On the level of public good provision in games of redistributive politics

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Abstract

This paper studies an electoral competition game between two candidates, building on Myerson (1993) and Lizzeri and Persico (2001 and 2005). Ex-ante identical voters first see their monetary endowment fully taxed and then receive promises from the candidates. Voters vote for the candidate offering them the more. Candidates have three different options : 1) offer a public good; 2) use individually targeted redistribution; and 3) use part of the available budget for redistribution and the rest for public good provision. We characterize all equilibria under proportional representation. We then prove that no equilibrium exists under plurality rule.

1 Introduction

Redistribution is an important aspect of a government's action. If well designed policies can help reduce income inequality and lead to more social justice, many studies have shown that redistributive politics can be used strategically by politicians to strengthen their electoral scores. A typical side effect of these strategic choices is the creation of inefficiencies and, often, the strengthening of inequalities.

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The present paper contributes to this debate, by building on the seminal contributions of Myerson (1993) and Lizzeri and Persico (2001 and 2005).[1] Myerson analyzes an electoral game between several candidates in which a population of identical citizens first see their monetary endowment fully taxed and then receive credible redistribution promises from candidates. Thus, candidates make (independently and simultaneously) fully targeted, individual-specific promises to voters. Citizens vote for the candidate who makes the most generous promise to them. The winner of the election is then determined through the specific electoral rule under analysis. Myerson analyzes this game for many different electoral rules. He shows that candidates have incentives to cultivate favored minorities. The result of redistributive politics is thus to create and increase inequalities, the extent of which depends on the electoral system.

Lizzeri and Persico (2001 and 2005) extend Myerson's model by allowing candidates to choose between targeted redistribution and offering a public good that benefits equally all voters. The benefits of this public good arenot individually targetable but the provision of the public good is assumed to be more efficient, in the sense that the utility voters receive from the good is greater than the tax cost they have to bear to finance its provision. Their models highlight the important trade-off between efficiency and targetability of various policies and show how this trade-off is influenced by the electoral system.

We extend the model of Lizzeri and Persico (2001) by giving politicians the choice between more options: pure redistribution, a public good that requires all the money in the economy to be produced and a partial public good that is less valuable but less costly to produce and that thus allows politicians to still have some funds for targeted redistribution. In what follows, we label this third option partial redistribution. The introduction of a choice in the level of production of the public good allows us

[1] The main alternative modelling strategy used in the literature to analyse electoral games builds on the probabilistic voting game of Lindbeck and Weibull (1987). Whereas both approaches have their merits and shortcomings, the approach of Myerson is more useful for the purposes of this paper because it allows to focus solely on the equilibrium effects of political forces and institutions. Indeed, in this world, voters are all ex-ante identical, not only in terms of their wealth but, more importantly, in terms of their preferences too. Thus, politicians cannot be reacting to differences within the electorate, by construction. This is not the case in probabilistic voting models, making it harder to disentangle the equilibrium effects of political forces from those of the polity's preferences and endowments.

to analyze whether the inefficiencies linked to targeted redistribution occur at the intensive or the extensive margin. This also allows us to analyze whether flexibility in the public good promises is going to increase or decrease the inefficiencies linked to redistributive politics.[2]

We analyze the game with two candidates under two electoral systems: proportional representation, in which candidates maximize their expected vote share; and a winner-take-all system, in which candidates maximize their probability of winning the election. To solve for equilibrium under proportional representation, we borrow from a key insight of the analysis of Lizzeri and Persico (2005): in equilibrium, the probability of winning a random vote with an offer worth x must be equal to $x/2$ if x is in the support of the equilibrium strategy of candidates, and less than $x/2$ if x falls outside the support of the candidates' equilibrium strategy.[3] Thus, in a nutshell, the equilibrium winning probability must be piecewise linear. A simple intuition for this fact is that, because the marginal cost of making an offer is linear,[4] so must be the marginal benefit of the offer, in equilibrium. Our detailed description of how to characterize the equilibria of such games with more than two options is the first contribution of this paper.

Starting with proportional representation, we characterize the Nash equilibrium in which candidates mix between the three options available (public good, partial and full redistribution) and pin down the parameter conditions for the existence of this fully mixed equilibrium. We then analyze for which values of parameters other equilibria can exist. Interestingly, we prove that whenever the three-option equilibrium exists, the two-option, mixed strategy Nash equilibrium (using the public good and full redistribution only) of Lizzeri and Persico (2001) fails to exist, and vice versa. We thus show that for any set of parameters, there exists a unique equilibrium. We then derive some results about welfare. In particular, we show that the introduction of a third option is efficiency-improving even though politicians have at their disposal another, more efficient policy (the public good). Thus, we show that improving the efficiency of an efficient policy leads to a welfare increase even when the policy that

[2] Crutzen and Sahuguet (2009) analyze a model of redistribution under distortionary taxation but focus on proportional representation only. It could be reinterpreted as a model of public good with a continuum of possible levels of provision.

[3] We have $x/2$ because we focus on two-candidate games. With n candidates, we would have x/n.

[4] As this is how an offer affects the budget available to each politician.

benefits from this efficiency gain is not the most efficient one.[5]

Turning to the winner-take-all system, we prove that there is *no* equilibrium for the parameter values that guarantee the existence of the three-option mixed strategy Nash equilibrium under proportional representation. This is the second main contribution of the paper. Our finding suggests that the game under winner-take-all system is not well behaved. In Myerson (1993) and Lizzeri and Persico (2001) this is not an issue. Myerson focuses on pure redistribution and thus, with two candidates, the mapping from votes to payoffs is not too important: the equilibria of the game under winner-take-all and proportional representation are actually one and the same. In Lizzeri and Persico (2001), the game under winner-take-all is similar to one of matching pennies. Intuition suggests that in equilibrium the probability of playing either option must be one half. Unfortunately this intuition does not carry through to a world with three options: one could think that, as in the rock-scissors-paper game, candidates would put equal weight on the three options. Yet, the analogy breaks down as in our model the choice of partial or full redistribution option also involves the choice of the individual promises schedules which determine how well this option does against the others. It is as if one could influence how well scissors would perform against rock, paper and scissors. With two options, this makes no substantial difference since there is no deviation from the optimal redistribution schedule that could improve on equilibrium. With three options, changing the redistribution schedule, a candidate can insure that he does better than *two* of the other options, thus breaking the candidate equilibrium.

Models of redistributive politics using a finite number of voters also exist using the theoretical framework of Colonel Blotto games.[6] This literature starts with Laslier and Picard (2002). The main difference between this literature and the one that builds on Myerson (1993) is that, because there is a finite number of voters, politicians must meet the budget constraint exactly, whereas in Myerson (1993) this constraint must be met in expectation only. Roberson (2008) treats the case of public good provision and his model is similar to the two-candidate model of Lizzeri and Persico (2001) under proportional representation. To the best of our knowledge, the game

[5] Crutzen and Sahuguet (2009) show that improving the efficiency of inefficient policies can lead to welfare losses.

[6] See Roberson (2006) for a recent treatment of the Colonel Blotto game, and Roberson (2011) for a survey of the field.

with at least two options à la Lizzeri and Persico (2001) under a winner-take-all system has not been analyzed in the finite number of voters framework.

2 The Model

2.1 Economy and players

There are two candidates, A and B. The economy contains a continuum of voters V of total mass 1. Each agent is endowed with one unit of money. Politicians tax voters' endowment and then make monetary promises. These promises are subject to an economy-wide budget constraint that constrains them to make balanced-budget policy pledges. These pledges are binding.

The game is as follows. First all citizens are fully taxed. Then, the two candidates can promise any of the three following policies: 1) a public good worth $1 < G < 2$ per euro, whose production requires all the economy's money (this is the most efficient policy), 2) a partial public good requiring a share α of the available budget and which is worth $G(\alpha)$ to voters with $1 < G(\alpha) + (1 - \alpha) < G$ and some individually targeted redistribution with the rest of the budget – in what follows we call this option partial redistribution; and 3) targeted redistribution only.

At the same time as they both announce their policy choice, politicians make individual binding promises to voters under the constraint that the sum of transfers must equal the total of money collected. All choices and offers by politicians are simultaneous and independent.

Citizens vote for the candidate who promises them the greatest utility. Under proportional representation, candidates maximize the share of total votes, given that each politician's share of spoils from office are divided proportionally to vote shares. Under the winner-take-all (majoritarian or plurality rule) system, candidates maximize their chance of winning a majority of the votes.

2.2 Game and politicians' strategies

A pure strategy for a candidate specifies the policy option he chooses. In the event he chooses either partial or full redistribution only, a pure strategy also specifies a promise of a transfer to each voter. Formally a pure strategy is a function $\mathcal{S} : V \rightarrow [0, +\infty)$, where $\mathcal{S}(v)$ represents the (after tax) utility promised to voter v. $\mathcal{S}$ must satisfy one of three conditions (depending on the chosen policy): either $\mathcal{S}(v) = G$ for all v's if the candidate chooses the public good, or $\int_V \mathcal{S}(v)\, dv = G(\alpha) + (1 - \alpha)$ with $S(v) \geq G(\alpha)$, which is the balanced budget condition when the partial redistribution is chosen, or $\int_V \mathcal{S}(v)\, dv = 1$, which is the balanced budget condition when a candidate choose full redistribution (each voter v is promised some $\mathcal{S}(v)$).

We follow the previous literature and focus on equilibria in simple symmetric strategies of the following form. Candidates choose simultaneously and independently the three policy options with probabilities p, p_α and $q = 1 - p - p_\alpha$ respectively for public good provision and partial and full redistribution. When candidates redistribute, they draw promises to all voters from the same distributions F and F_α. Due to the infinite number of voters, F and F_α also represent the empirical distributions of transfers.

3 Proportional Representation

3.1 Fully mixed strategy equilibrium

We first derive the equilibrium for the electoral game in which candidates mix between all three options. We characterize the values of the parameters $(\alpha, G(a), G)$ for which such an equilibrium exists. We then consider other parameter value configurations and characterize the corresponding equilibria. These equilibria were analyzed in Myerson (1993) and Lizzeri and Persico (2001) for games with at most two options.

The equilibrium with three options can be represented graphically. In the graph, the function $W^*(x)$ corresponds to the *equilibrium* probability of winning a vote when one promises x to a voter. The most fundamental characteristic of $W^*(x)$ is that it is piece-wise linear with a constant slope on the support of full redistribution and another constant slope on the support of partial redistribution.

There is a simple intuition for such linearity. When allocating monetary promises, marginal benefits in terms of won votes and the marginal cost of the allocation an extra dollar to a given voter must be equal. Since costs are linear, marginal benefits must also be linear. This intuition was pointed out in Lizzeri and Persico (2001 and 2005).

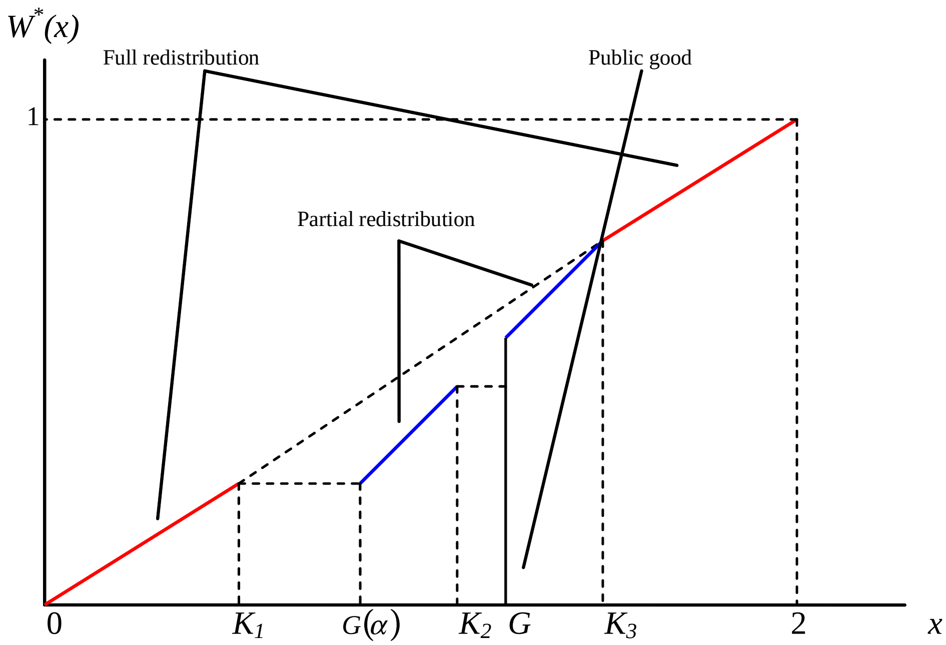

Figure 1: 3-option equilibrium

We prove in the appendix that an equilibrium must be characterized by a W^* function as depicted. Starting with partial redistribution, the minimum utility any voter can have under this pure strategy is $G(\alpha)$. The support of F^*_α is $[G(\alpha), K_2] \cup [G, K_3]$ with $K_2 < G$ and $K_3 < 2$ still to be pinned down. The support of $F^*$is $[0, K_1] \cup [K_3, 2]$ with $K_1 < G(\alpha)$ still to be determined. We now characterize the redistribution schemes under full and partial redistribution.

Full redistribution

By the linearity property of $W^*(x)$, we must have for all $x \leq K_1$:

$$\begin{aligned} W^*(x) &\equiv \frac{x}{2} = (1-p-p_\alpha) F^*(x) \\ &\Leftrightarrow \\ F^*(x) &= \frac{x}{2(1-p-p_\alpha)} \end{aligned}$$

and for all $x \geq G(\alpha) + 2(1-\alpha)$:

$$\begin{aligned} W^*(x) &\equiv \frac{x}{2} = (1-p-p_\alpha) F^*(x) + p + p_\alpha \\ &\Leftrightarrow \\ F^*(x) &= \frac{x/2 - (p+p_\alpha)}{(1-p-p_\alpha)}. \end{aligned}$$

The continuity of F^* yields:

$$F^*(K_1) = F^*[K_3].$$

leading to a first equilibrium condition:

$$K_1 = K_3 - 2(p+p_\alpha). \tag{1}$$

A second equilibrium condition is derived from the budget constraint associated with full redistribution:

$$\int_0^{K_1} x dF^*(x) + \int_{K_3}^2 x dF^*(x) = 1,$$

which yields, because $f(x) = \frac{1}{2(1-p-p_\alpha)}$:

$$(K_1)^2 - (K_3)^2 = -4(p+p_\alpha). \tag{2}$$

Together, conditions (1) and (2) yield:

$$\begin{cases} K_1 = 1 - p - p_\alpha \\ K_3 = 1 + p + p_\alpha \end{cases}. \tag{3}$$

Partial redistribution

There are also two conditions associated with the candidates' offer of partial redistribution with funds worth $(1-\alpha)$.

Remember that F_α^* is the cdf in terms of *utils* each voter gets from partial redistribution.

Looking at the graph above, the support of F_α^* is $[G(\alpha), K_2] \cup [G, K_3]$. Our fist task is to find the slope of the equilibrium probability of winning votes W^* when choosing partial redistribution. Looking at the graph, this slope s is such that:

$$s\left(K_3 - G(\alpha)\right) = p + p_\alpha$$

Thus, the slope we are looking for is given by:

$$s = \frac{p + p_\alpha}{K_3 - G(\alpha)}$$

Writing down the equilibrium probability of winning a vote with an offer in $[G(\alpha), K_2]$ and $[G, K_3]$ respectively, we have, for all $x \in [G(\alpha), K_2]$:

$$\begin{aligned} W^*(x) &\equiv s\left(x - G(\alpha)\right) = \left(1 - p - p_\alpha\right) F^*(K_1) + p_\alpha F_\alpha^*(x) \\ &\Leftrightarrow \\ F_\alpha^*(x) &= \frac{s\left(x - G(\alpha)\right) - \left(1 - p - p_\alpha\right) F^*(K_1)}{p_\alpha}, \end{aligned}$$

and for all $x \in [G, K_3]$:

$$\begin{aligned} W^*(x) &\equiv s\left(x - G(\alpha)\right) = \left(1 - p - p_\alpha\right) F^*(K_1) + p + p_\alpha F_\alpha^*(x) \\ &\Leftrightarrow \\ F_\alpha^*(x) &= \frac{s\left(x - G(\alpha)\right) - \left(1 - p - p_\alpha\right) F^*(K_1) - p}{p_\alpha}. \end{aligned}$$

The continuity of F_α^* implies:

$$F_\alpha^*(K_2) = F_\alpha^*(G),$$

that is:

$$\begin{aligned} K_2 - G(\alpha) &= G - G(\alpha) - \frac{p}{s} \\ &= G - G(\alpha) - \left(K_3 - G(\alpha)\right)\frac{p}{p + p_\alpha}. \end{aligned} \tag{4}$$

The budget constraint associated with partial redistribution yields:

$$\int_{G(\alpha)}^{K_2} x dF_{\alpha}^{*}(x) + \int_{G}^{K_3} x dF_{\alpha}^{*}(x) = G(\alpha) + (1-\alpha)$$

That is, because $f(x) = \frac{p+p_\alpha}{p_\alpha (K_3 - G(\alpha))}$:

$$(K_2 - G(\alpha))^2 + (K_3 - G(\alpha))^2 - (G - G(\alpha))^2 = \frac{2p_\alpha (1-\alpha)(K_3 - G(\alpha))}{p+p_\alpha}. \qquad (5)$$

The vote shares at equilibrium are given by:

$$\begin{aligned} S_{redist} &= 1/2(1-p-p_\alpha) + p(1-F^{*}(K_3)) + p_\alpha(1-F^{*}(K_3)) \\ S_G &= (1-p-p_\alpha) F^{*}(K_1) + 1/2p + p_\alpha F_\alpha^{*}(K_2) \\ S_{G_\alpha} &= (1-p-p_\alpha) F^{*}(K_1) + p(1-F_\alpha^{*}(K_2)) + 1/2p_\alpha \end{aligned}$$

and must all be equal to 1/2. They imply that:

$$\begin{aligned} F^{*}(K_3) &= 1/2 \\ F_\alpha^{*}(K_2) &= 1/2. \end{aligned}$$

This implies in turn that the distance between $G(\alpha)$ and K_2 is equal to that between G and K_3. We thus have:

$$K_2 - G(\alpha) = K_3 - G. \qquad (6)$$

The equilibrium is thus characterized by the system of five equations in five unknowns:

$$\begin{cases} K_1 = 1 - p - p_\alpha, \\ K_3 = 1 + p + p_\alpha, \\ K_2 - G(\alpha) = K_3 - G, \\ K_2 - G(\alpha) = G - G(\alpha) - (K_3 - G(\alpha))\dfrac{p}{p+p_\alpha}, \\ (K_2 - G(\alpha))^2 + (K_3 - G(\alpha))^2 - (G - G(\alpha))^2 = \dfrac{2p_\alpha(1-\alpha)(K_3 - G(\alpha))}{p+p_\alpha}. \end{cases},$$

The solution of the system is:

$$\begin{cases} K_1 = 2\alpha - G(\alpha), \\ K_2 = 2 - G - 2\alpha(1 - G(\alpha)), \\ K_3 = G(\alpha) + 2(1-\alpha), \\ p = \frac{(2\alpha - G(\alpha) - 1)(G + \alpha - G(\alpha) - 1)}{\alpha - 1}, \\ p_\alpha = \frac{(2\alpha - G(\alpha) - 1)(G + 2\alpha - G(\alpha) - 2)}{1-\alpha}. \end{cases}$$

Further, against a candidate who plays a fully mixed strategy, the distributions $F^*$and F^*_α identified above are the best ones by construction (because they are the unique solution to the player's maximization problem). Finally, because, by construction, $F^*$and F^*_α are the best cdf's one can use against an opponent using a fully mixed strategy, a deviation to a pure strategy or to a partially mixed strategy when you know your opponent is using a fully mixed strategy must yield a lower payoff. Hence, we have a fully mixed strategy Nash equilibrium. This implies too that, if the equilibrium exists (we identify existence conditions below), it is also unique in this class of equilibria.

Conditions for existence of a three-option equilibrium

For what parameter values does the above equilibrium exist? First, we need:

$$p = \frac{(2\alpha - G(\alpha) - 1)(G + \alpha - G(\alpha) - 1)}{\alpha - 1} > 0,$$

which requires $G + \alpha - G(\alpha) - 1 > 0$, since $\alpha - 1 < 0$ and $2\alpha - G(\alpha) - 1 < 0$. We thus need $G(\alpha) - \alpha < G - 1$. This is a measure of how efficient the public good system must be. This just boils down to a condition that states that the public good option is more efficient than partial redistribution in terms of the total size of the pie.

Second, we need:

$$p_\alpha = \frac{(2\alpha - G(\alpha) - 1)(G + 2\alpha - G(\alpha) - 2)}{1 - \alpha} > 0$$

that is:

$$G < G(\alpha) + 2(1 - \alpha).$$

This condition states that the public good must not be too efficient compared to $G(\alpha)$.

Finally, we need $K_3 \leq 2$, that is:

$$G(\alpha) + 2(1 - \alpha) \leq 2.$$

This conditions states that $G(\alpha)$ must not be too efficient compared to full redistribution. Note that combining the last two conditions also gives us the usual condition

$G \leq 2$, that states that the public good is not too efficient with respect to full redistribution.

To summarize,

Proposition 1 *Under proportional representation, when parameters are such that:*

$$\begin{cases} \alpha < G(\alpha) < 2\alpha \\ 1 \leq G \leq 2 \\ G(\alpha) + (1-\alpha) < G < G(\alpha) + 2(1-\alpha) \end{cases}$$

there exists a unique mixed strategy Nash equilibrium in which the candidates randomize between the three options. This equilibrium is defined by:

$$\begin{cases} K_1 = 2\alpha - G(\alpha), \\ K_2 = 2 - G - 2\alpha(1 - G(\alpha)), \\ K_3 = G(\alpha) + 2(1-\alpha), \\ p = \frac{(2\alpha - G(\alpha) - 1)(G + \alpha - G(\alpha) - 1)}{\alpha - 1}, \\ p_\alpha = \frac{(2\alpha - G(\alpha) - 1)(G + 2\alpha - G(\alpha) - 2)}{1-\alpha}. \end{cases}$$

3.2 Welfare considerations at the fully mixed strategy equilibrium

The welfare criterion we use is the total average utility in the economy. Given our assumptions, the welfare is maximized at value G when the public good is provided and is minimized at 1 when full redistribution only occurs.

Given the equilibrium strategies of candidates, the welfare is equal to:

$$p(G-1) + p_\alpha(G(\alpha) - \alpha) + 1$$

We then have:

Proposition 2 *Welfare increases as the partial public good $G(\alpha)$ becomes more efficient.*

Proof. Direct differentiation of social welfare with respect to $G(\alpha)$ yields:

$$\frac{\partial}{\partial G(\alpha)}\left(p(G-1)+p_\alpha(G(\alpha)-\alpha)+1\right)=\frac{\partial p}{\partial G(\alpha)}(G-1)+\frac{\partial p_\alpha}{\partial G(\alpha)}(G(\alpha)-\alpha)+p_\alpha$$

$$\begin{aligned}
&= \frac{2g-G+3\alpha+2}{\alpha-1}(G-1)+\frac{2g-G-4\alpha+3}{1-\alpha}(g-\alpha)+\frac{(2\alpha-g-1)(G+2\alpha-g-2)}{1-\alpha} \\
&= -(G-3g+4\alpha-2)\frac{G-g+2\alpha-2}{\alpha-1}>0.
\end{aligned}$$

■

3.3 Other Equilibria

We now turn to values of the parameters for which the strategies we described in which candidates randomize between the three strategies do not constitute an equilibrium. In these cases, we show that equilibria boil down to equilibria derived by Myerson (1993)or Lizzeri and Persico (2000 and 2001) in which one or two options are used.

3.3.1 Equilibria with 2 options

There are three possible 2-option equilibria. Let us pin down the conditions for their existence.

Starting with the 2-option equilibrium between the public good and full redistribution of Lizzeri and Persico (2001), for that equilibrium to exist, we need, obviously, $1<G<2$. We also need the public good to be relatively efficient compared to the option of providing the partial public good, that is, we need $G>G(\alpha)+2(1-\alpha)$. Indeed, in this case, offering the public good dominates offering the partial public good as G beats the partial public good even when one makes the best possible offer to a majority of the voters $(G(\alpha)+2(1-\alpha))$.

Finally, notice that, if both these conditions are met, then one of the existence conditions associated to the 3-option equilibrium is violated. Thus, whenever the 2-option equilibrium of Lizzeri and Persico exists, the 3-option equilibrium cannot exist and vice-versa.

Graphically, Lizzeri and Persico's 2-option equilibrium looks as follows.

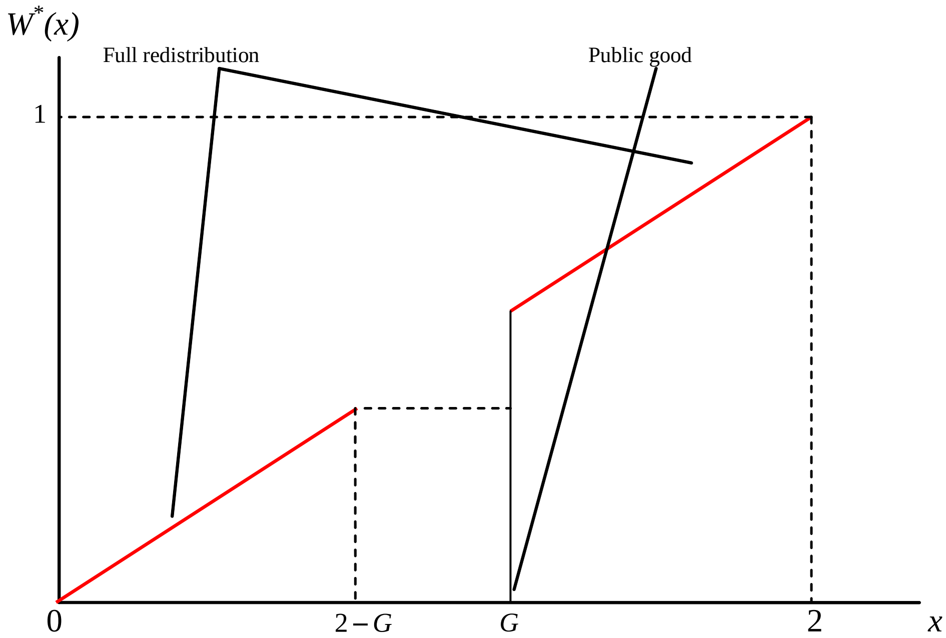

Figure 2: :public good and full redistribution

For an equilibrium in which only the partial public good and full redistribution are used, we need the partial public good to be efficient enough compared to G, so that relying on G is suboptimal: $G(\alpha) + (1 - \alpha) > G$. We also need the partial public good to be efficient but not too much, so that full redistribution is still used in equilibrium: $G(\alpha) + 2(1 - \alpha) < 2$.This equilibrium can be represented graphically as follows:

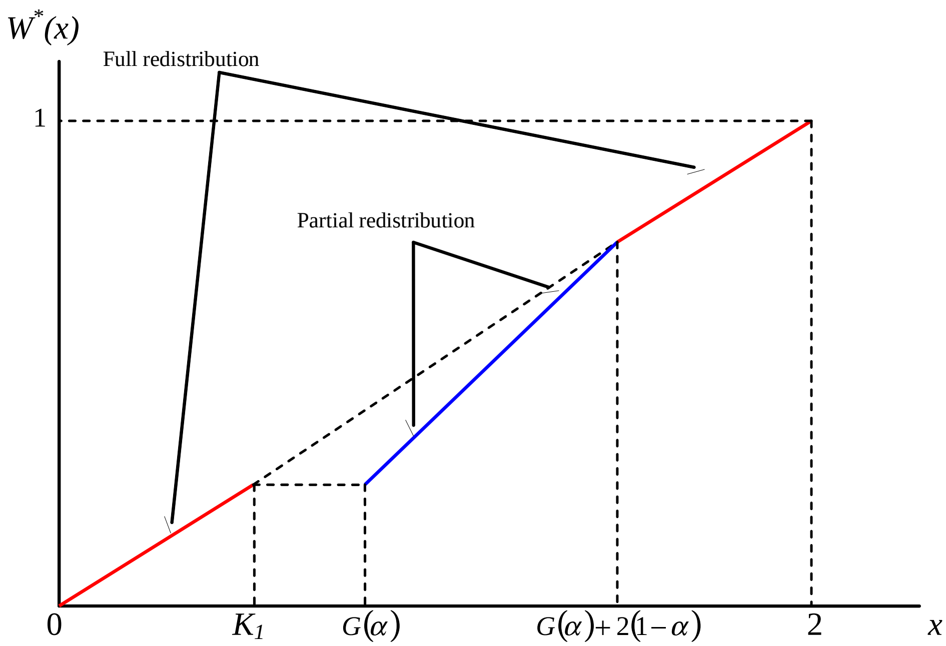

Figure 3: partial and full redistribution

For an equilibrium in which only the full and the partial public good are used, we need both G and $G(\alpha)$ to be valuable enough, namely we need $G > 1$ and

$$G(\alpha) + (1-\alpha) < G < 2 < G(\alpha) + 2(1-\alpha),$$

so that choosing full redistribution is suboptimal.

This equilibrium can be represented graphically as follows:

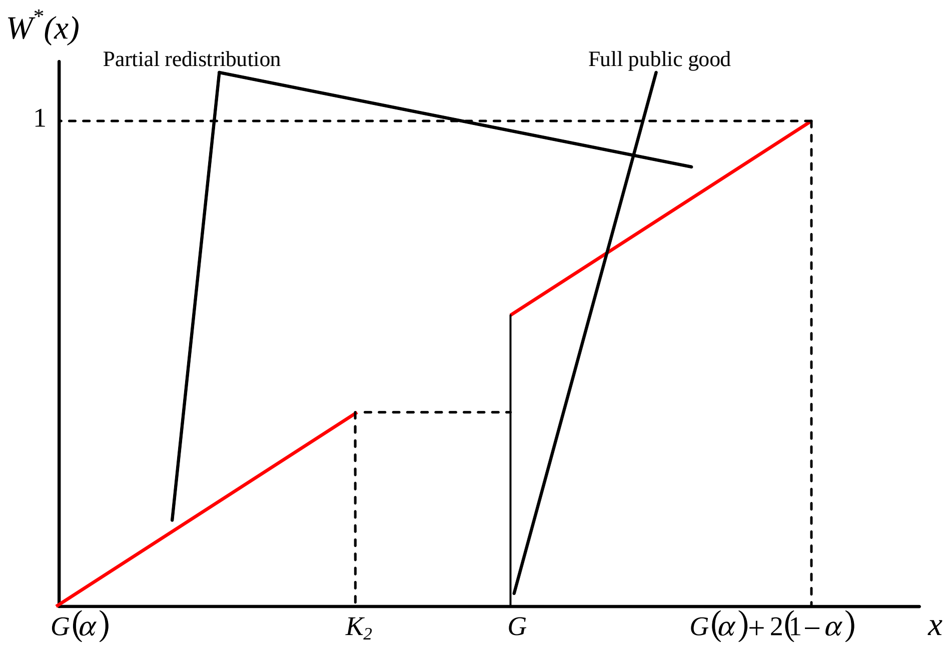

Figure 3: public good and partial redistribution

3.3.2 1-option equilibria

Playing G only is an equilibrium if and only if $G > G(\alpha) + 2(1-\alpha)$ and $G > 2$. The graphical representation of this equilibrium is obvious.

Playing full redistribution only is an equilibrium if and only if $G < 1$ and $G(\alpha) + (1-\alpha) < 2$. This equilibrium can be represented as follows:

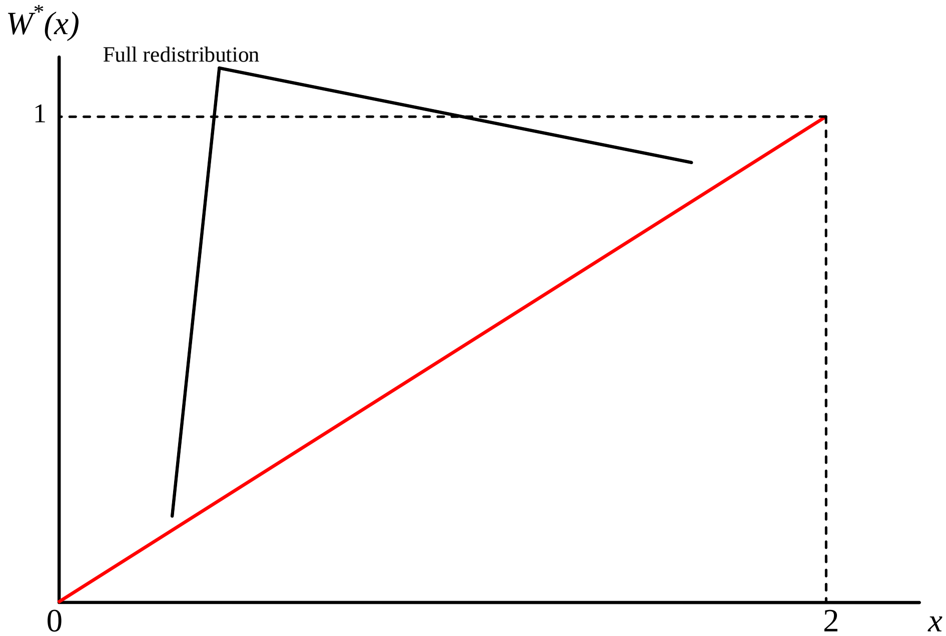

Figure 4: full redistribution

Playing the partial public good only is an equilibrium if and only if $G(\alpha)+(1-\alpha)>2$ and $G(\alpha)+2(1-\alpha)>G$. The graphical representation of this equilibrium is the same as the last one above, with the only difference that the support of $W^*(x)$ is $[G(\alpha), G(\alpha)+2(1-\alpha)]$ instead of $[0,2]$.

4 Winner-take-all election - no equilibrium

We now prove that there is no equilibrium for values of the parameter for which the three options are used under PR.

Starting with the candidate fully mixed strategy equilibrium, suppose by way of contradiction that, in that equilibrium, candidates use the public good with probability $p>0$, partial redistribution with probability $p_a>0$ and full redistribution with probability $q=1-p-p_\alpha>0$. We know that, whenever candidates redistribute, either fully or partially, they must make use in equilibrium of the distributions F and F_α identified above.

Given this, a candidate who redistributes fully can easily beat the other candidate if he knows the other candidate promises either the full public good or the intermediate one. Suppose first that $q < 1/2$; given that $G < 2$, it is possible to promise $G+\varepsilon$ (with ε small enough) to more than 50% of the voters using full redistribution. With this deviation, the candidate would win the election whenever the other candidate uses the public good or partial redistribution options, which happens with probability greater than 50%. Thus, in equilibrium, we need $q \geq 1/2$. Now, suppose $q > 1/2$, then a candidate using partial redistribution could deviate to promising 2 to as many voters as possible, which would insure his victory whenever the other candidate uses full redistribution. This leads to an immediate contradiction. Thus, in an equilibrium, we need $q = 1/2$.

We now prove that $p = 0$. The reason is that otherwise, a candidate could deviate using full redistribution, promising $G + \varepsilon$ to more than 50% of voters and promising 2 to as many voters as possible. This insures victory against the public good and full redistribution (and defeat against partial redistribution). We thus need $p = 0$.

We have thus proved that there can not be an equilibrium in a winner-take-all system in which the three options are used.

We now prove that there cannot be 2-option equilibria either. Suppose there were a 2-option equilibrium using partial and full redistribution only. Then the equilibrium would look like the one depicted in figure 3. But then deviating to offering the public good G would ensure victory against partial redistribution (since G is on the support of the partial redistribution option but is more efficient: $G(\alpha) + (1-\alpha) < G < G(\alpha) + 2(1-\alpha)$) and a tie against full redistribution. Thus there cannot be an equilibrium with partial and full redistribution only. Obviously, there cannot be a pure strategy Nash equilibrium either. We have thus proved that there is no equilibrium under a winner-take-all system.

Naturally, the two-option equilibrium of Lizzeri and Persico (2001) still exists, at least for the parameter configuration that is consistent with that equilibrium. Still, the fact that for a large set of parameter values there is no equilibrium under a winner-take-all system does point to the fact that this type of game is not well behaved when candidates maximize their probability of winning.

5 Conclusion

This paper extend Lizzeri and Persico (2001) to allow politicians to choose between several levels of public good provision. We show that how to construct equilibria under proportional representation and derive the welfare consequences of giving politicians more flexibility in their policy choices. We also show that under a winner take-all system the game does not have an equilibrium for values of the parameter for which the three option equilibrium under proportional representation exists.

One avenue for further research arises naturally from this last finding. Indeed, it would be worthwhile to check whether this non-existence result is specific to the continuous Colonel Blotto games considered in this paper or whther it extends also to discrete versions of this game, as analyzed by Laslier and Picard (2002) and Roberson (2006).

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7 APPENDIX

Proof of the linearity of W^* and characterization of equilibrium

To characterize the equilibrium, it is important to analyze the function $W^*(x)$ that gives the equilibrium probability of winning a vote when one promises x dollars to a voter. The function $W^*(x)$ is represented on figure 1. The first characteristic is that $W^*(x)$ is piece-wise linear with a constant slope on the support of full redistribution and another constant slope on the support of redistribution on top of the partial public good.

This linearity comes from a simple intuition. When allocating monetary promises, a candidate must equalize the marginal benefits and costs of allocation an extra dollar to a given voter. Since marginal costs are linear, marginal benefits must also be linear. This intuition was pointed out in Lizzeri and Persico (2000).

Let's call S the support of promises when there is full redistribution. In that case, a candidate chooses F :

$$Max_F \int_S W^* (x)\, dF(x) \text{ s.t. } \int_S x dF(x) = 1.$$

The associated Lagrangian is:

$$L = \int_S (W^* (x) + \lambda (1 - x))\, dF(x).$$

F is optimal if $W^* (x) + \lambda (1 - x)$ is maximal and constant on S. This directly proves the linearity of W^* on S.

Similarly, let's call S_α the support of promises when there is partial redistribution. In that case, a candidate chooses F_α :

$$Max_{F_\alpha} \int_{S_\alpha} W^* (x)\, dF_\alpha (x) \text{ s.t. } \int_{S_\alpha} (x - G(\alpha))\, dF_\alpha (x) = 1 - \alpha.$$

The associated Lagrangian is:

$$L = \int_{S_\alpha} (W^* (x) + \lambda_\alpha (1 - \alpha + G(\alpha) - x))\, dF_\alpha (x).$$

F_α is optimal if $W^* (x) + \lambda_\alpha (1 - \alpha + G(\alpha) - x)$ is maximal and constant on S_α. This directly proves the linearity of W^* on S_α.

We now need to prove that S is composed of two intervals $[0, K_1]$ and $[K_3, 2]$ and S_α is composed of two intervals $[G(\alpha), K_2]$ and $[G, K_3]$.

The minimal and maximal promise of full redistribution are easy to establish. It cannot be that the minimal promise is strictly larger than zero since this promise would win with probability zero at a strictly positive cost. The largest promise needs to be equal to 2. Using full redistribution it is possible to offer 2 to exactly 50% of the voters. If the maximal promise was smaller than 2, using full redistribution would insure a vote share of more than 50%. If it was larger than 2, using full redistribution would lead to a vote share smaller than 50%. Both situations are not consistent with equilibrium. This also gives us that $\lambda = 1/2$, that is the slope of the function W^* on the support of full redistribution.

Let's prove next that the intervals of promises under full redistribution and partial redistribution cannot overlap. The reason is the following. Suppose there exists promises $x, y \geq G(\alpha)$ in the support of both redistribution functions. We have that $W^*(G(\alpha)) < G(\alpha)/2$. Now, given the linearity of the W^* function on S, it has to be that $W^*(x) = 2x$ and $W^*(y) = 2y$. This leads to a contradiction since $W^*(G(\alpha))$ should be on the same line. This proves that S and S_α do not overlap and that $W^*(x) \leq 2x$ on S_α. This also proves that the slope of W^* on S_α is larger than 1/2.

W^* can only be discontinuous at G, otherwise it would pay to increase the offer to get a discrete increase in the probability of winning a vote at a negligible cost. This means that the largest promise used with partial redistribution is also used in the full redistribution.

We thus get that S is composed of two intervals $[0, K_1]$ and $[K_3, 2]$ and S_α is composed of two intervals $[G(\alpha), K_2]$ and $[G, K_3]$ as depicted on figure 1.