A war of attrition with endogenous effort levels

Johannes Hörner   Nicolas Sahuguet
Yale University*   HEC Montréal†

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Abstract

This paper extends the classic war of attrition to allow for a wide range of actions. Players alternate making arbitrary payments, and their opponent may either match this payment, or concede. We analyze both cases of complete and incomplete information. As opposed to the classic war of attrition, the equilibrium is unique, rent-dissipation is only partial, and weaker (lower valuation) players concede more quickly than stronger players.

*Department of Economics, Yale University, Box 208281, New Haven, CT 06520-8281

†Institute of Applied Economics, 3000, chemin de la Côte-Sainte-Catherine, Montréal (Québec) Canada H3T 2A7

1. Introduction

In the war of attrition, learning is trivial. Because players are restricted to two actions, quitting or waiting, each point in time corresponds to a unique information set, leaving little room for strategic posturing. Indeed, the war of attrition is strategically equivalent to a static game, the second-price all-pay auction. Yet it is difficult to think of a single application of the war of attrition in which players would not dispose of various instruments allowing for signalling. In industrial organization, the war of attrition has been used to model the dynamics of industry exit in oligopoly. Yet whichever way one models short-run competition, the firms' actions affect their flow payoffs. In political economy, it has been used to model lobbying. Yet here as well, there is no institution that would fix the amount per unit of time that lobbyists can contribute, and in fact all empirical evidence suggests that contributions vary over time. In biology, where the war of attrition was first introduced, animal fights are known to involve intricate strategic patterns, with each tactic entailing specific fitness costs to both parties. And finally, in the most striking application of wars of attrition, attrition warfare studies the most appropriate means to wear down one's enemy to the point of collapse, by continuous losses in personnel and material.

Costly signalling is documented in various settings, from economics to biology. In economics, there is a broad class of bargaining models in which actions can vary over time, providing ample opportunity for signalling and learning. However, offers that have been turned down do not affect payoffs. Bargaining does not generalize the war of attrition, and none of the standard applications of the war of attrition can meaningfully be studied as bargaining models. Yet in all of these applications, players can affect payoffs and beliefs through their actions over time.

This paper examines the robustness of the predictions of the war of attrition to such opportunities. We generalize the classic war of attrition by allowing players to vary the amount of resources expanded during the game. At any point in time, players choose how much to spend, just as in an all-pay auction. However, unlike an all-pay auction, the game is dynamic, so that bids convey information about valuations. Bids are incremental and sunk, and players must either keep matching their opponent's total expenditures, or drop out of the contest.

Taking into explicit consideration such jump bids changes predictions dramatically, both under complete and incomplete information. Relative to the classic war of attrition, we show that the identity of the winner changes, as does expected delay and rent dissipation.

Under complete information about valuations, investment leads to a preemption motive. The player with the larger value may bid an amount of resources that the other player will not will to match. This threat is sufficient to guarantee that the lower-valued player has no interest in starting competing in the contest.[1] We show that these differing results about rent and delay generalize to the case of incomplete information. Further, investing resources also has a signalling effect. Players have the incentive to invest resources to show their opponent that their valuation is high and that continuing the war is not profitable.

1.1. Model and Results

We analyze a war of attrition (or dynamic all-pay auction) in which two impatient players compete for an indivisible prize. Time is discrete and each player chooses one of a continuum of possible effort levels, or bids, in turn. Bids are sunk. After a player's bid, his opponent must either concede or at least match this bid. Therefore, our model differs from the standard discrete-time version of the war of attrition in two respects. First, our model has alternating moves. More importantly, players choose the amount of resources to expand at each period of time. As usual, the game ends when a player drops out. Our extensive form is related to the games analyzed in Harris and Vickers (1985) and Leininger (1991) in the context of patent races under complete information. More recently, Dekel, Jackson and Wolinsky (2006 b,c) have analyzed dynamic all-pay auctions with a similar sequential structure, in the context of vote buying.[2] Finally, our model can be viewed as a natural extension of the dollar auction to the case of private values.

[1] McAfee (2000) makes a similar point: "From an economic perspective, the defect in the theory arises because the low cost player is forbidden by assumption from fully exploiting his low cost. The low cost player might like to present a show of force so large that the high cost player is forced to exit, but the usual game prohibits such endogenous effort. In most actual wars of attrition, players have the ability to increase their effort, so as to force the other side out."

[2] See also Dekel, Jackson and Wolinsky (2006, a) for an interpretation of this extensive form as an all-pay auction with jump bidding.

To put our contribution in perspective, it is important to compare our results to those of the classic war of attrition. For the comparison to be meaningful, the appropriate benchmark is the war of attrition with alternating moves. The war of attrition is better known in its simultaneous-move version, but differences between the two versions vanish as the length between periods converges to zero, which is the focus of the analysis.

Results under complete information typically focus on the equilibria in mixed strategies. (The game admits other asymmetric equilibria, in which one player gives up immediately with high probability.) On the one hand, such equilibria exhibit delay, a desirable property in applications. On the other hand, as has long been recognized, some of its predictions fail to be convincing. In equilibrium, neither player gets an expected benefit from the game, no matter how asymmetrical players' valuations are. Further, the higher a player's valuation, the more unlikely he is to win the war.

We show that these conclusions are reversed when players can modulate their effort levels. First of all, the subgame-perfect equilibrium is unique under complete information. Second, delay essentially disappears, as one of the players quits no later than in the second period. Third, rent is only partially dissipated. In fact, if players are sufficiently patient and have unequal valuations for the prize, there is no rent dissipation whatsoever. The stronger player wins then at no cost. Taken together, these results cast serious doubts about the appropriateness of the war of attrition in many of its traditional applications.

Our model generates interesting and intricate strategic behavior under incomplete information. In the classic war of attrition, incomplete information is often viewed as a purification device of the mixed-strategy equilibrium under complete information. That is, the outcome of both formulations are distributionally equivalent (see Fudenberg and Tirole (1991) and Milgrom and Weber (1985)), with the added finding that best-replies are increasing, e.g. the higher a player's valuation, the longer he stays in the race. In our model, the *undefeated equilibrium* (Mailath, Okuno-Fujiwara and Postlewaite, 1993) is unique.[3] Our predictions differ

[3] The larger the set of actions, the larger the signalling opportunities: therefore, under incomplete information, it is necessary to impose a refinement that rules out "implausible" updating rules. Undefeated equilibrium has been previously defined only for signalling games (with one-sided incomplete information); we therefore provide an extension of its definition to games with two-sided incomplete information. Observe that this solution concept -as all other belief-based refinements- does not refine the set of equilibria in the classic

from the classic war of attrition: (i) the expected delay is shorter; (ii) rent dissipation is also smaller.[4] Finally, (iii) our model allows us to study the variation of contributions over time, which cannot be done, by assumption, in the classic war of attrition. If perceived (ex-ante) differences are large enough, instant concession takes place without the need of signaling. If these differences are small, however, signaling does take place. A player with the hand gains from being perceived as strong, but loses from such a perception if his opponent has the hand. Escalation is usually understood as the possibility that the resources spent in a contest may end up exceeding the value of winning.[5] Our model allows us to identify conditions under which, in equilibrium, bids themselves are increasing over time. To understand this last result, observe that, as time passes, all but the highest valuations drop out, so that effective preemption calls for increasing bids.

1.2. Literature

First and foremost, this paper relates to the literature on the war of attrition, which we review first. As players are allowed to vary their effort levels, the present model is also connected to contests and in particular to all-pay auctions. Finally, it is also useful to link our contribution to other papers developing ideas of preemption, signalling and escalation in other contexts.

The literature on the war of attrition is vast. Standard expositions can be found in Maynard Smith (1974), Riley (1980) and Hendricks, Weiss and Wilson (1988). The discrete-time version has been thoroughly investigated by Hendricks and Wilson (1985). Various extensions are discussed in Fudenberg and Tirole (1986), Kornhauser, Rubinstein and Wilson (1989), Ponsatí and Sákovics (1995) and Myatt (2005). Because the war of attrition provides an explanation for delay, the war of attrition has been very successfully applied to various economic issues: strikes in labor economics (Kennan and Wilson, 1989); exit in declining industries (Fudenberg and Tirole (1986) and Ghemawat and Nalebuff (1985, 1990) for instance); macroeconomic stabilization (Alesina and Drazen, 1991); standard adoption (Farrell and Saloner, 1988).

war of attrition.

[4] This result is in line with other models of jump-bidding in auctions. The revenue in auctions where jump-bidding is allowed is in general lower than in auctions in which jump-bidding is forbidden.

[5] This is a central message in the literature on the dollar auction.

However, the shortcomings of the basic war of attrition are well-known. For instance, in industrial organization, Ghemawat and Nalebuff (1990) argue that "there is a large payoff in extending the models of exit beyond the all-or-nothing production technology". A first step in this direction has been made by Whinston (1988), who considers firms that are allowed to shed capacity in small units. Not surprisingly, he shows that this possibility may reverse the usual conclusions drawn from the classic war of attrition.

The inability of the war of attrition to capture signaling has already been stressed by Maynard Smith (see, for instance, chapters 9 and 12 in Maynard-Smith (1982)), who argues that many animal contests are characterized by *assessment strategies*. Such strategies are observed during the first phase of a contest: early behavior signals differences between contestants and may settle the contest without further escalation. The classic war of attrition, as elegant as it is, fails to allow for such phenomena. Following Zahavi (1975, 1977), a large body of literature in biology has investigated the theoretical and empirical validity of the *handicap principle*: provided that it is costly to produce, a signal does provide useful information during animal conflicts.

There is a large literature about games in which players can choose the amount of resources to expand This literature started with Tullock (1980) who introduced the contest model. The probability of winning the contest depends on the endogenous levels of effort. The literature on contests is too vast so summarize it here. See Nitzan (1994) for a survey of the literature. Our model is closer to another branch of the literature, that all-pay auctions as the basic model for contests[6]. The main difference with contests à la Tullock is that the larger bid in an all-pay auction insures a victory in the context with probability one. However, the literature on contests relies mostly on static games. Players choose once and for all the amount of resources to use in the contest. There are few models of dynamic contests, in which both the timing and the level of resources expanded are decision variables.

We are not the first to extend the war of attrition in that direction. Jarque, Ponsatí and Sákovics (2003) analyze a war of attrition in which there are various possible level of concessions. The game is resolved when concessions by both players are sufficient for an agreement to take place. The extensive form is less restrictive and allows for a richer information struc-

[6] See Moulin (1986), Hillman (1988) and Baye et al. (1996) for early models of the all-pay auction.

ture. However, a mediator is used, and players do not observe the concessions made by their opponent. The only information available to players is whether the war is still ongoing or it has ended. This limits the strategic options and behavior based on preemption or signalling. Another way to model dynamic contests with endogenous efforts are tugs-of-war. In McAfee (2000) and Konrad and Kovenock (2005), the contest is modelled as a sequence of fights. With every fight won, a player edges closer to victory. These models differ from the war of attrition in the sense that a player needs to win a given number of contests in order to get the prize, while in a war of attrition, a player can give up at any time. As in our model, players can choose at each stage the amount of resources they want to expand. Both papers restrict attention to complete information. Therefore, no signalling takes place.

The first paper to look at the strategic role of preemptive bids is Dixit (1987). O'Neil (1986) and Harris and Vickers (1985) also consider dynamic all-pay contests, in which bids are used for preemption purposes. Closer to our model is Leininger (1991) and Dekel, Jackson and Wolinsky (2006a). These two papers analyze a game with an extensive form very similar to the one we use but restrict attention to the case of complete information. They also emphasize the role of budget constraints and their importance in such dynamic contexts.

The preemptive motive of a jump bid is also reminiscent of the strategic burning of money in bargaining. On a superficial level, the two behaviors have a similar logic: expanding resources to force concessions from one's opponent. However, as analyzed in Avery and Zemsky (1994), the logic of money burning in bargaining comes from the multiplicity of equilibria. Burning money enables a player to go to a more favorable equilibrium. In dynamic contests, preemption comes from the fact that the opponent gives up because he is not able to match the initial bid.

Analyzing the impact of private information in a dynamic contest is an important contribution of the paper. Many models of the war of attrition allow for private information, but signaling is impossible, as each period corresponds to a unique information set. The role of signalling in a dynamic all-pay auction is also considered in Hörner and Sahuguet (2007). There, the game lasts only for two periods. This makes it easier to highlight the different types of signaling, including bluffing (pretending to be strong) and sandbagging (pretending to be weak). The infinite horizon of the present paper destroys all incentives to sandbag since on can no longer take advantage of being perceived as weak. This opportunity only arises with a finite horizon. The literature on signaling in dynamic contests is small and we are the first to analyze signaling issues in the war of attrition. Signaling in dynamic games has been mainly developed in the context of bargaining models, for instance by Admati and Perry (1987), Chatterjee and Samuelson (1987) and Grossman and Perry (1986). Admati and Perry (1987) is a bargaining model with one-sided incomplete information. The seller's valuation is common knowledge and the buyers has two possible valuations. Low valuations delay their response to the uninformed seller's proposal in order to communicate that their valuation is low and gain in subsequent negotiations. Compared to our model, the price proposed (the bid in our model) does not play the role of the signal. The informed buyer signals his valuation using the delay in his response. It is also interesting to note, in Admati and Perry, signalling tends to increase delay, while in our model of war of attrition, delay is reduced by the possibility to signal. Closer to our model are bargaining models in which the signalling takes place through the offers made. Chatterjee and Samuelson analyze a bargaining game that is very close to a war of attrition. Incomplete information is two-sided, but attention is restricted to a limited choice of offers. Grossman and Perry is does not restrict players in their offers, and the refinement used is almost identical to ours.

The dynamics of bids in our model is intricate. In particular, we can observe patterns of escalation - bids that are becoming larger over time. The phenomenon of escalation is not well understood. The dollar auction model is the usual way to think about escalation. This game, introduced by Shubik (1971), has very simple rules: the auctioneer auctions off a dollar bill to the highest bidder, with the understanding that every bidder pays his bid and that bidders can revise their bids at any point in time. Shubik points out that escalation may occur, as all bids are sunk. In his definition, escalation means that players may keep expanding resources even after they have spent more than the value of the prize they are competing for. The dollar auction has been further analyzed by O'Neill (1987) and Leininger (1989). In order to use backward induction, these papers assume that each bidder faces a (known) budget constraint. The equilibrium outcome does not display escalation (there is only one bid that is never covered), and depends very much on the levels of budget. Equilibria in mixed strategies are not studied. Demange (1992) studies a dollar auction with incomplete information. In her model, there exists a deadline, such that, if this deadline is reached, the value of their private information decides the winner. This game displays multiple equilibria, but stability and forward induction uniquely select an equilibrium outcome. The choice of bids is binary, so that this game resembles a finite-horizon war of attrition. In all these papers, escalation is also interpreted in terms of total resources becoming larger and larger. Since these models do not allow players to choose the amount of resources to expand, it is not possible to analyze the dynamics of bids – in particular whether resources expanding per period of time are increasing or decreasing over time.

Finally, Kambe (1999) and Abreu and Gul (2000) study bargaining models with incomplete information and behavioral types. Some of the comparative statics are similar. The weaker player exits at the beginning of the game with positive probability, and the stronger player gets a higher payoff. In our model, unlike in bargaining, real resources are spent with each incremental bid, so that delay is not the only source of inefficiency.

1.3. Outline

Section 2 sets up the model and summarizes the relevant results of the war of attrition. Section 3 analyzes the model, starting with complete information, through one-sided incomplete information and then with two-sided incomplete information. Section 4 offers some concluding comments. An appendix gathers the proof of most results. For the tedious proof in the case of two-sided incomplete information, the reader is referred to an additional appendix, available at http://neumann.hec.ca/pages/nicolas.sahuguet/attritionappendix.pdf.

2. The model

2.1. The set-up

Two players, indexed by $i \in \{1, 2\}$, compete for an indivisible prize. The winner is determined by the following game G. Time proceeds in discrete periods $t \in \{1, 2, ...\}$. Bidders alternate in their moves. In each odd period, $t = 1, 3, \ldots$, player 1 chooses a bid $b_1 \geqq 0$. In the same period, after observing this bid, player 2 must choose between *conceding* (or *quitting*) and *matching* (or *covering*). Covering means that player 2 pays exactly b_1 as well. If player 2 quits, the game ends. In even periods $t = 2, 4, \ldots$, the same extensive-form game is played, with the roles of players reversed[7]. Therefore, in odd periods, player 1 *has the hand*, while player 2 has the hand in even period. Players discount future periods at a common, constant rate $\delta \in (0, 1)$. If the game ends, the last player to bid wins the prize. All actions are observed and all bids are sunk[8].

This extensive form is equivalent to stopping the contest after successive rounds where each bidder has had a chance to increase his bid but has decided not to do so. Despite its specific extensive form, this model is a natural application of the dollar auction model in the context of private values. It is also very similar to the (complete information) game analyzed in Leininger (1991) in the context of patent races. Recently, in a series of papers on vote buying, Dekel, Jackson and Wolinsky (2006) have analyzed dynamic all-pay auctions with a similar structure.[9]

How much a player values the prize is private information. Player i's *valuation* (or *type*) v_i is either 1, with probability $\mu_i \in (0, 1)$, or $\lambda \in (0, 1)$ with complementary probability.[10] Valuations, or types, are independently distributed. A player's payoff is the difference between his (discounted) valuation, if the case arises, and the discounted sum of his bids.[11]

Formally, the set of histories for player 1 of length t, H_t^1, is defined to be the set of positive sequences $(b_1, b_2, \ldots, b_{t-1})$ of length $t-1$ if t is odd (with H_1^1 being a singleton set containing the "empty history"), or the set of positive sequences $(b_1, b_2, \ldots, b_t)$ of length t if t is even. If t is odd, the history is a list of all bids submitted and covered in the previous periods. If t is even, the history also includes the outstanding bid. Similarly, H_t^2 is the set of positive sequences $(b_1, b_2, \ldots, b_t)$ of length t if t is odd, or the set of positive sequences

[7] Equivalently, we could "merge" the decision nodes of quitting/covering and of bidding by requiring that a player must either quit, or bid at least as much as the previous bid submitted by his opponent.

[8] It should be clear that assuming instead that only a fraction of the bids is sunk is equivalent to rescaling valuations.

[9] See also Dekel, Jackson and Wolinsky (2006,a) for an interpretation of this extensive form as an all-pay auction with jump bidding.

[10] The two-type information structure is clearly a limit of the analysis. However, adding more types will not change the qualitative insights of the result. Most dynamic signalling models also restrict attention to a 2-type model.

[11] This is equivalent to assuming that valuations are known, while disutilities of payments (costs measured in "utils") are not.

$(b_1, b_2, \ldots, b_{t-1})$ of length $t-1$ if t is even.

A pure strategy for player 1 is a pair of mappings $\sigma_1 = (b_1, c_1)$, with $b_1 : V \times \cup_{2\mathbb{N}_0+1} H_t^1 \to \mathbb{R}_+$, $c_1 : V \times \cup_{2\mathbb{N}} H_t^1 \to \{c, q\}$, where c denotes covering, q denotes quitting, and $V = \{1, \lambda\}$. A strategy for player 2 is similarly defined. Mixed strategies are defined in the obvious way. An infinite history is a countably infinite sequence of elements of $\mathbb{R}_+$; the set of these is denoted H^∞. A terminal history is a finite sequence of elements of $\mathbb{R}_+$; the set of these is denoted $H^\dagger$. A strategy profile $\sigma = (\sigma_1, \sigma_2)$ defines a probability distribution over $H = H^\infty \cup H^\dagger$ in the usual manner. A terminal history is reached if a player chooses to quit at some point, and an infinite history is reached otherwise.

Given a history $h \in H$ of length t, let $b_\tau(h)$ denote the bid submitted in period $\tau < t$. If $h \in H^\infty$, player i's type $v_i \in V$'s payoff along history h is[12]

$$V_i(h; v_i) = \sum_{\tau=0}^{\infty} -\delta^{\tau-1} b_\tau(h),$$

which may or may not be infinite. If $h \in H^\dagger$ is of length t, then, if t is odd,

$$V_1(h; v_1) = \delta^{t-1} v_1 - \sum_{t=0}^{\infty} \delta^{\tau-1} b_\tau(h), \; V_2(h; v_2) = -\sum_{t=0}^{\infty} \delta^{\tau-1} b_\tau(h),$$

while if t is even,

$$V_1(h; v_1) = -\sum_{t=0}^{\infty} \delta^{\tau-1} b_\tau(h), \; V_2(h; v_2) = \delta^{t-1} v_2 - \sum_{t=0}^{\infty} \delta^{\tau-1} b_\tau(h).$$

Players choose strategies to maximize their expected discounted payoff. When information is complete, the solution concept used is subgame-perfectness. Under incomplete information, a stronger refinement will be introduced.

While the model is quite stylized, it is general enough to encompass a variety of situations.

Example 1 (Industrial Organization)

Consider a standard alternating-move Cournot duopoly. Because of the fixed costs involved, the market cannot accommodate both firms permanently. As quantities affect the market price, each firm's choice affects its rival's profit, as well as its own. After its rival's

[12] Note that the cost that corresponds to matching the effort of one's rival in period t is discounted with respect to cost of the effort chosen in period $t+1$. If we decided to merge both decisions, the payoff would have to be slightly modified.

choice, a firm can either leave the market, or take losses directly related to its rival's choice. Therefore, firms alternate inflicting losses to themselves and their rival, until one chooses to leave.

Example 2 (Political Economy)

Another example that matches the dynamics of the model would be a lobbying contest. Suppose that two lobbyists try to influence the choice of a politician. The lobbyists alternate increasing their offers to the politician. The politician would then award the prize whenever a lobbyist would decide not to outbid his rival, by at least matching the previous contribution. In this application, it would make sense to merge the matching decision and the bidding decision (see footnotes 3 and 6). See for instance, Leininger and Yang (1994) for a complete information model along those lines and Dekel, Jackson and Wolinsky (2006c) for a model of vote buying with the same sequential structure.

Example 3 (Biology)

It is well-known that, in most applications of the war of attrition to biology, the assumption of constant, exogenous cost is not satisfied (Maynard-Smith and Harper, 2003). Consider for instance the fights between male house crickets. Hack (1997) reports no less than thirteen tactics used during those fights, each involving a specific loss in fitness for the initiator and its rival. Fights turn out to involve fascinating tactical patterns that remain little understood.

2.2. The classic war of attrition

To facilitate comparison with the classic war of attrition, we first solve the game where at each round, players face a binary choice between quitting and paying a fixed cost $c > 0$. This is a special case of the model presented above, when the set of possible bids is the singleton $\{c\}$. As the analysis of this special case follows standard arguments, we omit the calculations.

To rule out trivialities, we assume that $c \leq v_i\delta / (1-\delta)$, $i = 1, 2$, which is always satisfied provided discounting is low enough.

Complete Information: For each i, there exists an asymmetric equilibrium in which player i always covers and player $-i$ immediately quits. In addition, there exists an equilibrium with delay, in which players randomize their covering decisions with probability $\beta_i = c(1+\delta) / (\delta v_{-i})$. A player's expected payoff (when it his turn to bid) equals c/δ.

The expected delay, finally, equals $(2-\beta_2)\,/\,(\beta_1+\beta_2-\beta_1\beta_2)$.

Denoting by Δ the length of each period, so that $c = C\Delta$ and $\delta = e^{-r\Delta}$, we observe that, as Δ tends to zero, a player's expected payoff also tends to zero and delay tends to infinity. Since both players' expected payoff tend to zero, it follows that surplus is fully dissipated in the war of attrition. Observe also as pointed out earlier that β_i decreases with v_{-i}: the stronger the opponent, the less likely a player quits.

One-sided Incomplete Information: As before, there exist two asymmetric equilibria in which one of the players quits immediately provided c is small enough. The equilibrium with delay displays at most two phases: in the initial phase, the informed player's low type randomizes his covering decision while his high type covers with probability 1. When the unconditional probability of the informed player's low type drops below a certain threshold, the second phase starts: the informed player's low type gives up immediately while his high type randomizes. In both phases, the uninformed player randomizes his covering decision. As before, both players' payoffs tend to zero as Δ decreases (independently of the type) and delay tends to infinity.

Two-sided Incomplete Information: As before, there exist two asymmetric equilibria in which one of the players quits immediately provided c is small enough. The equilibrium with delay displays now up to three phases. In the initial phase, both players' low types randomize their covering decision while the high types cover with probability 1. As soon as at least one of the players' low type's unconditional probability drops below a certain threshold, this low type quits immediately and the game proceeds as under one-sided incomplete information. Delay, payoffs and dissipation display the same limiting features as before.

3. The results

Because our interest is not primarily centered on the role of discounting, we focus in what follows on the equilibrium behavior with low discounting (i.e., discount factors close to one).[13] In the proofs, however, the (unique) equilibrium strategies are identified for arbitrary discount factors.

[13] As in Rubinstein (1982), other equilibrium outcomes exist for $\delta = 1$.

3.1. Complete Information

In this subsection, we briefly describe the outcome of the game of complete information. While straightforward, the analysis already confirms that the predictions of the classic war of attrition hinge upon the binary nature of bids. In addition, this characterization provides a first step in the analysis of the game with incomplete information. There are essentially two cases to distinguish, depending on whether the players' valuations are identical or not.

Theorem 3.1. *Suppose that $v_1 > 0$ and $v_2 > 0$ are known.*

1. $v_1 = v_2 =: v$*: along the unique equilibrium path, player* 1*'s initial bid induces player* 2 *to quit with probability one. This bid tends to* $v/2$ *and player* 1*'s payoff tends to* $v/2$ *as* δ *tends to* 1.

2. $v_1 < v_2$*: along the unique equilibrium path, for some* $\bar{\delta} < 1$*, player* 1*'s initial bid is* 0*, player* 2 *covers with probability one, and player* 1*'s payoff is* 0 *if* $\delta > \bar{\delta}$.

3. $v_1 > v_2$*: along the unique equilibrium path, for some* $\bar{\delta} < 1$*, player* 1*'s initial bid is* 0*, player* 2 *quits with probability one, and player* 1*'s payoff is* 1 *if* $\delta > \bar{\delta}$.

As mentioned, the proof and detailed description of equilibrium strategies, valid for all $\delta \in [0, 1)$, can be found in appendix.

This simple case demonstrates that the main predictions of the classical war of attrition rely on the lack of flexibility in the choice of effort:

- when players are of unequal strength, no effort is exerted all. All bids are 0 and the "weaker" player concedes as soon as possible. The outcome is therefore efficient, and there is no rent dissipation whatsoever.

- in the special case of equal valuations, the outcome is determined by the temporal monopoly. The first player to move wins the prize, but submits a bid equal to half the common valuation. Therefore, dissipation is only partial.

In either case, the game does not last more than two periods, since at least one player concedes as soon as this opportunity arises. In addition, it is plainly clear that (perceived) strength can only benefit a player: the stronger he is, the less likely he is to drop out.

To readers familiar with Rubinstein (1982), it will come as no surprise that players "agree" immediately (in fact, the argument proceeds along the same lines than Shaked and Sutton (1987)): a strictly positive bid only makes sense if it induces the rival to concede with positive probability. If the rival is willing to do so, his continuation payoff must be zero. Therefore, a slightly higher bid would induce him to concede with probability one, and the game must end in no more than two periods. The role of valuations is very similar to the role of steps in Fudenberg, Gilbert, Stiglitz and Tirole (1983) and Harris and Vickers (1985). When players are of equal strength, the first one to move "takes a step" sufficiently large to deter his opponent from mimicking him. When players are of unequal strength, the stronger one will find it advantageous to match any step (less than his own valuation) taken by his weaker opponent, and the latter may as well concede whenever it is possible.

3.2. Incomplete Information

3.2.1. Solution Concept

When information is incomplete, subgame-perfectness is not sufficient to refine the set of Nash equilibria. We thus impose a refinement on equilibrium strategies and beliefs that generalizes the concept of Undefeated Equilibrium (Mailath, Okuno-Fujiwara, Postlewaite (1993)).[14] Our refinement serves two purposes:

First, we wish that our solution concept be recursive: that is, when a player's belief is degenerate, the continuation strategy profile should be as in the equilibrium of the game where this player's information is complete. For instance, if the incomplete information is initially one-sided, the equilibrium play in continuation games for which the player's belief is degenerate should be the same as in the game with complete information. Similarly, with two-sided incomplete information, the continuation strategies once a player's belief is degenerate should be as in the equilibrium with one-sided incomplete information (provided the latter admits a unique equilibrium).

Second, we wish that our solution concept eliminates all sequential equilibria that do not satisfy the following idea, which motivates the concept of undefeated equilibrium (as well as the concept of Perfect Sequential Equilibrium of Grossman and Perry (1986)): roughly,

[14] Mailath, Okuno-Fujiwara and Postlewaite only define undefeated equilibria for pure signalling games.

a sequential equilibrium should be pruned if a player has an out-of-equilibrium action such that, provided his opponent interprets this action as evidence that the player's type precisely belongs to some subset M, then all types in M, and only those types, prefer playing this action to the equilibrium outcome. In addition, this alternate play should correspond to some sequential equilibrium as well. For excellent discussions of this idea, see either Grossman and Perry (1986), Mailath, Okuno-Fujiwara, Postlewaite (1993), or Van Damme (1991).

Formally, let $\rho_i : H_t^i \rightarrow [0,1]$ be the player i's belief function: to each history $h_t^i \in H_t^i$, it assigns a probability to player $-i$'s high type. A sequential equilibrium is a pair $e = (\sigma, \rho)$ where σ_i is optimal given ρ_i and ρ_i is consistent. Denote by $SE(G)$ the set of sequential equilibria such that the support of the players' beliefs is non-increasing along every history. Let $V_i^e\left(h_t^i; v_i\right)$ be the expected payoff of player i's type v_i, given strategy profile e, conditional on history h_t^i.

Given two sequential equilibria $e = (\sigma, \rho)$ and $e' = (\sigma', \rho')$, e' *defeats* e if there exists $i \in \{1,2\}$, $h_t^i \in H_t^i$, $a_i \in \mathbb{R}_+ \cup \{c, q\}$ such that:

i. h_t^i occurs with positive probability under e and e';

ii. $\forall v_i \in V : \sigma_i\left(v_i, h_t^i\right)$ assigns zero probability to a_i;

iii. $K := \left\{v_i \in V; \sigma_i'\left(v_i, h_t^i\right) \text{ assigns positive probability to } a_i\right\} \neq \emptyset$;

iv. $\rho_{-i}'\left(\left(h_t^i, a_i\right)\right) = \rho_{-i}\left(h_t^i\right)\pi(1) / \left(\rho_{-i}\left(h_t^i\right)\pi(1) + \left(1 - \rho_{-i}\left(h_t^i\right)\right)\pi(\lambda)\right)$, for any $\pi(1)$, $\pi(\lambda) \in [0,1]$ satisfying:

$$v_i \in K \text{ and } V_i^{e'}\left(\left(h_t^i, a_i\right); v_i\right) > V_i^e\left(\left(h_t^i, a_i'\right); v_i\right) \rightarrow \pi(v_i) = 1, \text{ and } v_i \notin K \rightarrow \pi(v_i) = 0,$$

where $a_i' \in \mathbb{R}_+ \cup \{c, q\}$ is an action assigned positive probability by $\sigma_i\left(v_i, h_t^i\right)$.

v. The continuation equilibria induced by e' after histories such that at least one player's belief support has strictly decreased are themselves undefeated.

The first requirement states that e and e' are two alternative equilibria for which the history h_t^i could occur. However, while a_i is inconsistent with e, it is consistent with e' (ii. and iii.). The fourth requirement states that the beliefs of player $-i$ under equilibrium e', conditional on observing $\left(h_i^t, a_i\right)$, are precisely the beliefs he would hold by updating his beliefs under equilibrium e, given h_i^t, by using Bayes' rule, assuming that the types choosing a_i are exactly those types which benefit from playing according to e' rather than according to e (iv. allows a type which is indifferent to randomize). Finally, the fifth requirement -along with the restriction to sequential equilibria with non-increasing belief supports- implies that, as soon as a player assigns probability zero to some type of his opponent, his continuation strategy must be equal to a strategy in some undefeated equilibrium of the game with one-sided incomplete (or, if the case occurs, complete) information. This allows us to solve the game "recursively", starting with the case of complete information. An element of $SE(G)$ which is not defeated by another element in $SE(G)$ is *undefeated*. Observe that, under complete information, an undefeated equilibrium reduces to a subgame-perfect equilibrium.

In the remainder of this paper, an *equilibrium* is an undefeated equilibrium.

3.2.2. One-sided Incomplete Information

In this subsection, we consider the case in which one player's valuation is known -player 1's- while player 2's valuation is uncertain.[15] There are two cases to consider, depending on whether player 1's valuation is high or low.

Assume first that player 1's valuation is high (1), while player 2's valuation is either high (1) as well, with probability $\mu \in [0, 1]$, or low ($\lambda < 1$) with complementary probability. Equilibrium strategies can be categorized in five classes, depending on parameters. Yet despite this variety of behaviors, the model delivers several robust predictions:

- Play never lasts more than two periods. Whenever player 1 has the hand, he submits a bid that induces at least his opponent's low type to concede. Therefore, if player 2 ever covers, his valuation must be high (player 2 is *strong*) and his next bid leads player 1 to concede. If player 2's valuation is low (player 2 is *weak*), and he has the hand, he either submits a bid that induces player 1 to concede, or he bids nothing and necessarily concedes in the next period. Therefore, in some cases (when μ and/or λ are sufficiently large), if player 2 has the hand, the outcome is inefficient (that is, he wins the prize even if his valuation is low).

- The larger μ and/or λ, the more likely it is that player 1 chooses a high bid that induces player 2 to concede independently of his type. Conversely, if player 2's valuation is likely to be low, or if the difference between high and low valuation is sufficiently large, player 1

[15] Because the equilibrium strategies also specify how player 2 behaves when it is his turn to bid, there is no loss of generality in doing so.

"takes his chances" and submits a low bid that deters player 2 only if his valuation is low.

- The larger μ and/or λ, the more likely it is that, whenever player 2 has a low valuation, he chooses to mimic the bid he would submit if his valuation was actually high.

- Player 1's payoff decreases in λ and in μ, whether he has the hand or not.

- Player 2's high type's payoff increases in μ when he has the hand. It may decrease or increase in λ. However, if he does not have the hand, his payoff decreases in both λ and μ.

- Player 2's low type's payoff increases in μ and λ.

- Rent dissipation is partial, as the expected revenue never exceeds $\frac{1}{2}$.

Most of these results are fairly intuitive. As mentioned, the effect of λ on the payoff of player 2's high type is ambiguous: a high λ decreases player 1's payoff when he has the hand (as he must then submit a higher bid to deter even if his opponent's low type), and this makes it easier to force him to concede. On the other hand, a high λ makes it harder for player 2's high type to credibly signal his type through his bid (as player 2 can easily mimic him), which decreases his payoff. Observe that player 2's high type would like to be perceived as strong (high μ) when he has the hand, but as weak (low μ) when he does not have the hand. This is rather natural: being perceived as strong makes it more likely that player 1 prefers to concede when player 2 has the hand. However, if player 1 has the hand, he may then prefer a high bid that deters player 2 independently of his valuation, a behavior that is clearly detrimental to player 2's high type.

A more formal, but tedious description of the equilibrium strategies follows, which the reader may choose to skip.

Equilibrium outcomes can be distinguished along two dimensions: if player 1 bids in such circumstances, does he submit a bid which induces player 2 to quit independently of his type, to cover if and only if his type is high (i.e., his valuation is 1), or to cover independently of his type?[16] In the first case, the equilibrium (outcome) is said to be with *full deterrence*; in the second, with *partial deterrence*, and in the third, with *no deterrence*. Second, if player 2 were to bid in the same circumstances, would he submit the same bid independently of his type, or would he submit a different bid depending on his type? In the first case, the equilibrium

[16] Of course, it is also conceivable that player 2 randomizes his decision to quit (for some type). However, such behavior does not arise in equilibrium.

(outcome) is *separating*, while in the second, it is *pooling*. If his bidding behavior is more intricate, the equilibrium is *semi-pooling*. The following theorem describes the equilibrium outcome systematically.

Theorem 3.2. *The equilibrium exists and the distribution over outcomes is unique.*[17] *If:*

1. $\lambda < \frac{1}{2}$, $\mu < \frac{1}{2}$, *the equilibrium is separating, with partial deterrence;*

2. $\lambda < \frac{1}{2}$, $\mu \geqq \frac{1}{2}$, *the equilibrium is separating, with full deterrence;*

3. $\lambda \geqq \frac{1}{2}$, $\lambda \geqq 1 - \mu$, *the equilibrium is pooling, with full deterrence;*

4. $1 - \mu > \lambda \geqq \frac{1}{4} + \frac{1}{2}(1 - \mu)$, *the equilibrium is pooling, with partial deterrence;*

5. $\frac{1}{4} + \frac{1}{2}(1 - \mu) > \lambda \geqq \frac{1}{2}$, *the equilibrium is semi-pooling, with partial deterrence,*

   when δ is close enough to one.

These five cases partition the parameter space. The five parameter regions, labeled respectively I-V, are represented in Figure 1.

[17] That is, except possibly for parameters lying on the boundary of two regions, for which two equilibria may exist. Recall also that this is a characterization for $\delta \to 1$.

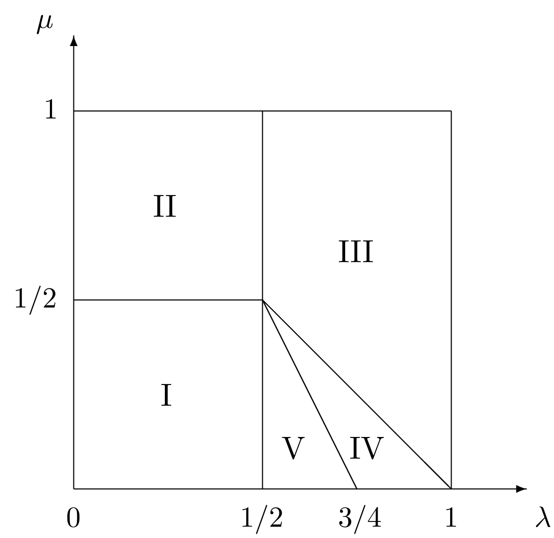


Figure 1: One-sided incomplete information with a known strong type

Thus, when the valuations of player 2 are dissimilar enough ($\lambda < \frac{1}{2}$), his bids are separating, as the low type cannot mimic the high type, who behaves as if the game were of complete information between players with valuation 1 (the low type is left with bidding 0). If in addition player 2's low type is likely enough ($\mu < \frac{1}{2}$), player 1 submits a low, risky bid of 0, which player 2 does not cover if he is of the low type; if he is of the high type however, player 2 covers and wins with his next bid. If player 2's low type is unlikely, ($\mu \geqq \frac{1}{2}$), player 1 bids as if the game were of complete information between players with valuation 1.

Equilibrium behavior is more sophisticated when the possible valuations of player 2 are similar enough ($\lambda \geq \frac{1}{2}$), so that player 2's low type may potentially mimic player 2's high type's behavior. If it is likely that player 2 is of the high type ($\lambda \geqq 1 - \mu$), play proceeds as in the game of complete information between two players with valuations 1, as player 2 finds it worthwhile to behave this way even if his valuation is lower. If the low type is sufficiently likely ($1 - \mu > \lambda$), deterring player 2's low type only becomes sufficiently attractive, despite the risk of being covered and losing to player 2's high type. However, because player 1's payoff increases in the probability that player 2 is of low type, the pooling bid which player 2 must submit, if he were to play first, to induce player 1 to quit increases in this probability as well. Therefore, when λ and μ are both sufficiently small ($\frac{1}{4} + \frac{1}{2}(1 - \mu) > \lambda \geqq \frac{1}{2}$), player 2's low type randomizes between a bid submitted by player 2's high type, and bidding 0, which reveals his type. Thereby, he lowers the bid necessary to induce player 1 to quit, as player 1 revises his belief about player 2 being of the low type downwards, upon observing the higher bid.

Assume now that player 1's valuation is low (λ), while player 2's valuation is either high (1) as well, with probability $\mu \in [0, 1]$, or low ($\lambda < 1$) with complementary probability. If discounting is sufficiently low, the equilibrium characterization is then trivial:

Theorem 3.3. *The equilibrium exists and the distribution over outcomes is unique. Along the equilibrium, player* 1 *always submits* 0*, a bid that is covered by player* 2 *independently his type. Player* 2 *bids* 0*, independently of his type, a bid that induces player* 1 *to concede with probability one.*

Therefore, player 1, who is known to be weak, bids nothing and concedes as soon as possible. This occurs despite the fact that player 2 may be weak as well with high probability. Thus, comparing this outcome with the outcome with complete information between weak players, it follows that player 2's low type's payoff drastically improves whenever there is a chance, no matter how small, that he may be strong (provided that players are sufficiently patient). This effect is reminiscent of some results in the literature on reputation (for instance, Fudenberg and Levine 1986)).

3.2.3. Two-Sided Incomplete Information

This subsection considers the case of two -sided incomplete information: that is, player 1's valuation is high (1) with probability μ_1, and low ($\lambda < 1$) otherwise, while player 2's valuation is high (1) with probability μ_2, and low (λ) otherwise. Even when the incomplete information is two-sided, the equilibrium outcome is unique. However, this outcome depends on parameters in a rather intricate way. Roughly, for each parameter region identified in the case of one-sided incomplete information, there are now several cases to distinguish, depending on the value of μ_2.

As before, several conclusions emerge:

- Delay may reach, but not exceed three periods. Indeed, if a player with a high valuation has the hand, the bid he submits induces his opponent to concede with positive probability (that is, his bid forces at least his opponent's lower type to concede). Thus, the only equilibrium bids that are covered independently of the opponent's type are submitted by the player's low type, who plans to concede in the next period.

- The higher μ_2, the more likely it is that player 1's high type's bid induces his opponent to concede independently of his type. This is only the case for player 1's low type's bid if λ is high enough: if λ is too low, player 1's low type cannot mimic player 1's high type. In general, the lower λ, the more similar the bids submitted by player 1's different types.

- Player 1's payoff when he has the hand increases with μ_1, and decreases with μ_2, independently of his own type.

- Rent dissipation is partial.

As with one-sided incomplete information, the effect of λ, the low valuation, on players' payoff is ambiguous. On the one hand, a higher λ implies that it is "cheaper" to deter the opponent, as his payoff from covering are lower. On the other hand, it also makes it "more expensive" for a player's high type to signal his valuation. If a player does not have the hand, his payoff may increase or decrease with μ_1 and μ_2, whether his valuation is high or low.

A more formal, but tedious description follows, which the reader may choose to skip.

To be more specific, suppose first that μ_1 and/or λ are sufficiently low. If μ_2 is low enough, player 1's initial bid is independent of his own type and deters only player 2's low type. If player 2's valuation is high, he will cover this bid and submit a lower bid that deters player 1's low valuation. If player 1's valuation is high, however, he will cover this bid as well, and submit a large bid that leads to concession.

For an intermediate range of values for μ_2, player 1's low type randomizes his initial bid, submitting a zero bid -and conceding next period- with positive probability. Thus, the bid common to both types leads player 2 to become more pessimistic, which makes it less appealing for him, after covering, to deter only player 1's low type. In this way, the common bid is attractive to both types of player 1.

Finally, if μ_2 is sufficiently high, player 1's high type prefers to submit a fully deterrent bid. Depending on how large λ is, player 1's low type may either choose to mimic player 1's high type, randomize between mimicking and bidding nothing, or bid nothing.

Suppose now that μ_1 and/or λ are sufficiently high. As in the previous case, player 1 submits partially deterrent bid if μ_2 is low enough, independently of his type. If player 2's type is actually high, he will cover this bid and submit then a large bid that induces player 1 to concede. If μ_2 is in an intermediate range of values, player 1's high type randomizes his bid. Therefore, the common bid leads player 2 to be more optimistic, so optimistic indeed that player 2's high type is precisely indifferent between a fully and a partially deterrent bid, and this indifference is resolved in a way that makes player 1's high type indifferent, in the first period, between the two bids he randomizes over. Finally, if μ_2 is large enough, player 1's high type submits a large bid that deters player 2, mimicked in this by player 1's low type provided λ is large enough.

The actual equilibrium strategies are more complex than this description may suggest. We categorize the equilibrium outcomes according to the deterrence effect of player 1's initial bid. As along any equilibrium path, the continuation games starting in period 1 turn out to be either of complete information or of one-sided incomplete information, we will not describe further the equilibrium outcomes.

For simplicity, let F.D. stand for full deterrence, P.D. for partial deterrence, and 0D. for no deterrence. Because player 1 may submit different bids depending on his type, we say that the equilibrium is (X.D.,Y.D.), where X , Y is either F, P or 0, if player 1's high type's bid is X.D., and player 1's low type's bid is Y.D.; for instance, the equilibrium is (F.D.,P.D.) if the initial bid of player 1's high type induces player 2 to quit independently of his type, while player 1's low type's bid induces player 2 to quit if and only if he is of the low type. For some parameters, player 1 may randomize between bids having different deterrence effects. If player 1's high type randomizes between a bid which is X.D., and a bid which is X'.D., while player 1's low type's bid is Y.D., we write ((X.D.,X'.D.) , Y.D.) (where X, X' and Y are either F, P or 0). The notation (X.D., (Y.D.,Y'.D.)) has a similar interpretation when player 1's low type randomizes.

Even when the discount factor is close to unity, the equilibrium behavior is rather intricate. Specifically, we have:

Theorem 3.4. *The equilibrium exists and the distribution over outcomes is unique. If:*

1. $\lambda < \frac{1}{2}$, $\mu_1 < \frac{1}{2}$, *then the equilibrium is* $(P.D., P.D.)$ *for* $\mu_2 < \mu_1$, $(P.D., (P.D., 0D.))$ *for* $\mu_1 \leqq \mu_2 < \frac{1}{2}$, *and* $(F.D., 0D.)$ *for* $\mu_2 \geqq \frac{1}{2}$;

2. $\lambda < \frac{1}{2}$, $\mu_1 \geqq \frac{1}{2}$, *then the equilibrium is* $(P.D., P.D.)$ *for* $\mu_2 < \frac{1}{2} - \lambda(1-\mu_1)$, $((F.D., P.D.), P.D.)$ *if* $\frac{1}{2} - \lambda(1-\mu_1) \leqq \mu_2 < \frac{1}{2}$ *and* $(F.D., 0D.)$ *for* $\mu_2 \geqq \frac{1}{2}$;

3. $\lambda \geqq \frac{1}{2}$, $\mu_1 \geqq \frac{1-\lambda}{\lambda}$, *then the equilibrium is* $(P.D., P.D.)$ *for* $\mu_2 < 1-\lambda$, *and* $(F.D., F.D.)$ *for* $\mu_2 \geqq 1-\lambda$;

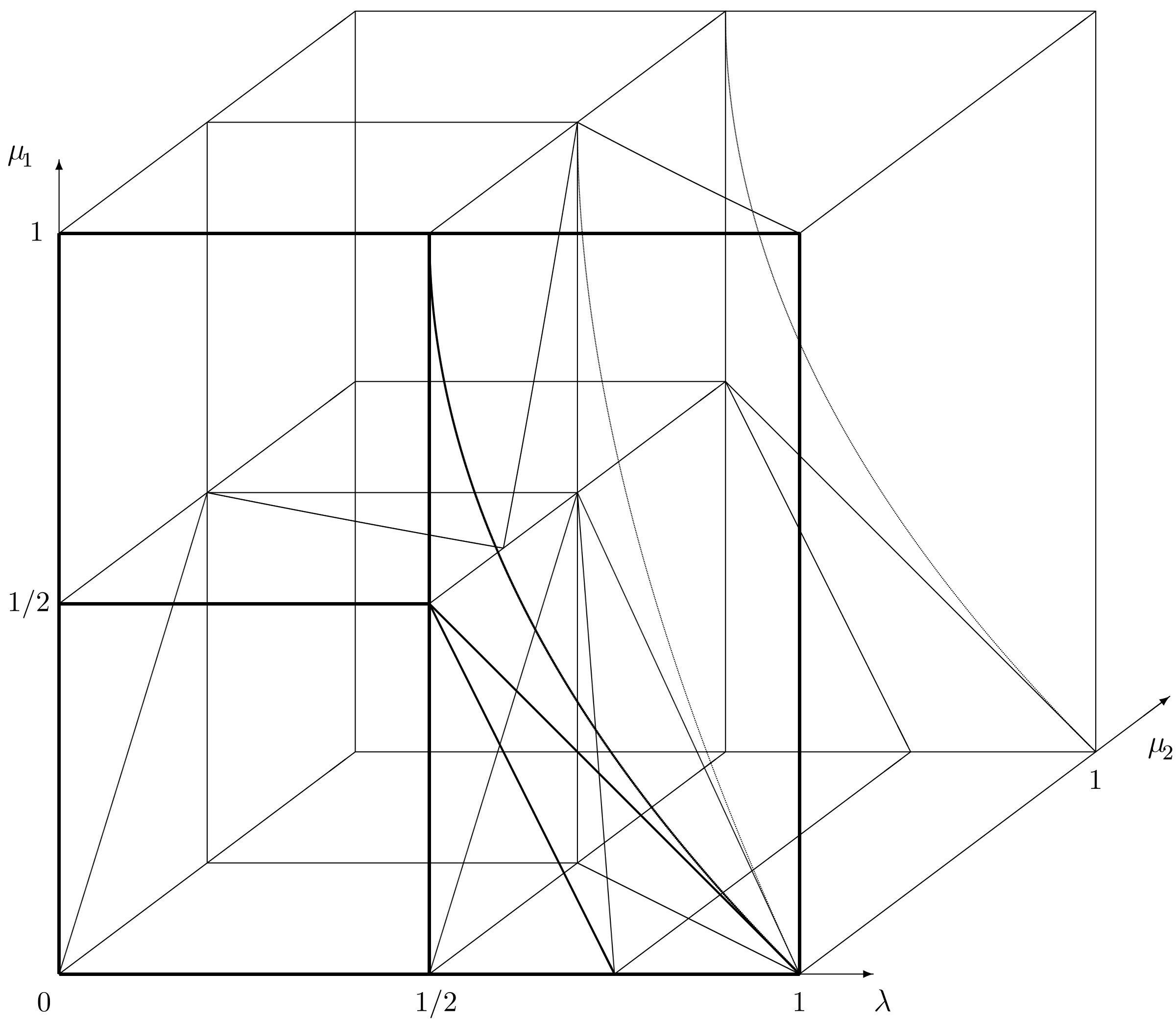
4. $\lambda \geqq \frac{1}{2}$, $\frac{1-\lambda}{\lambda} > \mu_1 \geqq 1-\lambda$, *then the equilibrium is* $(P.D., P.D.)$ *if* $\mu_2 < \mu_1\lambda$, $((F.D., P.D.), P.D.)$ *for* $\mu_1\lambda \leqq \mu_2 < 1-\lambda$, *and* $(F.D., F.D.)$ *for* $\mu_2 \geqq 1-\lambda$;

5. $\lambda \geqq \frac{1}{2}$, $1-\lambda > \mu_1 \geqq \frac{3-4\lambda}{2}$, *then the equilibrium is* $(P.D., P.D.)$ *if* $\mu_2 < \frac{1-\lambda}{\lambda}(1-\mu_1)$ *and* $(F.D., F.D.)$ *for* $\mu_2 \geqq \frac{1-\lambda}{\lambda}(1-\mu_1)$;

6. $\lambda \geqq \frac{1}{2}$, $\mu_1 < \frac{3-4\lambda}{2}$, *then the equilibrium is* $(P.D., P.D.)$ *if* $\mu_2 < \mu_1 + 1 - 1/(2\lambda)$, *and* $(F.D., (F.D., 0D.))$ *for* $\mu_2 \geqq \mu_1 + 1 - 1/(2\lambda)$,

   when δ *is close enough to one.*

The following figure delimits the main parameter regions.



Undefeated Equilibria when $\delta = 1$

4. Concluding remarks

This paper has shown that the predictions of the war of attrition are sensitive to the array of actions available to the players and describe the features of the signaling behavior that arises under incomplete information. Several testable implications are derived.

Attention has been limited to the simple, but restrictive case of two types. While it seems plausible that the case with finitely many types proceeds along similar lines, it is not clear what results would emerge with a continuum of types. We suspect that the conclusions would resemble those of Grossman and Perry's (1986) bargaining model, in which intervals of types pool on the same bid (with lower intervals pooling on smaller bids), and every bid induces a lower interval of the opponent's types to give up, but this remains an open problem.

It is worthwhile pointing out that, from a theoretical point of view, our set-up and results can be reinterpreted in terms of jump bidding in all-pay auctions: while the classic war of attrition does not allow players to submit jump bids, and the highest bid is continuously increased until all but one player drop out, such bids are available to players in our model. In this sense, we show that jump bids will be used in the unique subgame-perfect (respectively, undefeated) equilibrium, and that they dramatically decrease the expected revenue of the auctioneer. This result is to be compared to Daniel and Hirshleifer (1998) that derives similar results with (winner-only pays) ascending auctions when it is costly to submit or revise a bid. In their model, players alternate increasing their bids, until one bidder passes. They also find that jump bids are used, communicate bidders' private information rapidly and decrease revenue.

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Appendix

Complete Information

To avoid trivialities, we assume that $\delta \neq \lambda$. We consider two cases, depending upon players have the same valuation or not.

1. Suppose that both players' valuations, denoted λ, are identical: Let $\bar{V}_i$ ($\underline{V}_i$) denote the supremum (infimum) of i's payoff over all subgames of all (subgame-perfect Nash) equilibria [undefeated equilibrium obviously implies subgame perfectness] in which it his turn to make an offer. Obviously, $0 \leqq \underline{V}_i \leqq \bar{V}_i \leqq \lambda$. We have, for $i \in \{1, 2\}$:

$$\begin{cases} \underline{V}_i \geqq \lambda - \delta \bar{V}_i, \\ \bar{V}_i \leqq \lambda - \delta \underline{V}_i, \end{cases}$$

   because any offer strictly larger than $\delta \bar{V}_i$ is necessarily accepted by player i, while any offer strictly below $\delta \underline{V}_i$ is rejected. This implies that $\bar{V}_i = \underline{V}_i = \lambda / (1 + \delta)$; therefore, the unique equilibrium specifies that players bid $b^* = \delta\lambda / (1 + \delta)$ and cover if and only if $b < b^*$.

2. Suppose that players have different valuations. Specifically, assume that player 1's valuation is 1 while player 2's valuation is $\lambda < 1$: we have $\underline{V}_1 \geqq 1 - \delta \bar{V}_2$, $\bar{V}_1 \leqq 1 - \delta \underline{V}_2$, $\bar{V}_2 \leqq (\lambda - \delta \underline{V}_1)^+$, $\underline{V}_2 \geqq \lambda - \delta \underline{V}_1$. Simple manipulations yield that $V_1 := \bar{V}_1 = \underline{V}_1 = 1 \wedge (1 - \delta\lambda) / \left(1 - \delta^2\right)$ and $V_2 := \bar{V}_2 = \underline{V}_2 = (\lambda - \delta)^+ / \left(1 - \delta^2\right)$; indeed, if:

   1. $\lambda > \delta$: We get $V_1 \equiv \bar{V}_1 = \underline{V}_1 = (1 - \delta\lambda) / \left(1 - \delta^2\right)$, and $V_2 = (\lambda - \delta) / \left(1 - \delta^2\right)$. Player i offers $b_i^* = \delta V_{-i}$ and covers if and only if $b_{-i} < \delta V_i$.
   2. $\lambda < \delta$: We get $V_1 = 1$, $V_2 = 0$, and player i offers $b_i^* = \delta V_{-i}$ and covers if and only if $b_{-i} < \delta V_i$.

One-sided incomplete Information: $v_1 = 1$, $v_2 = \lambda$ or 1.

Assume that player 1 has valuation 1 for sure and player 2 has valuation 1 with probability μ (player 2_h) and valuation $\lambda \in (0, 1)$ with probability $1 - \mu$ (player 2_l). Player 2 is the *informed* player, and player 1 is the *uninformed* player. To avoid trivialities, we assume furthermore that $\lambda \neq \delta$, $\lambda \neq \delta / (1 + \delta)$. We denote by V_i the payoff of player i (the continuation game considered is clear from the context) when it is his turn to offer, and W_i his payoff when it is his turn to cover. A bid by player i is *fully deterrent* (F.D.) if it induces $-i$ to quit with probability one, independently of his type. It is *partially deterrent* (P.D.) if it induces $-i$ to quit if his type is λ, and to cover otherwise (both choices being made with probability one). It is *zero deterrent* (0D.) if it induces $-i$ to cover with probability one, independently of his type. Player i's equilibrium bid b is *separating* if both types of player i are assigned positive probability by player $-i$, and b is submitted with positive probability by one of player i's type, but not by the other. Player i's equilibrium bid b is *pooling* if both types of player i submit this bid b with probability one, and both types are assigned positive probability by $-i$. Finally, if $-i$ assigns positive probability to both types of i, any other bid is referred to as *semi-pooling*.

Narrowing down the set of possible equilibria

We proceed in steps. It is straightforward to verify that all deviations considered in the sequel are part of an equilibrium, as described is the next subsection.

1. $\mathbf{V}_1 \in \left[1/\left(1+\delta\right), 1 \wedge \left(1-\delta\lambda\right)/\left(1-\delta^2\right)\right]$: Let $\bar{V}_i$ ($\underline{V}_i$) denote the supremum (infimum) of i's payoff over all continuation games of all (Perfect Bayesian) equilibria *and over all possible beliefs of player* 1 in which it i's turn to make an offer. Because 2_h can always mimic the behavior of 2_l, it is clear that $\bar{V}_{2_h} \geqq \bar{V}_{2_l}$. Then $\underline{V}_1 \geqq 1 - \delta\bar{V}_{2_h}$ and $\bar{V}_{2_h} \leqq 1 - \delta\underline{V}_1$, which immediately implies $\underline{V}_1 \geqq 1/\left(1+\delta\right)$. Because $\underline{V}_{2_h} \geqq \underline{V}_{2_l}$, a similar reasoning yields that $\bar{V}_1 \leqq 1 \wedge \left(1-\delta\lambda\right)/\left(1-\delta^2\right)$.

2. $W_{2_l} = 0$ **in any equilibrium**: If $W_{2_l} > 0$, then 2_l covers with probability one, and must therefore win with positive probability in the continuation game. But since $\underline{V}_1 \geqq 1/\left(1+\delta\right)$, 2_l's subsequent bid has to be at least $\delta/\left(1+\delta\right)$. This implies that $\underline{V}_1 \leqq 0 - \delta^2/\left(1+\delta\right) + \delta^2 = \delta^3/\left(1+\delta\right) < \delta/\left(1+\delta\right)$ which is a contradiction.

3. **Any separating bid by 2_h must be F.D.**: Suppose not, and let b_q be a separating bid by 2_h which is covered by 1 with probability $1-q$. Similarly, pick a (not necessarily separating) bid b_p submitted with positive probability by 2_l which is covered with probability $1-p$. Because $W_{2_l} = 0$, $p\lambda - b_p \geqq q\lambda - b_q$, by revealed preference. Therefore, $b_q - b_p \geqq \left(q-p\right)\lambda > \left(1-p\right)\lambda - \left(1-q\right)$. Therefore, there exists a bid $b > b_q$ such that $1 - b > q - b_q$ and $p\lambda - b_p > \lambda - b$. Thus, the bid b constitute a profitable deviation

for 2_h, as it is preferred to the "equilibrium" bids by 2_h (after the separating bid b_q, $W_{2_h} = 0$), but not by 2_l, since if 1 perceives it as being only submitted by 2_h, 1 will quit with probability one ($b_q > b$); thus, the proposed bids cannot be part of a undefeated equilibrium.

4. **Any separating bid by 2_l must be 0D. and equal to 0**: It is clear that a bid which is 0D. and separating must be equal to 0. Suppose now that 2_l submits with positive probability a separating bid b which is not 0D.; this bid must be then at least $\bar{b} = \delta \left(1 \wedge (1 - \delta\lambda) / \left(1 - \delta^2\right)\right)$, which is the discounted payoff of 1 in the continuation game between 1 and 2_l, as 1 would cover with probability one any lower bid. However, by 1., player 1 quits with probability one if he observes any bid $b' > \bar{b}$. Thus, the bid b equals $\bar{b}$ and is F.D.. Consider any bid b_q submitted with positive probability by 2_h, which is covered with probability $1 - q$ by 1. Clearly, $b_q < b$, and the total *discounted* probability of winning of 2_h after bidding b_q is strictly less than 1 no matter the subsequent equilibrium play. This violates single-crossing, as either player 2_h strictly prefers to bid b rather than b_q (and any of its subsequent play), or 2_l has a strict preference for the latter over the former.

5. **In any (even) period, there exists at most one bid submitted with positive probability by both types**: In any undefeated equilibrium, a separating bid b by 2_h (F.D. by 3.) must be either $\delta/(1+\delta)$, or such that 2_l is indifferent between submitted b or not. Otherwise, if $b > \delta/(1+\delta)$ a bid $b - \varepsilon$, for $\varepsilon > 0$ small enough, is a profitable deviation for 2_h (it has to be interpreted as a separating bid from 2_h, and it is F.D. as it exceeds $\delta/(1+\delta)$). If $b = \delta/(1+\delta)$, then, as any strictly lower bid is covered with probability one by 1 according to 1., any other bid submitted must be 0D., and therefore unique and equal to 0. As it is necessary, by 1., for player 2_h to bid at least $\delta/(1+\delta)$ for player 1 not to cover with probability one, he will bid $\delta/(1+\delta)$ with probability one, and the bid 0 is a separating bid by 2_l. In equilibrium, these bids will be submitted in period 2. Suppose then from now on that, whenever 2_h submits a separating bid b, player 2_l is indifferent between this bid and any bid he submits with positive probability. If any of these bids is also submitted with positive probability by 2_h, a violation of single-crossing of preferences follows, as in 4.; therefore, in this case as

well, any other bid must be separating, submitted by 2_l, 0D. and equal to 0. Observe that any bid by player 1 is either F.D., 0D., or leaves player 2_l indifferent between covering or not. In particular, in case 2_l covers a bid he is not supposed to cover in equilibrium, but 2_h is, his next bid will be the same as the F.D. bid submitted by 2_h. Therefore, in any event, player 2_l is willing to mimic the behavior of 2_h. Hence, by single-crossing of preferences, if 2_l and 2_h are both willing to submit two distinct bids, the total discounted probability of winning must be the same across bids. In particular, it cannot be that one of these bids is F.D., for this would imply that all are, and players would strictly prefer the lowest such one. Let $q < 1$ be the highest total discounted probability of winning given such a bid submitted by both types of 2 (the maximum is taken over future bids as well). We can construct a F.D. bid $b > b_q$ such that 2_h strictly prefers b to b_q, and 2_l is indifferent between b and b_q. (pick the highest bid 2_l is willing to submit, if it is F.D., rather than bidding b_q and following equilibrium behavior afterwards). Therefore, such behavior cannot be part of an equilibrium. For future reference, this argument also establishes that any bid submitted with positive probability by both 2_l and 2_h must be either F.D. or 0D. and equal to 0, and shows that whenever 2_h submits a separating bid b, it must equal $b = \delta / (1 + \delta) \vee \lambda$

6. Five cases remain:

   1. 2_h and 2_l submit an F.D. bid b with probability one, where b is the discounted payoff of 1 if 1 were to cover.

   2. 2_l randomizes between the 0D. bid 0 and the F.D. bid λ while 2_h bids λ with probability one.

   3. 2_l submits the 0D. bid 0 with probability one and 2_h submits the F.D. bid $\delta / (1 + \delta) \vee \lambda$ with probability 1.

   4. 2_l and 2_h submit the 0D. bid 0 with probability one, and

   5. 2_l submit the 0D. bid 0 with probability one, while 2_h randomizes between the 0D. bid 0 and the F.D. bid $\delta / (1 + \delta) \vee \lambda$.

   Case 5, however is not possible; if $\delta / (1 + \delta) \geqq \lambda$, then any bid strictly below $\delta / (1 + \delta) \vee \lambda$ is covered by player 1 with probability one, and player 2_h would therefore have a strict incentive to bid $\delta/(1+\delta)$ with probability one. If $\delta/(1+\delta) < \lambda$, then both types of player 2 are indifferent between two bids, which is impossible given the argument in 6. Therefore, only cases 1-4 can be part of equilibrium behavior.

Existence and uniqueness of the equilibrium

We call *pooling* an equilibrium in which a pooling bid is submitted by 2 (in period 2), and *separating* an equilibrium in which only separating bids are submitted by 2 (in period 2). Similarly, we call an equilibrium in which the initial bid by 1 is covered only by 2_h and not by 2_l an equilibrium with *partial deterrence*, and an equilibrium in which both types of player 2 do not cover an equilibrium with *full deterrence* (all covering choices being made with probability one). In what follows, we denote V_1, V_{2_h}, V_{2_l} the equilibrium payoffs in a continuation game where the player is bidding and b_1, b_2^h, b_2^l the corresponding equilibrium bids, when player 1 assigns probability μ to 2_h. An equilibrium can be characterized by pooling, or separation, and by full deterrence, or partial deterrence, or it may involve some randomization. In light of the possibilities not rule out by the previous arguments, we investigate the necessary and sufficient conditions for existence of each kind of equilibrium in turn.

1. *Pooling and Full deterrence*

   When $\lambda \geq \delta/(1+\delta)$ and $\mu \geq \delta(1-\lambda)$, there exists a undefeated equilibrium with pooling and full deterrence with :

$$\begin{aligned} V_1 &= \frac{1}{1+\delta}, V_{2_h} = \frac{1}{1+\delta}, V_{2_l} = \lambda - \frac{\delta}{1+\delta}, \\ b_1 &= b_2^h = b_2^l = \frac{\delta}{1+\delta}. \end{aligned}$$

   Players cover any bid smaller than their discounted payoff in the continuation game.

   Proof: In order to get full deterrence, 2_h should have no incentive to cover : $\delta V_{2_h} \leq b_1$. This determines the payoff of the uninformed player : $V_1 = 1 - b_1 = 1 - \delta V_{2_h}$. Similarly, 2_h deters the uninformed player by bidding δV_1, which yields a payoff $V_{2_h} = 1 - \delta V_1$. Solving this system gives the values of V_1, V_{2_h}, b_1 and b_2^h. The values of b_2^l and V_{2_l} follow, since in a pooling equilibrium 2_l bids as 2_h.By construction, the covering decision are optimal. The necessary conditions are that $V_{2_l} \geqq 0$, which gives the first restriction

$\lambda \geq \delta/(1+\delta)$ and that the uninformed player prefers full deterrence to partial deterrence :

$$V_1 \geq 1-\mu-\delta V_{2_l} \Leftrightarrow \frac{1}{1+\delta} \geq 1-\mu-\delta\lambda+\frac{\delta^2}{1+\delta} \Leftrightarrow \mu \geq \delta(1-\lambda),$$

which gives the second restriction $\mu \geq \delta(1-\lambda)$.

2. *Pooling and Partial Deterrence*

When $\lambda \geq \delta(1-\mu-\delta(\lambda-\frac{\delta}{1+\delta}))$ and $\mu \leq \delta(1-\lambda)$, there exists a undefeated equilibrium with Pooling and Partial deterrence with:

$$V_1 = 1-\mu-\delta(\lambda-\frac{\delta}{1+\delta}), V_{2_h} = 1-\delta(1-\mu-\delta(\lambda-\frac{\delta}{1+\delta})),$$
$$V_{2_l} = \lambda-\delta(1-\mu-\delta(\lambda-\frac{\delta}{1+\delta})),$$
$$b_1 = \delta(\lambda-\frac{\delta}{1+\delta}), b_2^h = b_2^l = \delta(1-\mu-\delta(\lambda-\frac{\delta}{1+\delta})).$$

Players cover any bid smaller than their discounted payoff in the continuation game. Beliefs supporting such an equilibrium are the following: if the informed player bids more than b_2^h, then the uninformed player believes he is facing 2_h with probability one, and for any other bid he believes he assigns probability one to 2_l. Similarly, if the informed player covers a bid higher than the discounted equilibrium payoff of 2_l, the uninformed player assigns probability one to 2_h.

Proof: By construction. Partial deterrence means that on the equilibrium path, after the uninformed player bids, only the high type covers. The play in the ensuing continuation game is as under complete information. By covering, 2_l's payoff would be $\delta(\lambda-\frac{\delta}{1+\delta})-b_1$. Therefore, $b_1 = \delta(\lambda-\frac{\delta}{1+\delta})$ is the smallest bid deterring 2_l. The pooling bid is then easily computed, since it is the smallest bid which 1 does not cover. The two necessary conditions are that 2_l's payoff be positive, and that the uninformed player prefers partial deterrence to full deterrence.

3. *separating and Full deterrence*

When $\lambda \leq \frac{\delta}{1+\delta}$ and $\frac{1}{1+\delta} \geq 1-\mu$, there exists a undefeated equilibrium with separation and full deterrence with :

$$V_1 = \frac{1}{1+\delta}, V_{2_h} = \frac{1}{1+\delta}, V_{2_l} = 0,$$
$$b_1 = b_2^h = \frac{\delta}{1+\delta}, b_2^l = 0.$$

Players cover any bid smaller than their discounted payoff in the continuation game. Beliefs supporting such an equilibrium are the following: if the informed player bids less than $\frac{\delta}{1+\delta}$, then the uninformed player assigns probability one to 2_l. Beliefs after a higher bid are irrelevant since the game ends after any such bid.

Proof: In a separating equilibrium with full deterrence, 2_l's must be zero and $0D$, and 2_h's bid must be F.D.; bids and payoffs are easily computed. The necessary conditions are that 2_l cannot profitably deviate by mimicking 2_h and that 1 prefers full deterrence to partial deterrence.

4. *separating and Partial deterrence*

   When $\lambda \leq \frac{\delta}{1+\delta}$ and $\frac{1}{1+\delta} \leq 1-\mu$, there exists a undefeated equilibrium with separation and partial deterrence with :

$$V_1 = 1-\mu, V_{2_h} = \frac{1}{1+\delta}, V_{2_l} = 0,$$
$$b_1 = 0, b_2^h = \frac{\delta}{1+\delta}, b_2^l = 0.$$

   Players cover any bid smaller than their discounted payoff in the continuation game. Beliefs supporting such an equilibrium are the following: if the informed player bids less than $\frac{\delta}{1+\delta}$, then the uninformed player assigns probability one to 2_l. Beliefs after a higher bid are irrelevant since the game ends after any such bid.

   Proof: The same argument as above establishes that $b_2^l = 0$. To deter 2_l, Player 1 does not need to make a positive bid. To determine b_2^h, observe that player 1, if he were to cover the bid, would earn a payoff of $\frac{1}{1+\delta}$. As before, the two necessary conditions are that 2_l has no profitable deviation and that player 1 prefers partial to full deterrence.

5. *Mixed Strategy equilibrium : semi-pooling and Partial deterrence.*

When $\lambda \geqq \frac{\delta}{1+\delta}$ and $\lambda \leqq \frac{\delta\left(1-\mu+\frac{\delta^2}{1+\delta}\right)}{1+\delta}$, there exists such a undefeated equilibrium with

$$V_1 = \lambda/\delta, V_{2_h} = 1-\lambda, V_{2_l} = 0,$$

$$b_1 = \delta\left(\lambda - \frac{\delta}{1+\delta}\right), b_2^h = \lambda, b_2^l = \begin{cases} 0 \text{ with probability } \phi, \\ \lambda \text{ with probability } 1-\phi. \end{cases}.$$

Proof: We look for an equilibrium in which 2_l randomizes between a F.D. bid, also submitted by 2_h, and a 0D. bid of 0. Mixing is chosen so that, after observing the F.D. bid, the uninformed player updates his belief to $\mu' = \frac{\mu}{\mu+(1-\mu)(1-\phi)} > \mu$. This new belief turns out to be the smallest belief that leads to a pure strategy equilibrium. That is:

$$\lambda(1+\delta) = \delta\left(1-\mu' + \frac{\delta^2}{1+\delta}\right), \text{ or}$$

$$-\frac{\lambda(1+\delta)}{\delta} + \frac{\delta^2}{1+\delta} + 1 = \frac{\mu}{\mu+(1-\mu)(1-\phi)}, \text{ or}$$

$$\phi = \frac{1}{1-\mu} - \frac{\mu\delta(1+\delta)}{(1-\mu)\left((1+\delta)\delta + \delta^3 - \lambda(1+\delta)^2\right)}.$$

By construction, 2_l is indifferent between both bids,as his payoff is zero. The other necessary and sufficient conditions are readily verified.

These results are summarized in the following figure.

	Parameters	Payoffs:	Bids:
I: separating, Partial deterrence	$\lambda \leqq \frac{\delta}{1+\delta}$ $\mu \leqq \frac{\delta}{1+\delta}$	$\begin{cases} V_1 = 1-\mu \\ V_{2_l} = 0 \\ V_{2_h} = \frac{1}{1+\delta} \end{cases}$	$\begin{cases} b_1 = 0 \\ b_{2_l} = 0 \\ b_{2_h} = \frac{\delta}{1+\delta} \end{cases}$
II: separating, Full deterrence	$\lambda \leqq \frac{\delta}{1+\delta}$ $\mu \geqq \frac{\delta}{1+\delta}$	$\begin{cases} V_1 = \frac{1}{1+\delta} \\ V_{2_l} = 0 \\ V_{2_h} = \frac{1}{1+\delta} \end{cases}$	$\begin{cases} b_1 = \frac{\delta}{1+\delta} \\ b_{2_l} = 0 \\ b_{2_h} = \frac{\delta}{1+\delta} \end{cases}$
III: Pooling, Full deterrence	$\lambda \geqq \frac{\delta}{1+\delta}$ $\mu \geqq \delta(1-\lambda)$	$\begin{cases} V_1 = \frac{1}{1+\delta} \\ V_{2_l} = \lambda - \frac{\delta}{1+\delta} \\ V_{2_h} = \frac{1}{1+\delta} \end{cases}$	$\begin{cases} b_1 = \frac{\delta}{1+\delta} \\ b_{2_l} = \frac{\delta}{1+\delta} \\ b_{2_h} = \frac{\delta}{1+\delta} \end{cases}$
IV: Pooling, Partial deterrence	$\mu \leqq \delta(1-\lambda)$ $\lambda \geqq \frac{\delta\left(1-\mu+\frac{\delta^2}{1+\delta}\right)}{1+\delta}$	$\begin{cases} V_1 = 1-\mu \\ -\delta\left(\lambda - \frac{\delta}{1+\delta}\right) \\ V_{2_l} = \lambda - \delta V_1 \\ V_{2_h} = 1 - \delta V_1 \end{cases}$	$\begin{cases} b_1 = \delta\left(\lambda - \frac{\delta}{1+\delta}\right) \\ b_{2_l} = \delta\left(1-\mu-\delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) \\ b_{2_h} = b_{2_l} \end{cases}$
V: semi-Pooling, Partial deterrence	$\lambda \geqq \frac{\delta}{1+\delta}$ $\lambda \leqq \frac{\delta\left(1-\mu+\frac{\delta^2}{1+\delta}\right)}{1+\delta}$	$\begin{cases} V_1 = 1-\mu- \\ \delta\left(\lambda - \frac{\delta}{1+\delta}\right) \\ V_{2_l} = 0 \\ V_{2_h} = 1-\lambda \end{cases}$	$\begin{cases} b_1 = \delta\left(\lambda - \frac{\delta}{1+\delta}\right) \\ b_{2_l} = 0 \;\; (\phi) \text{ or } \lambda \;\; (1-\phi) \\ b_{2_h} = \lambda \end{cases}$

One-sided incomplete Information: $v_1 = \lambda,\ v_2 = \lambda$ or 1.

The arguments are similar to, but much simpler than the arguments used in the previous case. They are therefore omitted, and the following table summarizes the relevant results.

	Restrictions	Payoff:	Bids:
I: No deterrence	$\mu \geqq 1-\delta$ $\lambda \leqq \delta$	$\begin{cases} V_1 = 0 \\ V_{2_l} = \lambda \\ V_{2_h} = 1 \end{cases}$	$\begin{cases} b_1 = 0 \\ b_{2_l} = 0 \\ b_{2_h} = 0 \end{cases}$
II: Partial deterrence	$\mu \leqq 1-\delta$ $(\mu+\delta)\lambda \leqq \delta$	$\begin{cases} V_1 = (1-\delta-\mu)\lambda \\ \quad +\delta^2 \left(\frac{\lambda-\delta}{1-\delta^2}\right)^+ \\ V_{2_l} = \lambda - b_{2_l} \\ V_{2_h} = 1 - b_{2_h} \end{cases}$	$\begin{cases} b_1 = \delta\lambda - \delta^2 \left(\frac{\lambda-\delta}{1-\delta^2}\right)^+ \\ b_{2_l} = \delta V_1 \\ b_{2_h} = \delta V_1 \end{cases}$
III: Full deterrence	$\lambda \geqq \delta$ $(\mu+\delta)\lambda \geqq \delta$	$\begin{cases} V_{1_h} = \frac{\lambda-\delta}{1-\delta^2} \\ V_{2_l} = \lambda - b_{2_l} \\ V_{2_h} = \frac{1-\delta\lambda}{1-\delta^2} \end{cases}$	$\begin{cases} b_1 = \frac{\delta(1-\delta\lambda)}{1-\delta^2} \\ b_{2_l} = \delta V_{1_h} \\ b_{2_h} = \delta V_{1_h} \end{cases}$