Lattice schemes for option pricing, such as tree or grid/partial differential equation (p.d.e.) methods, are usually designed as a discrete version of an underlying continuous model of stock prices. The parameters of such schemes are chosen so that the discrete version “best” matches the continuous one. Only in the limit does the lattice option price model converge to the continuous one. Otherwise, a discretization bias remains. A simple modification of lattice schemes which reduces the discretization bias is proposed. The modification can, in theory, be applied to any lattice scheme. The main idea is to adjust the lattice parameters in such a way
that the option price bias, not the stock price bias, is minimized. European options are used, for which the option price bias can be evaluated precisely, as a template to modify and improve American option methods. A numerical study is provided. © 2006 Wiley Periodicals, Inc. Jrl Fut Mark 26:733–757, 2006

INTRODUCTION

The binomial tree introduced by Cox, Ross, and Rubinstein in 1979 (hereafter CRR) is one of the most important innovations to have appeared in the option pricing literature. Beyond its original use as a tool to approximate the prices of European and American options in the Black–Scholes (1973) framework, it has been adapted to numerous other instruments and is also widely used as a pedagogical device to introduce various key concepts in option pricing.

In the literature, many modifications have been proposed to improve the performance of the CRR approach. In Hull and White (1988), a control variate approach based on the European Black–Scholes price is suggested to improve the quality of the American price. Tian (1993) proposed to force higher moments of the discrete probability distribution of the stock price to match the moments of the underlying continuous distribution. Others, such as Broadie and Detemple (1996) and Tian (1999), proposed modifications smoothing out the jagged “price versus number of time steps” curve of the CRR approach, enabling the use of Richardson extrapolation. Tian’s (1999) method is essentially a CRR tree modified by a tilt parameter, whereas Broadie and Detemple (1996) use the Black–Scholes price instead of the usual continuation value at the penultimate time step before the option’s maturity. Other suggestions from Boyle (1988) and Kamrad and Ritchken (1991) are to replace the binomial tree by a trinomial tree. Figlewski and Gao (1999) for their part proposed a trinomial tree with sections of finer meshing, allowing a greater accuracy at the cost of negligible supplementary computations.

This short survey of the tree literature serves to emphasize that in all the approaches mentioned, the lattice is designed to minimize the discrepancies between the approximate (discrete) and target (continuous) probability distributions of the stock price by matching their first few moments. The rationale for this is that for any fixed number of time steps, a moment-matching lattice is believed to produce good option price estimates, because the estimate converges to the (continuous) model solution in the limit case of an infinite number of time steps.
Here a different avenue to lattice design which relies on a change of probability measure is suggested. Specifically, we show how to design lattice schemes when the design goal is not the usual moment matching. We build a lattice minimizing the difference between the lattice-based price and the Black–Scholes analytic price for a European option. This lattice is then used to price the corresponding American option. For a fixed number of time steps, such a modified lattice induces a discrete stock price distribution whose moments show some departure from the moments of the continuous, target distribution; in the limit however, the distribution associated to the modified lattice converges to the continuous distribution, as is the case for the unmodified lattice schemes. Figure 1 displays the typical improvement obtained with our modified CRR binomial tree for European and American put options. For the European option, the modified CRR approach yields much smaller price errors than the unmodified CRR approach. In the American case, the

\begin{figure}
\centering
\begin{subfigure}{0.5\textwidth}
\centering
\includegraphics[width=\textwidth]{European_put_price}
\caption{European put price}
\label{fig:European_put_price}
\end{subfigure}
\begin{subfigure}{0.5\textwidth}
\centering
\includegraphics[width=\textwidth]{American_put_price}
\caption{American put price}
\label{fig:American_put_price}
\end{subfigure}
\end{figure}

\begin{itemize}
\item CRR–BS are prices computed using the Cox, Ross, and Rubinstein (1979) binomial tree with the Black–Scholes price at the penultimate time step; modified CRR–BS are prices computed using the Cox, Ross, and Rubinstein (1979) binomial tree with a Black–Scholes price at the penultimate time step and the modification proposed in this study. True price is the Black–Scholes price for the European case and a 15,000 step CRR–BS tree for the American case. Parameters: $s_0 = 100$; $K = 100$; $T = 1$; $\sigma = 0.4$; $r = 0.05$.
\end{itemize}
biases are also improved by the modification, though the improvement is not as dramatic. As shown in Figure 2, the modified binomial tree computations may be carried out with a little more effort so that there is an important gain of precision at a low computational cost. Note that all lattice approaches in this article, whether with or without our modification, are implemented with the Broadie and Detemple (1996) idea of using Black–Scholes prices at the penultimate time step.

Technically, the modification is carried out by replacing the expectation under the risk neutral probability measure, by an equivalent expression under an alternative probability measure. This equivalent expression is specified by the Radon–Nikodym derivative. The family of alternative probability measures considered here is characterized by a unique parameter corresponding to the drift of the stock price process. A one-dimensional numerical search yields the parameter that minimizes the price discrepancy. This minimization can be performed with lattices of

![Image of Figure 2](image_url)
small dimension (say, 5 or 10 time steps); therefore, the additional work is usually small with respect to the precision gain.

The approach can be adapted to many lattice schemes. We describe its application to trees and grids, and provide full numerical experiments for both a binomial tree and a trinomial tree (respectively, that of Cox, Ross, Rubinstein, 1979, and Kamrad and Ritchken, 1991). The article is divided as follows. After this Introduction, background concepts on the changes of measures considered in this study are presented. It is then shown how to apply these measure changes to modify existing lattice schemes. The details on the numerical implementation and the results of this numerical study are given at the conclusion of the article.

**OPTION PRICING AND CHANGES OF PROBABILITY MEASURES**

Under the assumption of arbitrage-free and complete markets, Harrison and Kreps (1979) have shown the existence of a risk neutral probability measure \( Q \), which allows the computation of European style option prices as expected values discounted at the risk-free rate. More formally, the price of a European option, under a constant risk-free rate assumption, can be written as:

\[
V(S_0, 0) = E^Q[e^{-rT}f(S_T, K)]
\]

where \( S_T \) is the stock price at the maturity date \( T \) of the option, \( K \) is the strike price, and \( r \) is the constant risk-free rate. We denote the payoff function of the option as \( f(S_T, K) = \max(\phi(S_T - K), 0) \) with \( \phi = 1 \) for call options and \(-1\) for put options. For an American style option, the situation is more complex because of the early exercise possibility. It can however still be written as an expected value:

\[
V(S_0, 0) = \sup_{\tau \leq T} E^Q[e^{-\tau r}f(S_{\tau}, K)]
\]

where the supremum is over all stopping times \( \tau \leq T \).

This way of expressing the expected values in the option price formula is not unique. Indeed, it is well known that, through the Radon–Nikodym theorem, the expectation under one probability measure \( Q \) can be expressed as an expectation under another equivalent probability measure \( Q^* \). The European and American option prices can thus be written as

\[
V(S_0, 0) = E^Q\left[e^{-rT} f(S_T, K) \frac{dQ}{dQ^*}\right]
\]
where $\frac{dQ}{dQ^*}$ is the Radon–Nikodym derivative.

Theoretically, the expressions under any equivalent probability measure lead to identical prices. In practice, these expectations are often assessed numerically; therefore, different measures may yield different prices. For example, in a Monte Carlo simulation approach, prices computed with the risk-neutral measure may have a larger variance than the prices obtained under alternative measures. This is the rationale for the importance sampling approach in the Monte Carlo simulation literature.

To compute option prices using an alternative measure such as the one given by Equation (3), an expression for the Radon–Nikodym derivative must be available. One way to get such an expression is through the stochastic process specified for the underlying security. In the Black–Scholes context, the dynamics of the stock price under the risk neutral measure $Q$ is specified as

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with $S_0 = s_0$ and $W$ a standard Brownian motion under the risk neutral measure $Q$. To keep the problem tractable, we restrict the dynamics for the stock price under an alternative measure $Q^\lambda$ to the class

$$dS^\lambda_t = \lambda S^\lambda_t dt + \sigma S^\lambda_t dW^\lambda_t$$

with $S^\lambda_0 = s_0$. That is, we consider only measures belonging to the class $\{Q^\lambda : \lambda \in \mathbb{R}\}$ that preserves the geometric Brownian motion structure of the stock price. In this case, the change of measure is accomplished using a likelihood ratio $L(S^\lambda_t, \lambda)$ where

$$L(S^\lambda_t, \lambda) = \exp \left[ \frac{r - \lambda}{\sigma^2} \ln \left( \frac{S^\lambda_t}{s_0} \right) + \frac{1}{2} (\lambda - r) \frac{r + \lambda - \sigma^2}{\sigma^2} t \right]$$

The details are provided in Appendix A.

Equipped with these expressions for the likelihood ratio and the dynamics of the stock price, it is now possible to compute the expectations under alternative probability measures as in Equations (3) and (4). In the next section, we see how these formulas can be applied to modify lattices such as binomial or trinomial trees.
APPLYING CHANGES OF MEASURE TO LATTICE-BASED PRICING APPROACHES

It is described here how the change of measure can be implemented in tree lattices (such as binomial trees) and grid lattices (used with finite-differences approaches). The optimization of the change of measure, i.e. the selection of a proper value for the drift parameter \( \lambda \) is then discussed.

Changes of Measure for Tree Approaches

Consider a tree lattice with \( n \) time steps of length \( T/n \). The stock price at time \( iT/n \) (\( i = 0, 1, \ldots, n \)) and at the \( j \)th node of the lattice is denoted by \( s_{i,j} \). The probability, under the risk neutral measure \( Q \), of reaching node \( s_{i+1,k} \) from node \( s_{i,j} \) is represented by \( q_{i,j \rightarrow k} \). A tree-like lattice scheme simply specifies how the stock and option prices and probabilities can be computed given the length of the time step, the initial stock price, the interest rate and the volatility parameter. These specifications are usually obtained by matching the first two moments of, respectively, the approximate and target distributions.

To adapt a lattice to the alternative probability measure \( Q^\lambda \), it is sufficient to note that the change of measure from \( Q \) to \( Q^\lambda \) preserves the geometric Brownian motion structure of the stock price. The only difference is that the drift coefficient is no longer \( r \), but \( \lambda \). It is therefore straightforward to determine the stock prices and the transition probabilities of the \( Q^\lambda \)-lattice by replacing \( r \) by \( \lambda \) in the design for \( s_{i,j} \) and \( q_{i,j \rightarrow k} \). We shall denote by \( s^\lambda_{i,j} \) the stock price at the \( i \)th time step and the \( j \)th node of the \( Q^\lambda \)-lattice and \( q^\lambda_{i,j \rightarrow k} \) as the \( Q^\lambda \)-probability of reaching node \( s^\lambda_{i+1,k} \) from node \( s^\lambda_{i,j} \).

In the original \( Q \)-lattice, European and American option prices can be obtained by computing expectations sequentially from the end of grid. Indeed, using the familiar dynamic programming principle, the American option value at the \( i \)th time step and the \( j \)th node is

\[
v_{i,j} = \max \left\{ \max \{ \phi(s_{i,j} - K), 0 \}, e^{-rT/n} \sum_k v_{i+1,k} q_{i,j \rightarrow k} \right\}, \quad i < n
\]

while the (special case) European option value at the \( i \)th time step and the \( j \)th node is

\[
v_{i,j} = e^{-rT/n} \sum_k v_{i+1,k} q_{i,j \rightarrow k}, \quad i < n
\]

with the terminal nodes taking values \( v_{n,j} = \max \{ \phi(s_{n,j} - K), 0 \} \) in both cases.
A well-known drawback of lattices such as the binomial or trinomial tree is the jagged convergence pattern of the computed price plotted against the number of time steps. We will therefore adopt the simple modification proposed in Broadie and Detemple (1996) which considerably smooths out the convergence pattern. Specifically, we compute the Black–Scholes price at time step $n/1$, instead of the continuation value from Equation (8), so that the (American) option prices are given by

$$v_{i,j} = \max \left\{ \max \{ \phi(s_{i,j} - K), 0 \}, e^{-r \frac{j}{2}} \sum_k v_{i+1,k} q_{i,j \to k} \right\}, \quad i < n - 1 \quad (10)$$

with $v_{n-1,j} = BS(s_{n-1,j}, r, K, \frac{T}{n}, \phi)$ denoting the Black–Scholes price for a European style option with stock price $s_{n-1,j}$, risk-free rate $r$, volatility coefficient $\sigma$, strike price $K$, and time to maturity $\frac{T}{n}$; recall that $\phi$ is used to denote calls ($\phi = 1$) and puts ($\phi = -1$).

It is straightforward to adapt this tree approach to the $Q^\lambda$-lattice. Indeed, the discretized equivalent for the likelihood ratio (7) is

$$l_{i,j}(\lambda) = \exp \left[ r - \lambda \frac{1}{\sigma^2} \ln \frac{s_{i,j}}{s_0} + \frac{1}{2} (\lambda - r) r + \lambda - \frac{\sigma^2}{2} \frac{T}{n} \right] \quad (11)$$

and the option values will be computed with

$$v_{i,j}^\lambda = \max \left\{ \max \{ \phi(s_{i,j}^\lambda - K), 0 \}, e^{-r \frac{j}{2}} \sum_k v_{i+1,k} q_{i,j \to k}^\lambda \right\}, \quad i < n - 1 \quad (12)$$

where $v_{n-1,j}^\lambda = BS(s_{n-1,j}^\lambda, r, K, \frac{T}{n}, \phi) l_{n-1,j}(\lambda)$. Note that for the American case, because the prices in the tree are those with respect to $Q^\lambda$, it is important to multiply the early exercise value with the likelihood ratio in the dynamic programming equation. The European option is, again, a special case of the American option, and is valued as

$$v_{i,j}^\lambda = e^{-r \frac{j}{2}} \sum_k v_{i+1,k} q_{i,j \to k}^\lambda, \quad i < n - 1 \quad (13)$$

with $v_{n-1,j}^\lambda = BS(s_{n-1,j}^\lambda, r, K, \frac{T}{n}, \phi) l_{n-1,j}(\lambda)$. Appendix B shows how specific lattice schemes such as that of CRR (1979) and Kamrad and Ritchken (1991), can be implemented using the above equations.

**Changes of Measure for Grid/Partial Differential Equation Approaches**

Grid lattices are used to solve the pricing partial differential equations (p.d.e.s) obtained through the classical Black–Scholes analysis, often in
conjunction with finite-difference methods. We describe first the European option case, then its American counterpart.

The impact of the change of measure on a traditional grid/p.d.e. approach for European options is best shown with the help of the Feynman–Kac solution (cf. Duffie, 1996, or Tavella and Randall, 2000). Under the stock price process (6) corresponding to the probability measure $Q^\alpha$, the Feynman–Kac theorem indicates that the European option price given by the expectation (3)

$$V(S, 0) = E^{Q'}\left[e^{-rT} f(S_T, K) \frac{dQ}{dQ^\alpha}\right]$$

is also the solution of the pricing p.d.e.

$$\frac{\partial V(S, t)}{\partial t} + \lambda S \frac{\partial V(S, t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(S, t)}{\partial S^2} - r V(S, t) = 0$$  \hspace{1cm} (14a)

with the (terminal) boundary condition

$$V(S, T) = e^{-rT} f(S_T, K) \frac{dQ}{dQ^\alpha}$$  \hspace{1cm} (14b)

Applying the change of variables

$$U(S, t) = e^{\gamma t} V(S, t)$$

to Equations (14a) and (14b), we obtain the equivalent p.d.e. system

$$\frac{\partial U(S, t)}{\partial t} + \lambda S \frac{\partial U(S, t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U(S, t)}{\partial S^2} - r U(S, t) = 0 \hspace{1cm} (15a)$$

$$U(S, T) = f(S_T, K) \frac{dQ}{dQ^\alpha} \hspace{1cm} (15b)$$

Note that the option price at $t = 0$ is given by $U(S_0, 0) = V(S_0, 0)$. The impact of the change of measure in grid/p.d.e. methods for European options is thus twofold and simple. In comparison with the Black–Scholes p.d.e., the coefficient of the second derivative term changes from $r$ to $\lambda$; and the usual terminal boundary condition must be multiplied by the likelihood ratio.

The American option case cannot be treated per se with a standard Feynman–Kac solution, because of the supremum in the formulation (4). The theory linking the optimal stopping time problems and variational inequalities, in the context of American option pricing, was developed by...
Jaillet, Lamberton, Lapeyre (1990) and Myneni (1992). Using theorem 5.1 from Myneni (pp. 19–20), the stopping time problem can be computed as the solution of the partial differential, complementarity system:

\[
\left( \frac{\partial U(S,t)}{\partial t} + \lambda S \frac{\partial U(S,t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U(S,t)}{\partial S^2} - rU(S,t) \right)
\]

\[
\cdot \left( U(S,t) - f(S,K) \frac{dQ}{dQ^\lambda} \right) = 0 \quad (16a)
\]

\[
\frac{\partial U(S,t)}{\partial t} + \lambda S \frac{\partial U(S,t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U(S,t)}{\partial S^2} - rU(S,t) \leq 0 \quad (16b)
\]

\[
\left( U(S,t) - f(S,K) \frac{dQ}{dQ^\lambda} \right) \geq 0 \quad (16c)
\]

\[
U(S,T) - f(S_T,K) \frac{dQ}{dQ^\lambda} = 0 \quad (16d)
\]

Using appropriate spatial boundary conditions, the systems (15) and (16) can be solved using standard finite differences methods.

**Optimization of the Change of Measure**

The change of measure, parameterized by \( \lambda \), must be chosen appropriately to be beneficial. As mentioned in the Introduction, the crucial idea underlying this article is that the freedom offered by the change of measure technique can be used to decrease the error inherent to discrete methods (i.e., methods on trees and grids). The details are discussed in this section.

**On the Optimization Criteria**

For European option pricing, \( \lambda \) should be set to a value \( \lambda^* \) that minimizes the “bias,” i.e., the difference between the estimated European option price \( v_{0,0}^\lambda \) and the Black–Scholes price given by \( BS(s_0, r, \sigma, K, T, \phi) \). In other words, we seek a zero \( \lambda^* \) of the function:

\[
\text{Bias}(\lambda) = v_{0,0}^\lambda - BS(s_0, r, \sigma, K, T, \phi) \quad (17)
\]

where \( v_{0,0}^\lambda \) is the European option price obtained from the lattice approach.

For lack of an analytical formula for \( \lambda^* \), the value must be computed numerically. One keypoint of the approach proposed in this article
is that the value $\lambda^*$ is not sensitive to the discretization level; that is, $\lambda^*$ can be computed on a coarse (and thus fast) tree or grid, and then used successfully to improve the error of a much finer tree/grid. For example, a value $\lambda^*$ is found that suppresses the bias on a 10-time step tree; the same $\lambda^*$ is then used on a 200-time step tree, yielding a much more precise solution than the 200-time step tree without change of measure.

For American option pricing, where no analytical formula exist to provide a “target,” two approaches are possible. First, the $\lambda^*$ derived from the European case above often works well for the American case. Simply setting $\lambda$ to this value is thus a first possibility. As an alternative approach, one can compute prices of American options with two lattices using $n_1$- and $n_2$-time steps ($n_1 = 10$ and $n_2 = 20$ for example) and choose a $\lambda$ minimizing the difference between the two computed prices; this is done with the goal of “flattening” the convergence pattern.

**Numerical Solution of the Optimization**

As noted, a solution of Equation (17) must be found numerically. The difficulty of finding a zero of a function is dependent on the quality of the bracketing interval (the bounds between which a zero is known to lie), and on the function’s characteristics. The characteristics of $\text{Bias}(\lambda)$ are not easily analyzed mathematically. Empirically however, we observe that $\text{Bias}(\lambda)$ is invariably quasi-concave within a “rough interval” (which is lattice-specific and discussed below), with $\text{Bias}(\lambda)$ negative at both endpoints; this characterization holds for a wide variety of option parameters for both puts and calls (see Figure 3). The interval is called “rough” in that the actual bracketing interval is defined within it. $\text{Bias}(\lambda)$ also has the property that the function’s maximum is located at approximately the same relative position within the rough interval, for all observed options. Finally, in all but extreme sets of parameters, this maximum is positive. As a result, it is easy to devise a root-finding bracketing interval with one endpoint having positive $\text{Bias}(\lambda)$ value (it is the approximate maximum of the function), and the other endpoint having negative $\text{Bias}(\lambda)$ value (it is one of the two endpoints of the rough interval).

Equation (17) typically has many solutions, and two of them are within the rough interval. It is clear, though only empirically, which of the two endpoints gives a bracketing interval whose zero brings the best end results. The rough intervals are lattice-specific and can sometimes be derived analytically; for example, the interval for the tree lattices
is determined as the values of \( \lambda \) which insure positive probabilities in the lattice. These intervals are defined in the Appendix dealing with the specific lattice descriptions.

The bisection algorithm was used to find the zero within the bracketing interval. Experimentation with less-basic algorithms (be it secant, false position, or Ridder’s) left bisection a winner in terms of functions evaluations, and thus computing time. Note that for some option problems (e.g., deep out-of-the-money and very low volatility puts), no zero of \( \text{Bias}(\lambda) \) exists within the rough interval, as the function values lie completely in the negatives. In such cases, an approximate maximum of the function was used as proxy for the non-existent zero. In the numerical search routine, trees with 5 time steps were used to find the optimal value for \( \lambda \).

**COMPUTATIONAL RESULTS**

In this section, we first give a numerical illustration of the benefits of the change of measure; the sample option is priced on the CRR binomial tree with a Black–Scholes price at the penultimate time step (CRR–BS hereafter). Using a CRR–BS binomial tree and a Kamrad–Ritchken...
trinomial tree, we then provide more general results from tests performed on a large pool of options.

In our tests with explicit and implicit finite differences methods, improvements similar to that of the tree-based methods were observed. However, bracketing intervals for the zeros of the bias function are more difficult to obtain, and the values of $\lambda^*$ found for the European option, do not apply readily to the American option with the same parameters. No numerical results are provided for these grid/p.d.e. methods.

**Illustrative Computational Results**

The results provided in this section will offer some intuition on how the expected values and probability distributions are altered under the alternative measure. The following parameter values are used: $s_0 = 100$, $r = 0.05$, $\sigma = 0.4$, $T = 1$ and $n = 3$. These and the formulas for the CRR (1979) lattice given in Appendix B give $u = 1.2598$, $d = 0.7938$, $q^r = 0.4786$, and $q^\lambda = 0.6021$ for $\lambda^* = 0.2152$. This last value was found using the approach described in the Numerical Solution of the Optimization subsection.

In this framework the stock prices remain identical under all measures; hence, it is instructive to compute the discrete distribution at time step $n - 1$ for the stock price using both $\lambda = r$ and $\lambda = \lambda^* = 0.2152$. Using these distributions, it is then possible to make helpful comparisons showing how the proposed modification changes the computed values. Simple calculations show that the probability to reach state $j$ from the initial stock price in $n - 1$ steps can be written as:

$$
\chi_j^n = \frac{(n - 1)!}{j!(n - 1 - j)!}(q^\lambda)^j(1 - q^\lambda)^{n-1-j} \times l_{n-1,j}(\lambda)
$$

Table I reports the computed probabilities $\chi_j^n$ and $\chi_j^{\lambda^*}$ with the associated option prices at step $n - 1$. From the numbers in this table it is easy to see that the expected value and standard deviation (discounted) are 100 and 32.7981 when computed with $\chi_j^n$ while they are 99.9861 and 32.8125 when computed with the distribution $\chi_j^{\lambda^*}$. The European put option price computed with both measures are, 13.3989 and 13.0816 while the Black–Scholes (BS) price is 13.1459 (The slight discrepancy between 13.0816 and 13.1459 comes from the 5-time step optimization procedure). This shows that the discrete distribution that was originally
built to match the first moment of the target distribution is modestly altered to remove the large bias in terms of price from the original CRR–BS approach. This alteration of the probability distribution has a small effect on the discounted expected stock price. Indeed, the changes in probability for large stock prices are offset by the changes in probabilities for low stock prices. However, this is not the case when computing the expected value of the put option payoff, which has a low sensitivity to the probability changes associated to large stock prices and a high sensitivity to the changes associated with low stock prices.

Table II looks at European and American put option prices as the number of time steps is increased, using the CRR–BS binomial tree and the $Q^\lambda$-CRR–BS binomial tree. These numbers can be compared to benchmark values obtained from the Black–Scholes formula for the European put and a 15,000 step CRR–BS binomial tree for the American put. As it can be seen, the option prices computed with a 20- or 30-step $Q^\lambda$-CRR–BS lattice are more accurate than those obtained with 100-time step CRR–BS lattice.

The results presented in this table are specific to the chosen example. Furthermore, the additional work required by the algorithm is not taken into account. In the next section, a numerical study assessing the performance of the algorithm in terms of computer time and precision for a large test pool of option contracts is presented.

### Computational Results on a Pool of Options

To obtain a more general assessment on the quality of the method to compute American option prices, an analysis is performed similar to that of Broadie and Detemple (1996). The analysis begins by choosing a large
### TABLE II

$Q^{\lambda^*}$-CRR–BS Prices for Put Options

<table>
<thead>
<tr>
<th>$n$</th>
<th>$v_{0,0}^{\lambda^*}$</th>
<th>$v_{0,0}$</th>
<th>$v_{0,0}^{\lambda^*}$</th>
<th>$v_{0,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>13.2563</td>
<td>13.1507</td>
<td>13.7544</td>
<td>13.6698</td>
</tr>
<tr>
<td>60</td>
<td>13.1652</td>
<td>13.1463</td>
<td>13.6864</td>
<td>13.6719</td>
</tr>
<tr>
<td>90</td>
<td>13.1588</td>
<td>13.1461</td>
<td>13.6799</td>
<td>13.6714</td>
</tr>
<tr>
<td>100</td>
<td>13.1576</td>
<td>13.1461</td>
<td>13.6779</td>
<td>13.6712</td>
</tr>
<tr>
<td>200</td>
<td>13.1517</td>
<td>13.1460</td>
<td>13.6742</td>
<td>13.6699</td>
</tr>
<tr>
<td>400</td>
<td>13.1488</td>
<td>13.1459</td>
<td>13.6712</td>
<td>13.6690</td>
</tr>
<tr>
<td>3000</td>
<td>13.1463</td>
<td>13.1459</td>
<td>13.6682</td>
<td>13.6679</td>
</tr>
</tbody>
</table>

Note: $v_{0,0}^{\lambda^*}$ is the option price computed with the original CRR–BS lattice, while $v_{0,0}$ is the option price computed with the $Q^{\lambda^*}$-CRR–BS lattice. The benchmark values for the European and American put options are 13.1459 and 13.6677 and are obtained with the Black–Scholes formula and a 15,000-step CRR–BS lattice. Parameter values: $s_0 = 100, K = 100, r = 0.05, \sigma = 0.40, T = 1, \lambda^* = 0.2152.$

A test pool of options using randomly selected parameter values based on predetermined distributions. For each option, the prices using the $Q$-lattice and the $Q^{\lambda^*}$-lattice are computed and compared to a benchmark value which is obtained from a CRR (1979) binomial tree with 15,000 steps.

The following distributions are used for the parameter values of the options, with each parameter value drawn independently of the others: $T$ is uniformly distributed between 0.1 and 1.0 year; $K$ is uniformly distributed between 70 and 130; $r$ is uniformly distributed between 0 and 0.1; $\sigma$ is uniformly distributed between 0.1 and 0.6. The value $S_0$ is fixed at 100. Ten different lattice sizes (in terms of time steps) are examined for each method: 100, 200, . . . , 1000.
Using the simulated parameter values, the lattice prices and the benchmarks are computed and compared. The root mean squared error is used as the measure of aggregate pricing error. Specifically,

\[ RMSE(m) = \sqrt{\frac{1}{m} \sum_{i=1}^{m} e_i^2} \]  

(18)

where \( e_i = |P_i(\text{lattice}) - P_i|/P_i \); \( P_i \) is the benchmark obtained for the \( i \)th option; and \( P_i(\text{lattice}) \) is the \( i \)th price obtained with the \( Q \)-lattice or the \( Q^A \)-lattice. The variable \( m \) stands for the size of the test pool. In this study, \( m = 2,000 \) was used and the cases were eliminated where \( P_i < 0.50 \) to avoid large relative errors caused by a small divider.

Tables III and IV report the results about the gain in precision associated with the use of the proposed method for European and American put with a CRR (1979) lattice. In these tables, the gain in precision is measured by the ratio of RMSE. The gains for European options are substantial. Using the proposed method will reduce the average pricing error by a factor of roughly 12 when compared to the original CRR (1979) binomial tree and by a factor of approximately 3.8 when compared to the CRR (1979) with a Black–Scholes price at the penultimate time step. For the American case, the gains are also significant.

### Table III

<table>
<thead>
<tr>
<th>( n )</th>
<th>All samples</th>
<th>( T &lt; 0.5 )</th>
<th>( T \geq 0.5 )</th>
<th>At-money</th>
<th>Out-money</th>
<th>( \sigma &lt; 0.35 )</th>
<th>( \sigma \geq 0.35 )</th>
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<tbody>
<tr>
<td>Ratio of RMSE for the CRR and ( Q^A )-CRR–BS lattices</td>
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<td></td>
<td></td>
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<td>12.11</td>
<td>9.67</td>
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<td>22.51</td>
<td>10.80</td>
<td>13.41</td>
<td>11.19</td>
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<td>Ratio of RMSE for the CRR–BS and ( Q^A )-CRR–BS lattices</td>
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<td>9.14</td>
<td>2.99</td>
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Note. CRR is the CRR (1979) binomial tree; CRR–BS is the CRR (1979) binomial tree with the Black–Scholes price at the penultimate time step; \( Q^A \)-CRR–BS is the CRR (1979) binomial tree with a Black–Scholes price at the penultimate time step and the modification proposed in this study. “At-money” is for the subsample for which \( 0.9 < S_0/K < 1.1 \), whereas “Out-money” is for \( S_0/K \geq 1.1 \) and \( S_0/K \leq 0.9 \).
## TABLE IV
Ratio of RMSE for American Put Options

<table>
<thead>
<tr>
<th>n</th>
<th>All samples</th>
<th>$T &lt; 0.5$</th>
<th>$T \geq 0.5$</th>
<th>At-money</th>
<th>Out-money</th>
<th>$\sigma &lt; 0.35$</th>
<th>$\sigma \geq 0.35$</th>
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<td></td>
</tr>
<tr>
<td>100</td>
<td>7.17</td>
<td>6.85</td>
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<td>7.52</td>
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<td>6.72</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ratio of RMSE for the CRR and $Q^{l*}$-CRR–BS lattices</td>
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<td>2.98</td>
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<td>2.90</td>
<td>2.70</td>
<td>2.71</td>
<td>2.56</td>
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</table>

Note. CRR is the CRR (1979) binomial tree; CRR–BS is the CRR (1979) binomial tree with the Black–Scholes price at the penultimate time step; $Q^{l*}$-CRR–BS is the CRR (1979) binomial tree with a Black–Scholes price at the penultimate time step and the modification proposed in this study. "At-money" is for the subsample for which $0.9 < S_0/K < 1.1$, whereas "Out-money" is for $S_0/K \geq 1.1$ and $S_0/K \leq 0.9$.

with ratios of approximately 7 and 2.5. When the sample is broken in various classes we can see that the largest gains with respect to the CRR–BS case for American options are obtained for at-the-money options (which are defined by $0.9 < S_0/K < 1.1$) and high volatility options ($\sigma \geq 0.35$).

These results do not account for the additional effort associated with the proposed modifications; Figures 2 and 4 plot the log of the RMSE as a function of the log of the computing time for $n = 100, 200, \ldots, 1000$. As shown by the graph, the gain in precision is worth the additional effort associated to the numerical optimization and computation of the $Q^{l*}$ lattice. In this graph, the preferred method in terms of cost–benefit is in the lower left corner. The CRR–BS–$Q^{l*}$ clearly dominates the other two methods (CRR and CRR–BS) for both the European and American cases.

Tables V and VI present the results for the Kamrad and Ritchken (1991) trinomial tree. For this lattice scheme, the results are qualitatively similar, i.e., the $Q^{l*}$ lattice obtains sensible gains in terms of accuracy when measured in terms of RMSE ratio. Again, taking into account the computational effort, we see that the KR-BS–$Q^{l*}$ clearly dominates the other two methods (KR and KR-BS) for both the European and American cases as shown in Figures 5 and 6.
### TABLE V

Ratio of RMSE for European Put Options

<table>
<thead>
<tr>
<th>n</th>
<th>All samples</th>
<th>$T &lt; 0.5$</th>
<th>$T \geq 0.5$</th>
<th>At-money</th>
<th>Out-money</th>
<th>$\sigma &lt; 0.35$</th>
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**Ratio of RMSE for the KR and $Q^1$-KR–BS lattices**

<table>
<thead>
<tr>
<th>n</th>
<th>All samples</th>
<th>$T &lt; 0.5$</th>
<th>$T \geq 0.5$</th>
<th>At-money</th>
<th>Out-money</th>
<th>$\sigma &lt; 0.35$</th>
<th>$\sigma \geq 0.35$</th>
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<td>9.62</td>
<td>2.55</td>
<td>3.45</td>
<td>3.21</td>
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</table>

**Ratio of RMSE for the KR–BS and $Q^1$-KR–BS lattices**

**Note.** KR is the Kamrad-Ritchken (1979) trinomial tree; KR–BS is the Kamrad-Ritchken (1991) trinomial tree with the Black–Scholes price at the penultimate time step; $Q^1$-KR–BS is the Kamrad-Ritchken (1991) trinomial tree with a Black–Scholes price at the penultimate time step and the modification proposed in this study. “At-money” is for the subsample for which $0.9 < S_0/K \leq 1.1$, whereas “Out-money” is for $S_0/K \geq 1.1$ and $S_0/K \leq 0.9$.  

---

**FIGURE 4**

RMSE is the relative error defined by $\sqrt{\frac{1}{m} \sum_{i=1}^{m} e_i^2}$, where $e_i = \frac{|P_i(\text{lattice}) - P_i|}{P_i}; P_i$ is the benchmark obtained for the $i$th option; and $P_i(\text{lattice})$ is the $i$th price obtained with the corresponding lattice (CRR: Cox, Ross, and Rubinstein, 1979, binomial tree; CRR–BS: Cox, Ross, and Rubinstein, binomial tree with the Black–Scholes price at the penultimate time step; Modified CRR–BS: Cox, Ross, and Rubinstein lattice with the Black–Scholes price at the penultimate time step and the $Q^1$-lattice modification proposed in this study. The preferred method in terms of cost–benefit is in the lower left corner of the graph.
TABLE VI
Ratio of RMSE for American Put Options

<table>
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<tr>
<th>n</th>
<th>All samples</th>
<th>T &lt; 0.5</th>
<th>T ≥ 0.5</th>
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<th>Out-money</th>
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<th>σ ≥ 0.35</th>
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<tbody>
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<td>3.17</td>
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<td>2.95</td>
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Ratio of RMSE for the KR and Q\(1\)-KR–BS lattices

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<th>All samples</th>
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<th>T ≥ 0.5</th>
<th>At-money</th>
<th>Out-money</th>
<th>σ &lt; 0.35</th>
<th>σ ≥ 0.35</th>
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</thead>
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<td>1.61</td>
<td>2.45</td>
<td>1.58</td>
<td>1.68</td>
<td>1.56</td>
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<td>2.51</td>
<td>1.68</td>
<td>3.92</td>
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</tbody>
</table>

Ratio of RMSE for the KR–BS and Q\(1\)-KR–BS lattices

Note. KR is the Kamrad-Ritchken (1979) trinomial tree; KR–BS is the Kamrad-Ritchken (1991) trinomial tree with the Black–Scholes price at the penultimate time step; Q\(1\)-KR–BS is the Kamrad-Ritchken (1991) trinomial tree with a Black–Scholes price at the penultimate time step and the modification proposed in this study. “At-money” is for the subsample for which 0.9 \(\leq S_0/K \leq 1.1\), whereas “Out-money” is for \(S_0/K \geq 1.1\) and \(S_0/K \leq 0.9\).

RMSE is the relative error defined by \(\sqrt{\frac{1}{n} \sum_{i=1}^{n} e_i^2}\) where \(e_i = |P_i(lattice) - P_i|/P_i\); \(P_i\) is the benchmark obtained for the \(i\)th option; and \(P_i(lattice)\) is the \(i\)th price obtained with the corresponding lattice (KR: Kamrad and Ritchken, 1991, trinomial tree; KR–BS: Kamrad and Ritchken trinomial tree with the Black–Scholes price at the penultimate time step; modified KR–BS: Kamrad and Ritchken trinomial lattice with the Black–Scholes price at the penultimate time step and the \(Q\)-lattice modification proposed in this study. The preferred method in terms of cost–benefit is in the lower left corner of the graph.
CONCLUSION

We have introduced a modification to lattices often used to price American options in the Black–Scholes context. The modification replaces the expectation under the risk neutral probability measure by an equivalent expression under an alternative probability measure, which is specified by the Radon–Nikodym derivative. The proper change of measure is obtained by an optimization procedure on trees of small dimension. The results show that the additional work associated with the optimization step is largely offset by the gain in precision.

The approach should be adaptable to other pricing models for which the underlying driving process is a Brownian motion. In these cases, the change of measure can still be characterized by a unique parameter that...
enables the use of simple numerical search routines for the calibration.
It should also be mentioned that our approach, in its current form, relies
on a theoretical price for the European option. This price is needed to
smooth out the convergence of the lattice prices. An extension of the
approach proposed in this research could be performed by examining dif-
ferent smoothing devices such as, for example, the adaptive mesh tree
suggested in Figlewski and Gao (1999).

APPENDIX A

The Change of Measure

Define

$$W^\lambda_i = W_i - \frac{\lambda - r}{\sigma} t \quad \text{for any } t \geq 0 \quad (19)$$

and note that the replacement of $W_i$ by $W^\lambda_i + \frac{\lambda - r}{\sigma} t$ in Equation (5) leads
to Equation (6). We would like the distribution of $W^\lambda_i$ to be $N(0, t)$ under
the measure $Q^\lambda$, which is equivalent to ask that $W_i$ is $N(\frac{\lambda - r}{\sigma} t, t)$
under the measure $Q^\lambda$. Since $W_i$ is $N(0, t)$ under measure $Q$ and
$N(\frac{\lambda - r}{\sigma} t, t)$ under the measure $Q^\lambda$, the change of measure is obtained
using a ratio of density functions:

$$L^*(W_i, \lambda) = \frac{\frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{1}{2} \frac{W_i^2}{t}\right]}{\left(\frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{1}{2} \frac{(W_i - \lambda r)}{\sigma^2 t}\right]\right]} = \exp\left(-\frac{\lambda - r}{\sigma} W_i + \frac{1}{2} \left(\frac{\lambda - r}{\sigma}\right)^2 t\right)$$

By the simple change of variable presented in Equation (19), the like-
lihood ratio is expressed as a function of the $Q^\lambda$–Brownian motion:

$$L^{**}(W^\lambda_i, \lambda) = \exp\left[-\frac{\lambda - r}{\sigma} W^\lambda_i - \frac{1}{2} \left(\frac{\lambda - r}{\sigma}\right)^2 t\right]$$

Finally, since

$$S^\lambda_i = s_0 \exp\left[\left(\lambda - \frac{\sigma^2}{2}\right)t + \sigma W^\lambda_i\right]$$

implies that

$$W^\lambda_i = \frac{\ln(S^\lambda_i) - (\lambda - \frac{\sigma^2}{2})t}{\sigma}$$
the likelihood ratio can also be expressed as a function of the stock price under $Q$:

$$ L(S_t^\lambda, \lambda) = \exp \left[ -\frac{\lambda - r}{\sigma} \frac{\ln(S_t^\lambda/s_0)}{\sigma} - \frac{1}{2} \left( \frac{\lambda - r}{\sigma} \right)^2 t \right] $$

$$ = \exp \left[ \frac{r - \lambda}{\sigma^2} \ln \left( \frac{S_t^\lambda}{s_0} \right) + \frac{1}{2} (\lambda - r) \frac{\ln(\lambda - r) - \sigma^2}{\sigma^2} t \right] \quad (20) $$

APPENDIX B

Lattice Schemes Under Changes of Measure

Binomial Trees

Many binomial trees are designed on a similar basis. The stock price at $i$th time step and the $j$th node is

$$ s_{i,j} = s_0 u^i d^{j-i}, \; i \in \{0, 1, \ldots, n\}, \; j \in \{0, 1, \ldots, i\} $$

where $u$ and $d$ are the multiplicative constants for up and down movements in the tree. The probability, under the risk neutral measure $Q$, of an upward move is $q$, i.e.,

$$ q_{i,j \rightarrow k} = \begin{cases} 
q & \text{if } k = j + 1 \\
1 - q & \text{if } k = j \\
0 & \text{otherwise} 
\end{cases} $$

To choose the different constants $u$, $d$, and $q$, many authors proposed to match the first two moments of the binomial stock price with those of the target continuous time stochastic process, which leads to the two following equations related to the expectation and the variance:

$$ qu + (1 - q)d = e^{rT}/n $$

$$ qu^2 + (1 - q)d^2 - e^{2rT}/n = e^{2rT}(e^{\sigma^2T}/n - 1) $$

Because there are three variables and two equations, there is some freedom to assess a value to one of the variables. This leads to the different versions of the binomial tree.

The Binomial Tree of Cox, Ross, and Rubinstein

In the binomial tree proposed by CRR (1979), we have

$$ u = \exp \left[ \sigma \sqrt{T/n} \right] \quad \text{and} \quad d = \exp \left[ -\sigma \sqrt{T/n} \right] \quad (21) $$
and
\[
q = \frac{\exp[r^T_n] - d}{u - d} = \frac{\exp[r^T_n] - \exp[-\sigma \sqrt{T_n}/n]}{\exp[\sigma \sqrt{T_n}/n] - \exp[-\sigma \sqrt{T_n}/n]}
\] (22)

To adapt this tree to the measure \(Q^\lambda\), note that the discrete stock prices
\[
s^\lambda_{i,j} = s_0 u^i d^{j-i} = s_0 \exp[(2j - i)\sigma \sqrt{T/n}]
\] (23)
do not depend on \(r\) and thus remain identical to those of the original lattice. The probability \(q^\lambda\) of an upward move becomes
\[
q^\lambda = \frac{\exp[\lambda^T_n] - d}{u - d} = \frac{\exp[\lambda^T_n] - \exp[-\sigma \sqrt{T_n}/n]}{\exp[\sigma \sqrt{T_n}/n] - \exp[-\sigma \sqrt{T_n}/n]}
\] (24)

The likelihood ratio for the \(i\)th time step and the \(j\)th node is
\[
l_{i,j}(\lambda) = \exp \left[ \frac{r - \lambda}{\sigma} (2j - i) \sqrt{T/n} + \frac{1}{2}(\lambda - r) \frac{\lambda + r - \sigma^2}{\sigma^2} i \right] \] (25)

The rough interval in which appropriate values of \(\lambda\) can be searched for (discussed in the Computational Results section) is derived from the inequalities \(0 \leq q^\lambda \leq 1\). Indeed, replacing \(q^\lambda\) by Equation (24) in \(0 \leq q^\lambda \leq 1\) and isolating \(\lambda\) leads to
\[
-\frac{\sigma}{\sqrt{T/n}} \leq \lambda \leq \frac{\sigma}{\sqrt{T/n}}
\]

The Trinomial Tree of Kamrad and Ritchken

The trinomial tree proposed by Kamrad and Ritchken (1991) matches the two first moments of the stock return to their theoretical values. In this particular set up, the stock price at \(i\)th time step and the \(j\)th node is
\[
s_{i,j} = s_0 \exp \left[ (i - j)\sigma \sqrt{\frac{3}{2} \sqrt{T/n}} \right], \quad i \in \{0, 1, \ldots, n\}, \quad j \in \{0, 1, \ldots, 2i\}
\] (26)

The transition probabilities, under the risk neutral measure \(Q\), are
\[
q_{i,j \rightarrow k} = \begin{cases} 
\frac{1}{3} + \frac{1}{\sqrt{6} \sigma} \left( r - \frac{\sigma^2}{2} \right) \sqrt{T/n} & \text{if } k = j + 1 \\
\frac{1}{3} & \text{if } k = j \\
\frac{1}{3} - \frac{1}{\sqrt{6} \sigma} \left( r - \frac{\sigma^2}{2} \right) \sqrt{T/n} & \text{if } k = j - 1 \\
0 & \text{otherwise}
\end{cases}
\] (27)
To adapt this tree to the change of measure, note that the stock price remains the same $s_{ij}^\lambda = s_{ij}$ because it does not depend on $r$. The transition probabilities become

$$q_{ij}^{\lambda} = \begin{cases} 
\frac{1}{3} + \frac{1}{\sqrt{2\pi}} (\lambda - \frac{\sigma^2}{2}) \sqrt{\frac{T}{n}} & \text{if } k = j + 1 \\
\frac{1}{3} + \frac{1}{\sqrt{2\pi}} (\lambda - \frac{\sigma^2}{2}) \sqrt{\frac{T}{n}} & \text{if } k = j \\
\frac{1}{3} - \frac{1}{\sqrt{2\pi}} (\lambda - \frac{\sigma^2}{2}) \sqrt{\frac{T}{n}} & \text{if } k = j - 1 \\
0 & \text{otherwise}
\end{cases}$$ (28)

The likelihood ratio for the $i$th time step and the $j$th node is

$$l_{ij}(\lambda) = \exp \left[ \frac{r - \lambda}{\sigma} \sqrt{\frac{3}{2}} (i - j) \sqrt{\frac{T}{n}} + \frac{1}{2}(\lambda - r) \frac{r + \lambda - \frac{\sigma^2}{2} i T}{\sigma^2 n} \right]$$ (29)

The rough interval discussed in the Computational Results section is derived from the inequalities $0 \leq q_{ij,j-1}^{\lambda}$ and $0 \leq q_{ij,j+1}^{\lambda}$:

$$\frac{\sigma^2}{2} - \frac{\sqrt{6}}{3} \frac{\sigma}{\sqrt{T/n}} \leq \lambda \leq \frac{\sigma^2}{2} + \frac{\sqrt{6}}{3} \frac{\sigma}{\sqrt{T/n}}$$

Note that $q^{\lambda} \leq 1$ gives the looser bounds $\frac{\sigma^2}{2} - 2\frac{\sqrt{6}}{3} \frac{\sigma}{\sqrt{T/n}} \leq \lambda \leq \frac{\sigma^2}{2} + 2\frac{\sqrt{6}}{3} \frac{\sigma}{\sqrt{T/n}}$.

**BIBLIOGRAPHY**


