A War of Attrition with endogenous effort levels: Additional Appendix

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1. Two-sided Incomplete Information

Assume that player 1 has valuation 1 (player 1_h) with probability μ_1, and valuation λ otherwise (1_l). Player 2 has valuation 1 with probability μ_2 (player 2_h) and valuation λ otherwise (2_l). To avoid trivialities, assume that $1 > \lambda \neq \delta$, $\lambda \neq \delta / (1 + \delta)$. We denote by V_i the payoff of player i (the subgame considered is clear from the context) when it is his turn to offer, and W_i his payoff when it is his turn to cover. Two additional definitions are convenient. A bid b by player i is *p-partially deterrent* (p-P.D.) if $-i$ covers with positive probability $p \in (0, 1)$ if he is of the low type, and covers with probability one otherwise. Similarly, a bid b by player i is *p-fully deterrent* (*pF.D.*) if $-i$ covers with positive probability $p \in (0, 1)$ if he is of the high type, and quits with probability one otherwise.

To narrow down the set of equilibrium candidates, we proceed in steps:

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1.1. Narrowing down the set of possible equilibria

1. **player** i_h $(i \in \{1,2\})$ **does not submit a separating** p-P.D. **or** p-F.D. **bid** b: Consider first the case of a p-P.D. bid b. Suppose i_h submits such a separating bid. Observe that, as this bid is such that $-i_h$ is indifferent between covering or not, any bid $b' > b$ is F.D., if it is believed to be submitted by i_h only. Let β be the minimal bid i_l would be willing to submit instead any of his equilibrium bid, assuming that β is F.D.; then i_h must gain by bidding $(\beta \vee b) + \varepsilon$, for $\varepsilon > 0$ small enough: it is consistent for $-i$ to believe that such a bid is submitted only by i_h, and rational, given this belief, for $-i$ to quit with probability one. If i_l is indifferent between his equilibrium bid and β, then i_h strictly gains by submitting $\beta \vee b + \varepsilon$.

   Consider next a separating p-P.D. bid b. Let q_h be the total discounted probability of winning associated with the bid b (or the maximum thereof, if i_h may randomize over bids in future periods). Let q_l be the maximal total discounted probability of winning of player i_l (over his equilibrium actions). Because of single-crossing, $q_h \geqq q_l$. Observe from the analysis when information is one-sided that, after the bid b, player $-i_h$ will submit a bid inducing player i_h to quit. Observe also, from the same analysis, that if b is such that $-i_l$ is indifferent between covering or not, any bid $b' > b$ is P.D. if it is believed to be submitted by i_h only. Letting β be the minimal bid i_l would be willing to submit if it were P.D. and interpreted as being submitted by i_h only, the same argument as before establishes that bidding $(\beta \vee b) + \varepsilon$ is a profitable deviation for i_h.

2. **Player** i_l **does not submit a separating** $p-$**F.D. or F.D. bid** b: suppose this bid b is p-F.D.: since this bid b must equal the discounted payoff of $-i_h$ in the game of complete information (which is of course independent of p), any bid $b+\varepsilon$, whether it is interpreted as being submitted by i_l or not, is F.D.; therefore, such a bid constitutes a profitable deviation (whether or not it is a profitable

deviation for i_h as well). Suppose now that this bid is F.D.; then i_h must also submit an F.D. bid $b' \neq b$, because of the single-crossing property. However, it follows from the analysis under complete or one-sided incomplete information that $b' \leqq b$, so that i_l has a strict incentive to bid b' rather than b.

3. **Player i_h does not submit a $p-$P.D. bid** b: In view of 1., it is enough to show that no such a bid b can be submitted by both i_h and i_l with positive probability. Suppose such a bid b did exist, and let q be the total, discounted probability of winning of i_h among all continuation strategies of i_h specifying the bid b. Suppose first that $q < \mu_{-i}$. Observe that any bid $b' > b$, if perceived as coming from i_h uniquely, is P.D. (or even p-F.D. or F.D.). Pick the smallest bid b' which makes i_l indifferent between bidding b (and following his continuation strategy) and bidding b', being perceived in the latter case as i_h. Bidding $(b \vee b') + \varepsilon$ is a profitable deviation for i_h, since it is consistent, by construction, for $-i$ to believe that such a bid is submitted by i_h only. If $q \geqq \mu_{-i}$, the same construction applies, with b' being the minimal F.D. bid which makes i_l indifferent between bidding b (and following his continuation strategy) and bidding b', being perceived in the latter case as i_h.

4. **Player i does not submit a p-F.D. bid** b: In view of 1. and 2., it is sufficient to show that there is no p-F.D. bid submitted by both types of player i. The argument, analogous to 2., is omitted.

   The previous arguments establish that play must finish in finite time (Player cannot submit a pooling bid 0 twice in a row, in periods t and $t+1$, as otherwise there would be a profitable deviation in period t, replicating the behavior in behavior $t+2$). Although we have not ruled out so far a separating bid $p-$P.D. to do so, it is easy to derive a contradiction for every parameter region considered below. In view of this, only finitely many possibilities remain, successively analyzed in what follows.

The parameter regions identified in the case of one-sided incomplete information play an important role in what follows, and are referred to as regions I, II, III, IV and V (see the table and figure of the one-sided case). We first determine the sequential equilibria which are not defeated by some strategy profile (and beliefs), without verifying whether this strategy profile is itself part of a sequential equilibrium. This procedure is already lengthy enough, it is simple, and it turns out that there are only two instances in which the argument uses a strategy profile which is not itself part of a sequential equilibrium. In a second step, we summarize the results obtained, and characterize the (undefeated) equilibria for all parameters.

1.2. Separating Equilibria

1.2.1. High type plays Full Deterrence (F.D.), low type plays No Deterrence (0D.)

Let b_1^l and b_1^h be the bids of player 1, V_{1_l} and V_{1_h} their payoffs. Then

$$\begin{aligned} b_1^l &= 0, b_1^h = \frac{\delta}{1+\delta}, \\ V_{1_l} &= 0, V_{1_h} = \frac{1}{1+\delta}. \end{aligned}$$

The first obvious condition for this to be an equilibrium is that the low type has no interest in mimicking the high type or: $\lambda \leq \frac{\delta}{1+\delta}$.

Let's now check possible deviations: Let φ be the belief of player 2 (that player 1 is of high type) upon observing a deviation.

Deviation to Partial Deterrence Since $\lambda \leq \frac{\delta}{1+\delta}$ is needed, we only have to consider two zones of parameters for deviations.

Case 1: (λ, μ_1) belong to zone I

- $\varphi = \mu_1$:

  The deviating bid is $\hat{b}_1 = \delta (1 - \mu_1) \lambda$. For the beliefs to be consistent both types must have an incentive to deviate :

$$(1 - \mu_2) \lambda - \hat{b}_1 \geq V_{1_l},$$
$$(1 - \mu_2) - \hat{b}_1 + \mu_2 \frac{\delta^2}{1 + \delta} \geq V_{1_h}.$$

To rule out this deviation, it must be the case that at least one constraint is violated. Hence a necessary condition for equilibrium existence is that:

$$\mu_2 > \min \left(1 - \delta (1 - \mu_1), \frac{\delta (1 - \lambda (1 - \mu_1)(1 + \delta))}{1 + \delta - \delta^2} \right).$$

- $0 < \varphi < \mu_1$:

  The deviating bid is $\hat{b}_1 = \delta (1 - \varphi) \lambda$. For the beliefs to be consistent, the low type must prefer the deviation while the high type is indifferent:

$$(1 - \mu_2) \lambda - \delta (1 - \varphi) \lambda \geq 0,$$
$$(1 - \mu_2) - \delta (1 - \varphi) \lambda + \delta^2 \frac{\mu_2}{1 + \delta} = \frac{1}{1 + \delta}.$$

  Solving for $\delta (1 - \varphi) \lambda$ and plugging in the first constraint we get:

$$(1 - \mu_2)(\lambda - 1) - \delta^2 \frac{\mu_2}{1 + \delta} + \frac{1}{1 + \delta} \geq 0$$
$$\text{or } \mu_2 \geq \frac{(1 - \lambda)(1 + \delta) - 1}{(1 - \lambda)(1 + \delta) - \delta^2}.$$

  We also need to make sure that the beliefs for which these incentive constraints are satisfied belong to the zone we are analyzing:

$$0 < \varphi < \mu_1$$
$$\Rightarrow \delta (1 - \mu_1) \lambda < \delta (1 - \varphi) \lambda < \delta \lambda$$
$$\Rightarrow \delta (1 - \mu_1) \lambda < \frac{\delta}{1 + \delta} - \mu_2 + \delta^2 \frac{\mu_2}{1 + \delta} < \delta \lambda.$$

Hence we get the following constraint:

$$\frac{\delta\left(1-\lambda\left(1+\delta\right)\right)}{1+\delta-\delta^2}<\mu_2<\frac{\delta\left(1-\lambda\left(1+\delta\right)\left(1-\mu_1\right)\right)}{1+\delta-\delta^2}.$$

- $\varphi = 0$:

  This means that only the low types deviate to this partially deterrent bid. Partial deterrence then implies that $\hat{b}_1 = \delta\lambda$.

  Necessary and sufficient conditions for this deviation are:

$$\begin{gathered}\left(1-\mu_2\right)\lambda-\delta\lambda\geq 0,\\ \left(1-\mu_2\right)-\delta\lambda+\delta^2\mu_2(1-\delta)\leq\frac{1}{1+\delta}.\end{gathered}$$

  Hence, the conditions for this deviation are :

$$\begin{gathered}\mu_2\leq 1-\delta,\\ \mu_2\geq\frac{\delta\left(1-\lambda\left(1+\delta\right)\right)}{\left(1+\delta\right)\left(1-\delta^2\left(1-\delta\right)\right)}.\end{gathered}$$

- $\frac{\delta}{1+\delta} > \varphi > \mu_1$:

  The deviating bid is $\hat{b}_1 = \delta\left(1-\varphi\right)\lambda$. Low types have to be indifferent between the deviation and their original bid, and the high types prefer to deviate:

$$\begin{aligned}\left(1-\mu_2\right)\lambda-\delta\left(1-\varphi\right)\lambda &= 0,\\ \left(1-\mu_2\right)-\delta\left(1-\varphi\right)\lambda+\delta^2\frac{\mu_2}{1+\delta} &\geq \frac{1}{1+\delta}.\end{aligned}$$

  Solving for $\delta\left(1-\varphi\right)\lambda$ and plugging in the second constraint, we get:

$$\mu_2\leq\frac{\left(1-\lambda\right)\left(1+\delta\right)-1}{\left(1-\lambda\right)\left(1+\delta\right)-\delta^2}.$$

  We also need that φ lies in the interval we are analyzing:

$$1-\delta\left(1-\mu_1\right)<\mu_2<\frac{1}{1+\delta}.$$

- $\varphi = \frac{\delta}{1+\delta}$:

  The deviating bid is $\hat{b}_1 = \delta(1-\varphi)\lambda = \frac{\delta\lambda}{1+\delta}$.We are now on the frontier between Zone I and Zone II. Hence, the high type of the second player is indifferent between partial and full deterrence after covering the deviating bid. Let's call ψ the probability with which he chooses a fully deterrent bid.

  The incentive constraint for this deviations is for the low type :

  $$(1-\mu_2)\lambda - \frac{\delta\lambda}{1+\delta} = 0.$$

  This does not depend on ψ and can not occur but for non-generic values of the parameters.

- $1 > \varphi > \frac{\delta}{1+\delta}$:

  Low types have to be indifferent and the high types prefer the deviating bid:

  $$(1-\mu_2)\lambda - \delta(1-\varphi)\lambda = 0,$$
  $$(1-\mu_2) - \delta(1-\varphi)\lambda \geq \frac{1}{1+\delta}.$$

  We need that φ lies in the interval we are analyzing:

  $$1 > \varphi > \frac{\delta}{1+\delta}$$
  $$\Rightarrow \frac{1}{1+\delta} < \mu_2 < 1.$$

  Now the second constraint can be written:

  $$\mu_2 < 1 - \frac{1}{(1+\delta)(1-\lambda)}.$$

  but that is never the case since $\frac{1}{1+\delta} > 1 - \frac{1}{(1+\delta)(1-\lambda)}$.

- $\varphi = 1$:

  Only the high types deviate in this case. But then the deviating bid would be zero which would lead the low type to strictly prefer the deviation.

Case 2: (λ, μ_1) belong to zone II The deviating bid to P.D. is $\hat{b}_1 = \delta\,(1-\varphi)\,\lambda$.

- $\varphi = \mu_1$:

  The deviating bid is $\hat{b}_1 = \delta\,(1-\mu_1)\,\lambda$. For beliefs to be consistent, we need both types to be better off.

$$(1-\mu_2)\,\lambda - \delta\,(1-\mu_1)\,\lambda \geq 0,$$
$$(1-\mu_2) - \delta\,(1-\mu_1)\,\lambda \geq \frac{1}{1+\delta}.$$

  Hence this deviation is not possible if,

$$\mu_2 > \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1).$$

- $\frac{\delta}{1+\delta} < \varphi < \mu_1$:

  This requires the high type to be indifferent and the low type to prefer the deviation:

$$(1-\mu_2) - \delta\,(1-\varphi)\,\lambda = \frac{1}{1+\delta},$$
$$(1-\mu_2)\,\lambda - \delta\,(1-\varphi)\,\lambda \geq 0.$$

  Combining them we get:

$$(1-\mu_2)\,(1-\lambda) \leq \frac{1}{1+\delta},$$
$$or\ \mu_2 \geq \frac{(1-\lambda)\,(1+\delta)-1}{(1-\lambda)\,(1+\delta)}.$$

  We also need the beliefs φ to remain in the zone of analysis:

$$\frac{\delta}{1+\delta} < \varphi < \mu_1$$
$$\Rightarrow \delta\lambda\,(1-\mu_1) < \delta\,(1-\varphi)\,\lambda < \frac{\delta\lambda}{1+\delta}$$
$$\Rightarrow \frac{\delta\,(1-\lambda)}{1+\delta} < \mu_2 < \frac{\delta\,(1-\lambda\,(1+\delta)\,(1-\mu_1))}{1+\delta}.$$

- $\varphi = \frac{\delta}{1+\delta}$:

  Those beliefs are at the frontier between zone I and zone II. As a consequence, the second player, upon observing the deviating bid, is indifferent between full and partial deterrence. Let us denote by ψ the probability with which he chooses full deterrence.

  The low types must prefer the deviating bid and the high types must be indifferent:

$$\left(1-\mu_2\right)\lambda - \frac{\delta}{1+\delta}\lambda \geq 0,$$
$$\left(1-\mu_2\right) - \frac{\delta}{1+\delta}\lambda + \left(1-\psi\right)\delta^2 \frac{\mu_2}{1+\delta} = \frac{1}{1+\delta}.$$

  Since the $0 < \psi < 1$, we get:

$$\frac{\delta\left(1-\lambda\right)}{1+\delta} < \mu_2 < \frac{\delta\left(1-\lambda\right)}{1+\delta-\delta^2},$$
$$\text{and } \mu_2 \leq \frac{1}{1+\delta} \text{ from the low type's constraint.}$$

- $0 < \varphi < \frac{\delta}{1+\delta}$:

  The deviating bid is of $\delta\left(1-\varphi\right)\lambda$. The low types prefer the deviation while the high types are indifferent:

$$\left(1-\mu_2\right)\lambda - \delta\left(1-\varphi\right)\lambda \geq 0,$$
$$\left(1-\mu_2\right) - \delta\left(1-\varphi\right)\lambda + \delta^2 \frac{\mu_2}{1+\delta} = \frac{1}{1+\delta}.$$

  Combining the constraints yields:

$$\mu_2 \geq \frac{\left(1-\lambda\right)\left(1+\delta\right)-1}{\left(1-\lambda\right)\left(1+\delta\right)-\delta^2}.$$

We also need the beliefs φ to remain in the zone of analysis:

$$0 < \varphi < \frac{\delta}{1+\delta}$$

$$\Leftrightarrow \frac{\delta\lambda}{1+\delta} < \delta\left(1-\varphi\right)\lambda < \delta\lambda$$

$$\Leftrightarrow \frac{\delta}{1+\delta} - \delta\lambda < \frac{\delta}{1+\delta} - \delta\left(1-\varphi\right)\lambda < \frac{\delta}{1+\delta} - \frac{\delta\lambda}{1+\delta}$$

$$\Leftrightarrow \frac{\delta - \delta\lambda\left(1+\delta\right)}{1+\delta-\delta^2} < \mu_2 < \frac{\delta\left(1-\lambda\right)}{1+\delta-\delta^2}.$$

- $\varphi = 0$:

  The deviation bid is $\delta\lambda$. The low type must prefer the deviation while the high type prefers his equilibrium bid:

$$\left(1-\mu_2\right)\lambda - \delta\lambda \geq 0$$

$$\Leftrightarrow \mu_2 \leq 1-\delta$$

$$\left(1-\mu_2\right) - \delta\lambda + \delta^2\mu_2\left(1-\delta\right) \leq \frac{1}{1+\delta}$$

$$\Leftrightarrow \mu_2 \geq \frac{\delta\left(1-\lambda\left(1+\delta\right)\right)}{\left(1+\delta\right)\left(1-\delta^2\left(1-\delta\right)\right)}.$$

- $1 > \varphi > \mu_1$:

  This requires the low type to be indifferent between the deviation and the equilibrium bid, while the high type strictly prefers the deviation:

$$\left(1-\mu_2\right) - \delta\left(1-\varphi\right)\lambda \geq \frac{1}{1+\delta},$$

$$\left(1-\mu_2\right)\lambda - \delta\left(1-\varphi\right)\lambda = 0.$$

  Combining them yields:

$$\mu_2 \leq \frac{\left(1-\lambda\right)\left(1+\delta\right)-1}{\left(1-\lambda\right)\left(1+\delta\right)} = 1 - \frac{1}{\left(1-\lambda\right)\left(1+\delta\right)}.$$

  We also need that beliefs be in the right interval:

$$1 \geq \varphi > \mu_1$$

$$\Rightarrow 1-\delta\left(1-\mu_1\right) < \mu_2 \leq 1.$$

But

$$1-\delta\left(1-\mu_1\right)\leq 1-\frac{1}{\left(1-\lambda\right)\left(1+\delta\right)}$$
$$\Leftrightarrow 1\leq\left(1-\lambda\right)\left(1+\delta\right)\delta\left(1-\mu_1\right).$$

This is not possible since $\mu_1\geq\frac{\delta}{1+\delta}$ implies that $\left(1-\mu_1\right)\left(1+\delta\right)\leq 1$.

- $\varphi=1$:

  Only the high types deviate in this case. But then the deviating bid would be zero which would lead the low type to strictly prefer the deviation.

1.2.2. High type plays Full Deterrence (F.D.), Low type plays Partial Deterrence (P.D.)

Bids reveal types perfectly. The bids are:

$$b_1^h=\frac{\delta}{1+\delta},b_1^l=\delta\left(\lambda-\delta\frac{\left(\lambda-\delta\right)^+}{1-\delta^2}\right)$$

For full deterrence, the high type needs to bid the discounted payoffs of the high type of the second player if he had covered, i.e. $\delta\left(\frac{1}{1+\delta}\right)$. Similarly, for partial deterrence, the low type has to bid the payoff of the low type of the second player who would mimic the high type and cover. His payoff would be λ minus the bid of the high type in this situation which is $\delta\frac{(\lambda-\delta)^+}{1-\delta^2}$.

Necessary and sufficient conditions for this to be an equilibrium are:

The high type prefers the fully deterrent bid to the partially deterrent bid:

$$\begin{aligned}\frac{1}{1+\delta} &\geq \left(1-\mu_2\right)-\delta\left(\lambda-\delta\frac{\left(\lambda-\delta\right)^+}{1-\delta^2}\right)+\delta\mu_2\left[-\delta\frac{\left(\lambda-\delta\right)^+}{1-\delta^2}+\delta\left(1-\delta\left(1-\delta\frac{\left(\lambda-\delta\right)^+}{1-\delta^2}\right)\right)\right]^+\\ &= \left(1-\mu_2\right)-\delta\left(\lambda-\delta\frac{\left(\lambda-\delta\right)^+}{1-\delta^2}\right)+\delta\mu_2\left[\delta\left(1-\delta\right)-\delta\left(\lambda-\delta\right)^+\right].\end{aligned}$$

The low type prefers the equilibrium bid to the fully deterrent bid and to a bid of zero:

$$(1-\mu_2)\,\lambda-\delta\left(\lambda-\delta\frac{(\lambda-\delta)^+}{1-\delta^2}\right)\geq 0,$$
$$(1-\mu_2)\,\lambda-\delta\left(\lambda-\delta\frac{(\lambda-\delta)^+}{1-\delta^2}\right)\geq \lambda-\frac{\delta}{1+\delta}.$$

The three conditions can be summarized as:

$$\mu_2\left[1-\delta^2\,(1-\delta)+\delta^2\frac{(\lambda-\delta)^+}{1-\delta^2}\right]\geq\frac{\delta}{1+\delta}-\delta\lambda+\delta^2\frac{(\lambda-\delta)^+}{1-\delta^2}.$$
$$\mu_2\lambda\leq\lambda\,(1-\delta)+\delta^2\frac{(\lambda-\delta)^+}{1-\delta^2}-\left(\lambda-\frac{\delta}{1+\delta}\right)^+.$$

1.2.3. High type plays Partial Deterrence (P.D.), Low type plays no deterrence (0D.)

Case 1 : (λ,μ_1) belong to zone I or II. Then the partially deterrent bid of the high type would be 0, since the low type of player 2 would have to bid $\frac{\delta}{1+\delta}$ in the continuation game to mimic high types, which is higher than their valuation. But then the low type would also play the partially deterrent bid.

Case 2 : (λ,μ_1) belong to zone III or IV. Then the partially deterrent bid of the high type would be $b_1^h=\delta\left(\lambda-\frac{\delta}{1+\delta}\right)$. Necessary conditions for this to be an equilibrium is that the high type does not want to deviate to full deterrence and that the low type prefers his bid of zero to the partially deterrent bid:

$$(1-\mu_2)-\delta\left(\lambda-\frac{\delta}{1+\delta}\right)\geq\frac{1}{1+\delta},$$
$$(1-\mu_2)\,\lambda-\delta\left(\lambda-\frac{\delta}{1+\delta}\right)\leq 0.$$

Combining these two inequalities yield:

$$(1-\mu_2)\,(1-\lambda)\geq\frac{1}{1+\delta}.$$

But we have in these zones:

$$\lambda \geq \frac{\delta}{1+\delta} \text{ or}$$
$$(1-\lambda) \leq \frac{1}{1+\delta} \text{ and}$$
$$(1-\mu_2)(1-\lambda) < \frac{1}{1+\delta}.$$

which leads to a contradiction.

1.3. Pooling Equilibria

1.3.1. Pooling, Full-Deterrence

Let b_1 denote the pooling bid made by player 1. Let $\hat{b}_1$ denote a deviating bid by player 1 and let φ be the belief of Player 2 (that Player 1 is of high type) upon observing $\hat{b}_1$. Finally let V_{1_l}, V_{1_h} be the equilibrium payoffs of player 1's low type and high type, respectively.

Case 1 and 2: (λ, μ_1) belong to zones I or II A fully deterrent bid is of at least $\frac{\delta}{1+\delta}$ which exceeds the valuation of a low type. So these cases are impossible

Case 3: (λ, μ_1) belong to zone III Then F.D. requires that $b_1 = \frac{\delta}{1+\delta}$. Payoffs are $V_{1_l} = \lambda - \frac{\delta}{1+\delta}$ and $V_{1_h} = \frac{1}{1+\delta}$.

In order to have an equilibrium we have to check possible deviations to a partially deterrent bid $\hat{b}_1$ with associated beliefs φ. Deviation to a non-deterrent bid can not be profitable.

- $\varphi = \mu_1$:

  We have $\hat{b}_1 = \delta \max\left\{\lambda - \frac{\delta}{1+\delta}, \lambda(1-\mu_1) - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right\}$. The value of the deviating bid depends on the sign of $\lambda\mu_1 - \delta(1-\lambda)$.

Now both the low type and the high type must prefer that bid:

$$\begin{aligned}(1-\mu_2)\,\lambda-\hat{b}_1 &\geq \lambda-\frac{\delta}{1+\delta},\\ (1-\mu_2)-\hat{b}_1 &\geq \frac{1}{1+\delta}.\end{aligned}$$

The second constraint implies the first, so we just need:

$$\mu_2 \leq \frac{\delta}{1+\delta}-\hat{b}_1.$$

- $\mu_1 < \varphi < 1$:

  The deviating bid is still $\hat{b}_1 = \delta \max\left\{\lambda-\frac{\delta}{1+\delta}, \lambda\,(1-\varphi)-\delta\left(\lambda-\frac{\delta}{1+\delta}\right)\right\}$. This would require the low type to be indifferent between b_1 and $\hat{b}_1$ while the high type prefers $\hat{b}_1$. Simple inspection of the incentives constraints rules out this possibility.

- $\varphi = 1$:

  Only the high type chooses to deviate to a partially deterrent bid which would be of $\hat{b}_1 = \lambda - \frac{\delta}{1+\delta}$. But now we need:

  $$\begin{aligned}(1-\mu_2)\,\lambda-\hat{b}_1 &\leq \lambda-\frac{\delta}{1+\delta},\\ (1-\mu_2)-\hat{b}_1 &\geq \frac{1}{1+\delta}.\end{aligned}$$

  which is not possible.

- $\varphi = 0$:

  We have $\hat{b}_1 = \delta(\lambda - \frac{\delta}{1-\delta^2}(\lambda-\delta)^+)$. So the value of the deviating bid depends on the sign of $\lambda - \delta$.

  The incentive constraints for the deviation to be profitable are:

  $$\begin{aligned}(1-\mu_2)\,\lambda-\hat{b}_1 &\geq \lambda-\frac{\delta}{1+\delta},\\ (1-\mu_2)-\hat{b}_1+\delta^2\mu_2\,(1-\delta) &\leq 1-\frac{\delta}{1+\delta}.\end{aligned}$$

We can rewrite them as:

$$\begin{aligned} \mu_2 \lambda &\leq \frac{\delta}{1+\delta} - \hat{b}_1, \\ \mu_2 &\geq \frac{\frac{\delta}{1+\delta} - \hat{b}_1}{1 - \delta^2 (1-\delta)}. \end{aligned}$$

If $\lambda \leq \delta$, $\hat{b}_1 = \lambda\delta$. The incentive constraints for the deviation to be profitable are:

$$\begin{aligned} \mu_2 &\leq \frac{\delta}{(1+\delta)\lambda} - \delta, \\ \mu_2 &\geq \frac{\frac{\delta}{1+\delta} - \lambda\delta}{1 - \delta^2 (1-\delta)}. \end{aligned}$$

If $\lambda \geq \delta$, $\hat{b}_1 = \delta(\lambda - \frac{\delta}{1-\delta^2}(\lambda - \delta))$. The incentive constraints for the deviation to be profitable are :

$$\begin{aligned} \mu_2 &\geq \frac{\delta\left(1-\delta-\delta^2\right)}{\left(1-\delta^2\right)} \frac{(1-\lambda)}{\left(1-\delta^2(1-\delta)\right)}, \\ \mu_2 &\leq \frac{\delta\left(1-\delta-\delta^2\right)}{\left(1-\delta^2\right)} \frac{(1-\lambda)}{\lambda}. \end{aligned}$$

- $\delta(1-\lambda) < \varphi < \mu_1$:

  We are still in zone III. The partially deterrent deviating bid is $\hat{b}_1 = \delta \max\left\{\lambda - \frac{\delta}{1+\delta}, \lambda(1-\varphi) - \delta\left(\right.\right.$
  The high type must be indifferent and the low type must prefer the deviation.

  If $\varphi > \frac{\delta(1-\lambda)}{\lambda}$, indifference of the high type requires$\mu_2 = \delta(1-\lambda)$ is required which happens only for non-generic values of the parameters.

  If $\varphi < \frac{\delta(1-\lambda)}{\lambda}$, $\varphi = \frac{\mu_2 - \delta(1-\delta)(1-\lambda)}{\delta\lambda}$ is required. Plugging this in $\delta(1-\lambda) < \varphi < \min(\mu_1, \frac{\delta(1-\lambda)}{\lambda})$ yields the following necessary condition:

$$\delta(1-\lambda)(1-\delta(1-\lambda)) < \mu_2 < \min\left(\delta(1-\lambda), \delta(1-\lambda)(1-\delta) + \delta\lambda\mu_1\right).$$

- $\varphi = \delta(1-\lambda)$:

  Given those beliefs, Player 2 is indifferent between full and partial deterrence after observing the deviating bid. Let ψ be the probability with which he plays a fully deterrent bid. The deviating bid is the discounted payoff of the second player's low type that would mimic the high type and be covering. Hence we get :$\hat{b}_1 = \delta \max\left(\lambda - \frac{\delta}{1+\delta}, (1-\delta(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) = \delta\left((1-\delta(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$. The high type is indifferent between full and partial with beliefs $\varphi = \delta(1-\lambda)$, hence the low type will prefer partial deterrence.

  Now the incentive constraints for the deviation are:

$$
\begin{gathered}
(1-\mu_2)\,\lambda - \delta\left((1-\delta(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) \geq \lambda - \frac{\delta}{1+\delta}, \\
(1-\mu_2) - \delta\left((1-\delta(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) + \delta\mu_2\,(1-\psi)\left(-\delta\left(\lambda - \frac{\delta}{1+\delta}\right) + \delta\left(1 - \frac{\delta}{1+\delta
\end{gathered}
$$

  These can be rewritten as:

$$
\begin{aligned}
\mu_2 &\leq \frac{\delta\,(1-\lambda)\,(1-\delta\,(1-\lambda))}{\lambda}, \\
\mu_2 &= \frac{\delta\,(1-\lambda)\,(1-\delta\,(1-\lambda))}{1-\delta^2\,(1-\psi)\,(1-\lambda)}.
\end{aligned}
$$

  Since $\psi \in [0,1]$, we need:

$$
\delta\,(1-\lambda)\,(1-\delta\,(1-\lambda)) < \mu_2 < \frac{\delta\,(1-\lambda)\,(1-\delta\,(1-\lambda))}{1-\delta^2\,(1-\lambda)}.
$$

- $0 < \varphi < \delta(1-\lambda)$:

  The deviating bid is $\hat{b}_1 = \delta\left((1-\varphi)\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$.

  Now the incentive constraints for the deviation are:

$$
\begin{gathered}
(1-\mu_2)\,\lambda - \delta\left((1-\varphi)\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) \geq \lambda - \frac{\delta}{1+\delta}, \\
(1-\mu_2) - \delta\left((1-\varphi)\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) + \delta^2\mu_2\,(1-\lambda) = \frac{1}{1+\delta}.
\end{gathered}
$$

The second constraint implies the first.

To get the beliefs in the zone of analysis, we need:

$$\begin{gathered} 0 < \varphi < \delta(1-\lambda) \\ \Leftrightarrow \frac{\delta\left(1-\delta\right)\left(1-\lambda\right)}{1-\delta^{2}\left(1-\lambda\right)} < \mu_{2} < \frac{\delta\left(1-\delta+\delta\lambda\right)\left(1-\lambda\right)}{1-\delta^{2}\left(1-\lambda\right)}. \end{gathered}$$

Case 4: (λ, μ_1) belong to zone $IV-V$ F.D. requires that $b_1 = \delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right)$, $V_{1_h} = 1-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right)$,$V_{1_l} = \lambda-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right)$.

To make sure payoffs are positive, the following condition is needed :

$$\lambda \geq \delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right).$$

In order to have an equilibrium we have to check possible deviations to a partially deterrent bid $\hat{b}_1$ with associated beliefs φ.

- $\varphi = \mu_1$:

  We have $\hat{b}_1 = \delta((1-\mu_1)\lambda-\delta(\lambda-\frac{\delta}{1+\delta}))$. The incentive constraints for this type of deviation are:

$$\begin{aligned} (1-\mu_2)-\hat{b}_1+\delta^2\mu_2(1-\lambda) &\geq 1-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right). \\ (1-\mu_2)\lambda-\hat{b}_1 &\geq \lambda-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right), \end{aligned}$$

  The first constraint implies the second; hence, all is needed is:

$$\mu_2 < \frac{\delta(1-\mu_1)(1-\lambda)}{1-\delta^2(1-\lambda)},$$

- $\varphi = 0$:

  We have $\hat{b}_1 = \delta(\lambda-\frac{\delta(\lambda-\delta)^+}{1-\delta^2})$. The incentive constraints for this type of deviation are:

$$(1-\mu_2)-\delta(\lambda-\delta\frac{(\lambda-\delta)^+}{1-\delta^2})+\delta\mu_2(-\delta\frac{(\lambda-\delta)^+}{1-\delta^2}+\delta\left(1-\delta(1-\delta\frac{(\lambda-\delta)^+}{1-\delta^2})\right)\leq 1-\delta\left(1-\mu_1-\delta(\right.$$

$$(1-\mu_2)\lambda-\delta(\lambda-\delta\frac{(\lambda-\delta)^+}{1-\delta^2})\geq\lambda-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right).$$

If $\lambda\geq\delta$, the constraints become:

$$\begin{aligned}\mu_2(1-\delta^2(1-\lambda)) &\geq \delta(1-\mu_1)-\delta\lambda-\delta^4\frac{1-\lambda}{1-\delta^2},\\ \mu_2\lambda &\leq \delta(1-\mu_1)-\delta\lambda-\delta^4\frac{1-\lambda}{1-\delta^2}.\end{aligned}$$

If $\frac{\delta}{1+\delta}\leq\lambda\leq\delta$, the constraints become:

$$\begin{aligned}\mu_2(1-\delta^2(1-\lambda)) &\geq \delta(1-\mu_1)-\delta(1+\delta)\lambda+\frac{\delta^3}{1+\delta},\\ \mu_2\lambda &\leq \delta(1-\mu_1)-\delta(1+\delta)\lambda+\frac{\delta^3}{1+\delta}.\end{aligned}$$

- $0<\varphi<\mu_1$:

  We have $\hat{b}_1=\delta((1-\varphi)\lambda-\delta(\lambda-\frac{\delta}{1+\delta}))$. The incentive constraints for this type of deviation are:

$$(1-\mu_2)\lambda-\hat{b}_1\geq\lambda-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right),$$

$$(1-\mu_2)-\hat{b}_1+\delta^2\mu_2(1-\lambda)=1-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right).$$

  The first constraint can be rewritten as:

$$\begin{aligned}\mu_2 &= \frac{\delta\left(1-\mu_1-(1-\varphi)\lambda\right)}{1-\delta^2(1-\lambda)},\\ \mu_2 &\leq \frac{\delta\left(1-\mu_1-(1-\varphi)\lambda\right)}{\lambda}.\end{aligned}$$

  This clearly implies that if the constraint is satisfied for the high types, it is satisfied for the low types. For the beliefs to be in the zone of analysis, we need:

$$\begin{aligned}&0<\varphi<\mu_1\\ \Leftrightarrow&\frac{\delta\left(1-\mu_1-\lambda\right)}{1-\delta^2(1-\lambda)}<\mu_2<\frac{\delta\left(1-\mu_1\right)\left(1-\lambda\right)}{1-\delta^2(1-\lambda)}.\end{aligned}$$

- $\mu_1 < \varphi < \delta\,(1-\lambda)$:

  We have $\hat{b}_1 = \delta((1-\varphi)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}))$. The low types are indifferent while the high types prefer the deviation. The incentive constraints are:

$$(1-\mu_2) - \hat{b}_1 + \delta^2\mu_2(1-\lambda) \geq 1 - \delta\left(1 - \mu_1 - \delta(\lambda - \frac{\delta}{1+\delta})\right),$$

$$(1-\mu_2)\lambda - \hat{b}_1 = \lambda - \delta\left(1 - \mu_1 - \delta(\lambda - \frac{\delta}{1+\delta})\right).$$

  But this is not possible. If the low type is indifferent between full and partial deterrence, then the high type strictly prefers full deterrence.

- $\varphi = \delta\,(1-\lambda)$:

  The low type is indifferent, high types prefer the deviation. After observing the partially different bid, the high type of player 2 is indifferent between full and partial deterrence. He chooses a fully deterrent bid with probability ψ.

  The deviating bid is $\hat{b}_1 = \delta\left((1-\delta\,(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$.

  The constraints are:

$$(1-\mu_2)\,\lambda - \delta\left((1-\delta\,(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) = \lambda - \delta\left(1 - \mu_1 - \delta(\lambda - \frac{\delta}{1+\delta})\right),$$

$$(1-\mu_2) - \delta\left((1-\delta\,(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) + \delta^2\mu_2\,(1-\lambda)\,(1-\psi) \geq 1 - \delta\left(1 - \mu_1 - \delta(\lambda -\right.$$

  These two constraints cannot hold together. If the low types are indifferent between full and partial deterrence, then the high types strictly prefer full deterrence.

- $\delta\,(1-\lambda) < \varphi < 1$:

  Low types are indifferent, high types prefer the deviation.

  The deviating bid is $\hat{b}_1 = \delta\left((1-\delta\,(1-\lambda))\,\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$.

The constraints are:

$$(1-\mu_2)\,\lambda-\delta\left((1-\delta\,(1-\lambda))\,\lambda-\delta\left(\lambda-\frac{\delta}{1+\delta}\right)\right)=\lambda-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right),$$
$$(1-\mu_2)-\delta\left((1-\delta\,(1-\lambda))\,\lambda-\delta\left(\lambda-\frac{\delta}{1+\delta}\right)\right)\geq 1-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right).$$

These two constraints cannot hold simultaneously.

- $\varphi = 1$:

  Only the high type is deviating to partial deterrence. The deviating bid is $\hat{b}_1 = \delta\left(\lambda-\frac{\delta}{1+\delta}\right)$.

  The incentive constraints are:

$$\begin{aligned}(1-\mu_2)\,\lambda-\delta\left(\lambda-\frac{\delta}{1+\delta}\right) &\leq \lambda-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right),\\ (1-\mu_2)-\delta\left(\lambda-\frac{\delta}{1+\delta}\right) &\geq 1-\delta\left(1-\mu_1-\delta(\lambda-\frac{\delta}{1+\delta})\right).\end{aligned}$$

  For the same reason as before, this is not possible.

1.3.2. Pooling, Partial-Deterrence

Case 1 : (λ, μ_1) belong to zone I : Then $b_1 = \delta\lambda(1-\mu_1)$, $V_{1_l} = (1-\mu_2)\lambda - \delta\lambda(1-\mu_1), V_{1_h} = (1-\mu_2) - \delta\lambda(1-\mu_1) + \delta^2\frac{\mu_2}{1+\delta}$.

The equilibrium payoffs need to be positive so $(1-\mu_2) \geq \delta(1-\mu_1)$ is needed.

We need to check a possible deviation to a fully deterrent bid $\hat{b}_1$ associated with beliefs φ.

- $\varphi = \mu_1$:

  Then $\hat{b}_1 = \delta(1-\mu_1)$, but then $V_{1_l} = \lambda - \delta(1-\mu_1) \leq \lambda - \frac{\delta}{1+\delta} \leq 0$ which is a contradiction.

- $\varphi < \mu_1$:

  This deviation is also impossible for the same reason.

- $\varphi = 0$:

  Only the low types deviate. Their deviating bid must deter the high type of player 2, which is clearly not possible since $\lambda \leq \delta$.

- $\varphi = 1$:

  Then $\hat{b}_1 = \frac{\delta}{1+\delta}$. The low type has no incentive to deviate whatsoever. The high type prefers the deviation if:

$$\begin{aligned} \frac{1}{1+\delta} &\geq (1-\mu_2) - \delta\lambda\,(1-\mu_1) + \mu_2 \frac{\delta^2}{1+\delta} \\ &\Leftrightarrow \mu_2 \geq \frac{\delta\,(1-(1+\delta)\,(1-\mu_1)\,\lambda)}{1+\delta-\delta^2}. \end{aligned}$$

- $\mu_1 < \varphi < 1$:

  This requires the low type to be indifferent between deviation and the equilibrium bid. If φ is such that we remain in zone 1, we need:

$$\begin{aligned} \lambda - \delta(1-\varphi) &= (1-\mu_2)\lambda - \delta\lambda(1-\mu_1) \\ \Leftrightarrow \varphi &= \frac{\delta\,(1-\lambda) - \lambda\mu_2 + \delta\lambda\mu_1}{\delta}. \end{aligned}$$

  Hence we need:

$$\frac{\delta\,(1-\lambda\,(1+\delta)\,(1-\mu_1))}{(1+\delta)\,\lambda} < \mu_2 < \frac{\delta\,(1-\lambda)\,(1-\mu_1)}{\lambda}.$$

  If φ is such that parameters are in zone II, then the payoff of the low type is $\lambda - \frac{\delta}{1+\delta} < 0$, which is impossible.

Case 2 : (λ, μ_1) belong to zone II : Then $b_1 = \delta\lambda(1-\mu_1)$, $V_{1_l} = (1-\mu_2)\lambda - \delta\lambda(1-\mu_1), V_{1_h} = (1-\mu_2) - \delta\lambda(1-\mu_1)$.

The equilibrium payoffs need to be positive so $(1-\mu_2) \geq \delta(1-\mu_1)$ is needed.

We need to check possible deviation to a fully deterrent bid $\hat{b}_1$ associated with beliefs φ.

- $\frac{\delta}{1+\delta} \leq \varphi < 1:$

  The low type should (weakly) prefer the deviating bid $\hat{b}_1 = \frac{\delta}{1+\delta}$, but since $\lambda \leq \frac{\delta}{1+\delta}$, this is not possible.

- $\varphi = 0:$

  The low types cannot fully deter player 2 when $\lambda \leq \frac{\delta}{1+\delta}$.

- $0 < \varphi < \frac{\delta}{1+\delta}:$

  The payoff of player 2's high type who covers is larger than $\frac{1}{1+\delta}$. Hence the deviating bid has to be larger than $\frac{\delta}{1+\delta}$ which is clearly not possible.

- $\varphi = 1:$

The incentive constraint for the high type requires:

$$\frac{1}{1+\delta} \geq (1-\mu_2) - \delta\lambda(1-\mu_1)$$
$$\Leftrightarrow \mu_2 \geq \frac{\delta\left(1-\lambda\left(1+\delta\right)\left(1-\mu_1\right)\right)}{1+\delta}.$$

Case 3 : (λ, μ_1) **belong to zone** III : The value of the pooling bid is $b_1 = \delta \max\left[\lambda - \frac{\delta}{1+\delta}, \left(1-\mu_1\right)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})\right]$. So we need to examine two subcases depending on the sign of $\mu_1 - \delta\frac{(1-\lambda)}{\lambda}$.

Subcase 1 : $\mu_1 \geq \delta\frac{(1-\lambda)}{\lambda}$ Then $b_1 = \delta(\lambda - \frac{\delta}{1+\delta})$, $\Pi_l^1 = (1-\mu_2)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})$, $\Pi_h^1 = (1-\mu_2) - \delta(\lambda - \frac{\delta}{1+\delta})$.

The necessary condition for positive payoffs is:

$$\mu_2 \leq (1-\delta) + \frac{\delta^2}{(1+\delta)\lambda}.$$

We need to check possible deviation to a fully deterrent bid $\hat{b}_1$ associated with beliefs φ.

- $\varphi = 0$:

  Only the low types deviate to a fully deterrent bid of $\hat{b}_1 = \delta \left(1 - \delta \left(\frac{\lambda-\delta}{1-\delta^2}\right)^+\right)$. The constraints are:

$$\begin{aligned}
\lambda - \hat{b}_1 &\geq (1-\mu_2)\,\lambda - b_1, \\
1 - \hat{b}_1 &\leq (1-\mu_2) - b_1.
\end{aligned}$$

  But this is clearly not possible.

- $\varphi < \mu_1$:

  This implies that the high types are indifferent while the low types prefer the fully deterrent deviating bid. Let's call $\hat{b}_1$ the deviating bid which may depend on the value of the beliefs. The constraints are:

$$\begin{aligned}
\lambda - \hat{b}_1 &\geq (1-\mu_2)\,\lambda - b_1, \\
1 - \hat{b}_1 &= (1-\mu_2) - b_1.
\end{aligned}$$

  They clearly cannot hold simultaneously.

- $\varphi = \mu_1$:

  The deviating bid is $\hat{b}_1 = \frac{\delta}{1+\delta}$. The two incentive constraints are:

$$\begin{aligned}
1 - \frac{\delta}{1+\delta} &\geq (1-\mu_2) - \delta(\lambda - \frac{\delta}{1+\delta}) \\
&\Leftrightarrow \mu_2 \geq \delta\,(1-\lambda)\,, \\
\lambda - \frac{\delta}{1+\delta} &\geq (1-\mu_2)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}) \\
&\Leftrightarrow \mu_2 \geq \frac{\delta\,(1-\lambda)}{\lambda}.
\end{aligned}$$

- $\mu_1 < \varphi < 1$:

The deviating bid is $\hat{b}_1 = \frac{\delta}{1+\delta}$. The two incentive constraints are:

$$\begin{aligned} 1 - \frac{\delta}{1+\delta} &\geq (1-\mu_2) - \delta(\lambda - \frac{\delta}{1+\delta}) \\ &\Leftrightarrow \mu_2 \geq \delta(1-\lambda), \\ \lambda - \frac{\delta}{1+\delta} &= (1-\mu_2)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}) \\ &\Leftrightarrow \mu_2 = \frac{\delta(1-\lambda)}{\lambda}. \end{aligned}$$

This can only be satisfied for non-generic values of the parameters.

- $\varphi = 1$:

  The incentive constraints are:

$$\begin{aligned} 1 - \frac{\delta}{1+\delta} &\geq (1-\mu_2) - \delta(\lambda - \frac{\delta}{1+\delta}), \\ \lambda - \frac{\delta}{1+\delta} &\geq (1-\mu_2)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}). \end{aligned}$$

  This implies

$$\frac{\delta(1-\lambda)}{\lambda} < \mu_2 < \delta(1-\lambda).$$

Subcase 2 : $\mu_1 \leq \delta\frac{(1-\lambda)}{\lambda}$ Then $b_1 = \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}))$, $\Pi_l^1 = (1-\mu_2)\lambda - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}))$, $\Pi_h^1 = (1-\mu_2) - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}))$.

The necessary condition for positive payoff is:

$$\mu_2 \leq 1 - \delta(1-\mu_1) + \delta^2 - \frac{\delta^3}{(1+\delta)\lambda}.$$

We need to check possible deviation to a fully deterrent bid $\hat{b}_1$ associated with beliefs φ.

- $\varphi = \mu_1$:

  The deviating bid is $\hat{b}_1 = \frac{\delta}{1+\delta}$. The incentive constraints are:

$$\begin{aligned} 1 - \frac{\delta}{1+\delta} &\geq (1-\mu_2) - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})). \\ \lambda - \frac{\delta}{1+\delta} &\geq (1-\mu_2)\lambda - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})). \end{aligned}$$

They can be rewritten as:

$$\mu_2 \geq \frac{\delta (1-\delta)(1-\lambda) + \delta \mu_1 \lambda}{\lambda}.$$

- $\mu_1 < \varphi < 1$:

  The deviating bid is $\hat{b}_1 = \frac{\delta}{1+\delta}$. The incentive constraints are:

$$\begin{aligned} 1 - \frac{\delta}{1+\delta} &\geq (1-\mu_2) - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})), \\ \lambda - \frac{\delta}{1+\delta} &= (1-\mu_2)\lambda - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})). \end{aligned}$$

  Clearly this is possible only for nongeneric values of the parameters.

- $\varphi = 0$:

  Only the low type deviates to a fully deterrent bid of $\hat{b}_1 = \left(1 - \delta \left(\frac{\lambda - \delta}{1-\delta^2}\right)^+\right)$. The incentive constraints are:

$$\begin{aligned} 1 - \hat{b}_1 &\leq (1-\mu_2) - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})), \\ \lambda - \hat{b}_1 &\geq (1-\mu_2)\lambda - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})). \end{aligned}$$

  This is clearly not possible.

- $0 < \varphi < \delta(1-\lambda)$:

  After observing the deviating bid, the second player has beliefs in zone IV. The deviating bid corresponds to the discounted payoff of the high type in the subgame with beliefs $\varphi : \hat{b}_1 = \delta\left((1-\varphi) - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$. The incentive constraints are:

$$\begin{aligned} 1 - \hat{b}_1 &= (1-\mu_2) - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})) \\ \lambda - \hat{b}_1 &\geq (1-\mu_2)\lambda - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})) \end{aligned}$$

  This is not possible.

- $\varphi = \delta(1-\lambda)$:

  Impossible for the same argument as before.

- $\delta(1-\lambda) < \varphi < \mu_1$:

  The high types are indifferent and the low types prefer the deviation. The deviating bid is $\hat{b}_1 = \frac{\delta}{1+\delta}$. The incentive constraints are

$$\begin{aligned} 1 - \frac{\delta}{1+\delta} &= (1-\mu_2) - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})), \\ \lambda - \frac{\delta}{1+\delta} &\geq (1-\mu_2)\lambda - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})). \end{aligned}$$

  Clearly this is not possible.

- $\varphi = 1$:

  Only high types prefer the deviation. The incentive constraints are:

$$\begin{aligned} 1 - \frac{\delta}{1+\delta} &\geq (1-\mu_2) - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})), \\ \lambda - \frac{\delta}{1+\delta} &\leq (1-\mu_2)\lambda - \delta((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})). \end{aligned}$$

  They can be rewritten as:

$$\delta(1-\delta)(1-\lambda) + \delta\mu_1\lambda < \mu_2 < \frac{\delta(1-\delta)(1-\lambda) + \delta\mu_1\lambda}{\lambda}.$$

Case 4 : (λ, μ_1) **belong to zone** $IV - V$: Then we have $b_1 = \delta \max\left\{0, (1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})\right\}$.

Note that we have $\lambda \geq \frac{\delta}{1+\delta}$, and $\mu_1 \leq \delta(1-\lambda)$. This implies that $\mu_1 \leq \frac{\delta}{1+\delta}$.

Now we have $(1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta}) = (1-\mu_1-\delta)\lambda + \frac{\delta^2}{1+\delta}) \geq \left(\frac{1}{1+\delta} - \delta\right)\frac{\delta}{1+\delta} + \frac{\delta^2}{1+\delta} = \frac{\delta}{1+\delta} > 0$. Hence, $b_1 = \delta\left((1-\mu_1)\lambda - \delta(\lambda - \frac{\delta}{1+\delta})\right)$.

Now,. $V_{1_l} = (1-\mu_2)\lambda - \delta(1-\mu_1)\lambda + \delta^2(\lambda - \frac{\delta}{1+\delta})$.

Positive payoffs for the low types are necessary which requires:

$$\mu_2 \leq (1 - \delta + \delta^2 + \delta\mu_1) - \frac{\delta^3}{(1+\delta)\lambda}.$$

We now have to check a possible deviation to a fully deterrent bid $\hat{b}_1$ associated with beliefs φ.

- $\varphi = \mu_1$:

We have $\hat{b}_1 = \delta(1-\mu_1) - \delta^2(\lambda - \frac{\delta}{1+\delta})$. The incentive constraints related to this deviation are:

$$\begin{aligned} \lambda - \delta(1-\mu_1) + \delta^2(\lambda - \frac{\delta}{1+\delta}) &\geq (1-\mu_2)\lambda - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})), \\ 1 - \delta(1-\mu_1) + \delta^2(\lambda - \frac{\delta}{1+\delta}) &\geq (1-\mu_2) - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})) \\ &\quad + \delta\mu_2(-\delta(\lambda - \frac{\delta}{1+\delta}) + \frac{\delta}{1+\delta}). \end{aligned}$$

We can rewrite them as:

$$\begin{aligned} \mu_2\lambda &\geq \delta\,(1-\mu_1)\,(1-\lambda)\,, \\ \mu_2\left(1 - (1-\lambda)\,\delta^2\right) &\geq \delta\,(1-\mu_1)\,(1-\lambda)\,. \end{aligned}$$

Since $1-\lambda > \delta^2\,(1-\lambda)$, the second constraint is implied by the first.

- $\mu_1 < \varphi < \delta\,(1-\lambda)$:

  We have $\hat{b}_1 = \delta(1-\varphi) - \delta^2(\lambda - \frac{\delta}{1+\delta})$. The incentive constraints related to this deviation are:

$$\begin{aligned} \lambda - \delta(1-\varphi) + \delta^2(\lambda - \frac{\delta}{1+\delta}) &= (1-\mu_2)\lambda - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})), \\ 1 - \delta(1-\varphi) + \delta^2(\lambda - \frac{\delta}{1+\delta}) &\geq (1-\mu_2) - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})) + \delta^2\mu_2(1-\lambda). \end{aligned}$$

  As before the second constraint is implied by the first. So the only thing we need to worry is that the beliefs remain in the interval we analyze. We get:

$$\begin{aligned} &\mu_1 < \varphi < \delta\,(1-\lambda) \\ \Rightarrow\ &\frac{\delta}{\lambda}\,(\mu_1\lambda + (1-\delta)\,(1-\lambda)) < \mu_2 < \frac{\delta}{\lambda}\,(1-\mu_1)\,(1-\lambda)\,. \end{aligned}$$

- $\varphi = 0$:

Only the low types deviate to fully deterrent bid of $\hat{b}_1 = \left(1 - \delta\left(\frac{\lambda-\delta}{1-\delta^2}\right)^+\right)$. The incentive constraints are :

$$\lambda - \hat{b}_1 \geq (1-\mu_2)\lambda - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})),$$
$$1 - \hat{b}_1 \leq (1-\mu_2) - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})) + \delta^2\mu_2(1-\lambda).$$

But this is not possible since $\left(1-\delta^2(1-\lambda)\right) > \lambda$.

- $0 < \varphi < \mu_1$:

  The same argument as in the case $\varphi = 0$ makes this impossible.

- $\varphi = 1$:

  We have $\hat{b}_1 = \frac{\delta}{1+\delta}$. The incentive constraints related to this deviation are:

$$1 - \frac{\delta}{1+\delta} \geq (1-\mu_2) - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})) + \delta^2\mu_2(1-\lambda),$$
$$\lambda - \frac{\delta}{1+\delta} < (1-\mu_2)\lambda - \delta((1-\mu_1)\,\lambda - \delta(\lambda - \frac{\delta}{1+\delta})).$$

  These can be rewritten as :

$$\begin{aligned} \mu_2 &\geq \frac{\delta\,(1-\mu_1)\,(1-\lambda)}{1-\delta^2\,(1-\lambda)}, \\ \mu_2 &< \frac{\delta\,(1-\mu_1)\,(1-\lambda)}{\lambda}. \end{aligned}$$

- $\delta(1-\lambda) < \varphi < 1$:

  The deviating bid is $\frac{\delta}{1+\delta}$. Hence, the indifference condition for the low type does not depend on μ which yields a nongeneric condition on parameters.

- $\varphi = \delta(1-\lambda)$:

  The deviating bid is $\frac{\delta}{1+\delta}$. Hence, the indifference condition for the low type does not depend on μ which yields a nongeneric condition on parameters.

1.4. semi-Pooling Equilibria

1.4.1. Low type randomizes...

Between Full Deterrence (F.D.) and Partial Deterrence (P.D.) Let φ be the belief of Player 2 (that Player 1 is of high type) upon observing the pooling bid. Obviously, $\varphi \geqq \mu_1$. Denote the pooling bid by $\bar{b}$ and the separating bid by $\underline{b}$.

Case 1: (λ, φ) **belong to zone** I. Then $\bar{b} = \delta(1-\varphi)$ (F.D.), which is larger than $\frac{\delta}{1+\delta}$, which in turn is larger than λ, implying that the high type cannot make such a bid.

Case 2: (λ, φ) **belong to zone** II. F.D. requires then that $\bar{b} = \delta/(1+\delta)$, which, in this zone, is larger than λ, so that a low type cannot make such a bid.

Case 3: (λ, φ) **belong to zone** III. F.D. requires then that $\bar{b} = \delta/(1+\delta)$. Obviously, $\underline{b} = \delta\left(\lambda - \delta\frac{(\lambda-\delta)^+}{1-\delta^2}\right)$. payoffs do not depend on φ, and indifference for the low types obtains then at best for a nongeneric set of types.

Case 4: (λ, φ) **belong to zone** $IV - V$. Bids are then $\bar{b} = \delta\left(1-\varphi-\delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$ and $\underline{b} = \delta\left(\lambda - \delta\frac{(\lambda-\delta)^+}{1-\delta^2}\right)$. In the case in which $\lambda \leqq \delta$.

Indifference requires that $(1-\mu_2-\delta)\lambda = \lambda - \delta\left(1-\varphi-\delta\left(\lambda-\frac{\delta}{1+\delta}\right)\right)$, i.e.

$$\varphi = 1 - \frac{(\mu_2+\delta)\lambda + \delta^2\left(\lambda - \frac{\delta}{1+\delta}\right)}{\delta}.$$

Since φ is in zone $IV - V$, it must be that $\mu_1 \leqq \varphi \leqq \delta(1-\lambda)$, implying:

$$\mu_2 \geqq \delta(1-\lambda) - \frac{\delta^2}{1+\delta}, \tag{A1-A2}$$
$$\mu_2 \leqq \frac{\delta(1-\mu_1) - \delta(1+\delta)\lambda + \delta^3/(1+\delta)}{\lambda}.$$

Finally, nonnegativity of payoffs implies:

$$\mu_2 \leqq 1-\delta. \tag{A3}$$

By the same arguments as before, the only deviations to worry about are deviations to P.D., with a positive probability $\psi \leqq \mu_1$ of a high type. Such a deviation involves a bid $\tilde{b} = \delta \max\left[0, (1-\psi)\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right]$. Such a deviation does not exist iff:

$$\mu_2 \geqq \frac{\delta\lambda - \delta \max\left\{0, (1-\mu_1)\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right\}}{(1-\lambda)\left(1-\delta^2\right)}. \tag{A4}$$

It is easy to verify that conditions A1-A4 are neither redundant nor contradictory, and these are the necessary and sufficient conditions for existence of such a undefeated equilibrium.

In the case in which $\lambda \geqq \delta$: indifference of the low type translates into: $(1-\mu_2)\lambda - \delta\left(\lambda - \delta\frac{\lambda-\delta}{1-\delta^2}\right) = \lambda - \delta\left(1 - \varphi - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$, that is:

$$\varphi = 1 - \frac{\delta^2\left(\lambda - \frac{\delta}{1+\delta}\right) + \delta\left(\lambda - \delta\frac{\lambda-\delta}{1-\delta^2}\right) + \mu_2\delta}{\delta},$$

and $\mu_1 \leqq \varphi \leqq \delta(1-\lambda)$ becomes:

$$\begin{aligned} \mu_2 &\leqq \frac{\delta(1-\mu_1) - \delta^2\left(\lambda - \frac{\delta}{1+\delta}\right) - \delta\left(\lambda - \delta\frac{\lambda-\delta}{1-\delta^2}\right)}{\lambda}, \\ \mu_2 &\geqq \frac{1 - \delta(1-\lambda) - \delta^2\left(\lambda - \frac{\delta}{1+\delta}\right) - \delta\left(\lambda - \delta\frac{\lambda-\delta}{1-\delta^2}\right)}{\lambda}. \end{aligned} \tag{A5}$$

Conditions regarding deviations are unchanged.

Between Full Deterrence and No Deterrence (0D.) Let φ be the belief of Player 2 (that Player 1 is of high type) upon observing the pooling bid. Obviously, $\varphi \geqq \mu_1$. Denote the pooling bid by $\bar{b}$ and the separating bid by $\underline{b}$. By direct inspection, the payoff of the high type of Player 2 being always larger than $1/(1+\delta)$ when it is his turn to play, F.D. requires a bid at least as large as $\delta/(1+\delta)$, and hence λ should exceed such a threshold, which immediately implies that such equilibria cannot arise in zones I and II. In zone III, the bid for F.D., and the bid for no deterrence do not depend on φ, and thus equilibria can only arise for pathological parameters.

(λ, φ) **belong to zone** $IV - V$. Bids are then $\bar{b} = \delta\left(1 - \varphi - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)$ and $\underline{b} = 0$. Indifference requires that $\lambda - \delta\left(1 - \varphi - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right) = 0$, that is:

$$\varphi = \frac{\delta - \lambda - \delta^2\left(\lambda - \frac{\delta}{1+\delta}\right)}{\delta},$$

which must be within $[\mu_1, \delta(1-\lambda)]$. This is equivalent to:

$$\mu_1 \leqq \frac{\delta - \lambda - \delta^2\left(\lambda - \frac{\delta}{1+\delta}\right)}{\delta}, \tag{B1}$$

while the second condition boils down to $\lambda \geqq \frac{\delta}{1+\delta}$, trivially satisfied.

Deviations to F.D. can be dismissed for the same reasons as before. Deviations to P.D. have to be considered. Denote by ψ the belief of Player 2 after such a deviation $\tilde{b}$. Suppose that $\psi = 1$. Then $\tilde{b} = \delta\left(\lambda - \frac{\delta}{1+\delta}\right)$, and this would require both $1 - \mu_2 - \tilde{b} \geqq 1 - \lambda$ and $(1-\mu_2)\lambda - \tilde{b} \leqq 0$, a contradiction. Suppose next that $\psi = 0$. Then $\tilde{b} = \delta\left(\lambda - \delta\frac{(\lambda-\delta)^+}{1-\delta^2}\right)$, and it is necessary that $(1-\mu_2)\lambda \geqq \tilde{b} \geqq \lambda - \mu_2 + \delta^2\mu_2\left(1 - \delta - (\lambda-\delta)^+\right)$. This is equivalent to:

$$\mu_2 \leqq 1 - \delta + \delta\frac{(\lambda-\delta)^+}{1-\delta^2}, \tag{B2}$$

$$\mu_2 \geqq \frac{\lambda(1-\delta) + \delta^2\frac{(\lambda-\delta)^+}{1-\delta^2}}{1 - \delta^2\left(1 - \delta - (\lambda-\delta)^+\right)},$$

conditions that should not hold simultaneously for the equilibrium to exist. Next, suppose that $0 < \psi < 1$. Then $\tilde{b} = \delta\left((1-\psi)\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)^+$, and it is necessary that $1 - \mu_2 - \tilde{b} + \delta^2\mu_2(1-\lambda) \geqq 1 - \lambda$ and $(1-\mu_2)\lambda - \tilde{b} \geqq 0$. The first inequality always implies the second, so that $\psi \leqq \mu_1$ and the condition collapses to $\tilde{b} \leqq \lambda - \mu_2 + \delta^2\mu_2(1-\lambda)$ with equality if $\psi \neq \mu_1$, that is, to:.

$$\mu_2 \leqq \frac{\lambda - \delta\left((1-\psi)\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)^+}{1 - \delta^2(1-\lambda)}.$$

To prevent such a deviation, it is hence to necessary and sufficient to require that:

$$\mu_2 > \frac{\lambda - \delta\left((1-\mu_1)\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)^+}{1 - \delta^2(1-\lambda)}. \tag{B3}$$

To summarize, such a undefeated equilibrium exists if and only if Equations B1 and B3 do hold, and B2 does not hold.

Between Partial Deterrence and No Deterrence Let φ be the belief of Player 2 (that Player 1 is of high type) upon observing the pooling bid. Obviously, $\varphi \geqq \mu_1$. Denote the pooling bid by $\bar{b}$ and the separating bid by $\underline{b}$.

Case 1: (λ, φ) **belong to zone** I. Bids are then $\bar{b} = \delta\lambda(1-\varphi)$ and $\underline{b} = 0$. Indifference requires that $(1-\mu_2)\lambda - \delta\lambda(1-\varphi) = 0$, i.e., $\varphi = 1 - \frac{1-\mu_2}{\delta}$, which belongs to $\left[\mu_1, \frac{\delta}{1+\delta}\right]$ iff:

$$1 - \delta(1-\mu_1) \leqq \mu_2 \leqq \frac{1}{1+\delta}. \tag{C1}$$

The only deviations to verify are deviations to Full Deterrence. If such a deviation $\tilde{b}$ is perceived as coming for sure from the high type, then $\tilde{b} = \frac{\delta}{1+\delta}$ and the low type has no incentives to deviate. To prevent this deviation, it is necessary that $1 - \mu_2 - \delta\lambda(1-\varphi) + \mu_2\frac{\delta^2}{1+\delta} \geqq \frac{1}{1+\delta}$, which holds provided that:

$$\mu_2 \leqq \frac{1-\lambda-\frac{1}{1+\delta}}{1-\lambda-\frac{\delta^2}{1+\delta}}, \tag{C2}$$

which guarantees also that the high type does not want to deviate to F.D. no matter the beliefs of Player 2, and hence Player 1 has no such incentives either. Conditions C1-C2 are thus necessary and sufficient for such an equilibrium to exist.

Case 2: (λ, φ) **belong to zone** II. Bids are then $\bar{b} = \delta\max\left\{(1-\varphi)\lambda, \lambda - \frac{\delta}{1+\delta}\right\} = (1-\varphi)\lambda$ and $\underline{b} = 0$. Consider a deviation to a bid $\tilde{b} = \lambda$, accompanied with beliefs $\psi = \varphi$. This bid is fully deterrent. Indeed, the low type of Player 1 realizes zero payoff and is thus willing to randomize. However, a high type of Player realizes a payoff of $1-\lambda > (1-\mu_2)(1-\lambda)$, the latter being his payoff in the proposed equilibrium. Hence, no such undefeated equilibrium exists in zone II.

Case 3: (λ, φ) **belong to zone** III. Bids are then $\bar{b} = \delta\max\left\{(1-\varphi)\lambda, \lambda - \frac{\delta}{1+\delta}\right\}$ and $\underline{b} = 0$. The deviation used in the previous argument can be used verbatim in this case too.

Case 4: (λ, φ) **belong to zone** $IV-V$. Bids are then $\bar{b} = \delta\left((1-\varphi)\lambda - \delta\left(\lambda - \frac{\delta}{1+\delta}\right)\right)^+$ and $\underline{b} = 0$. Obviously, this requires $(1-\varphi)\lambda > \delta\left(\lambda - \frac{\delta}{1+\delta}\right)$ for otherwise low types of player 1 prefer partial deterrence to no deterrence. For deviations, it is again necessary and sufficient to check only for fully deterrent deviations, and more precisely to deviations in which $\psi = 1$ and $\tilde{b} = \frac{\delta}{1+\delta}$. Such a deviation is not profitable iff:

$$(\lambda - 1)\left(1 - \delta^2\right)\mu_2 \geqq \lambda - \frac{\delta}{1+\delta},$$

which is impossible. Therefore, no undefeated equilibrium exists in zone $IV - V$.

1.4.2. High type randomizes..

Between Full Deterrence (F.D.) and Partial Deterrence (P.D.) Let φ be the belief of Player 2 (that Player 1 is of high type) upon observing the pooling bid. Obviously, $\varphi \leqq \mu_1$. Denote the separating bid by $\bar{b}$ and the pooling bid by $\underline{b}$.

Case 1: (λ, φ) **belong to zone** I. Bids are then $\bar{b} = \frac{\delta}{1+\delta}$ and $\underline{b} = \delta(1-\phi)\lambda$. Suppose first that $\mu_1 \leqq \frac{\delta}{1+\delta}$. Consider a partially deterrent deviation $\tilde{b}$ with associated beliefs $\psi = \mu_1$. Hence, $\tilde{b} = \delta(1-\mu_1)\lambda < \underline{b}$ and the ensuing subgames having not been modified, both types of Player 1 strictly benefit from the suggested deviation. Suppose next that $\mu_1 > \frac{\delta}{1+\delta}$. If $\frac{1}{1+\delta} \leqq 1 - \mu_2 - \delta(1-\mu_1)$, the same argument applies, although the outcome of the game is modified (i.e., the high type of Player 2 fully deters, instead of partially deterring). If $\frac{1}{1+\delta} > 1 - \mu_2 - \delta(1-\mu_1)$, then the deviation does not work anymore, since the reduction in the partially deterrent bid is offset by the consequences of the behavior of the high type of Player 2, who prefers F.D. rather than P.D. Consider then instead a deviation to $\tilde{b} = \delta(1-\psi)\lambda$, with $\psi \in \left[\frac{\delta}{1+\delta}, \mu_1\right]$, such that Player 1's high type is indifferent between deviating and not deviating. Allowing, if necessary, Player 2's high type to randomize between F.D. and P.D. when $\psi = \frac{\delta}{1+\delta}$, this is certainly possible, since Player 2's high type prefers $\psi = \frac{\delta}{1+\delta}$ (along with Player 2's high type doing P.D.) to his equilibrium outcome, which in turn is preferred to $\tilde{b} = \delta(1-\mu_1)\lambda$, $\psi = \mu_1$, and Player Í's high type's payoff is continuous as we vary continuously ψ and when for $\psi = \frac{\delta}{1+\delta}$, we continuously vary the randomization probabilities of P.D. and F.D. of Player 2's high type. Since the bid of such a partially deterrent deviation is lower than the equilibrium partially deterrent bid, Player 1's low type certainly benefits from this deviation. Hence there is no undefeated equilibrium in zone I.

Case 2: (λ, φ) **belong to zone** $II, III,\ IV - V$: The previous arguments can be replicated, *mutatis mutandis*, to show that no undefeated equilibrium exists in these zones.

Between Full Deterrence (F.D.) and No Deterrence, or Between Partial Deterrence (P.D.) and no deterrence. Those equilibria would require that Player 1's high type's payoff be zero. However, by bidding δ, this type can ensure himself a strictly positive payoff, concluding the argument. Hence, no such equilibria exist.

1.5. Summarizing the results

1. $(\lambda, \mu_1) \in \Omega_1 \triangleq \{(\lambda, \mu_1) \mid \lambda \leqq \delta/(1+\delta), \mu_1 \leqq \delta/(1+\delta)\}$

   In this zone, all undefeated equilibria listed below are -for μ_2 outside the specified region- either not even a sequential equilibrium, or defeated by precisely the equilibrium which is undefeated for those parameters.

   1. $1 - \delta(1-\mu_1) < \frac{(1-\lambda)(1+\delta)-1}{(1-\lambda)(1+\delta)-\delta^2}$:

      The unique undefeated equilibrium is :

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < 1 - \delta(1-\mu_1) \\ (\text{P.D.}, (\text{P.D.},0\text{D.})) \text{ for } 1 - \delta(1-\mu_1) \leq \mu_2 < \min\left(\frac{1}{1+\delta}, \frac{(1-\lambda)(1+\delta)-1}{(1-\lambda)(1+\delta)-\delta^2}\right) \\ (\text{F.D.}, 0\text{D.}) \text{ for } \min\left(\frac{1}{1+\delta}, \frac{(1-\lambda)(1+\delta)-1}{(1-\lambda)(1+\delta)-\delta^2}\right) \leq \mu_2 \leq 1 \end{cases}$$

   2. $1 - \delta(1-\mu_1) > \frac{(1-\lambda)(1+\delta)-1}{(1-\lambda)(1+\delta)-\delta^2}$:

The unique undefeated equilibrium is :

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \frac{\delta(1-(1+\delta)(1-\mu_1)\lambda)}{1+\delta-\delta^2} \\ (\text{F.D.}, 0\text{D.}) \text{ for } \frac{\delta(1-(1+\delta)(1-\mu_1)\lambda)}{1+\delta-\delta^2} \leq \mu_2 \leq 1 \end{cases}$$

2. $(\lambda, \mu_1) \in \Omega_2 \triangleq \{(\lambda, \mu_1) \mid \lambda \leqq \delta / (1+\delta), \mu_1 \geqq \delta / (1+\delta)\}$

In this zone, all undefeated equilibria listed below are -for μ_2 outside the specified region- either not sequential, or defeated by precisely the equilibrium which is undefeated for those parameters.

1. $\left(\frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1)\right) \vee (1-\delta) \leq \frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2}$:

   The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1), \\ ((\text{F.D.}, \text{P.D.}), \text{P.D.}) \text{ for } \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1) \leq \mu_2 < \frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2}, \\ (\text{F.D.}, 0\text{D.}) \text{ for } \frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2} \leq \mu_2 < 1, \end{cases}$$

2. $\frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1) \leq \frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2} \leq (1-\delta)$:

   The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1), \\ ((\text{F.D.}, \text{P.D.}), \text{P.D.}) \text{ for } \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1) \leq \mu_2 < \frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2}, \\ (\text{F.D.}, \text{P.D.}) \text{ for } \frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2} \leq \mu_2 < 1-\delta, \\ (\text{F.D.}, 0\text{D.}) \text{ for } 1-\delta \leq \mu_2 < 1. \end{cases}$$

3. $\frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2} \leq \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1) \leq (1-\delta)$:

   The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1), \\ (\text{F.D.}, \text{P.D.}) \text{ for } \frac{\delta}{1+\delta} - \delta\lambda(1-\mu_1) \leq \mu_2 < 1-\delta, \\ (\text{F.D.}, 0\text{D.}) \text{ for } 1-\delta \leq \mu_2 < 1. \end{cases}$$

4. $(1-\delta) \vee \left(\frac{1}{1+\delta} \wedge \frac{\delta(1-\lambda)}{1+\delta-\delta^2}\right) \leq \frac{\delta}{1+\delta} - \delta\lambda\left(1-\mu_1\right)$:

   The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \frac{\delta}{1+\delta} - \delta\lambda\left(1-\mu_1\right), \\ (\text{F.D.}, 0\text{D.}) \text{ for } \frac{\delta}{1+\delta} - \delta\lambda\left(1-\mu_1\right) \leq \mu_2 < 1. \end{cases}$$

3. $(\lambda, \mu_1) \in \Omega_3^a \triangleq \{(\lambda, \mu_1) \mid \lambda \geqq \delta/(1+\delta), \mu_1 \geqq \delta(1-\lambda)/\lambda\}$

   1. $\delta(1-\lambda) < \frac{\frac{\delta}{1+\delta} - \delta\lambda + \delta^2 \frac{(\lambda-\delta)^+}{1-\delta^2}}{\lambda}$:

      The unique undefeated equilibrium is :

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \delta(1-\lambda) \\ (\text{F.D.}, \text{P.D.}) \text{ for } \delta(1-\lambda) \leq \mu_2 < \frac{\frac{\delta}{1+\delta} - \delta\lambda + \delta^2 \frac{(\lambda-\delta)^+}{1-\delta^2}}{\lambda} \\ (\text{F.D.},\text{F.D.}) \text{ for } \frac{\frac{\delta}{1+\delta} - \delta\lambda + \delta^2 \frac{(\lambda-\delta)^+}{1-\delta^2}}{\lambda} \leq \mu_2 < 1 \end{cases}$$

      (P.D., P.D.) is defeated by (F.D.,F.D.) for $\mu_2 \in [\delta(1-\lambda), 1]$. (F.D.,F.D.) is defeated by (P.D., P.D.) for $\mu_2 \in [0, \delta(1-\lambda)]$ and is defeated by (F.D., P.D.) for $\mu_2 \in \left[\delta(1-\lambda), \frac{\frac{\delta}{1+\delta} - \delta\lambda + \delta^2 \frac{(\lambda-\delta)^+}{1-\delta^2}}{\lambda}\right]$. (F.D., P.D.) is defeated by (P.D., P.D.) for $\mu_2 \in [0, \delta(1-\lambda)]$ and is defeated by (F.D.,F.D.) for $\mu_2 \in \left[\frac{\frac{\delta}{1+\delta} - \delta\lambda + \delta^2 \frac{(\lambda-\delta)^+}{1-\delta^2}}{\lambda}, 1\right]$.

   2. $\delta(1-\lambda) > \frac{\frac{\delta}{1+\delta} - \delta\lambda + \delta^2 \frac{(\lambda-\delta)^+}{1-\delta^2}}{\lambda}$:

      The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \delta(1-\lambda) \\ (\text{F.D.},\text{F.D.}) \text{ for } \delta(1-\lambda) \leq \mu_2 < 1 \end{cases}$$

      (P.D., P.D.) is defeated by (F.D.,F.D.) for $\mu_2 \in \left[\frac{\delta(1-\lambda)}{\lambda}, 1\right]$. (F.D.,F.D.) is defeated by (P.D., P.D.) for $\mu_2 \in [0, \delta(1-\lambda)]$.

      For $\mu_2 \in \left[\delta(1-\lambda), \frac{\delta(1-\lambda)}{\lambda}\right]$, (P.D., P.D.) is defeated by the following S.E. denoted (F.D.,F.D.)$'$: Player 1_h randomizes between a bid $\bar{b} = \psi + \frac{\delta}{1+\delta} > \frac{\delta}{1+\delta}$

and the bid $\underline{b} = \frac{\delta}{1+\delta}$. Player 1_l bids $\underline{b}$ for sure. Let φ be the probability that Player 1 is of the high type conditional on observing the bid $\underline{b}$. Obviously, $\varphi \leq \mu_1$. Pick φ to make sure that upon covering a bid $\underline{b}$, Player 2^h chooses full deterrence. The trick is that after the bid $\underline{b}$, Player 2^h covers with probability ψ/μ_2 ($\psi \leq \mu_2$), while after a bid $\overline{b}$, he does not. Of course, 2^l does not cover irrespective of the equilibrium bid observed. The choice of $\overline{b}$ ensures that 1_h is indifferent between both bids. Except for boundary values, there exists ψ small enough such that (F.D.,F.D.)$'$ is sequential whenever (F.D.,F.D.) is.

4. $(\lambda, \mu_1) \in \Omega_3' \triangleq \{(\lambda, \mu_1) \mid \lambda \geqq \delta/(1+\delta), \delta(1-\lambda)/\lambda \geqq \mu_1 \geqq \delta(1-\lambda)\}$

   1. It turns out to be convenient to specify here a few sequential Equilibria which are used to defeat others. Denote (F.D.,F.D.)$'$ the following sequential equilibrium: Player 1_h randomizes between a bid $\overline{b} = \psi + \frac{\delta}{1+\delta} > \frac{\delta}{1+\delta}$ and the bid $\underline{b} = \frac{\delta}{1+\delta}$. Player 1_l bids $\underline{b}$ for sure. Let φ be the probability that Player 1 is of the high type conditional on observing the bid $\underline{b}$. Obviously, $\varphi \leq \mu_1$. Pick $\varphi \in [\delta(1-\lambda), \mu_1]$, which is possible in Ω_3'. This ensures that upon covering a bid $\underline{b}$, Player 2^h chooses full deterrence. The trick is that after the bid $\underline{b}$, Player 2^h covers with probability ψ/μ_2 ($\psi \leq \mu_2$), while after a bid $\overline{b}$, he does not. Of course, 2^l does not cover irrespective of the equilibrium bid observed. The choice of $\overline{b}$ ensures that 1_h is indifferent between both bids. Except for boundary values, there exists ψ small enough such that (F.D.,F.D.)$'$ is sequential whenever (F.D.,F.D.) is. Moreover, there exists such equilibria that defeat (P.D., P.D.) as long as $\mu_2 > \frac{\delta\mu_1\lambda + \delta(1-\delta)(1-\lambda)}{\lambda}$. Since (P.D., P.D.) is undefeated in Ω_3' for smaller values of μ_2, no further comment is devoted to (P.D., P.D.). It is also necessary to define (F.D., P.D.)$'$, a sequential equilibrium in which Player 1_h makes the bid $(\mu_2 + \delta)\lambda - \frac{\delta^2}{1-\delta^2}(\lambda - \delta)^+$, while 1_l makes the usual partially deterrent bid. All the subgames are the usual ones.

2. $\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)\leq\delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)}\leq\frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}$:

   The unique undefeated equilibrium is:

$$\begin{cases}(\text{P.D.},\text{P.D.}) \text{ for } 0\leq\mu_2<\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda),\\ ((\text{F.D.,P.D.}),\text{P.D.}) \text{ for } \delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)\leq\mu_2<\delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)},\\ (\text{F.D.},\text{P.D.}) \text{ for } \delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)}\leq\mu_2<\frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda},\\ (\text{F.P.},\text{F.P.}) \text{ for } \frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}\leq\mu_2<1.\end{cases}$$

   All these equilibria (except -as mentioned before- (P.D., P.D.) whose case has been handled before) but one are either not sequential outside the specified interval, or defeated by the equilibrium which is undefeated for these parameters. The exception is (F.D., F.D.) which is defeated for $\delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)}\leq\mu_2<\frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}$ by $(\text{F.D.},\text{P.D.})'$.

3. $(\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda))\vee\frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}\leq\delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)}$:

   The unique undefeated equilibrium is:

$$\begin{cases}(\text{P.D.},\text{P.D.}) \text{ for } 0\leq\mu_2<\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda),\\ ((\text{F.D.,P.D.}),\text{P.D.}) \text{ for } \delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)\leq\mu_2<\delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)},\\ (\text{F.D.,F.D.}) \text{ for } \delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)}\leq\mu_2<1.\end{cases}$$

   All these equilibria are either not sequential outside the specified interval, or defeated by the equilibrium which is undefeated for these parameters.

4. $\delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)}\leq\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)\leq\frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}$:

   The unique undefeated equilibrium is:

$$\begin{cases}(\text{P.D.},\text{P.D.}) \text{ for } 0\leq\mu_2<\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda),\\ (\text{F.D.},\text{P.D.}) \text{ for } \delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)\leq\mu_2<\frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda},\\ (\text{F.D.,F.D.}) \text{ for } \frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}\leq\mu_2<1.\end{cases}$$

   All these equilibria are either not sequential outside the specified interval, or defeated by the equilibrium which is undefeated for these parameters.

5. $\delta(1-\lambda)\frac{1-\delta(1-\lambda)}{1-\delta^2(1-\lambda)} \vee \frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda} \leq \delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)$:

   The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \delta\mu_1\lambda+\delta(1-\delta)(1-\lambda), \\ (\text{F.D.}, \text{F.D.}) \text{ for } \delta\mu_1\lambda+\delta(1-\delta)(1-\lambda) \leq \mu_2 < 1. \end{cases}$$

   The equilibrium (P.D., P.D.) does indeed defeat (F, F) for $\mu_2 < \delta\mu_1\lambda + \delta(1-\delta)(1-\lambda)$.

5. $(\lambda, \mu_1) \in \Omega_4 \triangleq \left\{(\lambda, \mu_1) \mid \lambda \geqq \frac{\delta}{(1+\delta^2)}\left(1-\mu_1+\delta^2/(1+\delta)\right), \mu_1 \leqq \delta(1-\lambda)\right\}$

   1. $\frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)} < \frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}$:

   The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)}, \\ (\text{F.D.}, \text{P.D.}) \text{ for } \frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)} \leq \mu_2 < \frac{\frac{\delta}{1+\delta}-\delta\lambda+\frac{\delta^2}{1-\delta^2}(\lambda-\delta)^+}{\lambda}, \\ (\text{F.D.}, (\text{F.D.}, \text{P.D.})) \text{ for } \frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda} \leq \mu_2 < \frac{\delta(1-\mu_1)-\delta l+\frac{\delta^2(\lambda-\delta)^+-(1-\delta)}{1-\delta^2}}{\lambda}, \\ (\text{F.D.}, \text{F.D.}) \text{ for } \frac{\delta(1-\mu_1)-\delta l+\frac{\delta^2(\lambda-\delta)^+-(1-\delta)}{1-\delta^2}}{\lambda} \leq \mu_2 < 1. \end{cases}$$

6. $(\lambda, \mu_1) \in \Omega_5 \triangleq \left\{(\lambda, \mu_1) \mid \frac{\delta}{1+\delta} \leqq \lambda \leqq \frac{\delta}{(1+\delta^2)}\left(1-\mu_1+\delta^2/(1+\delta)\right)\right\}$

   1. $\frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)} < \frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda}$:

   The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 < \frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)}, \\ (\text{F.D.}, \text{P.D.}) \text{ for } \frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)} \leq \mu_2 < \frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda}, \\ (\text{F.D.}, (\text{F.D.}, \text{P.D.})) \text{ for } \frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda} \leq \mu_2 < 1-\delta, \\ (\text{F.D.}, (\text{F.D.}, 0\text{D.})) \text{ for } 1-\delta \leq \mu_2 < 1. \end{cases}$$

   To prove this, denote by (F.D., P.D.)$'$ a sequential equilibrium (S.E.) in which 1_h fully deters by bidding $(\mu_2+\delta)\lambda$ while 1_l partially deters by bidding $\delta\lambda$ (the complete specification of strategies and beliefs is left to the reader, as for the following S.E.). Denote by (F.D., 0D.)$'$ the S.E. in which 1_h fully deters by bidding λ while 1_l deters no type and bids 0.

The S.E. (P.D., P.D.) is defeated for $\mu_2 \in \left(\frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)}, \frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda}\right]$ by (F, P), for $\mu_2 \in \left(\frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda}, 1-\delta\right]$ by (F.D., P.D.)$'$ and by (F.D., 0D.)$'$ for $\mu_2 \in \left(1-\delta, 1-\delta+\delta^2+\delta\mu_1-\frac{\delta^3}{(1+\delta)\lambda}\right]$. For $\mu_2 > 1-\delta+\delta^2+\delta\mu_1-\frac{\delta^3}{(1+\delta)\lambda}$, (P.D., P.D.) is not sequential. For $0 \le \mu_2 < \frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)}$ exists and is obviously undefeated.

The S.E. (F.D., P.D.) is defeated by (P.D., P.D.) for $\mu_2 \in \left[0, \frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)}\right)$, and is not sequential for $\mu_2 > \frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda}$. It is undefeated for $\mu_2 \in \left[\frac{\delta\mu_1\lambda+\delta(1-\delta)(1-\lambda)}{1-\delta^2(1-\lambda)}, \frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda}\right)$.

The S.E. (F.D., (F.D.,P.D.)) only exists for $\mu_2 \in \left[\frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda}, 1-\delta\right)$, and is undefeated on this interval.

The S.E. (F.D., (F.D.,0D.)) requires $\mu_2 \ge 1-\delta$, and is undefeated under this condition.

2. $\frac{\frac{\delta}{1+\delta}-\delta\lambda}{\lambda} < \frac{\delta\mu_1\lambda+\delta^2(\lambda-\delta/(1+\delta))}{(1-\delta^2)(1-\lambda)} \le 1-\delta$:

The unique undefeated equilibrium is:

$$\begin{cases} \text{(P.D., P.D.)} \text{ for } 0 \le \mu_2 \le \frac{\delta\mu_1\lambda+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{(1-\delta^2)(1-\lambda)}, \\ \text{(F.D., (F.D.,P.D.))} \text{ for } \frac{\delta\mu_1\lambda+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{(1-\delta^2)(1-\lambda)} < \mu_2 < 1-\delta, \\ \text{(F.D., (F.D.,0D.))} \text{ for } 1-\delta \le \mu_2 < 1. \end{cases}$$

To see this, notice that (P.D., P.D.) is defeated by (F.D., P.D.)$'$ for $\mu_2 \in \left(\frac{\delta\mu_1\lambda+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{(1-\delta^2)(1-\lambda)}, 1-\delta\right]$ and by (F.D.,0D.)$'$ for $\mu_2 \in \left(1-\delta, 1-\delta+\delta^2+\delta\mu_1-\frac{\delta^3}{(1+\delta)\lambda}\right]$, above which it is not sequential. Obviously, it is undefeated when $\mu_2 \le \frac{\delta\mu_1\lambda+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{(1-\delta^2)(1-\lambda)}$. The S.E. (F.D., (F.D.,P.D.)) is defeated by (P.D., P.D.) for $\mu_2 < \frac{\delta\mu_1\lambda+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{(1-\delta^2)(1-\lambda)}$ and does not exist for $\mu_2 > 1-\delta$, and $(F, F0)$ is undefeated for the same reasons than before.

3. $1-\delta \leq \frac{\delta\mu_1\lambda+\delta^2(\lambda-\delta/(1+\delta))}{\left(1-\delta^2\right)(1-\lambda)}$:

The unique undefeated equilibrium is:

$$\begin{cases} (\text{P.D.}, \text{P.D.}) \text{ for } 0 \leq \mu_2 \leq \frac{(1-\delta)\lambda+\delta\lambda\mu_1+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{1-\delta^2(1-\lambda)}, \\ (\text{F.D.}, (\text{F.D.,0D.})) \text{ for } \frac{(1-\delta)\lambda+\delta\lambda\mu_1+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{1-\delta^2(1-\lambda)} < \mu_2 < 1. \end{cases}$$

Indeed, (P.D., P.D.) is defeated by $(\text{F.D.,0D.})'$ for

$$\mu_2 \in \left(\frac{(1-\delta)\lambda+\delta\lambda\mu_1+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{1-\delta^2(1-\lambda)}, 1-\delta+\delta^2+\delta\mu_1-\frac{\delta^3}{(1+\delta)\lambda}\right],$$

above which it is not sequential, and (F.D., (F.D.,0D.)) is defeated by (P.D., P.D.) for $\mu_2 < \frac{(1-\delta)\lambda+\delta\lambda\mu_1+\delta^2\left(\lambda-\frac{\delta}{1+\delta}\right)}{1-\delta^2(1-\lambda)}$. These S.E. are furthermore undefeated on the specified intervals.