# Costly Signaling in Auctions[1]

Johannes Hörner

Northwestern University[2]

Nicolas Sahuguet

HEC Montréal[3]

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[2]Kellogg School of Management, Department of Managerial Economics and Decision Sciences, 2001 Sheridan Road, Evanston, IL 60208, USA e-mail: j-horner@kellogg.northwestern.edu

[3]Institute of Applied Economics, HEC Montréal, 3000, chemin de la Côte-Sainte-Catherine Montréal (Québec) Canada H3T 2A7 email: nicolas.sahuguet@hec.ca.

## Abstract

This paper analyzes a dynamic auction in which a fraction of each bid is sunk. Jump bidding is used by bidders to signal their private information. *Bluffing* (respectively *Sandbagging*) occurs when a weak (respectively strong) player seeks to deceive his opponent into thinking that he is strong (respectively weak). A player with a moderate valuation bluffs by making a high bid and drops out if his bluff is called. A player with a high valuation should vary his bids and should sometimes sandbag by bidding low, to induce lower bids by his rival.

Keywords: Auctions, Asymmetry, Jump Bidding, Bluffing, Sandbagging

JEL Classification Numbers: C71 - D44 - D82

"Another issue related to bidding strategy is whether to be bold or cautious in opening bidding. The man who strongly desires an item will jump in with both feet, as it were, or try to rout the enemy by starting out with a high, possibly loud, bid intended to "knock out" his opponents. Sometimes he even tops his own bid. This approach may discourage competitors at the outset and prevent them from ever getting caught up in the spirit of the bidding. In a very different strategy, a prospective buyer, even though determined to purchase an item, bids tentatively and cautiously in order to feel out the opposition. He hopes that by indicating a low regard for the offering he will lull opponents into a false sense of security." - Ralph Cassady, Jr.

# 1  Introduction

The received view in auction theory is that, in English auctions, a bidder should submit a bid that barely exceeds the previous one by the minimum increment, unless his valuation is reached, at which point he should stop bidding. This view, however, fails to account for *bluffing* and *sandbagging*, the two well-known bidding tactics described by Cassady in the introductory quotation. It also fails to account for *jump bidding*, the phenomenon that, in many auctions, bidding occurs in repeated jumps.

There are two theories to explain jump bidding. The first theory interprets such bids as coordination devices: Avery (1998), in a common values setting, shows that the opening bid may be used to coordinate upon the asymmetric equilibrium to be played in a second round.

According to the second theory, jump bidding may follow from the costs of submitting and revising bids (Fishman (1988), Hirshleifer and Png (1989), Daniel and Hirshleifer (1998)). This second theory also provides an explanation for the bidding delays one observes in "spontaneous auctions" such as takeover contests. In takeover contests, or in larger auctions, such as the U.S. government P.C.S. spectrum auction, bidding costs may be substantial. As a result, the costs of bidding must be weighed carefully by bidders; hence the delay and the jump bids.[1]

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[1]The cost of bidding for big private finance initiatives in the U.K. is so high, for instance, that it has led companies to be much more selective about the projects for which they tender, a phenomenon described by Timmins et al. (2002) for the Financial Times. Besides the cost of obtaining financing for the bids, Hirshleifer and Png mention several kinds of cost of takeover bidding: "(...) fees to counsel, investment bankers, and other outside advisors, the opportunity cost of executive time."

The element common to these two theories is that jump bidding is used to signal one's strength in the auction. In Avery's model, a high opening bid is followed by a less aggressive strategy by the competitor. When bidding is costly, a high bid is used to deter other bidders from entering, or remaining in, the bidding contest. In both theories, signaling is monotonic in the sense that bidders with higher types (valuation or signal) bid more.

We propose an alternative way of looking at signaling in bidding contests. Our main contribution is to show that non-monotonic signaling can exist. Bluffing and sandbagging strategies are used by bidders.

We analyze two-round auctions. Two bidders compete for an object. Each player knows what the object is worth to him, but this valuation is private information. In the first period, one of the bidders has the opportunity to make an opening bid. His rival must then either match the bid, or quit the auction. If the first bid is matched, a sealed bid auction determines the winner in the second round. In the simplest version of the model, the first bidder may initially submit one of two bids, either high or low.

Bidding high has two potential effects. 1) It may deter the other player from continuing the auction, allowing the first player to win with no further bidding. This is the *deterrence effect*. 2) The bid might be covered, which can lead to an *escalation effect*. If the opening bid is interpreted as a sign of strength, the second player correctly infers that to have a chance of winning he has to bid aggressively in the second round. While the deterrence effect benefits the first player, escalation makes it more expensive for him to win.

Bidding low, the alternative option, has a *sandbagging effect*. This kind of bid certainly does not deter the second player. However, if he interprets the low bid as a sign of weakness in the first player, he may decide to weaken his own bid in the second round, so as not to waste resources. This reduces the costs of winning and makes sandbagging an attractive option for players with a high valuation.

In standard signaling games, the sender always tries to convince the receiver that he is strong; in our terms, that he has a high valuation. In our game, however, the incentives to signal are more sophisticated. In particular, a player with a high valuation can benefit both from being perceived as very strong and from being perceived as very weak. This leads to complex equilibrium behavior where both direct and inverted signaling are present. Players with a weak valuation will make low opening bids, while those with intermediate level valuations will "bluff" by making high opening bids to achieve deterrence, but withdraw from

the auction if their bid is called, thereby avoiding escalation. Players with high valuations will choose randomly between high and low opening bids, enjoying both the deterrence effect of a high bid and the sandbagging effect of a low one. Thus, the bidding strategies are not monotonic. The second player's decision whether or not to cover is less interesting. If the prize is worth enough to him he will cover; if not he will pass.

Our results apply to any contest in which expenditures or investments are sunk as part of the bidding process, for example, auctions, legal contests, lobbying contests and technology races.[2] Bluffing is a tactic often used in auctions (See Avery (1998) for several examples). Sandbagging is often used in legal proceedings, when a litigant withholds legal arguments until they reach the courts of appeal (see for instance U.S. AirWaves, Inc. vs. F.C.C., United States Courts of Appeal, No. 98-1266). By waiting to raise arguments and present evidence in the reply not contained or raised in the moving papers, a defense counsel might "sandbag" the trial court.

Our result sheds light on two-stage auctions. The importance of two-stage auctions is now well recognized. Ye (2004) documents a variety of examples of two-stage auctions in which bidding takes two rounds. The auctioneer typically uses the first bid to select the participants of the second round. For instance, Central Maine's Power placed its entire 2.110 megawatt asset portfolio for sale in such a two-stage auction. Similarly, in California, Pacific Gas and Electricity divested all of its fossil generation plants using such a procedure. Ye discusses the players' incentives to misrepresent their valuations in the first stage. Ye's focuses on the impact of the first stage on the entry decision (which is assumed to be costly) of bidders. One way to reinterpret our model is to consider first-stage bids as entry costs and see costly entry as a way to manipulate the beliefs of other potenital bidders.

While Ye models the first bid as non-binding, Perry, Wolfstetter and Zamir (2000) consider instead the case in which the first period bid serves as the minimal allowable bid in the second stage, as we do. Caffarelli (1998) documents another example of such a two-stage auction, used for the privatization of the Italian industrial conglomerate ENI. In the first round, all agents submitted sealed bids and the highest two were selected for the second round. All first-round bids were made public, so that signaling becomes potentially impor-

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[2]Consider, for instance, the contest between Boeing and Airbus to develop a super jumbo. The niche is an appropriate market to be occupied by a single company, and the development costs amount to $12 billion for Airbus, which are sunk whether or not its plane ends up being preferred to Boeing's rival offering or not (The Economist, June 1995).

tant. Yet Perry, Wolfstetter and Zamir assume that only losing bids are revealed, destroying thereby incentives to signal. However, they mention that revealing all the bids can have interesting consequences, for instance of revealing the ranking of valuations. Landsberger, Rubinstein, Wolfstetter and Zamir (2001) analyze such a game in which the ranking of valuations is known to bidders This does not really address the problem of signaling in two-stage auctions since it implicitly assumes that first-stage bidding has to be monotonic. One of our main results is that monotonic equilibria would not occur in such situations and we show that players incentives to misrepresent their valuations in the first stage are complex, since both sandbagging and bluffing strategies are used in equilibrium.

The importance of bluffing and sandbagging strategies is well established in the game of poker. A detailed comparison with poker models can be found in Section 7. To summarize briefly: the incentives are reversed in poker. A player with a strong hand would like his opponent to submit high bids, and a player who suspects his opponent to hold a strong hand has an incentive to submit low bids. Auctions exhibit the opposite features. As a result, in the poker game most closely resembling ours (Newman (1959)), the lowest and the highest types submit high bids, while intermediate types submit low bids. By submitting high bids with low types, a player "jams" his opponent's inferences, so as to encourage bidding. In our model, it is the intermediate types who are the most aggressive in their initial bids, while high types are more likely to submit a low bid, so as to win more cheaply afterwards.

The non-monotonic equilibrium is reminiscent, to a certain extent, of the results of Baliga and Sjöström (2003). They analyze a model of an arms race with a simultaneous cheap talk stage beforehand. In the two message cheap talk game, they derive an equilibrium in which both weak and very tough types use the "dove" message while intermediate types use the "hawk" message. The context is very different since the arms race model is a game of coordination and communication is two-sided. Players want to coordinate if possible, and messages help in that respect. The non-monotonicity comes from the presence of very tough types who do not look for coordination; they mimic low types (sandbagging) and take advantage of the situation when their opponent is fooled in believing they are facing a dove.

Another related paper on non-monotonic signaling is Feltovich and al. (2002). In the context of a standard signaling model, they show that when some additional signal is exogenously provided, there might exist non-monotonic equilibria in which low and high types do not use the standard, endogenous signal, while intermediate types do. The logic is however

quite different. In their model, high types are not trying to fool the other party into believing that they are a weak type. Rather, they rely on the exogenous signal to be distinguished from low types and rely on the counter-signal to separate from intermediate types, while saving on the cost of the endogenous signal.

The particular auction we analyze is related to the dollar auction. The dollar auction is also a dynamic auction but the prize has a common value for all bidders. However, Demange (1992) introduces private information in the dollar auction and analyzes "escalation equilibria" in which players can end up paying more than the value for the prize. This is a standard result in the war of attrition. This definition of escalation is not what we have in mind: in our setup, a jump bid triggers escalation in the sense that it leads to more aggressive bidding. After a jump bid, players bid more aggressively than after an ordinary bid. Another difference is that Demange analyzes a game with two possible types and two possible bids. Our set-up allows for more complex signaling strategies that can not be analyzed in her framework.

The structure of the paper is as follows. In the next section we provide some specific examples to illustrate our results. We first show that in a two-stage-auction, the incentives to manipulate beliefs are such that monotonic equilibria cannot exist under the first-price winner-only-pays rule. We then show how signaling through jump bidding takes place in dynamic auctions under various formats (all-pay first-price, winner-only-pays first-price) in the case of discrete types. In Section 3, we introduce the general model. Section 4 characterizes the different kinds of signaling that might take place in equilibrium. Section 5 shows that under a number of additional assumptions, there exists a unique equilibrium satisfying a common refinement. Section 6 shows that our main findings also hold when players are not restricted to a binary choice of opening bid, by deriving an equilibrium in non-partitioning strategies. Section 7 compares our results to the literature on auctions and bluffing. Section 8 concludes. Proofs are in appendix.

## 2   Illustrating the Results: Examples and Counter-Examples

Consider a dynamic auction with two *bidders*. Bidders have private, independent valuations for an item. Bidder 1 initially submits a bid, either an ordinary bid of 0, or a jump bid of $K > 0$. If he bids $K$, bidder 2, upon observing this bid, decides either to bid $K$ as well (to

*cover*, or *match*) or to quit. If he quits, bidder 1 wins the item. If bidder 2 covers, or if bidder 1 chooses the ordinary bid, a second bidding stage begins: both players simultaneously submit an unrestricted (non-negative) bid, and the high bidder wins the item. Ties are randomly broken. A more detailed description of this game is provided in Section 3. The game tree is depicted in Figure 1.
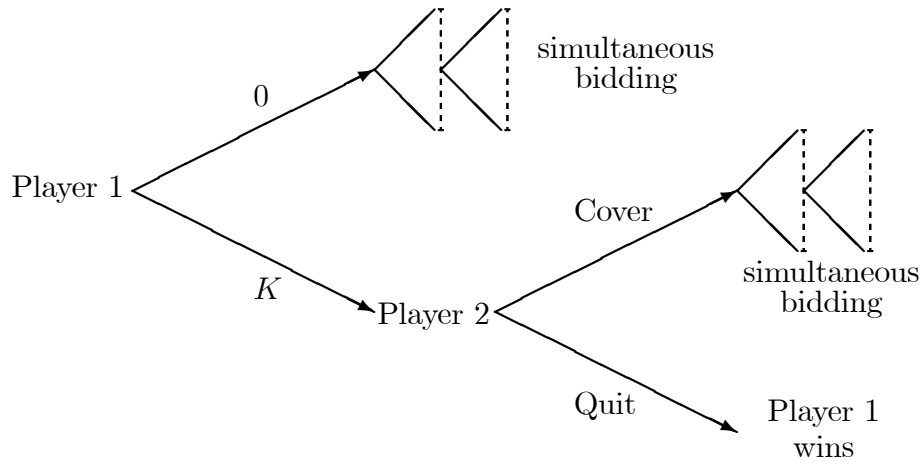


Figure 1 : Game tree

Player 1 has a first-mover advantage in this game due to the signaling possibilities that a jump bid offers. The main message of the paper is to show that, under various auction formats, if the jump bid is not fully deterrent (that is, player 2 covers with some probability after a jump bid), then the equilibrium signaling is typically non-monotonic, exhibiting intriguing strategic features. Before describing those features, it is helpful to understand why the equilibrium cannot be in threshold strategies, even in the familiar case of the (winner-only-pays) first price auction.

## 2.1 First-price auction: Non-existence of equilibria in threshold strategies

Since the auction is dynamic, it is necessary to be more specific about what we mean by a winner-first-price auction in this context. If Player 1 does not bid $K$, the winner is determined by a winner-only-pays first-price auction. If Player 1 bids $K$, and Player 2 quits, Player 1 wins and pays $K$, while Player 2's payoff is 0. If Player 1 bids $K$ and Player 2

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covers, the item is assigned in the second stage by a winner-only-pays first-price auction with nonnegative bids. The winner pays the sum of his bids. The loser's payoff is zero. Ties are broken randomly.

Suppose that the players' valuations are drawn from some common, positive and continuous distribution over the support $[0,1]$, $K < 1$. We argue that no equilibrium in which Player 1 submits the jump bid with positive probability, and Player 2 covers such a bid with positive probability, can be an equilibrium in threshold strategies: that is, an equilibrium in which Player 1 submits the jump bid if and only if his valuation exceeds some threshold $\alpha < 1$, while Player 2 covers the jump bid if and only if his valuation exceeds some threshold $\gamma < 1$. Suppose indeed that such an equilibrium existed.

Observe first that Player 1 cannot submit the jump bid with probability one. Indeed, if Player 2 does not cover the jump bid for sure, submitting the jump bid yields a negative expected payoff to Player 1 if his valuation is low enough, as he may forced to pay $K$ in the event that Player 2 concedes. If Player 2 were to cover for sure, then it cannot be optimal for Player 1 to submit the jump bid for sure as well: otherwise, at least one player must win with positive probability with an arbitrarily low valuation, yielding again a negative expected payoff. Therefore, it must be that $\alpha > 0$.

We are thus led to consider two possible "continuation games". In one of them, Player 1's valuation is drawn from $[0, \alpha]$, while Player 2's valuation is drawn from $[0, 1]$. In the second one, Player 1's valuation is drawn from $(\alpha, 1]$, while Player 2's valuation is drawn from $[\gamma, 1]$.[3] It follows from standard arguments that, in the first case, Player 1 with valuation $v_1 = \alpha$ must submit with positive probability a total bid $0 + b_1$ that wins with probability $p_1 = 1$, while if his valuation $v_2$ slightly exceeds $\alpha$, he submits a total bid $b_2$ (including $K$) that wins with probability $p_2 < 1$. As usual, incentive compatibility then requires:

$$
\begin{cases}
p_1 \left( v_1 - b_1 \right) \geq p_2 \left( v_1 - b_2 \right), \\
p_2 \left( v_2 - b_2 \right) \geq p_1 \left( v_2 - b_1 \right),
\end{cases}
$$

or $(p_1 - p_2) v_1 \geq p_1 b_1 - p_2 b_2 \geq (p_1 - p_2) v_2,$

which is impossible since $p_1 > p_2$ and $v_2 > v_1$.

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[3]Type $\alpha$ must be indifferent across jump bids, so it is irrelevant for the argument whether he submits 0 or $K$. For definiteness, we assume he submits 0.

This means that, if Player 2 were to follow the corresponding strategy, Player 1 would want to deviate from his own strategy in one of two cases: either it is sufficiently cheap and deterrent to submit the jump bid that it is worth doing so with a signal just below $\alpha$, or it is worth 'lowballing' in the first period by bidding 0 rather than $K$ when the valuation is just above $\alpha$ in order to win in the second period with a relatively low bid.

We will argue in this paper that such behavior (*sandbagging*) is in fact part of the equilibrium. Tractability, however, prevents us from solving for the equilibrium in this general set-up, at least with a first-price auction. We can do so in a discrete example (see below), and we can also solve more generally in the case of the all-pay auction format. In each case, the equilibrium will be non-monotonic. In such a non-monotonic equilibrium high types (the highest valuations) randomize between the jump bid and the ordinary bid. Intermediate types use the jump bid with probability one, thus creating the non-monotonicity in the bidding strategies.

To make high types indifferent, it is necessary that an ordinary bid and a jump bid have benefits and costs that cancel out. A jump bid has a clear benefit, which is that it is sometimes not covered and enables the first player to win the auction without further competition. The relative cost of the jump bid compared to the ordinary bid is the result of a more aggressive bidding strategy from player 2 after a jump bid. This comes from the fact that on average, the ordinary bid is used by lower types than the jump bid. A signaling equilibrium obtains when the deterrence effect and the escalation effect associated with the jump bid have the same cost/benefits ratio as the sandbagging effect associated with the ordinary bid. Even if player 1 is indifferent between initial bids, it does not mean that the signaling has no effect. His payoff is strictly higher when he uses the signaling strategy than when he does not. The following examples illustrate this result under various auction rules.

## 2.2  The first-price, all-pay auction

We first consider the case of a first-price, all-pay auction. That is, both winner and loser forfeit all their bids, including $K$ if that case occurs.

Specifically, we assume that bidder 1 has one of three possible types: a low valuation of $\frac{1}{4}$ with a probability of 1/10, an intermediate valuation of $\frac{1}{2}$ with a probability of 1/10, or a high valuation of 1. Bidder 2 has either a low valuation of 3/5 or a high valuation of 3/2, with equal probability. Let the jump bid be $K = 1/10$. If bidder 2 does not cover, bidder

1 wins the object and pays $K$, and bidder 2's payoff is 0. If bidder 2 covers (or if bidder 1 chooses to bid 0), a simultaneous first-price all-pay auction takes place in the second stage.

How should bidder 1 choose his initial bid? Suppose he makes a jump bid of $K$. If bidder 2 perceives it as a sign of strength upon observing it, he may then prefer to quit, allowing bidder 1 to win at low cost. However, if bidder 2 covers the jump bid, more aggressive bidding will ensue in the second stage. Suppose, on the other hand, that bidder 1 bids 0. This bid has no deterrence effect; but if Player 2 perceives it as a sign of weakness, he will believe that he can win with a "less aggressive" bid in the second stage, thus saving costs.

The following strategy profile (along with the corresponding beliefs) is a sequential equilibrium. Player 1's low type bids 0, the intermediate type bids $K$ and the high type randomizes between bids, bidding $K$ with probability $7/8$. Player 2's low type covers a bid of $K$ with probability $1/5$ and the high type covers for sure. After bidding $K$, Player 1's intermediate type submits a losing bid (0 in this example) in the second round, whenever it is reached. An equilibrium displaying such features is referred to as an *equilibrium with covering*, formally defined in Section 4.

Figure 2 summarizes this equilibrium. Player 1 submits a jump bid with probability $4/5$ and Player 2 covers with probability $3/5$. Because different types bid and cover differently, the players' beliefs vary across subgames. For instance, Player 2 assigns zero probability to Player 1's intermediate type in the subgame following an ordinary bid (subgame 0), and assigns zero probability to Player 1's low type in the subgame following a jump bid which is covered (subgame $K$). Player 1 assigns a higher probability to Player 2's high type in subgame $K$ than in subgame 0.

The bidding in the subgames reflects these beliefs:

- In subgame 0, bids only range from 0 to $7/10$. Player 1's low type bids either 0 or continuously randomizes over $[0, 1/10]$, while the high type continuously randomizes over $[1/10, 7/10]$. Meanwhile, Player 2's low type continuously randomizes over $[0, 2/10]$, while the high type continuously randomizes over $[2/10, 7/10]$.

- In subgame $K$, bids range from 0 to 1. Player 1's intermediate type bids 0, and the high type bids either 0, or continuously randomizes over $[0, 1]$. Player 2's low type continuously randomizes over $[0, 1/6]$, while the high type continuously randomizes over $[1/6, 1]$.

From the bidding distributions given in the appendix, it follows that, by bidding $1/10$ ($= K$) in the subgame 0, Player 1 wins with probability $2/5$. This implies that Player 1's

high type is indeed indifferent between both an ordinary and a jump bid: by bidding $0$ and then $K$, he wins with probability $2/5$, while by bidding $K$ and then $0$, he wins with probability $2/5$ as well (the probability that Player 2 quits).

It follows also from the bidding distributions that, by bidding $0$ in subgame $K$, Player 2's low type wins with probability $1/6$: this implies that his payoff from doing so equals $\frac{3}{5} \cdot \frac{1}{6} = K$, so that, at the covering stage, he is indeed indifferent between quitting or not.

Finally, observe that, if Player 1's intermediate type were to bid in subgame $0$, he would bid $K$ (since this is the intersection of the low type's and the high type's bidding supports), and win with probability $2/5$; as we have seen, this is also his expected payment and probability of winning if he submits the jump bid. Therefore, it is optimal to submit a jump bid.
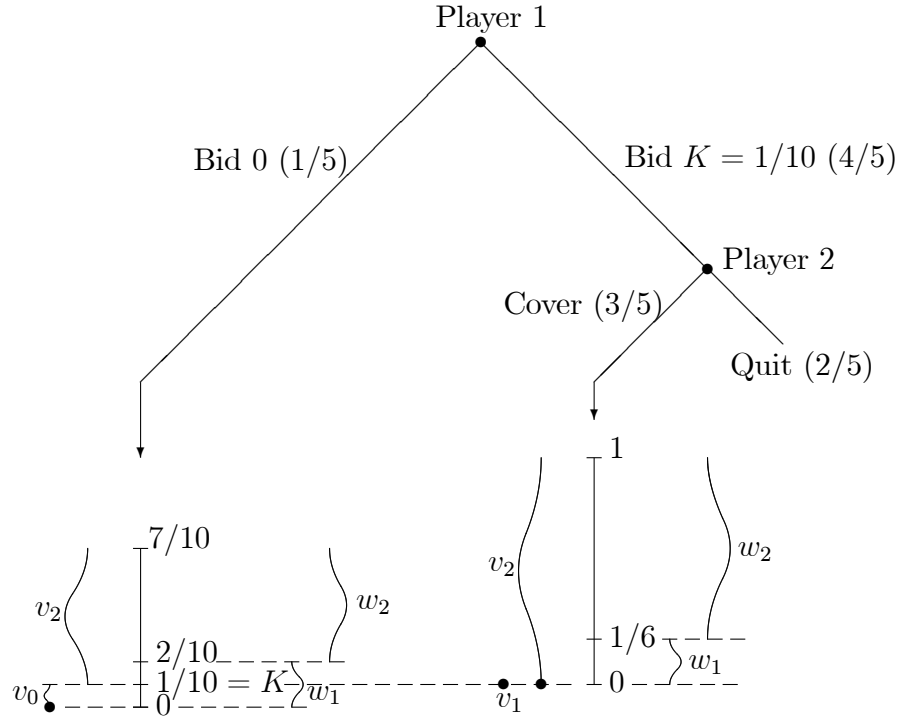


Figure 2 : Equilibrium with covering

That Player 1's initial bid is not monotonic in his valuation is a novel and rather surprising feature of the equilibrium. Somewhat paradoxically, Player 1's initial bidding strategy cannot be monotonic precisely because his "overall" probability of winning in the auction must be

monotonic in his valuation. To see this implication, suppose, for the sake of argument, that the high type $v_H$ always uses a jump bid, and suppose that Player 2 covers (at least with positive probability) when his valuation is high. Then Player 1's high type is willing to follow a course of action (among others) which has a probability of winning strictly less than one (say $\lambda$): he bids $K$ first, and then, if necessary, submits a bid in the interior of his bidding support in the second stage say $b_K$.[4] At the same time, either of Player 1's low type or intermediate type, $v_L$, is willing to follow a course of action that has a winning probability of one: he bids 0 first, and then submits the highest bid ever submitted in the ensuing auction (since Player 1's high type always bids $K$ by assumption, one of the lower types must be Player 1's highest type in the auction following a bid of 0), say $b_0$. Because preferences satisfy the single-crossing condition, this yields a contradiction. We would have $v_L - b_0 \geq \lambda v_L - b_K$ and $v_H - b_0 \leq \lambda v_H - b_K$).

In fact, in a monotonic equilibrium, the incentive for high types to deviate would be too great because the benefit of sandbagging would be very large. Therefore, either Player 2 never covers a bid of $K$, or, if such a bid is ever observed, the high type must be randomizing between both initial bids.

In addition, whenever covering occurs with positive probability, some types of Player 1 must be "bluffers", that is, must submit a losing bid in the second round. Otherwise, Player 2's lowest type who covers would not recoup the cost of covering. Hence, the equilibrium has to be non-monotonic.

Since the high type randomizes between both initial bids, he is indifferent between them. The benefits of a jump bid are that with probability 2/5, Player 2 quits and Player 1 wins with a bid of only $K$. However a covered jump bid leads to an escalation, while an ordinary bid leads to a softening of the bidding competition. Bidding is less aggressive in subgame 0 than in subgame $K$, because players perceive their opponent as weaker in the former than in the latter. For example, a bid of 7/10 is sufficient to win for sure in subgame 0, while for a sure win in subgame $K$ it is necessary to bid 1. In the 'static' game, the minimum bid required for a sure win is 173/200, which is strictly in between that of subgames 0 and $K$.

The strength of Player 1 in a subgame can be measured by the reversed hazard rate order: as a random variable, Player 1's valuation in subgame 0 is smaller than in the static

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[4]Recall that, in a first-price all-pay auction, the bid distribution has no atom at the upper extremity of the players' bidding supports.

game, which in turn is smaller than in subgame $K$.[5]

In addition to the equilibrium with covering just described, there exists another sequential equilibrium, in which Player 1 uses the ordinary bid with probability 1. Such an equilibrium, in which Player 1 never makes a jump bid and thus Player 2 never needs to cover, is termed a *non-revealing equilibrium.* The strategy profile supporting such a *non-revealing equilibrium* are given in the appendix. Such a profile is legitimate if out-of-equilibrium actions are interpreted as trembles. If bids are interpreted as rational signals, this equilibrium seems less reasonable, because Player 1's higher types have an incentive to submit a jump bid.

The idea that out-of-equilibrium actions should be interpreted as rational signals underlies, for instance, the concept of Perfect Sequential Equilibrium, defined by Grossman and Perry (1986), and further discussed in Section 5. Roughly speaking, in our set-up, a sequential equilibrium fails to be a Perfect Sequential Equilibrium if there exists an out-of-equilibrium action for Player 1, and associated beliefs for Player 2, such that, if Player 2 were to take a best-response to these beliefs after observing this action, Player 1 would have an incentive to deviate from the equilibrium and take this action if and only if his type is an element of the support of these beliefs. (The definition of Perfect Sequential Equilibrium, given in Section 5, imposes additional requirements that are derived from Bayes' rule) Indeed, Player 1's intermediate and high types strictly prefer the equilibrium with covering described above to the non-revealing equilibrium that is described in the appendix: Player 1's high type's payoff is 60/200, compared to 27/200, and Player 1's intermediate type's payoff is 20/200, compared to 5/200. (Player 1's low type is indifferent, as his payoff is 0 in both equilibria.) While this is insufficient evidence *per se* to rule out the non-revealing equilibrium as not being perfect sequential, it suggests that such beliefs can be found, because Player 1's higher types have, in a sense, a strong incentive to bid $K$. Indeed, we prove in the appendix that, for these parameters, the non-revealing equilibrium is not a Perfect Sequential Equilibrium, and therefore, the equilibrium with covering is the only such equilibrium. However, if $K$ is much larger (say, $K = 1$), it is then clearly in the best interest of Player 1 never to use the jump bid.

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[5] For discrete random variables $X$ and $Y$ taking on values in a set $A$, $X$ is smaller than $Y$ in the reversed hazard rate order if, for all $n \in A$,

$$\frac{P\{X = n\}}{P\{X \leq n\}} \leq \frac{P\{Y = n\}}{P\{Y \leq n\}}.$$

For this specific example, observe that Player 2's high type covers in any equilibrium, because his valuation exceeds the sum of $K$ and Player 1's high valuation. Nevertheless, *equilibria with assured deterrence*, in which Player 2 never covers, exist for other parameters. For instance, it is straightforward to check that, if $K$ equals $\frac{1}{2}$ instead of $1/10$, it is an equilibrium for Player 1's to bid $K$ for sure if his type is high, and to bid 0 otherwise, and for Player 2 never to cover (supported by the belief that, if the bid is covered, Player 2's type is high).

In the rest of the paper, we generalize the following ideas to more general environments:

1. For small enough values of $K$, there exists an equilibrium with covering, in which Player 1 bids $K$ for sure if his type falls into an intermediate range, and loses for sure in the second round when covering occurs (*bluffing*). For higher valuations, Player 1 randomizes between bidding 0 (*sandbagging*) and bidding $K$, with a probability independent of his type. For lower types, Player 1 bids 0 for sure.

2. There exists a unique Perfect Sequential Equilibrium. For small values of $K$, it is the *equilibrium with covering*. For higher values, it is an *equilibrium with assured deterrence*, in which Player 1 submits $K$ if his type is high enough, and Player 2 never covers. For even higher values of $K$, the equilibrium is *non-revealing*, as Player 1 always submits a bid of 0.

The expected revenue generated by the equilibrium with covering ($187/300 \simeq 0.62$) is smaller than the expected revenue in the static (or non-revealing) auction ($319/450 \simeq 0.71$, calculations omitted). This result is mainly driven by the inefficient allocation of the item, whenever bluffing succeeds. As we will see in a more general set-up, when players are symmetric, the equilibrium with covering always raises a lower expected revenue than the non-revealing equilibrium, but the loss is not monotonic in the value of the jump bid.

## 2.3 Other auction formats

This paper is primarily based on first-price all-pay auctions, or a slight generalization thereof. All-pay auctions are remarkably tractable when distributions are asymmetric. It is not possible, for instance, to analyze this dynamic auction using a (winner-only-pays) first-price

auction with absolutely continuous distributions.[6] It is however possible to do so with discrete types, as we now show.[7]

Preliminary to providing the example, we should note that two important conditions must be satisfied if non-monotonic bidding featuring bluffing and sandbagging is to be achieved.

1. Because the jump bid must credibly signal strength, it must involve a cost that prevents it from being used by Player 1's low type. This cost need not be incurred automatically, as in an all-pay auction, or in an auction with a fixed cost of bidding; it can be of a probabilistic nature, as in a first-price winner-only-pays auction, where the jump bid has to be paid in the event that Player 2 quits.

2. A player's bidding strategy must reflect the perceived strength of his opponent. The stronger he believes his opponent to be, the more aggressive the bids a player submits.

The second condition is necessary both for bluffing to act as a deterrent and for sand-bagging to be effective; since a jump bid signals Player 1's strength, Player 2's low type may prefer to quit, even if covering is not costly (as in a winner-only-pays auction), because it expects a zero continuation payoff from covering. Conversely, after an ordinary bid, both players submit moderate bid, which makes an ordinary bid attractive to Player 1's high type. This second condition is a well-known characteristic of winner-only-pays first-price auctions, when distributions are ordered by reversed hazard rate (see Krishna (2002), p. 47). However, this condition fails in a winner-only-pays second-price auction, where a weakly dominant strategy is to bid one's value, independently of one's beliefs about the other player. Therefore, sandbagging cannot be effective in a winner-only-pays second-price auction. Hence, in this case, non-monotonic bidding (and hence equilibrium with covering) is not possible[8].

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[6]In fact, with a standard first-price auction and absolutely continuous distributions, the existence of a Bayesian Nash equilibrium in relevant subgames is unknown, as current existence results require that the lower extremity of the player's type support be identical across players. See Krishna (2002), Appendix G, for details and references.

[7]A similar equilibrium can be constructed with a second-price, all-pay auction. The details are available from the authors.

[8]Arozamena and Cantillon (2004) make a somewhat related point. They analyze a dynamic game in which a firm can invest to lower its cost distribution prior to a procurement auction. They show than in the case of a first price auction, this will make the other firm bid more agressively, while in the case of a second price auction, investments do not change bidding behavior since firms bid their true cost.

The equilibrium with covering is robust to most equilibrium refinements, including Perfect Sequential Equilibrium, but it may not be unique. Details are in Appendix.

**Example 1 (first price winner-only-pays auction)**:

Each player has three possible valuations: $v_0 = 0$, $v_1 = 1/2$ and $v_2 = 1$. Each valuation is equally likely for player 1: $p_0 = p_1 = p_2 = 1/3$, where $p_i$ is the probability of valuation $v_i$. As for player 2, his valuation is either $v_2$ with probability $1/2$, or $v_0$, $v_1$ with probability $1/4$ each (the main features of the equilibrium only depend on the sum of the low and intermediate valuation probabilities of player 2). Assume $K = 1/10$.

There is a (non-monotone) equilibrium with covering: Player 1 bids 0 if his valuation is $v_0$; he bids $K$ with probability $q \in (0, 1)$ if his valuation is $v_1$ and randomizes with probability $p \in (0, 1)$ between 0 and $K$ if his valuation is $v_2$, where $q > p$. Player 2 covers if and only if his valuation is 1 (and submits then a strictly positive bid). Finally, if Player 1's valuation is $v_1$, he submits a losing bid whenever his initial bid $K$ is covered. As before, intermediate types $v_1$ are willing to bluff in an attempt to deter their opponent, and lose for sure if covering occurs. The high type $v_2$ is indifferent between a jump bid that is deterrent but triggers aggressive bidding and an ordinary bid that leads to cautious bidding. Of course, covering is not directly costly to Player 2's low or intermediate type, but the jump bid changes his beliefs in such a way that, if he were to cover, his payoff would then be zero. Therefore, he is willing to quit. Observe that in this example, player 1's intermediate type randomizes as well, although the equilibrium is non-monotonic ($q > p$). With a continuum of types, both the type and the number (measure) of bluffers become endogenous, so it is natural to interpret $q$ as a fraction, and $p$ as a true randomization. Examples where $q = 1$ can be constructed as well, but involve therefore a nondegenerate choice of parameters (unless one allows more than three types). Nevertheless, it is important to point out that equilibria with covering are always mixed equilibria, in the sense that intermediate types ("bluffers") are indifferent between both initial bids, even if in equilibrium they may bid $K$ with probability one. As with all mixed equilibria, the probabilities are determined by equilibrium considerations, not by optimality conditions alone.

The intuition for the non-monotonic equilibrium is the same as in the previous example. The bidding after a jump bid is more aggressive than after an ordinary bid, but with some probability a jump bid is not covered and the auction is won at a low cost. The non-monotonic structure of signaling is necessary. Player 1 must be perceived as weaker after an

ordinary bid. To achieve that, it must be that the intermediate type use the jump bid.

# 3    The Bidding Game

## 3.1    The Model

Two risk-neutral bidders (Player 1 and 2) compete for an object prize. The bidders' valuations, denoted respectively $v$ for Player 1 and $w$ for Player 2, are drawn independently from distributions $F$ and $G$ with support $[0, 1]$. We assume that $F$ and $G$ are continuously differentiable (with densities $f$ and $g$). Valuations are private information. As in the examples, the auction is modeled as a two-stage game. In the first stage of the game Player 1 makes an opening bid. For now, we restrict bid choices to be either low (an ordinary bid normalized to 0) or high (a jump bid $K > 0$). If Player 1 uses the jump bid, Player 2 decides whether to cover or not. Covering means bidding $K$ as well. If Player 2 does not cover, Player 1 wins the object. If he covers, or if Player 1's opening bid is 0, the game enters a second stage that consists of a simultaneous auction. This second bid is unrestricted: players may bid any amount at this stage. The winner pays the sum of his bids, while the loser pays a fraction $\rho \in (0, 1]$ of his opening bid and of his second bid.[9] This formulation is a slight generalization of the first-price all-pay auction, since we only require that at least a fraction of the opening bid is sunk. The prize is awarded to the highest bidder. In the case of a tie, the winner is chosen randomly. There is no discounting, and players' payoffs are quasi-linear.

## 3.2    Background to the Model

There are two crucial assumptions for this model: (i) bidding is costly, and (ii) in the last stage, bidders submit their bids simultaneously.

The first assumption has already been discussed in the examples. For a jump bid to credibly signal strength, submitting such a bid must be costly (recall that this cost can be in expected terms, as in the example of a winner-only-pays first-price auction).

The second assumption captures the idea that no bidder gets the opportunity to bid "last". In such an alternative situation, the game can be solved by backward induction. Given this unfair advantage, the "last" bidder knows precisely how much to bid in order to

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[9]It is only necessary, for our results, that a fraction of the initial bid is sunk.

win the auction, and his penalized rival has therefore no incentive to attempt to manipulate his beliefs. Therefore, we view our model as relevant for bidding contests in which there is a deadline of some sort, and no player is given the exclusive authority to make final decisions. Legal, lobbying and takeover contests are good examples of such situations.

In Section 6, we drop the assumption that the jump bid takes only one value. The other assumptions of the model are made for simplicity. Allowing for a longer, but finite, horizon does not change the qualitative results. In particular, allowing Player 2 to overbid, rather than only match, Player 1's jump bid is a special case of such a longer horizon. (The analysis of the game in which players get the opportunity to simultaneously submit a jump bid is available from the authors and does not add to the insights of this simpler model.)

## 3.3    Strategy and Equilibrium

A strategy for Player 1 specifies, as a function of his type, the probability of a jump bid in the first period, and the bid he makes in the two subgames in which he is called upon to bid again. That is, let $\Omega$ be the measure space that results when we impose Lebesgue measure on the unit interval $I$. Then the strategy for Player 1 consists of a measurable function $p_1 : \Omega \times [0,1] \rightarrow \{0,1\}$, $(s,v) \mapsto p_1(s,v)$, which maps a uniform draw from the unit interval and a valuation into an action, 0 or 1, that corresponds respectively to bidding 0 and to bidding $K$, and two measurable functions $b_1^i : [0,1] \rightarrow \mathbb{R}_+$, $i = 0, K$, which map a valuation into a nonnegative bid, for each of the subgames $i = 0, K$. To guarantee that the equilibria in the subgames may be characterized by first-order-conditions, we further assume that the distribution function in each subgame is piecewise continuously differentiable in the valuation, that is, $\int_0^v \int_0^1 p(s,t)\,ds\,dF(t)$ is piecewise continuously differentiable in $v$.

Player 2's strategy specifies whether or not he covers (if required) and how much he bids in the two subgames, as a function of his valuation $w$. A strategy for Player 2 consists of mappings $p_2$, $b_2^i$, $i = 0, K$, with the obvious interpretations.

If Player $i$'s valuation is $s$, his bid in the subgame following an opening bid of $k = 0$ or $K$ is denoted by $b_i^k(s)$. A (Perfect Bayesian) equilibrium consists of strategies and beliefs for each player, such that 1) strategies are sequentially rational in that the bid choices maximize the expected payoffs given beliefs about the other player's valuation and strategy, and 2) beliefs are correct and updated according to Bayes' rule. We call a winning bid one that has a positive probability of being the highest. A losing bid is any bid which is not a winning

bid.

# 4    Characterization of Signaling

In this section, we characterize the general features that signaling may exhibit. Player 1 can use bids to send information to Player 2 in two ways. (i) He can use a jump bid to deter Player 2 from entering the auction. (ii) By using an ordinary bid he can hope to induce Player 2 to believe that he is weak, thus softening the competition in the second stage of the game. Our main interest lies in the jump-bidding decision of Player 1, characterized by the function $p_1$. Of course, the bidding decision $p_1$ is closely related to the covering decision $p_2$ by Player 2.

We show that only three kinds of equilibrium exist: equilibrium with covering, non-revealing equilibrium, and equilibrium with assured deterrence. An *equilibrium with covering* is an equilibrium in which Player 1 makes a jump bid with positive probability for some of (i.e., a positive measure of) his valuations, whereupon Player 2 covers for some of his valuations. A *non-revealing equilibrium* is an equilibrium in which Player 1 never makes a jump bid, and thus Player 2 does not need to cover. Finally, an *equilibrium with assured deterrence* is an equilibrium in which Player 1 makes a jump bid with positive probability for some of his valuations, and Player 2 never covers.

**Theorem 1** *Every (Perfect Bayesian) equilibrium is either an equilibrium with covering, a non-revealing equilibrium, or an equilibrium with assured deterrence. More precisely, every (perfect Bayesian) equilibrium is characterized by numbers $\alpha$ $\beta$, $\gamma \in (0, 1]$, $\alpha \leqq \beta$, such that,*

$$\int_0^1 p(s, v)\, ds = \begin{cases} 0 \text{ for } v \in [0, \alpha], \\ 1 \text{ for } v \in (\alpha, \beta], \\ \in (0, 1) \text{ for } v \in (\beta, 1]. \end{cases}$$

$$\int_0^1 p_2(s, w)\, ds = \begin{cases} 0 \text{ for } v \in [0, \gamma], \\ 1 \text{ for } v \in (\gamma, 1]. \end{cases}$$

*In addition, $\gamma < 1$ if and only if $\alpha < \beta < 1$, in which case Player 1 makes a losing bid in the subgame $K$ if and only if $v \in (\alpha, \beta]$.*

In words, Player 1 makes an ordinary bid if his valuation is sufficiently low, a jump bid

for sure (i.e., with probability one) if his valuation falls within some intermediate interval, and randomizes between the ordinary and the jump bid for his highest valuations. If Player 2 covers, Player 1 bids nothing in the final stage if and only if his valuation falls within the intermediate interval. Player 2 covers if his valuation is large enough. Of course, the intervals $(\alpha, \beta)$, $(\beta, 1)$ and $(\gamma, 1)$ could be empty (a non-revealing equilibrium). When $(\alpha, \beta)$, $(\beta, 1)$ are non-empty but $(\gamma, 1)$ is empty we have an equilibrium with assured deterrence. Whenever some type of Player 1 bids $K$ and some type of Player 2 covers, the intervals $(\alpha, \beta)$ and $(\beta, 1)$ are non-empty (this characterizes an equilibrium with covering).

**Proof.** See appendix. ∎

This theorem shows that signaling, in equilibrium, can take only two forms. When the decision to submit a jump bid is monotone in Player 1's valuation, the only signaling that takes place is for deterrence. Players with high valuations use a jump bid that is not matched by Player 2. This form of signaling corresponds to the rationale for jump bidding already present in the literature of costly bidding, and is similar to the examples in Avery (1999) with degenerate second-stage equilibria.

The second form of signaling that may take place is more intricate. Bidders with intermediate valuations bluff. By choosing an early bid that only bidders with high valuations would otherwise make, they use the deterrence effect generated by bidders with high valuations, who will bid aggressively even if their bid is covered. Player 2 covers only if his valuation is high enough, so in that event it is in the bluffer's best interest to give up. Bidders with high valuations sandbag. When they bid low, they use the behavior of bidders with low valuations. Such a bid leads Player 2 to believe that there is a high probability that he faces a weak opponent. Acting on this belief, he bids less aggressively in the second round, thus conserving resources. Of course, both bluffing and sandbagging are rational. They correspond to the two different ways in which a player can try to manipulate his rival's beliefs. The chance that bluffing succeeds may satisfy a bidder with an intermediate valuation, but not one with a high valuation, who wants to have a high probability of winning, achieved either through repeated large bids, or through sandbagging. High types randomize and thus are indifferent between both initial bids. This does not mean that the signaling has no effect since through the manipulation of beliefs, player 1 achieves higher payoff than he would if he was never using the jump bid.

Apart from the case of an all-pay auction, where $\rho = 1$, it is difficult to show the

existence of any kind of equilibrium, because, in any asymmetric continuation game, first-order-conditions reduce to a second-order O.D.E.; this implies, in particular, that showing the existence of an equilibrium with covering means showing the existence of a solution for a third-order, nonlinear O.D.E., which we are unable to do.[10]

In the case of important special case of an all-pay auction ($\rho = 1$), it is possible to further characterize the equilibrium. In particular, the randomization of Player 1's high types takes a particularly simple form, as they do all use the same probability:

**Lemma 2** *If $\rho = 1$, then $\int_0^1 p(s,v)\,ds = p$ for some constant $p \in (0,1)$, for (almost all) $v \in (\beta, 1]$.*

**Proof.** See Appendix.

In addition, it is possible to show the existence and uniqueness of an equilibrium with covering in the case of the power distribution function, $F(v) = G(v) = v^\mu$, $\mu > 1$, provided the jump bid is not too large.[11]

**Theorem 3** *If $\rho = 1$, and the valuations are distributed according to a common power function distribution $F(v) = v^\mu$, there exists $\bar{K} > 0$, such that for any $K \in (0, \bar{K}]$ a unique equilibrium with covering exists.*

**Proof.** See Appendix. ∎

## 4.1 Sandbagging and bluffing

In an equilibrium with covering, bids sometimes escalate. According to circumstances, Player 1 may either sandbag or bluff. By definition, attempts by the first player to deter his opponent from competing will sometimes fail. When this happens, his jump bid is lost, the players adjust their beliefs and bidding starts afresh. The intuition behind the structure of equilibria with covering is the following.

Player 1's early bid depends on his valuation. (i) Players with low valuations simply cannot afford a jump bid of $K$, and hence make an ordinary bid. This is obvious for players

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[10]In fact, even if the distribution of valuations in a subgame happened to belong to some common family of distributions, a closed-form solution would still not follow (unless $\rho = 1$), since the lower ends of the support would not coincide.

[11]The uniform distribution is a special case when $\mu = 1$. Some arguments involve dividing by $\mu - 1$, but they are easily adapted to the uniform distribution.

whose valuation is smaller than $K$, but even players with higher valuations do not automatically use a jump bid, since Player 2 might cover. (ii) Players with intermediate valuations always make a jump bid. They are bluffing, hoping to deter Player 2 and inducing him to quit, thus winning. By choosing a jump bid that would otherwise be made only by players with high valuations, they use the deterrence effect generated by high valuations, who are prepared to bid aggressively even if their opponent covers. Player 2 will cover only if he has a sufficiently high valuation, so in that event it is in the bluffer's best interest to give up. Jump bidders with intermediate valuations never bid further if covering occurs. (iii) High types randomize. They are indifferent between a jump bid and an ordinary bid. Hence, they sometimes try to deter Player 2, and sometimes keep a low profile. This second behavior is motivated by sandbagging: Player 2 falsely believes that he is facing a weak opponent and hence bids moderately, allowing Player 1 to win at low cost.

The covering decision by Player 2 is simple. Players with low valuations do not cover, whereas those with high valuations do. Note that the "lowest" type who decides to cover beats all bluffers. Being the lowest type to cover, he only wins if Player 1 bids zero. Player 2's covering decision is thus, for some threshold $\gamma$: In the second period, players update their beliefs using Bayes' rule and play a first-price all-pay auction. There are two possible subgames: either Player 1 paid $K$ and Player 2 covered, or Player 1 has bid 0.

Figure 3 depicts the bids in the subgame after a jump bid for the case of the uniform distribution ($\mu = 1$). The dashed line is $b_2^K$ and the connected line is $b_1^K$.
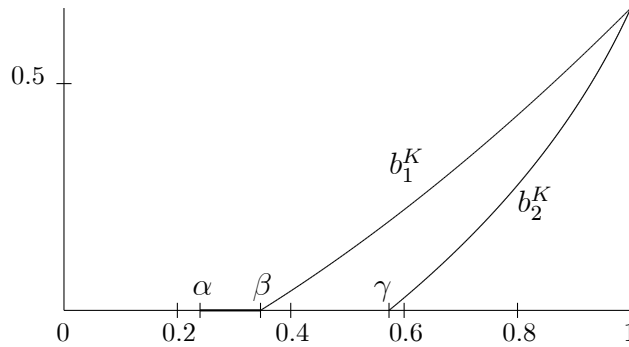


Figure 3 : Bid functions after a jump bid

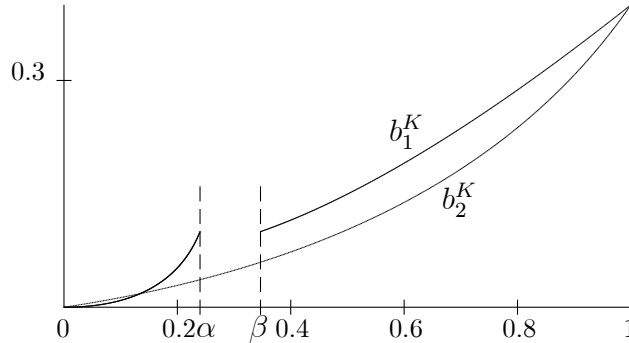Similarly, Figure 4 depicts bids in the subgame following an ordinary bid.

Figure 4 : Bid functions after an ordinary bid

These figures aid the understanding of why players with high valuations are indifferent between both bids. In subgame $K$, type $\beta$ of Player 1 wins against all types below $\gamma$ (who do not cover). In subgame 0, type $\beta$ bids $K$ and beats all types below $\gamma$, who also bids $K$. Those of Player 1's types that are larger than $\beta$ win against the same types of Player 2 in both subgames. Two incentives explain Player 1's behavior when he has a high valuation. On the one hand, he motivates less aggressive bidding by making an opening bid of 0. On the other hand, by bidding $K$, he could win without further bidding. The thresholds are such that, ex ante, players with high valuations find both bids equally attractive.

Note that intermediate and high types are in fact indifferent between initial bids, and that the initial bids do not change the outcome of the auction. However, the signaling strategy used by player 1 is efficient in the sense that he does better than in a static auction. Sandbagging and bluffing strategies are present for equilibrium reasons and not because there exist strict incentives to use them. We now analyze in more details how the signaling equilibrium changes the payoff of players.

## 4.2   Winners and Losers

Who wins and who loses in this game? The natural benchmark is the simultaneous, static, first-price all-pay auction, in which all bids are sunk and the highest bidder wins. In an equilibrium with signaling, Player 1's intermediate valuations are better off than in the static auction. They take advantage of high valuations since they can use the deterrence effect of the jump bid. To see that they are better off, note that the smallest type of Player

2 that covers is higher than the intermediate types (the bluffers). That means that in an equilibrium with covering intermediate types have a higher probability of winning than in the static auction, making them better off. High valuations are also better off. They randomize their initial bids. They take advantage of the deterrence effect when they use the jump bid and take advantage of the sandbagging effect when they use the ordinary bid. (The calculations justifying these claims are omitted - They are very similar to those in the proof of theorem 4). It is not surprising that the intermediate and high valuations benefit from the opportunity of bidding early. Losers are found among the low valuations, who are hurt by the high valuations who use the ordinary bid, as this exerts an upward pressure on bids in that subgame. The situation is reversed for the second player. High valuations may have to first reveal themselves through the cover, which is sunk. Low valuations, however, are able to better adjust their bid, which enables them to avoid wasting resources when a high bid reveals the first player to be at least of the intermediate valuation.

We believe that these results on winners and losers derived for all-pay auctions extend to other auction formats (as was illustrated in the examples of section 2).

In some applications, one may want to maximize the revenue of the game, i.e., the total expected payments of the players. An example of such an application is lobbying, from the politician's point of view. In others, one may want to minimize it; for example, in military conflicts. With respect to revenue maximization, as the static first-price, all-pay auction is an optimal auction, by the Revenue Equivalence Theorem (See Myerson (1981)), it is obviously best to set $K = 0$, so that the dynamic game essentially collapses to the static one.[12]  Of course, this is valid only as far as the initial distributions of valuations are the same for both players. When bidders are asymmetric, the static all-pay auction is not optimal and the dynamic auction could generate higher revenues. It is straightforward to show that a moderate, intermediate value of $K$ minimizes revenue.

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[12]The all-pay auction is the optimal auction when distributions are symmetric and regular and no reserve price can be used by the seller. Symmetry is assumed in our set-up and the power distributions we use in our analysis also satisfy the regularity (or increasing virtual valuations, $x - \frac{1-F(x)}{x}$) condition.

# 5 The structure of equilibria

Equilibria with covering are certainly the most interesting ones, displaying intriguing strategic features. In this section, we first characterize non-revealing equilibria and equilibria with assured deterrence.[13] We then introduce an equilibrium refinement (Perfect Sequential Equilibrium) that yields a striking existence and uniqueness result, where the selected equilibrium depends on the particular value of the jump bid $K$.

## 5.1 Non-revealing Equilibria

In a non-revealing equilibrium, the first player always makes an ordinary bid. The game is then essentially equivalent to a static, first-price all-pay auction. This equilibrium is reasonable when $K$ is very large, so that the high bid is unattractive. On equilibrium path, the bids are $b(v) = \frac{\mu}{\mu+1} v^{\mu+1}$. For $K \geq \frac{\mu}{\mu+1}$, this equilibrium does not even depend on out-of-equilibrium beliefs and bids. Indeed, bidding $K$ gives at most $v - K$ to type $v$ of Player 1 which is less than $\frac{v^{\mu+1}}{\mu+1}$, the payoff he receives in a non-revealing equilibrium. Hence, any belief would do to ensure that a non-revealing equilibrium exist.

## 5.2 Equilibria with Assured Deterrence

In an equilibrium with assured deterrence, the first player sometimes uses a jump bid that the second player never covers. The first player must follow a 'threshold' strategy: he bids high if and only if his valuation is sufficiently high. This kind of equilibrium makes sense for relatively high values of $K$. Although a high bid has an assured deterrence effect, it is costly enough to be chosen only by the highest valuations of Player 1. Player 2 has therefore two good reasons to give up after a high bid: covering is expensive, and the opponent is strong. Observe in particular that the second player does not cover even if his valuation is one, i.e. even if he is certain to have a higher valuation than the first player's. This is due to the asymmetry between players. When the second player has the opportunity to cover, his updated beliefs are pessimistic after the jump bid. Denote the threshold by $\alpha$. Solving for $h$, and computing the bid of type $\alpha$, after a change of variable and simplification it must

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[13]We retain the assumption of all-pay auctions and power-function distributions used in the previous section.

be that:

$$K = \frac{\mu}{\mu - 1} \int_{1-\alpha}^{1} x^{\frac{1}{\mu-1}} \left( \frac{x - (1-\alpha)}{\alpha} \right)^{\frac{1}{\mu-1}} dx.$$

Observe that the derivative with respect to $\alpha$ of the right-hand side is equal to

$$\frac{\mu}{(\mu-1)^2} \int_0^1 x^{\frac{1}{\mu-1}} \left( \frac{x - (1-\alpha)}{\alpha} \right)^{\frac{1}{\mu-1}-1} \frac{1-x}{\alpha^2} dx > 0.$$

Therefore, a solution $\alpha \in (0,1)$ to the equation exists provided that

$$\int_0^1 \mu x^\mu dx = \frac{\mu}{\mu+1} \gtreqless K.$$

To complete the description of the equilibrium we must specify beliefs held by Player 1 in case Player 2 covers that lead to a payoff for Player 2 smaller than $K$. When we specify that Player 1 hold beliefs that Player 2's type are distributed according to a power distribution on $[\gamma, 1]$, and compute the limit payoff after covering when $\gamma$ tends to 1, we get:

$$\lim_{\gamma \to 1} \frac{\beta^\mu - \alpha^\mu}{1 - \alpha^\mu} \cdot \gamma = \frac{\left( (1-\alpha^\mu) \frac{\mu-1}{\mu} + 1 \right)^{\frac{\mu}{\mu-1}} - \alpha^\mu}{1 - \alpha^\mu}.$$

It is then enough that these payoffs are smaller than $K$ for an equilibrium with assured deterrence to exist. The exact bounds on $K$ can be found in the appendix in the proof of theorem 4, in which we construct Perfect Sequential Equilibria that are also Perfect Bayesian Equilibria.

## 5.3  Existence and Uniqueness of a Perfect Sequential Equilibrium

Since all three types of equilibria are possible, which one is most likely to emerge? Obviously, this depends on the value of the parameter $K$. (Perfect Bayesian) Equilibria with covering exist if and only if $K \leqq \bar{K}$, equilibria with assured deterrence exist if and only if $K < \frac{\mu}{\mu+1}$, and non-revealing equilibria always exist. However, the beliefs used to construct non-revealing equilibria for low $K$ are not plausible. Such equilibria make sense for large $K$, when early bidding is not worthwhile, but seem unreasonable otherwise. The Intuitive Criterion does not have any bite in this game because, while it constrains the support of beliefs that can be held after a deviation, it does not impose any restriction on the relative likelihood of

the valuations that belong to this support. With a continuum of valuations, this leaves considerable leeway.

One might wish to impose the condition that, if a player has incentives to deviate for two distinct valuations, his opponent's beliefs after observing such a deviation should preserve the relative likelihood of these valuations. This is the main idea behind Perfect Sequential Equilibrium (P.S.E.), defined by Grossman and Perry (1986). The logic behind this refinement is straightforward. Fix a Perfect Bayesian Equilibrium and suppose that a player deviates. His opponent hypothesizes that the move was made by some subset $C$ of the player's valuations, and revises his belief according to Bayes' rule conditional upon the player's valuation being in $C$. If the Perfect Bayesian Equilibrium that follows given these beliefs is preferred to the original equilibrium by precisely the valuations in $C$, then the original equilibrium fails to be perfectly sequential. The point is that this deviation allows the player's valuations in $C$ to separate themselves convincingly from the other valuations, so that it is not credible for his opponent to hold any other belief after such a deviation. This eliminates equilibria based on such beliefs. This refinement is inspired by a forward induction argument. Deviations should be interpreted not as trembles, but as rational signals to influence beliefs.

For the definition of P.S.E., let $\psi_j$ $\left(\text{or } \phi_j\right)$ be the beliefs of Player $j$ about Player $i \neq j$, and $T_i$ and $T_j$ the types spaces.

**Definition 1** *A Perfect Bayesian Equilibrium (P.B.E.) is a Perfect Sequential Equilibrium (P.S.E.) if, for all Players $j$, and all their possible deviations, there exists no P.B.E. of the subgame following the deviation, with beliefs $\psi_j$ and $\psi_i$ immediately prior to the deviation, and beliefs $\phi_j$ and $\phi_i$ after the deviation such that:*

1. *$\phi_j(t) = \psi_j(t)$ for all $t \in T_i$,*

2. *$\phi_i(t) = 0$ if $\psi_i(t) = 0$ or if $t \in T_j$'s expected payoff in the P.B.E. of the subgame (following the deviation) is strictly smaller than his expected payoff in the original P.B.E.[14], **and** $\phi_i(t) > 0$ if $\psi_i(t) > 0$ and $t \in T_j$'s expected payoff in the P.B.E. of the subgame is strictly larger than his expected payoff in the original P.B.E.,*

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[14] *Here and in the remainder of the definition, the expected payoff in the original P.B.E. should be understood as Player 1's expected payoff, when he follows the strategy prescribed in the original P.B.E, conditional on the node where the considered deviation occurs is reached.*

3. $\frac{\phi_i(t)}{\psi_i(t)} \geqq \frac{\phi_i(t')}{\psi_i(t')}$ *whenever $\phi_i(t') > 0$ and $\phi_i(t) > 0$, for $t \in T_j$ whose payoff in the P.B.E. of the subgame is strictly larger than his expected payoff in the original P.B.E., with equality if $t' \in T_j$'s payoff in the P.B.E. of the subgame is strictly larger than his expected payoff in the original P.B.E..*

Condition (1) states that the deviator should not revise his beliefs, since he has not learnt anything about his opponent. Condition (2) places restrictions on the support of the beliefs to be considered: this support should (a) include players who are strictly better off in the P.B.E. following the subgame, given those beliefs, than in the original P.B.E., and (b) exclude those who are strictly worse off. Condition (3) states that, except possibly for deviators that are indifferent to the deviation, whose likelihood may possibly decrease, the deviators' relative likelihood should not be altered.

The following theorem establishes the structure of P.S.E. in this game.

**Theorem 4** *For each $K \geqq 0$, there exists a unique P.S.E. This P.S.E. is the equilibrium with covering for $K < \bar{K}$, the equilibrium with assured deterrence for $\bar{K} \leqq K < \frac{\mu}{\mu+1}$, and the non-revealing equilibrium otherwise.*

**Proof.** See Appendix. ∎

This result, illustrated in the following figure, is intuitive. For large $K$, deterrence is too costly and the first player does not take advantage of this opportunity. For intermediate $K$, deterrence is effective. Given the entry cost that it represents and the signal of strength that it conveys, a high bid is sure to deter the second player. Finally, for low values of $K$, a high bid is not always deterrent and the equilibrium exhibits covering.
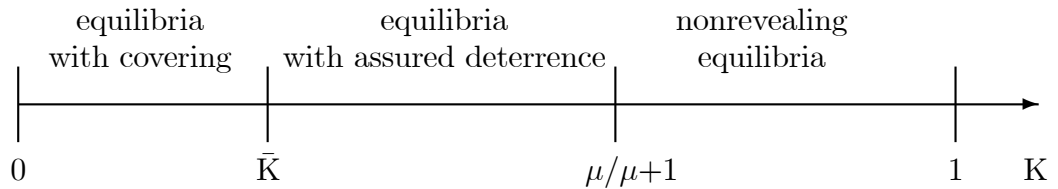
# 6 Endogenous choice of jump bid

An important limitation of the analysis so far is that the jump bid, $K$, is exogenous. Since the level of the jump bid determines which type of equilibrium obtains, it is important to understand whether jump bidding also occurs when the first player may submit any jump bid he pleases. Avery looks at this problem in Theorem 4.7.; given the monotonic structure of jump bidding in his framework, endogenizing the jump bid leads to an equilibrium in which the behavior in the limit is just a standard second-price auction without jump bid. We show that endogenizing the jump bid does not lead to a degenerate equilibrium in our framework. There exists a natural counterpart to the equilibrium with covering that was obtained previously. For simplicity, we analyze the case of uniform distribution ($\mu = 1$). Features of such an equilibrium can be deduced from the following observations:

First, Player 1 uses at most one jump bid that is deterrent with probability one. It would otherwise be profitable to reduce such a bid to the lowest level sufficient for sure deterrence. This deterrent bid is obviously an upper bound on all bids made.

Second, since Player 2 covers if and only if his valuation is sufficiently large, it must be that this threshold increases with the opening bid. Suppose, on the contrary, that there are two opening bids $k_1 < k_2$, and associated thresholds $\gamma_1 \geqq \gamma_2$. Then the lowest type $\underline{v}$ of Player 1 bidding with positive probability $k_2$ had better bid $k_1$, since his payoff is $\underline{v} \cdot \gamma_2 - k_2$ which is strictly smaller than $\underline{v} \cdot \gamma_1 - k_1$ (Notice that, being the lowest type of Player 1 in the subgame following an opening bid of $k_1$ which is covered, his payoff is zero in that subgame).

Third, given the single-crossing property of expected profits, the infimum over types of Player 1 who bids $k$ with positive probability is an increasing function of $k$. Moreover, it can be shown that these lowest types bid 0 with probability one in the subgame that might occur after a bid of $k$. They correspond to the bluffers of the previous section.

Fourth, for any $k$ that Player 1 bids without deterring his opponent, the support of the types of Player 1 bids $k$ with positive probability must include 1. Suppose instead that for such a level $k$, this support has maximum $\bar{m} < 1$. Suppose further, as can be shown, that expected profits $\Pi(\cdot)$ are continuously differentiable in types, and that there exists $\varepsilon > 0$ and a bid $k' \in [0,1]$ below the deterrent bid, such that all types in $(\bar{m}, \bar{m} + \varepsilon)$ bid $k'$ with positive probability. Consider $m = \bar{m} + dv$, where $dv < \varepsilon$. Since type $m$, by bidding $k$ and bidding then, if necessary, as much as $\bar{m}$, obtains $\Pi(\bar{m}) + dv$, it must be that $d\Pi(\bar{m})/dv$ is

larger than 1. On the other hand, by bidding $k'$, the marginal profit of type $m$ (which by virtue of the envelope theorem is his probability of winning) is strictly smaller than 1, for such a bid is covered with positive probability and type $m$ is not the largest type of Player 1 in such a subgame. Hence, if the support of types bidding $k$ does not include 1, higher types can profitably deviate by bidding $k$ and bidding afterwards, if necessary, the minimum to win.

Finally, conditional on a bid $k$, there must be a strictly positive probability that Player 1's bid in the subgame that possibly follows is 0, for otherwise the lowest types supposed to cover would not find it worthwhile. Given these considerations, we have the following theorem:

**Theorem 5** *There exists an equilibrium with endogenous choice of jump bid. The support of types making a given, non-deterrent, bid $k$ must be an interval $[\alpha(k), 1]$, where $\alpha(\cdot)$ is increasing in $k$. If $k$ is covered, type $\alpha(k)$ bids zero while higher types bid actively. The conditional distribution assigns strictly positive weight on $\alpha(k)$, but is atomless above. Thus, type $v$ of Player 1 randomizes his early bid: with positive probability $p(k)$, he bids $k = \alpha^{-1}(v)$. He continuously randomizes on $[0, \alpha^{-1}(k))$ according to a density $\psi(v, k)$.*

**Proof.** See Appendix. ∎

Let us define $\lambda(k)$ as

$$\lambda(k) = p(k) + \int_\alpha^1 \psi(s, k)\, ds.$$

Hence, $d\lambda$ is the density over Player 1's types who bid $k$, and is well defined as long as $p$ and $\int_\alpha^1 \psi(s, k)\, ds$ are of the same cardinality, that is, as long as $0 < \int_\alpha^1 \psi(s, k)\, ds < \infty$. $p(k)/\lambda(k)$ is thus the probability, conditional on observing $k$, that Player 1 is of type $\alpha(k)$. In the equilibrium that we derive, type $\alpha$ bids 0 in the subgame $k$, and no other type bidding $k$ bids 0 thereafter. Hence, $p(k)/\lambda(k)$ is the conditional probability that Player 1 bids 0 in the subgame $k$. In other words it represents the proportion of bluffers among types who bid $k$ in the first period.

It is interesting to note that $p(k)/\lambda(k)$ increases monotonically in $k$, ranging from 0 to $k$. This means that the higher the jump bid, the larger is the proportion of bluffers among the types who made this bid. Also, for any $k \in [0, \bar{k}]$, $\alpha(k) \geqq \gamma(k)$, which contrasts with the case where $K$ was unique and exogenous: the bluffers are of higher type than the lowest types of their opponent. In addition, one might expect second period bids of Player 1 to be

a decreasing function of the jump bid. This need not be so. For low $k$, small increases have large effects on $\gamma$. For a large type $v$ of Player 1, that means that increasing slightly the first period bid (starting from a small one) strongly increases the probability of winning without bidding in the second period. For him to be indifferent ex ante, it must be that in the case that Player 2 decides to cover, profits are lower, that is, his bid is larger. The randomization can be seen in Figure 5, where darker areas correspond to larger probabilities (that is, to larger values of $\psi(dx, dk)$).

In this equilibrium, signaling has a very continuous form. After observing the opening jump bid, the second Player has gained information about Player 1's type. He can rule out some very low types who never use that particular opening bid, but his beliefs remain imprecise: to every bid there corresponds an interval of possible types. Nevertheless, some information is revealed. The likelihood of the various types changes. After a given bid, some types appear more likely than others. This type of signaling was discussed in Weber (1994). Weber studies "non-partitioning strategies" which partially reveals information without inducing posterior partitioning of the players' type spaces. The reason that signaling takes this form is that with a continuum of potential jump bids, a monotonic equilibrium perfectly reveals Player 1's type. This leads to degenerate second period bidding in which Player 2 would certainly win the auction. This is exactly what happens in the model proposed by Daniel and Hirshleifer. They exhibit a separating equilibrium in which a continuum of jump bids is used. However, this perfectly reveals the bidder's valuation. The other bidder then reacts by giving up the bidding or by using a bid just large enough to win the auction. The logic is the same in Avery's model when he allows for a large number of potential jump bids (Theorem 4.7.). The equilibrium is monotonic and in the limit corresponds to the static second price auction.

We believe that the structure of signaling analyzed in our model captures well the information transmission that takes place in bidding auctions: it is a mixture of aggressive bidding to intimidate competitors (bluffing) and of cautious bidding to lull them into a false sense of security (sandbagging). This form of signaling remains present when the choice of jump bids is endogenous and thus seems quite robust.
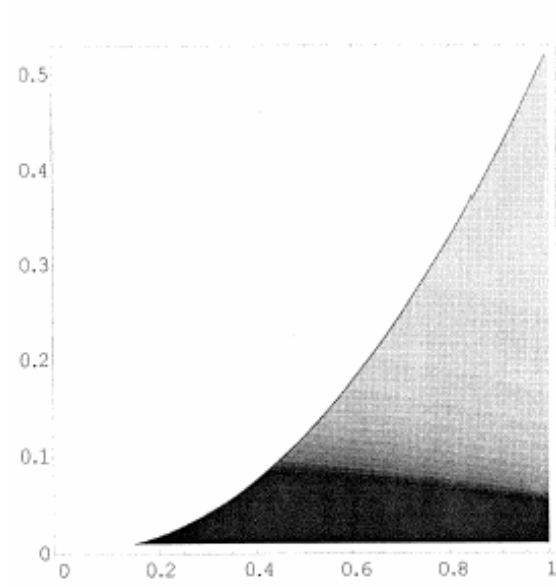
Figure 5: Randomization $\psi\left(x,k\right)+p\left(k\right)$

# 7    Discussion

The reader has probably noticed the resemblance between our game and the game of poker, and indeed, the main paradigm used by game theorists to discuss informational questions in a rigorous framework has been poker. Variations of this game have been studied by Borel (1938), Von Neumann and Morgenstern (1944), Nash and Shapley (1950), to name just a few. However, the game we have discussed here is based on different assumptions, played using different strategies and gives different results. Let us briefly review these differences in turn.

Poker is a zero-sum game. In the case of a showdown, the winner is decided by the players' hands, which are beyond their control. That is why poker is said to be not about managing cards, but money. Models which place restrictions on how much can be won or lost by a player do so only to ensure tractability. A poker player can win whatever his opponent spends; bets made in early rounds affect players' behavior not only by affecting their beliefs but by raising the stakes.

In our game, on the other hand, winning is about managing the trade-off between the probability and the cost of winning. It is a non-zero sum game in which what a player spends

31

is lost for everybody, and bids have no affect on the stakes. This makes it possible to study the expected bids of the auction and its efficiency properties.

To discuss and compare the results, it is helpful to recall a distinction introduced by Von Neumann and Morgenstern. In the *Theory of Games and Economic Behavior* (1944), they identify two motivations for bluffing in poker: the first one consists in bidding high or overbidding to created a (false) impression of strength, thus conceivably inducing one's opponent to pass. The second stems from the need to create uncertainty in the opponent's mind as to the correlation between bids and hands. In their own words, "The first is to give a (false) impression of weakness in (real) weakness, the second is the desire to give a (false) impression of weakness in (real) strength". Bluffing pays off because it sometimes induces the other player to believe that his opponent's hand is, in reality, strong, thereby exerting a deterrent effect, and because, at other times, it induces him to raise, hoping that the hand is terrible while it is actually very good. By bluffing, you can win a lot with a bad hand. Without, you will only win very little, even with a good hand.

In models of poker similar to that proposed by Von Neumann (see Karlin and Restrepo (1957), Newman (1959) and Sakai (1986) for extensions), bluffing is optimal when a player's hand is really bad, and not middle-range. By contrast, in our model players with low valuations do not bluff because it is too expensive to do so. As in poker, bluffing pays off because it may deter the opponent from competing further. Unlike poker, however, a player who bids high with a high valuation has nothing to gain by confusing his opponent's beliefs. This may lead the other player to "call", a disastrous outcome for both players. A player with a moderate valuation gains an advantage by mimicking the behavior of a high valuation player.

Sandbagging is relatively rare in poker. We find it in Nash and Shapley (1950)'s three-player poker model. In this model all hands belong to one of just two categories: high and low. The first two players may bid low, even if they have strong hands, so as to induce the third one to raise the stakes by bidding high. As with bluffing, the confusion of beliefs this generates continues to be beneficial even when their hands are actually low. Sandbagging in our model has the opposite motivation: the aim is persuade the other player to bid low, which allows the sandbagger to win at moderate cost. This strategy is particularly attractive to a player with a high valuation, for whom winning is especially important. Sandbagging exploits the behavior of players with low valuations. By copying this behavior,

the sandbagger induces his opponent to bid cautiously so as to avoid potentially unnecessary expenditure. The uncertainty created by sandbagging leads the second player to make a stronger response to an opening bid than he would if he could be sure he was facing a player with a low valuation. As a result sandbagging damages players with low valuations.

Avery considers a similar two-stage auction with affiliated values. In his model, players choose simultaneously between an ordinary bid and a jump bid. (In an extension, players may choose from a finite set of more than two possible opening bids.) For a given pair of an 'aggressive' bidding function triggered by an unmatched jump bid and of an 'accommodating' bidding function used in the second-stage in the event of a matched jump bid, Avery shows that there exists a unique symmetric equilibrium of the two-stage auction, in which signaling takes place in the first-period: a player's first-stage strategy is characterized by a threshold. The player submits the jump bid rather than the ordinary bid if and only if his signal exceeds this threshold.

While our model is very similar to Avery's, focusing on independent private values raises the difficulties that he points out (p. 186): there is no fear of the winner's curse and it remains a dominant strategy to bid up to one's true value in response to a jump bid. By introducing bidding costs, as in Daniel and Hirshleifer, we obtain results quite different from Avery's. In particular, the opening bid strategy is non-monotonic.

Daniel and Hirshleifer analyze an infinite horizon, alternating-move game in which players must either match (or overbid) their rival's latest bid, or pass. The auction ends when a player passes. Players' valuations are private and independently distributed. Submitting or revising a bid entails a fixed cost, paid independently of the final outcome. Daniel and Hirshleifer show that there exists an equilibrium in which the first bidder opens with a fully separating (monotonic) bid, which induces his rival either to pass, or to match if his valuation is high enough. If he matches, the first bidder then passes. See Daniel and Hirshleifer and discussions in the related literature on takeover bidding contests.

# 8 Concluding Remarks

The importance of jump bids is evident in the literature on auctions. Such bids are used to signal strength and deter competitors from bidding further. However, as Cassady points out, signaling in auctions can take another form and have another rationale. Cautious bidding

can lull competitors into a false sense of security. We analyze a model in which both forms of signaling are used in equilibrium. Costly bidding (in the sense that part of the bids are sunk), and a last stage in which bidding is simultaneous, are the two central ingredients necessary to obtain this type of signaling. Our theory complements Avery's model of jump bidding by introducing a new explanation for observing jump bids. We show that this type of signaling is robust to the endogenous choice of jump bids. Bluffing and sandbagging are used simultaneously in an equilibrium with a continuum of equilibrium jump bids. Revelation of information takes a restrictive and disjointed form. When players use non-partitioning strategies, à la Weber (1994), observing a jump bid changes the likelihood of types, making some types more likely following the observation of a certain bid. Note that Rosen's (1986) informal analysis leads to different predictions. Rosen conjectures that a strong player wants his rival to think his strength is greater than it truly is, thereby inducing him to exert less effort, and that the same applies to a weak player in a weak field; a weak player in a strong field seeks, on the other hand, to give out signals showing that he is even weaker than he actually is, thereby leading his rival to slacken off.

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# Appendix

## A    Examples and Counterexamples: Details

The first-price all-pay auction:

**The equilibrium with covering**    As described in Section 2, Player 1's low type bids 0, Player 1's intermediate type bids $K$, and Player 1's high type randomizes between 0 and $K$, bidding $K$ with probability 7/8. Player 2's low type covers with probability 1/5.

Let $v_0 = \frac{1}{4}$, $v_1 = \frac{1}{2}$, $v_1 = 1$, $w_1 = 3/5$ and $w_2 = 3/2$. Denote by $F^0\left(\cdot; v_i\right)$ $\left(F^K\left(\cdot; v_i\right)\right)$ the bidding distribution of Player 1's type $v_i$ in subgame 0 (respectively, in subgame $K$). Similarly, let $G^0\left(\cdot; w_i\right)$ $\left(G^K\left(\cdot; w_i\right)\right)$ denote the bidding distribution of Player 2's type $w_i$ in subgame 0 (respectively, in subgame $K$). We have:

$$F^0\left(b; v_0\right) = \frac{2}{3} + \frac{10}{3}b,\ b \in [0, 1/10],\ \ F^0\left(b; v_2\right) = \begin{cases} \frac{10}{3}b - \frac{1}{3},\ b \in [1/10, 2/10], \\ \frac{1}{15} + \frac{4}{3}b,\ b \in [2/10, 7/10], \end{cases}$$

$$G^0\left(b; w_1\right) = \begin{cases} 8b,\ b \in [0, 1/10] \\ \frac{3}{5} + 2b,\ b \in \left[\frac{1}{10}, \frac{2}{10}\right] \end{cases},\ \ G^0\left(b; w_2\right) = 2b - \frac{2}{5},\ b \in [2/10, 7/10].$$

$$F^K\left(0; v_1\right) = 1,\ F^K\left(b; v_2\right) = \begin{cases} \frac{1}{21} + \frac{40}{21}b,\ b \in [0, 1/6], \\ \frac{5}{21} + \frac{16}{21}b,\ b \in [1/6, 1] \end{cases},$$

$$G^K\left(b; w_1\right) = 6b,\ b \in [0, 1/6],\ G^K\left(b; w_2\right) = \frac{6}{5}b - \frac{1}{5},\ b \in [1/6, 1].$$

**The non-revealing equilibrium**    In the non-revealing equilibrium, Player 1 bids 0 independently of his type. Let $F\left(\cdot; v_i\right)$ $\left(G\left(\cdot; w_i\right)\right)$ denote the bidding distribution of Player $i$'s

type $v_i$ (respectively, Player 2's type $w_i$) in the subgame following a bid of 0. Then:

$$F(b; v_0) = \frac{50}{3}b + \frac{7}{12}, \ b \in \left[0, \frac{1}{40}\right], \ F(b; v_1) = \frac{50}{3}b - \frac{5}{12}, \ b \in \left[\frac{1}{40}, \frac{17}{200}\right],$$

$$F(b; v_2) = \begin{cases} \frac{25}{12}b - \frac{17}{96}, \ b \in \left[\frac{17}{200}, \frac{73}{200}\right], \\ \frac{5}{6}b + \frac{67}{240}, \ b \in \left[\frac{73}{200}, \frac{173}{200}\right]. \end{cases}$$

$$G(b; w_1) = \begin{cases} 8b, \ b \in \left[0, \frac{1}{40}\right], \\ 4b + \frac{1}{10}, \ b \in \left[\frac{1}{40}, \frac{17}{200}\right], \\ 2b + \frac{27}{200}, \ b \in \left[\frac{17}{200}, \frac{73}{200}\right], \end{cases} \quad G(b; w_2) = 2b - \frac{73}{100}, \ b \in \left[\frac{73}{200}, \frac{173}{200}\right].$$

Payoffs are $\pi^1_{v_0} = 0$, $\pi^1_{v_1} = \frac{1}{40}$, $\pi^1_{v_2} = \frac{27}{200}$, $\pi^2_{w_1} = \frac{7}{200}$ and $\pi^2_{w_2} = \frac{127}{200}$.

To support this equilibrium, we need to specify Player 2's beliefs if he observes $K$. Suppose that if he were to observe a jump, he would assign probability $1/8$ to $v_0$, $3/8$ to $v_1$ and thus $\frac{1}{2}$ to $v_2$. As we will verify, Player 2 finds it therefore optimal to cover independently of his type. To verify that this deters Player 1 to bid $K$, independently of his type, it is necessary to determine how Player 2 would bid in the second stage, given the aforementioned beliefs. Suppose so that Player 1 is either of type $v_0$, with probability $1/8$, of type $v_1$, with probability $3/8$, or of type $v_2$; Player 2 is either type $w_1$ or $w_2$, with equal probability. Using the previous notation, equilibrium bid distributions are:

$$F(0; v_0) = 1, \ F(b; v_1) = \frac{40}{9}b + \frac{1}{9}, \ b \in \left[0, \frac{1}{5}\right], \ F(b; v_2) = \begin{cases} \frac{10}{3}b - \frac{2}{3}, \ b \in \left[\frac{1}{5}, \frac{3}{10}\right], \\ \frac{4}{3}b - \frac{1}{15}, \ b \in \left[\frac{3}{10}, \frac{4}{5}\right]. \end{cases}$$

$$G(b; w_1) = \begin{cases} 4b, \ b \in \left[0, \frac{1}{5}\right], \\ 2b + \frac{2}{5}, \ b \in \left[\frac{1}{5}, \frac{3}{10}\right], \end{cases} \quad G(b; w_2) = 2b - \frac{3}{5}, \ b \in \left[\frac{3}{10}, \frac{4}{5}\right].$$

In particular, Player 1's payoff is $\pi^1_{v_0} = \pi^1_{v_1} = 0$ and $\pi^1_{v_2} = 1/5$. Observe that Player 1's intermediate has no incentive in bidding $K$ (paying $K$ does not induce Player 2 to quit, and yields 0 afterwards). As for Player 1's high type, he would get $1/5 - 1/10 = 1/10$ from doing so, while he gets $27/200 > 1/10$ on the equilibrium path. Player 2's low type's payoff is $\frac{1}{6} \cdot \frac{3}{5} \gtreqless \frac{1}{10}$, and covering is indeed optimal. Therefore, these beliefs support the non-revealing equilibrium.

**The non-revealing equilibrium is not a Perfect Sequential Equilibrium** Starting from the non-revealing equilibrium described above, suppose Player 2 observes a (out-of-equilibrium) bid $K$. If Player 2 believes (i) that this bid cannot have been submitted by the low type, (ii) that -if Player 1's type is intermediate- he has randomized and submitted this bid with probability 2/3, and (iii) that -if Player 1's type is high- he has submitted this bid with probability 1, then we will show that Player 2's intermediate type is indifferent between covering or not -in particular, he is willing to cover with probability $\frac{1}{2}$, which we assume henceforth- and Player 2's high type covers for sure. In addition, given this covering behavior, and the ensuing bidding described below, Player 1 would have indeed strictly preferred not to submit $K$ if his type is low (i.e., he would have stuck with the action prescribed the non-revealing equilibrium); he would have been indifferent between this deviation (i.e., bidding $K$) and equilibrium play if his type was intermediate, and he would have strictly preferred the deviation if his type was high. This yields then that the non-revealing equilibrium is not perfect sequential.

If the deviation occurs and is interpreted in the way described above, Player 2 believes that Player 1 is either of type $v_1$, with probability 1/3, or of type $v_2$, with probability 2/3. Similarly, given the covering behavior specified above, Player 1 assigns probability 1/3 to Player 2 being of type $w_1$, and 2/3 to Player 2 being of type $w_2$ (if covering occurs). Bidding distributions are then given by (using the notation introduced earlier):

$$F\left(b ; v_1\right)=5 b+\frac{1}{2}, b \in\left[0, \frac{1}{10}\right], \quad F\left(b ; v_2\right)=\left\{\begin{array}{l}\frac{5}{2} b-\frac{1}{4}, b \in\left[\frac{1}{10}, \frac{7}{30}\right], \\ \frac{1}{10}+b, b \in\left[\frac{7}{30}, 1\right],\end{array}\right.$$

$$G\left(b ; w_1\right)=\left\{\begin{array}{l}6 b, b \in\left[0, \frac{1}{10}\right], \\ 3 b+\frac{3}{10}, b \in\left[\frac{1}{10}, \frac{7}{30}\right],\end{array}\right. \quad \text{and } G\left(b ; w_2\right)=\frac{3}{2} b-\frac{7}{20}, b \in\left[\frac{7}{30}, \frac{9}{10}\right].$$

Player 1's high type payoff in this auction (that is, his continuation payoff in the game) is $1-9 / 10=1 / 10$. Therefore, his overall payoff is $\frac{1}{4}-\frac{1}{10}+\frac{3}{4} \frac{1}{10}=9 / 40$, which strictly exceeds 27/200, his payoff in the non-revealing equilibrium. Player 1's intermediate type's payoff is $\frac{1}{4} \cdot \frac{1}{2}-\frac{1}{10}=\frac{1}{40}$, which is exactly is payoff in the non-revealing equilibrium. Finally, if Player 1's low type were to bid $K$, his payoff would be $\frac{1}{4} \cdot \frac{1}{4}-\frac{1}{10}=-\frac{3}{80}$, which is strictly less than his payoff in the non-revealing equilibrium. Player 2's low type's payoff upon covering is 1/10, so that he is indeed indifferent between covering and quitting, and Player 2's high type has

a strict preference for covering.

**First-price winner-only-pays Auction** Both players have three possible valuations: either $0$, $v := 1/2$ or $1$, with probability $p_0$, $p_1$ and $1 - p_0 - p_1$ for player 1, and $s_0$, $s_1$ and $1 - s_0 - s_1$ for player 2. Let $s := s_0 + s_1$.

Player 1 may submit an early bid $K \in (0, v)$.

Consider the following strategy profile: player 1's type $v$ and 1 submit $K$ with probability $q$ and $p < q$ respectively, and player 2 covers if and only if his type is 1.

In the subgame following a bid $K$, the total bid submitted by player 1's type $v$ is $v$: that is, his second bid is $v - K$. His expected payoff is thus:

$$\pi_v^1 = s\left(v - K\right),$$

while:

$$\pi_1^1 = s\left(1 - K\right) + \left(1 - s\right)\frac{qp_1}{qp_1 + p\left(1 - p_0 - p_1\right)}\left(1 - v\right).$$

To see where the last summand is coming from, observe that player 1's and player 2's high type must have the same payoff, since they both are willing to submit the highest bid. However, by bidding slightly above $v - K$, player 2's high type wins if and only if player 1's type is $v$, that is, he wins with probability:

$$\frac{qp_1}{qp_1 + p\left(1 - p_0 - p_1\right)}.$$

[In particular, it follows that player 2's high type submits such a bid with this very probability.] The specification relative to the covering decisions for player 2 are clearly optimal. Also, it is plain that player 1's type 0 has no incentive to bid $K$, since his expected payoff from doing so is strictly negative.

If the first bid is 0, beliefs about player 1's type are updated to:

$$r_0 = \frac{p_0}{p_0 + \left(1 - q\right)p_1 + \left(1 - p\right)\left(1 - p_0 - p_1\right)}, \quad r_1 = \frac{\left(1 - q\right)p_1}{p_0 + \left(1 - q\right)p_1 + \left(1 - p\right)\left(1 - p_0 - p_1\right)}, \tag{A1}$$

and $1 - \mu_0 - \mu_1$ for types $0$, $v$ and $1$ respectively. We look for an equilibrium such that there exists $t_1 > t_2 > 0$, $G_2 \in (r_0, r_0 + r_1)$, with:

- both low types bid 0.

- player 1's intermediate type continuously randomizes over the interval $[0, t_1]$; player 2's intermediate type bids $0_+$ with positive probability and continuously randomizes over $[0, t_2]$.

- player 1's high type continuously randomizes over some nonempty interval $[t_1, \beta]$; player 2's high type continuously randomizes over the interval $[t_2, \beta]$.

We let $G_2$ denote the probability with which player 2 wins with a bid $t_2$, and $H_0$ the probability with which player 1 wins with a bid $\varepsilon > 0$ for $\varepsilon$ arbitrarily small. While we need $H_0 > s_0$ for the equilibrium strategies to be as described as above, this constraint need not bother us, as we can always specify $s_0$ small enough (instead, we will simply exhibit some $s$ satisfying the inequalities to be defined.). For instance, $s_0 = 0$ is fine.

Observe also that, since both players' high types have the same expected payoff (as they are willing to submit the same highest bid), it must be that player 1 wins with probability $r_0 + r_1$ by submitting a bid $t_1$, since this is the case for player 2.

We must have:

$$
\begin{aligned}
\pi_1^1 &= (1 - t_1)(r_0 + r_1), & \text{(A2)} \\
\pi_v^1 &= (v - t_1)(r_0 + r_1) \\
&= (v - t_2)s \\
&= vH_0, \\
\pi_1^2 &= (1 - t_1)(r_0 + r_1) \\
&= (1 - t_2)G_2, \\
\pi_v^2 &= (v - t_2)G_2 \\
&= vr_0.
\end{aligned}
$$

It is straightforward to solve (A1) and (A2), but the formulas are unwieldy. Using for instance the specifications:

$$
K = \frac{1}{10}, \; p_0 = \frac{1}{3}, \; p_1 = \frac{1}{3}, \; s = \frac{1}{2},
$$

we obtain

$$
q > p.
$$

More precisely, define $\kappa := \sqrt{7009} \simeq 83.7$, we have:

$$
\begin{aligned}
\pi_1^1 &= \pi_1^2 = \frac{193 + \kappa}{480} \simeq .58, \; \pi_v^1 = \frac{1}{5}, \; \pi_v^2 = \frac{193 + \sqrt{577}}{1080} \simeq .26, \; t_1 = \frac{\kappa - 72}{50} \simeq .23, \\
t_2 &= \frac{1}{10}, \; G_2 = \frac{193 + \kappa}{432} \simeq .64, \; r_1 = \frac{101 + 5\kappa}{2160} \simeq .24, \; r_0 = \frac{193 + \kappa}{540} \simeq .51, \\
q &= \frac{158 - \kappa}{140} \simeq .53, \; p = \frac{\kappa - 63}{40} \simeq .51.
\end{aligned}
$$

It is now clear that $0 < t_2 < t_1$, $G_2 \in (r_0, r_0 + r_1)$, and $\beta$ can then be determined.

# B  Proof of Theorem 1

If the jump bid is not submitted with positive probability, there is nothing to prove.

Suppose that the jump bid is submitted with positive probability in equilibrium (which requires $K < 1$), and such a bid is never covered. Since a fraction of this bid is sunk, observe that a positive measure of Player 1's types submit the ordinary bid as well. So consider the two corresponding auctions, auction 0 and auction $K$. Let $\bar{b}$ be the highest bid submitted by Player 1 in auction 0. Thus, Player 2 bids no more than $\bar{b} + \varepsilon$, for any $\varepsilon > 0$, and Player 1 can win for sure by (submitting an ordinary bid and then) submitting a bid $\bar{b} + \varepsilon$. Since he wins for sure by submitting the jump bid $K$ (and nothing beyond), it follows that $\bar{b} = K < 1$ (indeed, if $\bar{b} > K$, submitting the ordinary bid and bidding $\bar{b}$ is not optimal, while if $\bar{b} < K$, submitting the jump bid is not optimal), and Player 1 wins for sure if he submits such a bid $\bar{b}$ in auction 0. Now, Player 2's type 1 bids no less than $\bar{b} - \varepsilon$, for any $\varepsilon > 0$ in auction 0 (since Player 1 would not bid as much as $\bar{b}$ otherwise), and he must therefore win with probability 1 by bidding $\bar{b}$ (as bidding $\bar{b} + \varepsilon < 1$ would strictly dominate bidding $\bar{b} - \varepsilon$ otherwise). It follows that, among those types of Player 1 submitting an ordinary bid, all types but the largest bid strictly less than $\bar{b}$ (and win with probability strictly less than 1). Since Player 1 wins for sure if he submits the jump bid, it follows that types submitting such a bid must be larger than the types submitting an ordinary bid. Therefore, Player 1 submits an ordinary bid if and only if his type is less than some threshold $\alpha$.

The reasoning is similar, but slightly more involved, when the jump bid is submitted with positive probability in equilibrium (which requires $K < 1$) and also covered with positive probability. We will consider the *expected* total bid and probability of winning of Player 1

for each initial bid he may submit, where the expectation is taken with respect to Player 2's covering decision. Let $\bar{b}_0$ (respectively, $\bar{b}_K$) be the largest (total) expected bid submitted by any type of Player 1 among those submitting an ordinary (respectively a jump) bid. [This bid is an expected bid in the case of a jump bid, as it is the sum of the jump bid and of a bid that depends on the realization of the covering decision.] Thus, Player 1 wins with probability 1 if he submits an ordinary bid and then bids $\bar{b}_0 + \varepsilon$, for any $\varepsilon > 0$, or if he submits a jump bid and bids slightly more than the highest bid he would submit in case of covering, for a total expected bid of $\bar{b}_K + \varepsilon$, for any $\varepsilon > 0$. Since Player 1's type 1 must have a strictly positive payoff, it follows that either of those expected bids wins with probability 1 (for Player 1), and $\bar{b}_0 = \bar{b}_K$ (same argument as above). Let $P_0$ and $P_K$ denote the support of the ("expected") probability of winning of Player 1, corresponding respectively to the ordinary and the jump bid. By the previous argument, we have $\max\{p \mid p \in P_0\} = \max\{p \mid p \in P_K\} = 1$, and it is standard to show that each support is an interval. Consider $p \in (\min\{p \mid p \in P_0 \cap P_K\}, 1]$, and let $v_0$ and $v_K$ be any type winning with probability $p$ in the auction 0 and $K$ (Obviously, the corresponding (sum of expected) bids $b_0$ and $b_K$ are equal). We claim that $v_0 = v_K$. If not, by monotonicity of preferences, all types between the two valuations must bid $b_0 = b_K$ (with probability one), so that in at least one of the auctions, there is a positive measure of types making the same bid, contradicting the continuity of the support. Therefore, there exists $\beta \in (0,1)$ such that $v > \beta$ implies $\int_0^1 p_1(s,v)\, ds \in (0,1)$ (for almost all such $v$).

It is clear that Player 2's covering decision follows a threshold rule. Because $\gamma < 1$ and a fraction of the opening bid is sunk, if Player 2 covers, his second-round equilibrium bid must be the higher bid with positive probability (for any type of Player 2 that covers). Therefore, in such a subgame, Player 1 must submit a losing bid with positive probability. Let $E$ be the corresponding set of Player 1 types, which has positive measure. The set of types $E^0 \subseteq \bar{E}$ such that $\int_0^1 p(v,s)\, ds < 1$, $v \in \bar{E}$ if and only if $v \in E^0$ must be of measure 0: to see this, observe that the bid $K$ must be in the interior of Player 1's support of winning bids after bidding 0 (since such a bid must win with the same probability after an ordinary bid as does a jump bid $K$ only) and his distribution of such winning bids is continuous at $K$ (as Player 2 would strictly prefer $K + \varepsilon$ to $K - \varepsilon$ otherwise). This implies that the measure of Player 1's types bidding $K$ after an ordinary bid is zero. Therefore, $E$ differs from an interval by a set of measure zero; we denote its upper extremity by $\beta$, and Player 1 must bid 0 with probability one for any type below its lower extremity $\alpha$.

# C Proof of Lemma 2

For the proof of this lemma, as well as for the proof of Theorem 3, it is useful to recall a few facts about all-pay auctions (see Amann and Leininger (1996)). The support of bid distribution is identical across bidders and the distribution of bids is continuous on this support. Bidding is weakly increasing in types, and there cannot be any atom in the bid distribution except at the bid of zero. Only one player can have a bid distribution with a probability mass at zero.

In an all-pay auction, in which $F_1$ and $F_2$ represent the distribution of valuations, Player 1's objective is to maximize $v_1 \cdot F_2 \left( b_2^{-1} \left( x \right) \right) - x$ over $x \in \mathbb{R}^+$, while Player 2's objective is to maximize $v_2 \cdot F_1 \left( b_1^{-1} \left( y \right) \right) - y$ over $y \in \mathbb{R}^+$. First-order conditions are:

$$F_2' \left( b_2^{-1} \left( x \right) \right) \cdot \left( b_2^{-1} \right)' \left( x \right) = \frac{1}{b_1^{-1} \left( x \right)}, \text{ and}$$

$$F_1' \left( b_1^{-1} \left( y \right) \right) \cdot \left( b_1^{-1} \right)' \left( y \right) = \frac{1}{b_2^{-1} \left( y \right)}.$$

To determine the equilibrium bid functions, it is useful to introduce the mapping $h \left( \cdot \right) = b_2^{-1} \circ b_1 \left( \cdot \right)$, which maps Player 1's valuation into Player 2's valuation making the same bid. The first-order conditions can be rewritten as:

$$\left( b_2^{-1} \right)' \left( b_1 \left( v \right) \right) = \frac{1}{v \cdot F_2' \left( b_2^{-1} \circ b_1 \left( v \right) \right)} = \frac{1}{v \cdot F_2' \left( h \left( v \right) \right)},$$

$$b_1' \left( v \right) = \frac{1}{\left( b_1^{-1} \right)' \left( b_1 \left( v \right) \right)} = h \left( v \right) \cdot F_1' \left( v \right),$$

whenever the density is positive. Finally, since $h' \left( v \right) = \left( b_2^{-1} \right)' \left( b_1 \left( v \right) \right) \cdot b_1' \left( v \right)$, we obtain the following ordinary differential equation:

$$h' \left( v \right) = \frac{h \left( v \right) \cdot F_1' \left( v \right)}{v \cdot F_2' \left( h \left( v \right) \right)},$$

which along with the boundary condition $h \left( 1 \right) = 1$ fully determines the mapping $h$. In particular, the mapping $h$ indicates whether one of the players has an atom in his bid distribution. If $\max \left\{ h^{-1} \left( 0 \right) \right\} \neq 0$, then Player 1 bids 0 with positive probability. It is also useful to note that when $F_1 \left( F_2 \right)$ are power distribution functions, there exists a closed-form solution for the function $h$.

Going back to the analysis of the equilibrium with covering, let us define $p(v) = \int_0^1 p(s, v)\, ds$ and denote by $\lambda$ the probability that Player 1 uses a jump bid, that is:

$$\lambda \triangleq F(\beta) - F(\alpha) + \int_\beta^1 p(v)\, dF(v). \tag{A3}$$

Recall that if a player has a valuation between $\alpha$ and $\beta$ he makes a jump bid for sure, and that players with valuations between $\beta$ and 1 randomize.

Let $h^0$, $h^K$ be the mappings from Player 1's type to Player 2's type making the same bid in the subgames $0, K$. These $h$ mappings determine the probability of winning. Since the profits are directly related to the probability of winning, indifference between subgames implies equality of profits across subgames and hence the identity of the mapping $h$ across subgames. Because valuations above $\beta$ are indifferent between both subgames, we must have $h(v) := h^0(v) = h^K(v)$ for $v > \beta$, which implies that:

$$\frac{vh'(v)}{h(v)} = \frac{p(v)\, f(v)\, /\lambda}{g \circ h(v)\, /\, (1 - G(\gamma))} = \frac{(1 - p(v))\, f(v)\, /\, (1 - \lambda)}{g \circ h(v)}.$$

It follows directly that $p(v)$ is constant and

$$\frac{p}{1 - p}(1 - G(\gamma)) = \frac{\lambda}{1 - \lambda}. \blacksquare \tag{A4}$$

# D   Proof of Theorem 3

Using the notation introduced in the proof of Lemma 2, the mapping $h : [0,1] \to [0,1]$, uniquely determined by $h(1) = 1$, $h(\alpha) = h(\beta)$, and

$$h'(v) = \begin{cases} \dfrac{f(v)}{(1-\lambda)g \circ h(v)} \dfrac{h(v)}{v}, & v \leqq \alpha, \\[2ex] \dfrac{1 - \frac{F(\alpha)}{1-\lambda}}{1 - F(\beta)} \dfrac{f(v)}{g \circ h(v)} \dfrac{h(v)}{v}, & v \geqq \beta, \end{cases}$$

must satisfy

$$h(\alpha) = \gamma \quad \text{and} \quad \int_{\max\{h^{-1}(0)\}}^\alpha \frac{h(v)}{1 - \lambda}\, dF(v) = K. \tag{A5 and A6}$$

To see this, recall that the mapping $h$, which must be identical in both subgames on their common domain, represents the correspondence between players' valuations in the auction subgames. Since the mapping is the same in both subgames, this means that players with

45

valuations between $\alpha$ and $\beta$ do not bid in the auction after a jump bid since (by construction) they do not bid (in fact they are not present) in the subgame after an ordinary bid. The first equation is tantamount to the boundary conditions $h(\alpha) = h(\beta) = \gamma$ and the second is just that type $\alpha$ must be indifferent between both subgames, that is $b(\alpha) = K$ .

Because Player 2's type $\gamma$ is indifferent between covering or not, his payoff from covering must be equal to the payoff of zero he gets when he does not cover:

$$\frac{F(\beta) - F(\alpha)}{\lambda}\gamma = K. \tag{A7}$$

An equilibrium consists of values in $[0, 1]$ for the parameters $\alpha, \beta, \gamma, p$ and $\lambda$ that solve (A3-A7).

It is easy to see that equations (A3,A4, A7) admit a solution $\lambda$, $p$ and $\gamma$ in the unit interval, given $\alpha$, $\beta$, provided that:

$$K \lesseqgtr \frac{F(\beta) - F(\alpha)}{1 - F(\alpha)}.$$

In this case, $\lambda$ is the solution to:

$$G\left(\frac{K\lambda}{F(\beta) - F(\alpha)}\right) = \frac{F(\beta)\lambda - (F(\beta) - F(\alpha))}{(1 - \lambda)(\lambda - (F(\beta) - F(\alpha)))}. \tag{A8}$$

Therefore, an equilibrium with covering exists if and only if equations (A5-A6) (where $\lambda$ is given by (A8)) admit a solution $\alpha$ and $\beta$ that are in the unit interval. We now check this using the explicit solution for the function $h$ available in the case if a power distribution. In this case, the system of equations become:

$$\frac{p}{1 - p}(1 - \gamma^\mu) = \frac{\lambda}{1 - \lambda}, \; \lambda = \beta^\mu - \alpha^\mu + p(1 - \beta^\mu), \; K = \gamma\frac{\beta^\mu - \alpha^\mu}{\lambda},$$

$$h(v) = \begin{cases} \left(\frac{(1-p)v^{\mu-1}+p-\lambda}{1-\lambda}\right)^{\frac{1}{\mu-1}}, & v \gtreqqless \beta, \\ \left(\frac{v^{\mu-1}+\beta^{\mu-1}-\alpha^{\mu-1}+p\left(1-\beta^{\mu-1}\right)-\lambda}{1-\lambda}\right)^{\frac{1}{\mu-1}}, & v \lesseqqgtr \alpha, \end{cases}$$

$$(1 - p)\beta^{\mu-1} + p - \lambda = (1 - \lambda)\gamma^{\mu-1} \text{ and } \int_{\max\{h^{-1}(0)\}}^{\alpha} h(v)\,dF(v) = K.$$

46

We can actually express all variables as a function of $\gamma$ and $\beta$:

$$K = \gamma\left(1 - \frac{1-\beta^\mu}{1-\gamma^\mu}\frac{1-\gamma^{\mu-1}}{1-\beta^{\mu-1}}\right),$$

$$\lambda = \frac{1-\gamma^\mu}{\gamma^\mu}\left(\frac{1-\beta^{\mu-1}}{1-\gamma^{\mu-1}}-1\right),$$

$$p = \frac{1}{\gamma^\mu}\left(1 - \frac{1-\gamma^{\mu-1}}{1-\beta^{\mu-1}}\right) = \frac{1}{\gamma^\mu}\frac{1-\gamma^{\mu-1}}{1-\beta^{\mu-1}}\left(\frac{1-\beta^{\mu-1}}{1-\gamma^{\mu-1}}-1\right),$$

$$\alpha = \left(\beta^\mu - \frac{1-\gamma^\mu}{\gamma^\mu}\left(\frac{1-\beta^{\mu-1}}{1-\gamma^{\mu-1}}-1\right)\left(1 - \frac{1-\beta^\mu}{1-\gamma^\mu}\frac{1-\gamma^{\mu-1}}{1-\beta^{\mu-1}}\right)\right)^{\frac{1}{\mu}}.$$

Depending on $\mu \gtreqless 1$, all these variables are in the right domain if:

$$\beta \in [0,\gamma] \text{ for } 0 < \mu < 1$$

$$\beta \in \left[\left(1 - \frac{1-\gamma^{\mu-1}}{1-\gamma^\mu}\right)^{\frac{1}{\mu-1}},\gamma\right] \text{ for } \mu > 1.$$

We take $\gamma$ as the "exogenous" parameter and show that there is a $\beta$ in the right interval that satisfies the last constraint on the bid of type $\alpha$. We can show that $\max\{h^{-1}(0)\} > 0$ $\forall \mu > 0$. Let us focus on the case $\mu > 1$. The condition:

$$\int_{\max\{h^{-1}(0)\}}^\alpha h(v)\,d\frac{F(v)}{1-\lambda} = K$$

becomes, after a change of variable and some manipulations:

$$\frac{K}{\rho} = \frac{\mu}{\mu-1}\int_0^1 (1-x)^{\frac{1}{\mu-1}}\left(\delta^{\mu-1}-\rho x\right)^{\frac{1}{\mu-1}}dx, \text{ or}$$

$$\frac{K}{\rho\delta} = F\left(1, -\frac{1}{\mu-1}, 2+\frac{1}{\mu-1}, \frac{\rho}{\delta^{\mu-1}}\right),$$

where $\rho = \gamma^{-\mu}\left(1 - \frac{1-\gamma^\mu}{1-\gamma^{\mu-1}}(1-\beta^{\mu-1})\right)$, $\delta = \gamma\left[\rho\left(1 - \frac{1-\beta^\mu}{1-\beta^{\mu-1}}(1-\gamma^{\mu-1})\right)\right]^{1/\mu}$, and $F$ is the hypergeometric function. The existence and uniqueness of a solution $\beta \in \left[\left(1 - \frac{1-\gamma^{\mu-1}}{1-\gamma^\mu}\right)^{\frac{1}{\mu-1}},\gamma\right]$ is then obtained by considering both sides as functions of $z = \frac{1-\beta^{\mu-1}}{1-\gamma^{\mu-1}} \in \left[1, \frac{1}{1-\gamma^\mu}\right]$ and considering their variations: the left-hand side is strictly decreasing and onto $\mathbb{R}_+$, while the right-hand side is strictly increasing and is equal to zero for $z = 1$. The only nontrivial statement is the monotonicity of the right-hand side. However, manipulation of its deriva-

tive with respect to $z$ gives, for $z < \frac{1}{1-\gamma^\mu}$:

$$\frac{d}{dz}F\left(1, -\frac{1}{\mu-1}, 2 + \frac{1}{\mu-1}, \frac{\rho}{\delta^{\mu-1}}\right) \propto z\left[(z-1)(1-\gamma^\mu) + (1-\gamma)\gamma^{\mu-1}\left(1 - \left(1-\gamma^{\mu-1}\right)z\right)^{\frac{1}{\mu-1}}\right]$$

$$+ (\mu-1)(1 - (1-\gamma^\mu)z)\left[1 - \left(1 - \left(1-\gamma^{\mu-1}\right)(z-1)\right)\left(1 - \left(1-\gamma^{\mu-1}\right)z\right)^{\frac{1}{\mu-1}}\right] > 0.$$

Finally, because the left-hand side (resp. right-hand side) is everywhere increasing (resp. decreasing) in $\gamma$, the root $z$ is monotonically increasing in $\gamma$, that is, $\beta$ is decreasing in $\gamma$, which establishes that the total derivative of $K$ with respect to $\gamma$ is positive, and in particular, a solution to the system of equations exists if and only if $K$ is below some critical threshold $\bar{K}$. ∎

# E    Proof of theorem 4

$K \in [\mu/(\mu+1), 1]$ : the only P.S.E. involves no opening bid. The strategies of the non-revealing equilibria given in the text do indeed form a P.B.E. Since any deviation from those strategies is not profitable (we have that the payoff in this equilibrium equal to $\frac{v^{\mu+1}}{\mu+1}$ is larger than $v - K$, which is an upper bound on the payoff from a deviation), these equilibria are P.S.E..

Consider next a P.B.E. with $K < \mu/(\mu+1)$ in which Player 1 never bids $K$. We show that such an equilibrium cannot be a P.S.E.. Consider a deviation by all players with $v \in [\alpha, 1]$ to jump bid $K$. Suppose, first, that, for some $\gamma \in [0, 1)$, Player 2 always finds it worthwhile to cover when his valuation lies in $(\gamma, 1]$. Consider the subgame between players with those valuations, and let $h : [\alpha, 1] \to (\gamma, 1]$, $v \longmapsto h(v)$ such that $w = h(v)$ makes the same bid as a player with valuation $v$. Defining $\beta$ such that $\gamma_+ \triangleq h(\beta), \beta$ is necessarily larger than $\alpha$, since players with valuations arbitrarily close to $\gamma$ have profits arbitrarily close to $\frac{\beta^\mu - \alpha^\mu}{1-\alpha^\mu} \cdot \gamma$, which must exceed $K$. Also, $\gamma > \alpha$ and we can compute $h(v) = \left[\left(\frac{1-\gamma^\mu}{1-\alpha^\mu}\right)(v^{\mu-1} - 1) + 1\right]^{\frac{1}{\mu-1}}$. We have to verify that in this subgame all players with valuations in the interval $[\alpha, 1]$ achieve higher profits than in the original P.B.E., where players with valuation $v$ earn $\frac{v^{\mu+1}}{\mu+1}$. In the subgame following the deviation, players with $v \in [\beta, 1]$ achieve profits $\Pi(v) = \Pi(\beta) + \int_\beta^v \frac{h(s)-\gamma}{1-\gamma}ds$, while players with valuations in $[\alpha, \beta]$

make zero profit. Ex ante profits of valuations $[\beta, 1]$ must exceed $v^{\mu+1}/(\mu+1)$, that is:

$$\gamma^\mu \cdot v + \int_\beta^v \frac{(h(s)^\mu - \gamma^\mu)}{1 - \gamma^\mu} ds - K > v^{\mu+1}/(\mu+1).$$

Note that the derivative of the left-hand side with respect to $v$ is $\gamma^\mu + \frac{(h(v)^\mu - \gamma^\mu)}{1 - \gamma^\mu}$ which is larger than the corresponding derivative of the right-hand side, which is $v^\mu$. Hence, the inequality will hold if it holds for $v = \beta$. Consider players with valuations in $[\alpha, \beta]$. Ex ante profits from deviating are $\gamma^\mu \cdot v - K$. Marginal profits, $\gamma^\mu$, once again exceed marginal profits $v^\mu$ in the original P.B.E., since $v \leqq \beta < \gamma$. If players with valuation $\alpha$ are indifferent between deviating and not deviating, players with lower valuations will prefer not to deviate. This is equivalent to requiring that $\alpha \cdot \gamma - K = \frac{\alpha^{\mu+1}}{\mu+1}$. Hence, provided that there exist $\gamma, \alpha$ such that

$$\begin{cases} \alpha \cdot \gamma^\mu - K = \alpha^{\mu+1}/(\mu+1), \\ \frac{\beta^\mu - \alpha^\mu}{1 - \alpha^\mu} \cdot \gamma = K. \end{cases}$$

, we have found another P.B.E. in which the deviators are better off, the non-deviators worse off were they to deviate and the beliefs of Player 2 after a deviation correspond to the set of types who benefit from the deviation. Thus the equilibrium is not a P.S.E.

Suppose now that it is not worthwhile for Player 2 to cover after a deviation, regardless of valuation. It follows that the original P.B.E. is not a P.S.E. if

$$\begin{cases} \alpha - K = \alpha^{\mu+1}/(\mu+1), \\ \frac{\left(1 + (1-\alpha^\mu)\frac{\mu-1}{\mu}\right)^{\frac{\mu}{\mu-1}} - \alpha^\mu}{1 - \alpha^\mu} < K. \end{cases}$$

$(\frac{\left(1 + (1-\alpha^\mu)\frac{\mu-1}{\mu}\right)^{\frac{\mu}{\mu-1}} - \alpha^\mu}{1 - \alpha^\mu} = \frac{\beta^\mu - \alpha^\mu}{1 - \alpha^\mu} \cdot \gamma$. This system guarantees that it is indeed optimal for Player 2 not to cover, regardless of valuation; that if a player has a valuation in the interval $(\alpha, 1]$ he will strictly prefer the expected payoff from deviating to the original expected payoff, and that all players with valuations in the interval $[0, \alpha)$ will strictly prefer the expected payoff of the original P.B.E. to their expected payoff from deviating. Although it is not difficult to show which case obtains a function of $K$, this is not even necessary. It is enough to note that for $\alpha = 0$, expected profits from deviating are smaller than expected profits from not deviating, whereas in both cases, since expected profits from deviating are larger than $\alpha^\mu - K$, they are also larger than $\alpha^{\mu+1}/(\mu+1)$, provided that $\alpha$ is close enough to 1. This

result is based on the fact that if $K$ is strictly less than $\frac{\mu}{\mu+1}$. $\gamma$ being a continuous function of $K$, there does then necessarily exist, for any $K < \frac{\mu}{\mu+1}$, an $\alpha \in (0,1)$ satisfying one of the two systems.

We finally need to verify that there does not exist a P.S.E. providing assured deterrence outside the interval $[\bar{K}, \frac{\mu}{\mu+1}]$. It is then easy to show that an equilibrium with assured deterrence, as specified in the text, is a P.S.E.. Recall that in an equilibrium with assured deterrence, there exists an $\alpha \in (0,1)$ such that all Player 1's with valuations strictly smaller than $\alpha$ make a zero opening bid, while all players with valuations strictly larger than $\alpha$ bid $K$. In this equilibrium Player 2 never covers, regardless of valuation. In simultaneous bidding between players with valuation $v \in [0, \alpha]$ and players with $w \in [0, 1]$, after a zero opening bid, the expected profit of Player 1 with valuation $v = \alpha$ is $\alpha - \frac{\mu}{\mu-1} \int_{1-\alpha}^{1} x^{\frac{1}{\mu-1}} \left( \frac{x-(1-\alpha)}{\alpha} \right)^{\frac{1}{\mu-1}} dx$; since a player with valuation $\alpha$ will be indifferent between this expected profit and the expected profit following a bid of $K$, which is $\alpha - K$, it follows that $K = \frac{\mu}{\mu-1} \int_{1-\alpha}^{1} x^{\frac{1}{\mu-1}} \left( \frac{x-(1-\alpha)}{\alpha} \right)^{\frac{1}{\mu-1}} dx$. Consider a deviation by Player 2 in which he covers. More precisely, suppose that players with $w \in (\gamma, 1]$ cover while players with valuations lower than $\gamma \in (0,1)$ do not. Obviously, if players with valuations arbitrarily close to $\gamma$ from above have expected profits from the deviation that are arbitrarily small, players with valuations strictly above $\gamma$ obtain strictly positive expected profits from deviating while players with valuations strictly below achieve strictly negative profits. These two situations compare with the zero profit that Player 2 achieves in the original P.B.E., regardless of valuation. In the subgame following the deviation by Player 2. Player 1's with valuations $v \in (\alpha, 1]$ play against Player 2 with $w \in (\gamma, 1]$. In this subgame, Player 2 with valuation $\gamma_+$ (a valuation arbitrarily close to $\gamma$ from above) can achieve profit $K$, only if

$$\frac{h(\gamma)^\mu - \alpha^\mu}{1 - \alpha^\mu} \cdot \gamma = K.$$

Hence, the equilibrium with assured deterrence is a P.S.E. if and only if such a $\gamma \in (0,1)$ cannot be found. Since the left-hand side is increasing in $\gamma$, it is both necessary and sufficient that

$$\lim_{\gamma \to 1} \frac{\beta^\mu - \alpha^\mu}{1 - \alpha^\mu} \cdot \gamma = \frac{\left( (1-\alpha^\mu) \frac{\mu-1}{\mu} + 1 \right)^{\frac{\mu}{\mu-1}} - \alpha^\mu}{1 - \alpha^\mu}$$

$$\leqq \frac{\mu}{\mu-1} \int_{1-\alpha}^{1} x^{\frac{1}{\mu-1}} \left( \frac{x-(1-\alpha)}{\alpha} \right)^{\frac{1}{\mu-1}} dx.$$

50

The latter inequality corresponds to the condition for which an equilibrium with covering exists, which precisely states that $K \gneqq \bar{K}$.

Finally, when $K > \frac{\mu}{\mu+1}$, there is no equilibrium with assured deterrence, as we saw in section 5. Finally, equilibria with covering, as specified in the text for $K < \bar{K}$, are obviously P.S.E., since they are P.B.E. and every subgame is on the equilibrium path ∎.

# F  Proof of theorem 5

The analysis that follows establishes the existence of equilibrium. We first need to compute payoffs in the subgames where Player 2 has covered. We reformulate the different functions at stake in those subgames using the mapping $h(\cdot) = b_2^{-1} \circ b_1(\cdot)$. This function maps Player 1's type into the type of Player 2 who makes the same bid. Let $F_1$ and $F_2$ be c.d.f.'s on $[0,1]$, with positive densities on $(0,1)$, $F_1$ being the distribution of types of Player 1 and $F_2$ the distribution of types of Player 2.

Consider a subgame following a bid of $k > 0$ by Player 1, and assume that types are distributed as follows: the support of types for Player 1 is $[\alpha, 1]$, $\alpha \in (0,1)$, and its distribution has an atom of size $p$ at $\alpha$, and is continuously distributed with positive density on $(\alpha, 1)$. Player 2's types are nonatomic, uniformly distributed on $(\gamma, 1]$. Since type $v$ of Player 1 maximizes $\Pi_1^P(v) = v \cdot \Pr\{x > b_2\} - x$ in the subgame, an immediate consequence of the envelope theorem is that $\partial \Pi_1^P(v)/\partial v = \Pr\{x > b_2\}$, and hence, by monotonicity of the bids in the types, $\Pi_1^P(v) = \int_\alpha^v \Pr\{b_1(t) > b_2\}\,dt$ (the superscript $P$ is mnemonic for ex post, since these are the profits in the subgame). In this context, defining the mapping $h(v) = b_2^{-1} \circ b_1(v)$, we get that:

$$\Pi_1^P(v) = \int_\alpha^v \frac{h(s) - \gamma}{1 - \gamma}\,ds.$$

Having a reduced form for Player 1's profits in the subgames, we address now the first period strategies. Given that a fraction $\gamma$ of Player 2's types has not covered, Player 1's expected profit is:

$$\Pi_1^A(v) = \gamma \cdot v + (1 - \gamma) \cdot \Pi_1^P(v) - k = \int_\alpha^v h(s)\,ds + \gamma \cdot \alpha - k.$$

Assume now that $\alpha(\cdot)$, $\gamma(\cdot)$, $h(v, \cdot)$, $p(\cdot)$ are differentiable mappings in $k$. Since Player 1 randomizes over early bids, his ex-ante profits must be equal across early bids. So if all types

51

$v$ of Player 1 on some interval $V$ are (overall) indifferent between all $k$'s in some common interval $K$, it must be that $\partial \Pi_1^A (v) / \partial v = h(v)$ is independent of $k \in K$, for $v \in V$. Also, it must be that $\partial \Pi_1^A (v) / \partial k = 0$ on $K \times V$. Using that $h(v)$ is independent of $k$, this simplifies to: $(\gamma(k) - h(\alpha(k))) \frac{d\alpha(k)}{dk} + \frac{d\gamma(k)}{dk} \cdot \alpha(k) - 1 = 0$, that is, $\frac{d\gamma(k)}{dk} \cdot \alpha(k) = 1$. Let us define $\lambda(k)$ as

$$\lambda(k) = p(k) + \int_{\alpha}^{1} \psi(s, k) \, ds. \tag{1}$$

Eq. (1) has the following interpretation. $\psi(\cdot, \cdot)$, which was defined above, depends both on $k$ and $v$; as a function of $k$, it is the density over bids made by type $v$; as a function of $v$, it is the density over types that, along with type $\alpha$, bid $k$. $d\lambda$ is thus the density over Player 1's types who bid $k$, and is well defined as long as $p$ and $\int_{\alpha}^{1} \psi(s, k) \, ds$ are of the same cardinality, that is, as long as $0 < \int_{\alpha}^{1} \psi(s, k) \, ds < \infty$. $p(k) / \lambda(k)$ is thus the probability, conditional on observing $k$, that Player 1 is of type $\alpha(k)$. In the equilibrium that we derive, type $\alpha$ bids 0 in the subgame $k$, and no other type bidding $k$ bids 0 thereafter. Hence, $p(k) / \lambda(k)$ is the conditional probability that Player 1 bids 0 in the subgame $k$. In other words it represents the proportion of bluffers among types who bid $k$ in the first period. Accordingly, it is necessary in equilibrium that:

$$\frac{p(k)}{\lambda(k)} = \frac{k}{\gamma(k)}, \tag{2}$$

which states that $\gamma(k)$ is the cut-off between types of Player 2 who have positive expected profits in the subgame and those who do not. Finally, we can rewrite equation (??) as:

$$\frac{v \cdot \partial h(v) / \partial v}{h(v)} = \frac{\psi(v, k)}{\lambda(k)} \cdot (1 - \gamma(k)) \tag{3}$$

Since $h(v)$ is independent of $k$ on any interval $V \times K$ where types of Player 1 randomize, it must also be that $\frac{1 - \gamma(k)}{\lambda(k)} \cdot \psi(v, k)$ is independent of $k$. In particular, we can define $g(v)$ such that $g(v) = \frac{1 - \gamma(k)}{\lambda(k)} \cdot \psi(v, k)$, which is independent of $k$ on any such interval. Multiplying equation (1) by $\frac{1 - \gamma(k)}{\lambda(k)}$ and using equation (2), we get

$$\left(1 - \frac{k}{\gamma(k)}\right) (1 - \gamma(k)) = \int_{\alpha(k)}^{1} g(s) \, ds.$$

Differentiating this identity with respect to $k$, and using that $g$ is independent of $k$, we obtain

$$1 + \frac{k \cdot d\gamma\left(k\right)/dk}{\gamma\left(k\right)^2} - d\gamma\left(k\right)/dk - 1/\gamma\left(k\right) = -g\left(\alpha\left(k\right)\right) \cdot d\alpha\left(k\right)/dk.$$

Finally, since $\frac{d\gamma}{dk} \cdot \alpha = 1$, this last equation can be rewritten as

$$1 + \frac{k \cdot d\gamma\left(k\right)/dk}{\gamma\left(k\right)^2} - d\gamma\left(k\right)/dk - 1/\gamma\left(k\right) = g\left(\frac{1}{d\gamma\left(k\right)/dk}\right) \cdot \frac{d^2\gamma\left(k\right)/dk^2}{\left(d\gamma\left(k\right)/dk\right)^2}. \tag{4}$$

This equation only involves the unknowns $\gamma$ and $g$.

Conjecturing a solution of the form $\gamma\left(k\right) = c \cdot k^n$ for some $n$ and $c$, we have that $\alpha\left(k\right) = k^{1-n}/nc$, since $\frac{d\gamma}{dk} \cdot \alpha = 1$. Let $\bar{k}$ be the largest early bid made. Then $\alpha\left(\bar{k}\right) = 1$, which also implies that $\gamma\left(\bar{k}\right) = 1$. It follows that $n = \bar{k}$, and $c = n^{-n}$, so that $\gamma\left(k\right) = \left(\frac{k}{n}\right)^n$, while $\alpha\left(k\right) = \left(\frac{k}{n}\right)^{1-n}$. Eq.(4) then implies that

$$g\left(v\right) = n\left(1 + \frac{1-v}{1-n}v^{\frac{2n-1}{1-n}}\right).$$

We can now solve for $p$ and $\psi$. It must be that, for any $k$ and associated $\alpha$, $p + \int_0^k \psi\left(\alpha, \tilde{k}\right) d\tilde{k} = 1$. Differentiating this identity with respect to $k$ yields

$$\frac{dp\left(k\right)}{dk} = -\psi\left(\alpha\left(k\right), k\right) = -\frac{\lambda\left(k\right)}{1-\gamma\left(k\right)} \cdot g\left(\alpha\left(k\right)\right).$$

Using Eq. (2), this is equivalent to

$$\frac{dp\left(k\right)/dk}{p\left(k\right)} = -\frac{\gamma\left(k\right)}{1-\gamma\left(k\right)} \cdot \frac{g\left(\alpha\left(k\right)\right)}{k}.$$

Integration of the latter equation leads, for some constant $A \in \mathbb{R}^+$, to

$$p\left(k\right) = A \cdot \left(1 - \left(\frac{k}{n}\right)^{1-n}\right)^{\frac{1}{1-n}} e^{\frac{-\left(k/n\right)^n}{n\left(1-n\right)}}.$$

Since $\lambda\left(k\right)=p\left(k\right)\cdot\frac{\gamma\left(k\right)}{k}=p\left(k\right)\cdot\frac{k^{n-1}}{n^{n}}$, and $\int_{0}^{\bar{k}}\lambda\left(k\right)dk=1$, it must be that

$$A=\left(\int_{0}^{n}\frac{k^{n-1}}{n^{n}}\left(1-\left(\frac{k}{n}\right)^{1-n}\right)^{\frac{1}{1-n}}e^{\frac{-(k/n)^{n}}{n(1-n)}}dk\right)^{-1}.$$

The positive integrand is dominated by $\frac{k^{n-1}}{n^{n}}$, and the integral is thus well-defined for $n>0$. To complete the analysis, it remains to determine the deterrence level $n$. In fact, in the subgame following $\bar{k}=n$, no type of Player 2 follows. (The equilibrium of the subgame with perfect information, in which only the highest types compete, is easy to solve and entails zero profit ex post for Player 2. Hence, Player 2 should not enter the subgame). Hence, the profit of type 1 of Player 1 is $1-n$. On the other hand, that type must be indifferent between this bid and any other bid, say 0. It must thus be that

$$1-n=\int_{0}^{1}h\left(s\right)ds,\tag{5}$$

where the right-hand side is the player's ex ante profit from bidding 0. Since $g\left(v\right)=\frac{v\cdot\partial h(v)/\partial v}{h(v)}$, we can, upon integration, determine $h$. The integral of the former equation, denoted $I\left(n\right)$, is then equal to

$$I\left(n\right)=\int_{0}^{1}x^{n}\exp\left[\frac{x^{\frac{2n-1}{1-n}}}{2n-1}-\frac{x^{\frac{n}{1-n}}}{n}-\frac{1-n}{n\left(2n-1\right)}\right]dx.$$

It can then be shown that

$$
\begin{aligned}
I\left(0\right) &= \int_{0}^{1}\frac{e^{\frac{x-1}{x}}}{x}dx=-e\cdot E_{i}\left(-1\right)\simeq0.6,\\
I\left(1\right) &= \int_{0}^{1}xdx=1/2.
\end{aligned}
$$

Since $I\left(\cdot\right)$ is continuous in $n$, the existence of a solution to Eq. (5) then follows from the intermediate value theorem. Uniqueness of that kind of equilibrium is numerically obvious, and $\bar{k}\simeq0,53$, which is almost twice as much as the bound found in the case of exogenous $K$. To sum up,

$$\alpha\left(k\right) \;=\; \left(\frac{k}{\bar{k}}\right)^{1-\bar{k}}, \;\; \gamma\left(k\right) = \left(\frac{k}{\bar{k}}\right)^{\bar{k}},$$

$$p\left(k\right) \;=\; A \cdot \left(1 - \left(\frac{k}{\bar{k}}\right)^{1-\bar{k}}\right)^{\frac{1}{1-\bar{k}}} e^{\frac{-\left(k/\bar{k}\right)^{\bar{k}}}{\bar{k}\left(1-\bar{k}\right)}},$$

$$\psi(v,k) \;=\; \bar{k}(1 + \frac{1-v}{1-\bar{k}}v^{\frac{2\bar{k}-1}{1-\bar{k}}}) \cdot \frac{p\left(k\right)\gamma\left(k\right)}{k(1-\gamma\left(k\right))}, \;\; \text{on the relevant domain.}$$

To see that second period contributions need not increase with the early bid, differentiate Player 1's second period bid with respect to the early bid. It is straightforward to obtain that type $v$'s bid decreases with the early bid $k$ if and only if

$$\alpha\left(k\right) \geqq \int_{\alpha(k)}^{v} \frac{\psi\left(s,k\right) \cdot h\left(s\right)}{\lambda\left(k\right)} ds,$$

which is not satisfied for small enough $k$ and large enough $v$.