The consistency axiom in cost-allocation problems: a discussion*

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Abstract

We take the lack of consensus on what is an appropriate definition of the consistency axiom in cost-sharing problems as evidence that close examination is necessary. We contrast the two most popular definitions of consistency by recalling results in Sudhölter (GEB1998) for homogenous cost functions, and using several examples to argue that the definition found in Hart and Mas-Collel (1989) is the most relevant one.

1 Introduction

The *consistency axiom* has been widely discussed in situations where a fixed resource must be allocated between a number of individuals. These situations include the allocation of a single private good in the presence (Young, 1987) or absence (Sönmez, 1994) of conflicting claims, the assignment of indivisible commodities (Sasaki, 95; Ehlers and Klaus, 2005), the allocation of several commodities in exchange economies (Tadenuma and Thomson, 1991; Thomson and Zhou, 1993), the problem of land division (Chambers 2004), as well as matching (Sasaki and Toda, 1992; Toda, 2006) and bargaining problems (Lensberg, 1987; Thomson and Lensberg, 1989). We refer the reader to Thomson (2006) for a comprehensive survey of the literature on consistency in such allocation problems. The key principle behind the axiom is that of self-similarity: an allocation rule is consistent if the resource shares it (re-)allocates in any *reduced problem*, formed of a subset of the original population along with their allocation under this very rule, match the shares it allocates in the original problem (the *original shares*).

Consistency is an appealing axiom for at least two reasons. From a practical standpoint, only consistent allocation rules can actually be carried out. Indeed,

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because an inconsistent allocation rule could yield a reduced problem for which some agents receive (under the same rule) shares other than their original shares it is not clear how it should be implemented. Some individuals will prefer that it be applied to the original problem while others will claim that the reduced problem is the relevant level of application. Both arguments will be legitimate.

Consistency also bears a conceptual appeal. Since allocation rules are typically decided upon because they satisfy a number of desirable properties (e.g., fairness, efficiency, incentive compatibility...), ensuring that these underlying principles are respected when restricting attention to any subset of individuals is essential. The consistency axiom does precisely that. In this regard, there exists an analogy between consistent allocation rules and self-similar mathematical objects, like fractals (see, e.g., Hutchinson, 1981), whose structure is preserved regardless of the level of zoom. In our framework, "zooming in" amounts to considering smaller and smaller subsets of individuals, along with their joint original allocation.

Some authors have also given a sequential interpretation to the consistency axiom (e.g., Lensberg, 1987; see also Thomson, 2006, and references therein): after a subset of agents has left the procedure with their allotted share, (re-)applying the same (consistent) allocation rule to the reduced problem yields the same outcome as if the allocation had been reached in a single blow. Therefore, one could potentially save on computational complexity by treating the original allocation problem as a succession of two-person allocation problems.

Although quite attractive, one can argue that this sequential interpretation assumes that claims or renegotiations can only take place after production has ended. Hence, it is a very attractive interpretation when the resource to be distributed is a fixed. Yet, it is not quite as convincing when the resource to be allocated is output from a production process with possibly varying returns to scale. In fact, the literature on consistency in cost sharing (i.e., when the level of output to be shared is endogenous) lacks consensus: several definitions of the consistency axiom have been proposed, which differ in what is meant by a reduced problem. We shall contrast the two most popular definitions. The first one takes the view that the residual cost function for a subgroup of agents equals the total cost minus what the complement subgroup *has already paid* via the cost-sharing rule in effect. Thus, if C is the original cost function, x the actual demand vector, φ the sharing rule, S the subgroup of agents considered and z_S a potential demand subvector for agents in S, the residual cost function for subgroup S, given x_{-S}, is $C(z_S, x_{-S}) - \sum_{i \notin S} \varphi_i(x)$. We shall denote by CON1 this version of the consistency axiom.

On the other hand, the second version considers the residual cost to be the total cost minus what the complement subgroup *would have had to pay* under the sharing rule, potentially authorizing the agents in S to revise their demands: $C(z_S, x_{-S}) - \sum_{i \notin S} \varphi_i(z_S, x_{-S})$. We shall call it CON2.

The distinction between these two definitions is critical: CON1 ignores the behavior of the cost-sharing rule at demand profiles other than the actual one, while CON2 takes it into account. These two versions of the consistency axiom are not logically related (one does not imply the other). In fact, it is shown in Sudhölter (1998) that requiring the same set of mild normative axioms in addition to CON1 and CON2, characterizes two popular cost-sharing rules on the domain of homogenous cost-sharing problems: the nucelolus rule and the Shapley rule, respectively.

After relating the present discussion to the relevant literature in resource allocation and cooperative game theory in Section 2, we present the formal cost-sharing model (Section 3). Our contribution lies in the interpretation of CON1- and CON2-consistency (Section 4) and in our discussion of their relative relevance in the cost-sharing context (Section 5).

2 Relation to the literature

As mentioned, our discussion builds on results found in Sudhölter [20] establishing that the nucleolus and the Shapley rules can be seen to differ only in how they treat reduced sharing problems; i.e., in what version of consistency axiom they respect. In fact, the consistency axiom was originally introduced in the literature on cooperative games, and this tension between the two different definitions of what is a reduced game appears there as well. In Davis and Maschler (1965), the reduced game for a coalition considers what remains after the agents outside the coalition take their share of the grand coalition surplus (i.e. at the *actual* profile). Expression (RC1, see next section in the formal definition of CON1) is clearly the cost-sharing version of that interpretation.

By contrast, Hart and Mas-Colell (in Hart and Mas-Colell, 1989, hereafter HMC) chose to acknowledge the dependence of the shares of the "departing" individuals on the hypothetical participation of the "remaining" agents when defining the reduced game for a coalition. Expression (RC2, see next section in the formal definition of CON2) is the cost-sharing analog of their definition.

Nevertheless, while their views differ on what constitutes a reduced game, these authors stress that the definition adopted should be relevant to the particular question at hand (See HMC, and Maschler, 1990). In light of the present discussion, our interpretation of their warning is that CON1-consistency may be appropriate for allocating a fixed resource but its appeal is debatable when production externalities are present; CON2-consistency is appropriate in that case.

Except for the question of allocating costs, most allocation problems encountered in the literature on consistent allocation dealt with distributing a fixed resource (see Thomson 2006). The consensus, in this case, is to adopt a definition of the consistency axiom à la Davis-Maschler. This may help explain the fact that the early literature on consistency in cost sharing (see Moulin and Shenker, 1994, hereafter MS94) used CON1-consistency, perhaps out of habit, even though the total cost to be shared typically depends on the agents' demands.

Since then, attempts to properly define the consistency axiom in cost-sharing problems can be sorted into two categories: those akin to CON1-consistency (e.g., MS94; Tijs and Koster, 1998; Fleurbaey and Maniquet, 1999; Albizuri and

Zarzuelo, 2005a and 2005b; Koster, 2006) and those related CON2-consistency (Tijs and Koster, 1998[1]; Friedman, 2004; McLean, Pazgal and Sharkey, 2004)

Maurice Koster (see Koster, 2006) recently developed an intriguing axiom related to consistency in the case where the cost function is homogenous. He takes the view that the resource to be shared is the cost function itself and defines the residual cost function of a coalition to be the original one minus the portion of the original cost function virtually allocated to the "departing" agents at the original demand profile (it is, therefore, a definition of the Davis-Maschler type). This definition is intriguing for at least two reasons. First, it seems to incorporate a notion of responsibility of sorts, in the sense that it aims to hold agents responsible for the increase in costs due to their presence which, we argue, is not quite in line with the spirit of consistency. But, perhaps most importantly, it fails to recognize a sharing rule as straightforward as splitting costs equally to be consistent when externalities are present.

Note that a different approach altogether would be to view the production process as consisting of two stages: production, followed by distribution. The reduced problem would then amount to one of rationing (see Thomson, 2006, for a short discussion). We agree with this observation in the sense that the appropriate definition of consistency should depend on when renegotiations can actually be carried out in practice. Here, we take the view that the production process takes place in one single shot and cost shares must be agreed upoon before production even starts.

3 Cost allocation and consistency

We consider the situation where a group of individuals jointly utilize a single production process. Each agent i demands a positive amount of consumption, x_i, and the total cost of meeting the vector $x = (x_1, ..., x_n)$ of all demands, $C(x)$, must be exactly split between them. A *cost-sharing problem* is a triple, (N, C, x), where $N = \{1, ..., n\}$ is the set of individuals, $C : \mathbb{R}_+^N \to \mathbb{R}$ a cost function and $x \in \mathbb{R}_+^N$ a demand profile. We denote by $\mathcal{C}^N$ the class of cost functions, and by Π^N the set of all cost-sharing problems. A cost-sharing rule (or *sharing rule*) is a formula, $\varphi : \Pi \to \mathbb{R}^N$, dividing the total cost as a function of the demands of the agents; i.e., such that the *budget is balanced*:

$$\sum_{i \in N} \varphi_i(N, C, x) = C(x).$$

In order to define the main axiom of this paper, the consistency axiom, one must introduce the notion of a *reduced problem*, which is the (re-)allocation problem facing a subset of individuals, while ignoring the rest of the population.

[1] Tijs and Koster (1998) considers two different definitions of consistency, one of each type. While the authors seem to favor the definition of the HMC type, they do not provide a satisfactory justification for their inclination. Moreover, given that subsequent work by Koster (see Koster, 2006) reverts to a Davis-Maschler approach to consistency, one may wonder whether the conceptual distinction between the two approaches had been clearly identified.

We define the reduction of the cost-sharing problem (N, C, x) to coalition $S \subsetneq N$ to be a triple: (S, C^S, x_S), where $x_S \in \mathbb{R}_+^S$ is the restriction of x taking only the demands of the agents in S, and $C^S : \mathbb{R}_+^S \to \mathbb{R}$ is a *residual cost function* such that $C^S(x_S) = \sum_{i \in S} \varphi_i(N, C, x)$. Thus, the reduced problem can be thought of as a cost-sharing problem as well. Similarly, we denote by $\mathcal{C}^S$ the class of residual cost functions and by Π^S the set of all residual cost-sharing problems involving agents in S. The specification of how the residual cost function, C^S, is related to the original cost function, C, is central to our discussion. In the following definition of consistency, we take such a specification as given.

Definition 1 (Consistency) *A sharing rule, φ, is* consistent *if for any cost-sharing problem, (N, C, x), any subset $S \subsetneq N$, and any $i \in S$ the following holds:*

$$\varphi_i(S, C^S, x_S) = \varphi_i(N, C, x).$$

4 Interpreting CON1 and CON2

As mentioned in the introduction, several distinct definitions of the consistency axiom have been proposed in the cost-sharing literature. They differ in their interpretation of what is an appropriate residual cost function. The definition of a residual cost function given in MS94 is the following:[2]

$$C^S_{x_{-S}, y_{-S}}(z_S) = C(z_S, x_{-S}) - \sum_{i \in N \setminus S} y_i \tag{RC1}$$

for any $z_S \in \mathbb{R}_+^S$, where $y = \varphi(N, C, x)$ is the vector of original cost shares.

The reader may have noticed that we adapted the definition of MS94 to the heterogenous-goods case. Also, their definition involved a somewhat *ad hoc* truncation of the residual cost function by restricting attention to its non-negative part[3]. For the sake of exposition, we ignore this truncation operation as our criticism lies at a deeper, more conceptual level. Other truncations have been suggested in the literature (see, e.g., Albizuri and Zarzuelo, 2005a and 2005b); our critique applies to those as well.

At the heart of expression (RC1) is the idea that once the cost share of the agents outside S has been determined, they can "put money on the table and depart without leaving an address: the remaining division problem can be conducted entirely without [them]." The image is from Moulin (2002) and is germane to the sequential interpretation of consistency mentioned in the introduction.

One may suspect that Expression (RC1) is itself inconsistent, so to speak. On the one hand, requiring the reduced problem to be a cost-sharing problem implies the presence of a residual cost function. In turn, this implies that what

[2] For notational brevity, we write x_{-S} and y_{-S} instead of $x_{N \setminus S}$ and $y_{N \setminus S}$.

[3] The same truncation also appears in Tijs and Koster (1998) and in Fleurbaey and Maniquet (1999)

the joint cost of the agents in S *could have been*—had they made demands other than the ones actually observed—matters. Hence the dependence of the residual cost on a vector of *hypothetical* demands, z_S, which may differ from the *actual* vector of demands, x_S. On the other hand, the fact that the sum of cost shares for the agents outside of the coalition, $\sum_{i \notin S} y_i$, is solely dependent on the actual demand vector implies that hypothetical demand profiles are irrelevant. Therein lies a potential flaw: if one chooses to consider the joint cost that agents in S *should have to pay under a given sharing rule* if they demanded z_S, one must acknowledge the fact that the shares of the agents outside S might depend on this hypothetical demand profile (unless the cost function is additively separable, which is the case only for trivial cost-sharing problems).

In other words, Expression (RC1) could be modified in the following way:

$$C^S_{x_{-S},\varphi}(z_S) = C(z_S, x_{-S}) - \sum_{i \in N \setminus S} \varphi_i(N, C, (z_S, x_{-S})) \qquad \text{(RC2)}$$

for any $z_S \in \mathbb{R}^S_+$, where information on the sharing rule as well as on the cost function at hypothetical profiles is taken into account.

From now on, we shall call *CON1* (resp. *CON2*) the consistency axiom with respect to Expression (RC1) (resp. Expression RC2).

5 Discussion: Contrasting the two definitions

Theorems 2.5 and 3.4 in Sudhölter establish that CON1 and CON2 are not logically related as combining them with the same set of axioms characterizes two distinct rules in the case of homogenous cost functions[4]: the nucleolus rule and the Shapley rule, respectively. Nevertheless, while one does not logically imply the other, there is a sense in which one could argue that CON2 is less demanding than CON1. This sense is that of usage. Indeed, all the sharing rules considered seriously (i.e., not solely as cooked-up counterexamples) in the cost-sharing literature satisfy CON2, whereas many violate CON1. We illustrate our point on three prominent examples.

The Equal-Split Rule (ES) shares costs equally between the agents, such that each pays $C(x)/n$.

Average Cost Pricing (ACP) shares costs in proportion of one's demand relative to total demand: agent i's cost share equals $\frac{x_i}{\sum_{j \in N} x_j} C(x)$.

The Moulin-Sprumont Serial Rule (SER), a natural extension of the Moulin & Shenker serial rule (see Moulin & Shenker 92), which has recently received much attention in the cost-sharing literature, is characterized by many desirable properties of fairness and incentive compatibility (see Sprumont, 1998). At its core is the idea that all individuals demanding a given level of output are equally

[4] A cost function is *homogenous* if its inputs are perfect substitutes:

$$C(x_1, ..., x_n) \equiv c(x_1 + ... + x_n)$$

for some function c.

responsible for the cost increment up to their joint demand level. On a three-person example where individuals are ordered such that $x_1 \leq x_2 \leq x_3$, and denoting $x^1 = (x_1, x_1, x_1)$ and $x^2 = (x_1, x_2, x_2)$, the serial cost shares can be written as follows:

$$\begin{cases} y_1 = & \frac{1}{3}C(x^1) \\ y_2 = & \frac{1}{3}C(x^1) + \frac{1}{2}[C(x^2) - C(x^1)] \\ y_3 = & \frac{1}{3}C(x^1) + \frac{1}{2}[C(x^2) - C(x^1)] + [C(x) - C(x^2)] \end{cases}$$

Naturally, ACP and SER make the most sense when individual demands are expressed in comparable units and, in particular, when the cost function is homogenous (i.e., when the total cost depends only on the total demand level).

We argue that the above three sharing rules are each defined by a sensible normative principle: equality, proportionality and seriality. Whether one deems these principles compelling is beside the point of this work. We solely contend that each of these three rules is governed by a clear underlying logic and, hence, should pass any meaningful test of internal coherence, which is precisely what the consistency axiom should provide. As it turns out, all three of the above rules satisfy CON2 (see the Appendix), while the SER violates CON1 as the following example shows:

Consider three agents, and a demand vector $x = (1, 2, 3)$. demanding $q_1 \leq q_2 \leq q_3$ units of consumption, respectively. Suppose the total cost of meeting their demands equals $C(x) = (x_1 + x_2 + x_3)^2 = 36$. Splitting the total cost according to the serial mechanism yields the following vector of cost shares: $y = (3, 11, 22)$. If we consider agents 1 and 3 in isolation, expression (RC1) yields a residual cost function equal to $C^{13}_{x_2,y_2}(z_{13}) = (z_1 + 2 + z_3)^2 - 11$. Reapplying SER to this residual cost function yields "revised" cost shares of 2.5 and 22.5 for agents 1 and 3, respectively. In fact, at the root of the disagreement between CON1- and CON2-consistency is the fact that SER utilises information on the cost function, C, at demand vectors x^1 and x^2 other than the *actual* demand vector, x. By definition, CON2-consistency allows for rules to take such information into account whereas CON1-consistency does not. In fact, from Expression (RC1), it is clear that CON1-consistency requires a separability of sorts in the cost function.

In fact, many sharing rules satisfy CON2-consistency. The reader can check that sharing costs according to fixed proportions, to path methods (see Friedman, 2004), dictatorial and priority rules, and two-part pricing (in the presence of fixed costs) are all consistent sharing rules according to CON2 while most of them violate CON1. Thus, we claim that CON2 is a more permissive version of the consistency axiom, whose only role is to exclude "strange" rules, which is precisely what we mean in plain language when using the word "consistent". For instance, a rule allocating costs according to ES among the agents in N, but according to ACP among S when its cardinality is odd, and according to SER when it is even, is neither CON1- nor CON2-consistent.

For general cost-sharing problems, we believe the definition in Friedman (2004)—which exactly corresponds to Expression (RC2)—to be most faithful to the notion of consistency[5] and the results therein to be among the most meaningful contributions to the topic of consistency in cost sharing. Nonetheless, even Friedman fails to acknowledge the conceptual gap between his definition (of the HMC type) and that of MS94 (of the Davis-Maschler type) when justifying the use of his definition:

> "[It] is a natural extension of the version used in [HMC] for TU games and by [MS94] for cost sharing problems in which the cost function is required to be homogenous."

This oversight, although mentioned in passing, serves as further evidence that the present discussion is necessary.

[5] McLean et al. (2004) uses the same definition. However, their framework is less general because they focus on setting a constant per-unit price for each agent.

6 Appendix: Proof that ES, ACP and SER are CON2-consistent

Consider a cost-sharing problem $(N, C, x) \in \Gamma$ as well as a coalition $S \subset N$ of agents. Denote by $L = N \backslash S$ the set of agents who "leave" the procedure and by $(S, C^S_{x_L,\varphi}, x_S)$ the reduced problem of the agents in S (i.e, the agents who "stay"), with $C^S_{x_L,\varphi}$ defined as in Expression (RC2). We shall show that $\varphi_i(S, C^S_{x_L,\varphi}, x_S) = \varphi_i(N, C, x)$ for all $i \in S$ when φ is ES, ACP and SER, respectively.

6.1 Proof that ES is CON-2consistent

In this section, we let $\varphi \equiv ES$. It is immediate from Expression (RC2) that the residual cost function for coalition S equals the following:

$$C^S_{x_L,\varphi}(z_S) = \frac{n - |S|}{n} C(N, C, (z_S, x_L)).$$

Clearly, applying ES to the reduced problem, amounts to dividing this residual cost by $n - |S|$, which is equivalent to splitting the original cost equally between the n agents.

6.2 Proof that ACP is CON2-consistent

In this section, we let $\varphi \equiv ACP$. It follows from Expression (RC2) that the residual cost function for coalition S can be written as:

$$C^S_{x_L,\varphi}(z_S) = \frac{\sum_{i \in S} z_i}{\sum_{j \in L} x_j + \sum_{i \in S} z_i} C(N, C, (z_S, x_L)).$$

This residual cost, when split proportionally to the size of the z_i's yields exactly the original ACP shares when $z_S = x_S$.

6.3 Proof that SER is CON2-consistent

The consistency of SER is a corollary of a more general result in Friedman (2004) but we provide a proof nonetheless, for the sake of self-containedness. Also, because our proof is specific to SER, it may provide the reader with a better intuition for the result.

In this section, let $\varphi \equiv SER$. We first introduce some notation. Let $S = (s_1, ..., s_{|S|})$, and for any integer $j \in \{1, ..., |S|\}$, we denote by $x^j_S = (x_{s_1}, ..., x_{s_{j-1}}, x_{s_j}, x_{s_j}, ..., x_{s_j})$, the vector of size $|S|$ obtained by replacing the last $|S| - j + 1$ coordinates of x_S with x_{s_j}. Without loss of generality, we shall assume that the agents in N are ordered in increasing order of their demand levels: $x_1 \leq x_2 \leq ... \leq x_n$ and that the same holds for agents in S ($x_{s_j} \leq x_{s_{j+1}}$ for all j). To economize on notation, we shall often denote by $y_i = \varphi_i(N, C, x)$ agent i's cost share in the original problem, and by $y'_i = \varphi_i(S, C^S_{x_L,\varphi}, x_S)$ her share in the reduced problem of coalition S. We also abuse notation slightly and write $\varphi_S(N,C,x) = \sum_{i\in S}\varphi_i(N,C,x)$ and $\varphi_L(N,C,x) = \sum_{i\in L}\varphi_i(N,C,x)$.

Next, we turn to two characteristic properties of SER:

- Equal treatment of equals (ETE): for any $j,k \in S$, $x_j = x_k \implies [y_j = y_k$ and $y'_j = y'_k]$,
- Independence of higher demands (IHD): for any $j \in S$, $\varphi_{s_j}(N,C,(x_S^j,x_L)) = \varphi_{s_j}(N,C,(x_S,x_L))$.

We proceed by induction. We first consider agent s_1's cost share:

$$\begin{aligned}
y'_{s_1} &= \frac{1}{|S|}C^S_{x_L,\varphi}(x^1_S) \\
&= \frac{1}{|S|}\left(C(x^1_s,x_L) - \varphi_L(N,C,(x^1_S,x_L))\right) \\
&= \frac{1}{|S|}\varphi_S(N,C,(x^1_S,x_L)) \qquad \text{by budget balance} \\
&= \frac{1}{|S|}|S|\varphi_{s_1}(N,C,(x^1_S,x_L)) \qquad \text{by ETE} \\
&= \varphi_{s_1}(N,C,x) \qquad \text{by IHD} \\
&= y_{s_1}.
\end{aligned}$$

Now, fix $i \in S$, and suppose we established that $y'_{s_j} = y_{s_j}$ for all $j \leq i$. By the definition of SER,

$$y'_{s_i} = y'_{s_{i-1}} + \frac{1}{|S|+1-i}\left(C^S_{x_L,\varphi}(x^i_S) - C^S_{x_L,\varphi}(x^{i-1}_S)\right)$$

where, by budget balance, ETE and IHD, and the induction hypothesis, respectively, we get $C^S_{x_L,\varphi}(x^{i-1}_S) = \varphi_S(N,C,(x^{i-1}_S,x_L)) = \sum_{k=1}^{i-2}y'_{s_k} + (|S|-i+2)y'_{s_{i-1}} = \sum_{k=1}^{i-2}y_{s_k} + (|S|-i+2)y_{s_{i-1}}$. Similarly, $C^S_{x_L,\varphi}(x^i_S) = \varphi_S(N,C,(x^i_S,x_L)) = \sum_{k=1}^{i-1}y'_{s_k} + (|S|-i+1)y_i$. It follows immediately that $y'_{s_i} = y_{s_i}$. Therefore, SER is CON2-consistent.

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