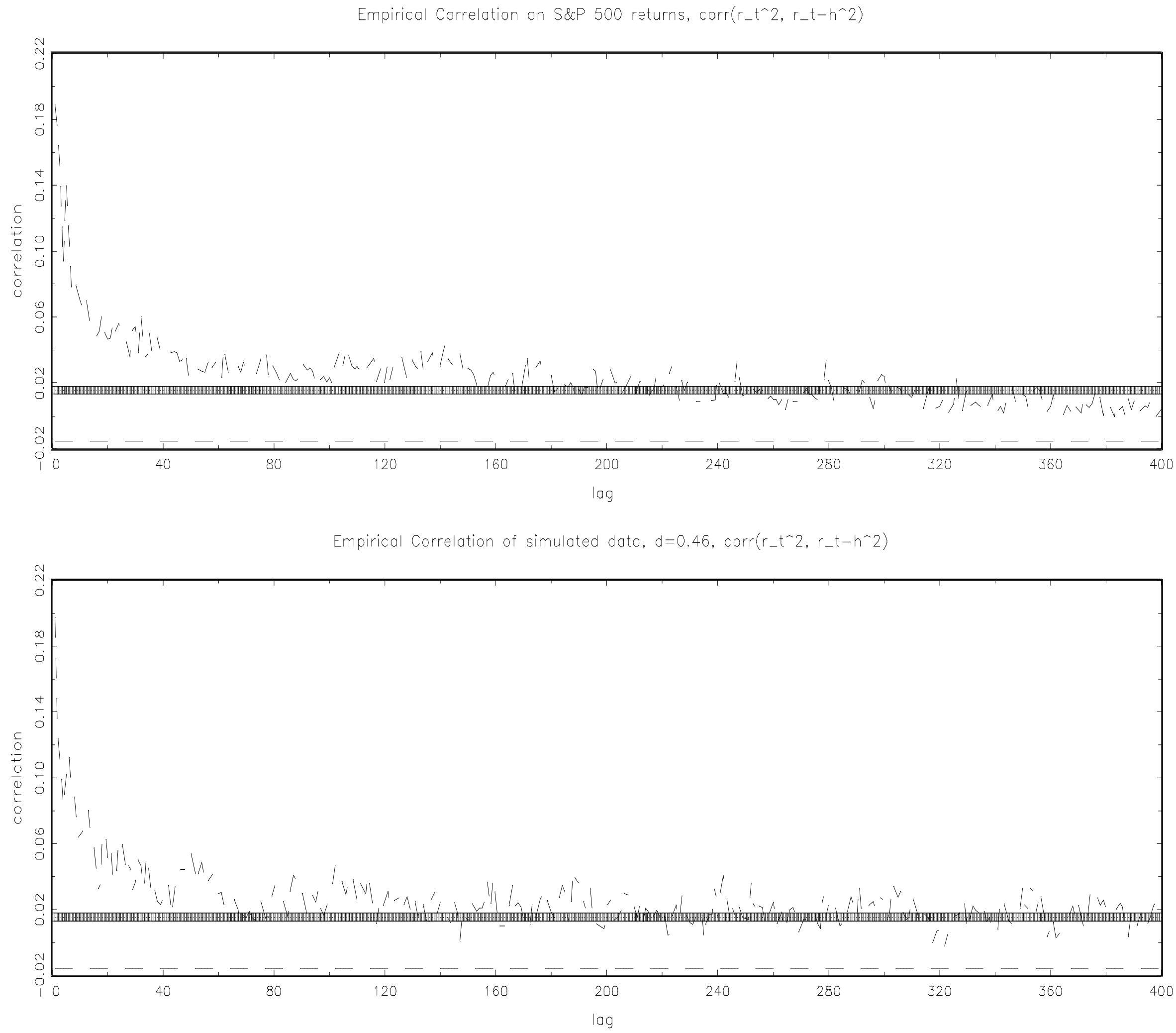
(B) Hypothesis test									
$H_0 : d = 0$ (no-fractional integration)									
ARFIMA(0,d,0)-SV: $c = 0.3,\ r_y = \sigma_\eta = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	0	8.3	0	0.1	8.9	0	0.2	6	0
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	0.4	5.8	0	0.2	4.7	0	0.9	4.2	0
$H_0 : d = 0.4$ (fractional integration)									
ARFIMA(0,d,0)-SV: $c = 0.3,\ r_y = \sigma_\eta = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	0	8.3	0.4	0	12.3	0.1	2.4	21.2	0.5
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR	10	16.8	0.9	25.8	16.9	0.9	60.3	12.8	1.4

Table 2. Empirical power of asymptotic, bootstrap and MMC tests.

(A) Specification test									
$H_0 : \theta = \theta_2 = (d, r_y, \sigma_\eta)'$ (ARFIMA(0,d,0)-SV)									
ARFIMA(0,d,0)-SV: $d = 0.3,\ r_y = \sigma_\eta = 0.5,$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR									
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR									6

(B) Hypothesis test									
$H_0 : d = 0$ (no-fractional integration)									
ARFIMA(0,d,0)-SV: $c = 0.3,\ r_y = \sigma_\eta = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR									
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR									
$H_0 : d = 0.4$ (fractional integration)									
ARFIMA(0,d,0)-SV: $c = 0.3,\ r_y = \sigma_\eta = 0.5$									
	$T = 50$			$T = 100$			$T = 500$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR									
	$T = 1000$			$T = 2000$			$T = 5000$		
	Asy	Boot	MMC	Asy	Boot	MMC	Asy	Boot	MMC
LR									

Figure 1. Correlation functions for the S& P 500 index and artificial data.



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