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Fair division with no information*

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Summary. We consider a situation in which a central authority must allocate non-tradeable and non-marketable goods between a group of individuals in a fair way. There are exogenous divisibility constraints imposed on the goods to be allocated. The authority has absolutely no information on the preferences of the recipients; moreover, no interaction is allowed among recipients or between the authority and the recipients. Envy-freeness is the equity criterion adopted. Using a remarkable property of simplices (which we introduce and prove) we argue that assigning bundles of equal expected value (forming what is called in this paper the class of "balanced allocations") is hardly fair unless extra effort is made to discriminate between these proposed allocations.

Keywords and Phrases: Fair division, Envy-freeness, Imperfect information.

JEL Classification Numbers: C65, D63, D89.

1 Introduction

We consider a situation in which a central authority, or an impartial arbitrator, must allocate non-tradeable and non-marketable goods between a group of individuals in a fair way. There are exogenous divisibility constraints imposed on the goods to be allocated. The authority has absolutely no information on the preferences of the recipients; moreover, no interaction is allowed among recipients or between the authority and the recipients.

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Such informational situations in which the arbitrator knows nothing of the recipients' preferences (and cannot entice their revelation) for goods with the aforementioned technical constraints often occur in practice. Two instances come to mind. First, imagine a central organ bank responsible for allocating different organs to various hospitals; it may be impossible to discriminate between claims because the urgency of the situation in allocating extremely scarce, vital and overdemanded goods is such that priority considerations lose their meaning. Another case of ignorance about the preferences of the recipients occurs when the existing demand procedure is not strategyproof. Indeed, when it would be disadvantageous for the recipients to not disproportionately modify their reports, the arbitrator is left with demands which are so distorted and insincere that they become unreliable; this scheme is notably the one encountered by the central administration when teaching positions in various fields must be assigned to different universities in European countries where universities are government-run.

In such a context, it is clear that the arbitrator alone must resolve this problem of fair division with no information. For example, consider a situation where there are 5 goods to be allocated between 3 recipients, denoted by A, B and C, under the following divisibility constraints: good 1 is indivisible, good 2 is divisible in three identical shares, good 3 is divisible in seven identical shares and, finally, goods 4 and 5 are perfectly divisible. Given the lack of information that the arbitrator faces, she must behave as if the preferences of the recipients (which we assume to be linear for simplicity) are random, independent and identically distributed across recipients. Let $\mu = \frac{1}{6} \times (7, 6, 7, 6, 4)$ be the arbitrator's expectation of this distribution or, equivalently, it can be interpreted as a vector of relative values that the arbitrator decides to assign to the n goods. That is to say that the recipients are expected to value receiving the totality of goods 1 and 3 equally and assign it a value of $\frac{7}{6}$ that of the totality of goods 2 and 4 (which are in turn equally valued), the relative value of the totality of good 5 is expected to be $\frac{4}{7}$ of that of good 1.The coefficient $\frac{1}{6}$ is merely introduced so that the sum of the coordinates of μ be equal to five, the number of goods to be allocated; or, equivalently, the total expected value of the goods to be distributed is normalized to equal to five.[1]

How should the arbitrator allocate these five goods? A quite natural approach would be to try to equalize the expected subjective value or "expected utility" of each bundle. (Note that when we usually speak of expected utility, uncertainty is placed on the outcomes of the allocations whereas preferences are fixed; in our case, uncertainty bears on the realization of the recipients' preferences.) Throughout the paper, such an allocation will be qualified as "balanced". For example, the following allocation is a balanced one; i.e. the expected subjective value of each bundle is equal to $\frac{5}{3}$:

[1] Moreover, in this example, we will assume that the arbitrator believes the marginal utilities of the recipients to be uniformly distributed on a simplex of $\mathbb{R}^5_+$ such that μ is the expected vector of this distribution.

	A	B	C
good 1	1	0	0
good 2	0	$2/3$	$1/3$
good 3	$1/7$	$2/7$	$4/7$
good 4	$1/6$	$1/3$	$1/2$
good 5	$1/4$	$1/2$	$1/4$

Balanced allocations equalize the expected utility of the recipients. Such a feature makes ethical sense when all the participants of a given procedure (here, the recipients) agree that the procedure in use is fair *ex ante* (procedural justice) or whenever the plays are repeated a sufficient number of times so that the average result approaches the expected one (endstate justice). However, given the informational restrictions in the context considered here, the arbitrator alone decides how to conduct the allocation whether or not the recipients agree; besides, even if the recipients did agree on a specific procedure, they have no evidence that the arbitrator actually applies it fairly or that the dice are not loaded. Hence, in the distinction between procedural justice and endstate justice our approach clearly calls for the latter. And because the situations that we have in mind when describing the problem are ones where the division problem may not be repeated often, if at all, the notion of expected utility is inappropriate.

Instead, we recommend a different equity criterion, based on envy-freeness. We can show thanks to a tool developed in the text of the paper that A has only a 42% chance of not being envious of C under the above allocation (whereas C has a 58% chance of not being envious of A!). Moreover, there exists an allocation at which the probability that any given agent be not envious of any other given recipient equals 50% (which is easily shown to be the theoretical maximum). It can be well approximated by the following allocation:

	A	B	C
good 1	1	0	0
good 2	0	$2/3$	$1/3$
good 3	$1/7$	$2/7$	$4/7$
good 4	$21/100$	$3/10$	$49/100$
good 5	$23/60$	$9/20$	$1/6$

Such an allocation will be qualified as "equitable" throughout the paper. However, note that this allocation is not balanced. Indeed, the "expected utility" of recipient A is equal to $\frac{1}{6} \times \left(7 + 7 \times \frac{1}{7} + 6 \times \frac{21}{100} + 4 \times \frac{23}{60}\right) \approx 1.8$ which is different from that of recipient B: $\frac{1}{6} \times \left(6 \times \frac{2}{3} + 7 \times \frac{2}{7} + 6 \times \frac{3}{10} + 4 \times \frac{9}{20}\right) = 1.6$. Therefore, an equitable allocation need not be balanced. Furthermore, in this particular situation, there doesn't exist any allocation which is both balanced and equitable.[2]

[2] The least inequitable balanced allocation is the following:

	A	B	C
good 1	1	0	0
good 2	0	0	1
good 3	0	1	0
good 4	$1/2$	$1/2$	0
good 5	0	0	1

yielding a 50% chance of A not envying B, a 49% chance of A not envying C and a 49% chance of B not envying C.

This kind of results typically holds for any number of goods and recipients as well as for any vector μ; although, the class of balanced allocations and that of equitable allocations may reduce to the empty set depending on the divisibility constraints imposed on the goods.

Nevertheless, we claim that the class of balanced allocations remains appealing. First of all, equitable allocations, when they exist, are very difficult to find in practice.[3] Also, we show that in the more general context, where n goods are to be allocated between m recipients, any balanced allocation satisfies the property that, given any two agents i and i', the probability that i be not envious of i' is a least equal to $\alpha_n = \left(\frac{n-1}{n}\right)^{n-1}$ (as opposed to $\frac{1}{2}$ for an equitable allocation). The sequence (α_n) is a decreasing one converging to approximately 0.37. In the particular example considered so far, this lower bound is equal to $\left(\frac{4}{5}\right)^4 \approx 0.41$. Hence, by restricting our attention to the class of balanced allocations we may not always find an equitable one, but we are sure to avoid allocations which are ridiculously inequitable; this fact is a valuable one, especially given how hard it is to find an equitable allocation, even for small values of m and n. Also, we develop a mathematical tool allowing one to determine for any given allocation the probability of envy occurring between two given agents. Our conclusion consists in suggesting restricting one's attention to the class of balanced allocations and discriminating between them thanks to the tool we developed. As far as practicability is concerned, our procedure can be readily implemented.

The paper is organized as follows. The next section presents the theoretical framework and establishes a remarkable property of simplices that proves to be a useful tool for measuring the likelihood of envy arising from any given allocation. Section 3 analyses the structure of the class of balanced allocations and relates it to that of equitable allocations. Section 4 provides conditions on the divisibility constraints of the goods for balanced and equitable allocations to exist and show that these conditions are much more restrictive for the latter class. Section 5 concludes. Most proofs can be found in Appendix as well as a technical discussion stating that our qualitative results are robust to certain changes in the specification of the model, including the probability distribution of the preferences of the recipients.

2 The framework

Consider an arbitrator faced with the responsibility of dividing n goods between m recipients. The n goods may not be perfectly divisible and, for simplicity, we assume that all imperfectly divisible goods can be divided in shares of equal size. Moreover, the goods to be divided are assumed to be nonmarketable and nonexchangeable.

An *allocation* $\omega = (\omega^1, ..., \omega^m) \in \mathbb{R}^{n\times m}$ satisfies $\sum_{i=1}^{m} \omega_j^i = 1$ for every good j. ω^i is the bundle of shares received by agent i, and $\omega_j^i \in [0,1]$ represents the share of good j received by individual i. For instance, if three kidneys are to be allocated, a hospital receiving one kidney is said to receive a share equal to one third of the good "kidney". We denote by Ω the set of allocations.

[3] We are not able to solve analytically the system of equations defining the set of equitable allocations. And no appropriate algorithm for approximating these allocations is known to us.

For simplicity, the preferences of the recipients are assumed to be linear; we denote by $\lambda^i \in \mathbb{R}^n$ the vector of marginal utilities of recipient i, such that $\lambda^i \cdot \omega^i$ is the utility derived by agent i from allocation ω, where the scalar product "$\cdot$" is that associated with the Euclidean norm of $\mathbb{R}^n$. The arbitrator has no information about the preferences of the recipients. Communication or trade of any kind between recipients and between the arbitrator and recipients is not allowed or possible for some exogenous reasons.

Given this uncertainty, the arbitrator considers all the vectors of marginal utilities of the recipients to be generated by the same uniform random process. In this paper, we assume the distribution of the λ^i's to be uniform on the simplex $\Lambda_n = ch(\mu_1 e^1, ..., \mu_n e^n)$, where $(e^1, ..., e^n)$ is the canonical basis of $\mathbb{R}^n$; and $ch(\cdot)$ returns the convex hull of its arguments[4]. If $\mu = (1, 1, ..., 1)$, Λ_n becomes the unit simplex, denoted by $\Delta_n = ch(e_1, e_2, ..., e_n)$. The mean of the random process is precisely $\frac{1}{Area(\Lambda_n)} \int_{\Lambda_n} \lambda d\sigma = \mu$, where $Area(\Lambda_n)$ denotes the $(n-1)$-dimensional volume of Λ_n. Also, the vector $\mu = (\mu_1, ..., \mu_n) \in \mathbb{R}^n_+$ can be interpreted as a vector of relative values that the arbitrator decides to assign to the n goods; it is normalized such that $\sum_{j=1}^n \mu_j = n$.

By definition, the arbitrator considers that individual i is *likely to prefer* bundle a to bundle b if her probability of preferring a to b is greater than or equal to that of preferring b over a. Formally:

$$a \succsim_i^{prob} b \Longleftrightarrow \Pr\left[\lambda^i \cdot (a-b) \geqslant 0\right] \geqslant \Pr\left[\lambda^i \cdot (a-b) \leqslant 0\right] \tag{1}$$

when $a \neq b$.[5] If $a = b$, the recipient is necessarily indifferent between the two bundles.

If $a \neq b$, the hyperplane $\{q \in \mathbb{R}^n | q \cdot (a-b) = 0\}$ divides the simplex Λ_n into two surfaces: $\Lambda_n^+(a-b) = \{q \in \Lambda_n | q \cdot (a-b) \geqslant 0\}$ and $\Lambda_n^-(a-b) = \{q \in \Lambda_n | q \cdot (a-b) \leqslant 0\}$. Thus, the density of the random process generating λ being uniform, definition (1) becomes:

$$a \succsim^{prob} b \Longleftrightarrow \int_{\Lambda_n^+(a-b)} d\sigma \geqslant \int_{\Lambda_n^-(a-b)} d\sigma.$$

There is a clear geometric intuition behind this approach; definition (1) translates into:

$$a \succsim^{prob} b \Longleftrightarrow \frac{Area(\Lambda_n^+(a-b))}{Area(\Lambda_n)} \geqslant \frac{1}{2}.$$

Notice that the binary relation $\succsim^{prob}$ is not transitive, rendering efficiency consideration of the Pareto type inappropriate.

[4] Please refer to the Appendix 6.4 for a discussion of the choice of the normalizing surface.

[5] Given that all marginal utility vectors are identically distributed, $a \succsim_i^{prob} b$ if and only if $a \succsim_{i'}^{prob} b$ for any two agents i and i'. We shall simply write $a \succsim^{prob} b$ henceforth.

2.1 Equitable allocations

The equity criterion we seek to satisfy is that of envy-freeness. An allocation is envy-free if no individual prefers the endowment of another to his own; this definition, introduced in Foley [2], needs to be adapted to the present framework. We shall be satisfied with allocations which are envy-free in probability; we call these allocations "equitable":

Definition 1 *An allocation $\omega \in \Omega$ is equitable relative to μ if, for all $i, i' = 1, ..., m$*

$$\omega^i \succsim^{prob} \omega^{i'}$$

The egalitarian allocation is obviously the most natural solution to our problem; and we suppose that this trivial solution is not attainable (i.e. not all goods are divisible into m equal shares). Hence, for each pair of recipients with different bundles, the maximum guaranty in probability of envy not occurring between them is 50%; indeed, since λ^i and $\lambda^{i'}$ are identically distributed, we have $\Pr[\lambda^i \cdot \left(\omega^i - \omega^{i'}\right) \geqslant 0] + \Pr[\lambda^{i'} \cdot \left(\omega^{i'} - \omega^i\right) \geqslant 0] = 1$. The former definition can then be restated as follows:

Definition 2 *An allocation, $\omega \in \Omega$, is equitable relative to μ if*

$$\frac{Area(\Lambda_n^+(\omega^i - \omega^{i'}))}{Area(\Lambda_n)} \geqslant \frac{1}{2} \qquad \forall i, i' \in \{1, ..., m\} \quad s.t.\ i \neq i'.$$

We denote by $\mathcal{E}_\mu$ the set of equitable allocations relative to μ.

Similarly, for $0 \leq \alpha \leq \frac{1}{2}$, we say that an allocation $\omega \in \Omega$ is α-equitable relative to μ if $\frac{Area(\Lambda_n^+(\omega^i - \omega^{i'}))}{Area(\Lambda_n)} \geqslant \alpha$ for all pairs of different recipients i and i'.

In the following subsection we establish a mathematical property which proves to be a useful tool in measuring the equitableness of any allocation.

2.2 A tool for measuring the equitableness of an allocation

Measuring envy in probability amounts to computing the ratio $\frac{Area(\Lambda_n^+(x))}{Area(\Lambda_n)}$ for any vector $x \in \mathbb{R}^n$. This procedure is greatly facilitated thanks to an interesting property of simplices which we state and prove.

We first introduce a bit of notation which will only be relevant to this subsection. Let $x \in \mathbb{R}^n \backslash \{0\}$ and $K \in \mathbb{R}$. Let $H = \{q \in \mathbb{R}^n | q \cdot x - K = 0\}$ be a hyperplane of $\mathbb{R}^n$. For all $q \in \mathbb{R}^n$, denote $H(q) = q \cdot x - K$. Let Λ_n be an $(n-1)$-dimensional simplex of $\mathbb{R}^n$, denote by $s_1, s_2, ..., s_n$ its vertices. The hyperplane H divides Λ_n into two subsets: $\Lambda_n^+ = \{q \in \Lambda_n | H(q) \geqslant 0\}$ and $\Lambda_n^- = \{q \in \Lambda_n | H(q) \leqslant 0\}$. The following theorem provides an analytical formula for the ratio of the surfaces of Λ_n^+ and Λ_n^-.

Theorem 1 *If H is not parallel to any edge of Λ_n (i.e. $H(s_j) \neq H(s_{j'})$ for all $j \neq j'$), then:*

$$\frac{Area(\Lambda_n^+)}{Area(\Lambda_n)} = \sum_{j \in J} \prod_{j' \neq j} \frac{H(s_j)}{H(s_j) - H(s_{j'})}$$

where $J = \{j \mid H(s_j) > 0\}$.

Proof. In Appendix 6.1. □

In the case of the unit simplex ($\Lambda_n = \Delta_n$, i.e. $s_j = e_j$ for all $j = 1, ..., n$) where the hyperplane H passes through the origin (i.e. $K = 0$), the proposition becomes:

Corollary 1 *Let $x \in \mathbb{R}^n$, with $x_j \neq x_{j'}$ for all $j \neq j'$. The hyperplane $H = \{q \in \mathbb{R}^n | q \cdot x = 0\}$, divides the unit simplex Δ_n into two subsets, Δ_n^+ and Δ_n^-, such that:*

$$\frac{Area(\Delta_n^+)}{Area(\Delta_n)} = \sum_{j \in J} \prod_{j' \neq j} \frac{x_j}{x_j - x_{j'}} \quad \textit{where } J = \{j \mid x_j > 0\}.$$

Proof. Clear.[6] □

We can now infer the following proposition.

Proposition 1 *Let Λ_n be any $(n-1)$-dimensional simplex of $\mathbb{R}^n$, with vertices $s_j = \mu_j e_j$ for $j = 1, 2, ..., n$. The set Γ_n of vectors u such that the hyperplane $H = \{q \in \mathbb{R}^n | q \cdot u = 0\}$ divides Λ_n into two surfaces of equal area is diffeomorphic to $\mathbb{R}^{n-1} \setminus \{0\}$.*

Proof. In Appendix 6.2. □

Back to our problem of fair division, how should the arbitrator distribute the goods between the recipients?

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[6] This expression allows us to (re)discover a remarkable identity of the real numbers: Let $\{x_1, x_2, ..., x_n\}$ be n distinct real numbers, then

$$\sum_{j=1}^{n} \prod_{j' \neq j} \frac{x_j}{x_j - x_{j'}} = 1.$$

A possible geometric interpretation is then:

$$Area(\Delta_n^+) + Area(\Delta_n^-) = Area(\Delta_n).$$

3 Balanced allocations

We mentioned in the introduction that one might be tempted to resort to equalizing expected subjective values across recipients in response to our problem. We call "balanced allocations" the allocations resulting from such a concern .

Definition 3 *An allocation, $\omega \in \Omega$, is said to be balanced relative to μ if*

$$\sum_{j=1}^{n} \mu_j \omega_j^i = \frac{n}{m} \qquad \forall i = 1, ..., m.$$

We denote by $\mathcal{B}_\mu$ the set of balanced allocations relative to μ.

3.1 Fairness properties of balanced allocations

The intuition supporting the use of balanced allocations is that bundles of equal value should be equally desired by the recipients; this same reasoning gave rise to the competitive equilibrium with equal income solution and its envy-free outcome[7]. However, since the goods are nonmarketable and since no trade is permitted (directly or indirectly) between recipients, this intuition does not fare so well in terms of the equity criterion adopted here. Indeed, balanced allocations are not necessarily $\frac{1}{2}$-equitable:

Proposition 2 *For any vector, $\mu \in \mathbb{R}_+^n$ such that $\sum_{j=1}^n \mu_j = n$, any allocation balanced relative to μ is α-equitable, where*

$$\alpha = \left(\frac{n-1}{n}\right)^{n-1}.$$

Proof. Let $\omega = (\omega^1, ..., \omega^m) \in \mathcal{B}_\mu$. Let $0 \leqslant i, i' \leqslant m$, and let $u = \omega^i - \omega^{i'}$. The hyperplane $\{q \in \mathbb{R}^n | q \cdot u = 0\}$ contains the center of gravity, G, of the simplex Λ_n since $OG \cdot u = \frac{1}{n}\mu \cdot (\omega^i - \omega^{i'}) = 0$ from Definition 3. Therefore, thanks to a remarkable property of convex sets due to Grunbaum [3] first introduced in the economic literature by Caplin and Nalebuff (Lemma 2 in [1]), $\frac{Area(\Lambda_n^+(u))}{Area(\Lambda_n)} \geqslant \left(\frac{n-1}{n}\right)^{n-1}$, which completes the proof. □

Note that the lower bound only depends on the number of goods to be divided. Also, a straightforward application of Theorem 1 yields that this lower bound is achieved for any balanced allocation such that $\omega^i - \omega^{i'}$ is proportional to the vector $\left(\frac{1-n}{\mu_1}, \frac{1}{\mu_2}, ..., \frac{1}{\mu_n}\right)$, or any other vector obtained from the latter by a permutation of the j's, for some i and i'.

We are interested in evaluating how well balanced allocations perform, in terms of equity, compared to equitable allocations. It was pointed out earlier that, for

[7] See Moulin [4], ch. 4, for an extensive discussion of the CEEI solution.

equitable allocations, recipients have at least a 50% probability of not being envious of any given individual. For a balanced allocation, however, this lower-bound probability is equal to $\alpha_n = \left(\frac{n-1}{n}\right)^{n-1}$, this decreasing sequence converges to $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \approx 0.37$. The values of this sequence are noticeably different from $\frac{1}{2}$, suggesting that one should give second thoughts as to whether balanced allocations are the obvious answer to our question of fair division without any information. Or, one may want to discriminate between the "good" balanced allocations and the "bad" ones; the mathematical tool presented in Subsection 2.2 allows one to do just that.

3.2 Structure of $\mathcal{B}_\mu$ and $\mathcal{E}_\mu$

We now take a look at the structure of the sets of balanced and equitable allocations when they are non-empty. For the sake of generality, we assume in this section that all the goods are perfectly divisible.

3.2.1 $n = 2$

When $n = 2$, all balanced allocations are α-equitable with $\alpha = \frac{1}{2}$; i.e. $\mathcal{B}_\mu \subset \mathcal{E}_\mu$. Conversely, for every $\omega \in \mathcal{E}_\mu$, and $i, i' \in \{1, ..., m\}$ such that $u \equiv \omega^i - \omega^{i'} \neq 0$, Theorem 1 yields $\frac{\mu_1.u_1}{\mu_1.u_1-\mu_2.u_2} = \frac{\mu_2.u_2}{\mu_2.u_2-\mu_1.u_1} = \frac{1}{2}$. Therefore, $\mu \cdot u = 0$, i.e. $\mu \cdot \omega^i = \mu \cdot \omega^{i'}$ hence $\omega \in \mathcal{B}_\mu$. Thus $\mathcal{B}_\mu = \mathcal{E}_\mu$.

3.2.2 $n = 3$; $m = 2$

The complexity of the structure of $\mathcal{B}_\mu$ and $\mathcal{E}_\mu$ increases as m and n increase. We provide a description of the case where 3 goods are to be allocated between 2 recipients. It is easy to see in a 3-dimensional Edgeworth box that $\mathcal{B}_\mu = \{\omega \in \Omega | \mu_1\omega_1^1 + \mu_2\omega_2^1 + \mu_3\omega_3^1 = \mu_1\omega_1^2 + \mu_2\omega_2^2 + \mu_3\omega_3^2 = \frac{3}{2}\}$ is the intersection of the simplex $\frac{3}{2}\Lambda_3$ with the Edgeworth box (see Fig. 1). The set of equitable allocation becomes $\mathcal{E}_\mu = \{\omega \in \Omega | \omega^1 = (1/2, 1/2, 1/2) + u \text{ s.t. } u \in \Gamma_3\}$ and is illustrated in Figure 2, Γ_3 being the set of vectors u such that the hyperplane $\{q \in \mathbb{R}^3 | q \cdot u = 0\}$ divides Λ_3 in two surfaces of equal size; a way to construct the set Γ_3 can be found in the Appendix 6.2.

Now, let $v \in \mathbb{R}^3$ and let $\varepsilon > 0$ be small enough such that $\omega = (\omega^1, \omega^2) \in \Omega$ where $\omega^1 = \frac{1}{2}(e + \varepsilon v)$ and $\omega^2 = \frac{1}{2}(e - \varepsilon v)$ and $e = (1, 1, 1)$. Consider the three following possible values for v:

- if $v = \left(\frac{1}{\mu_1}, -\frac{1}{\mu_2}, 0\right)$, then ω is both balanced $(\mu.v = 0)$ and equitable (easily checked thanks to Theorem 1)
- if $v = \left(\frac{1}{\mu_1}, -\frac{1}{2\mu_2}, -\frac{1}{2\mu_3}\right)$ then ω is balanced $(\mu.v = 0)$ but not equitable (again, checked using Theorem 1)
- if $v = \left(\frac{1}{\mu_1}, -\frac{1}{2\mu_2}, -\frac{1}{3\mu_3}\right)$ then ω is not balanced $(\mu.v \neq 0)$ but equitable (checked using Theorem 1).

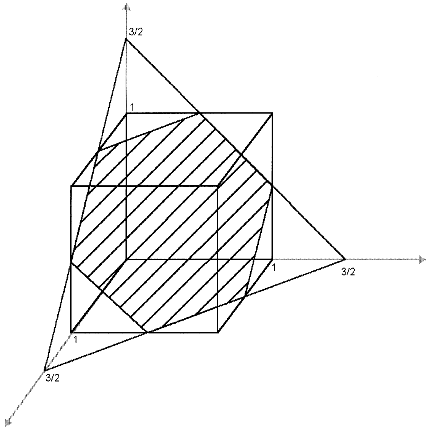

Figure 1. $\mathcal{B}_\mu$ (shaded hexagon)

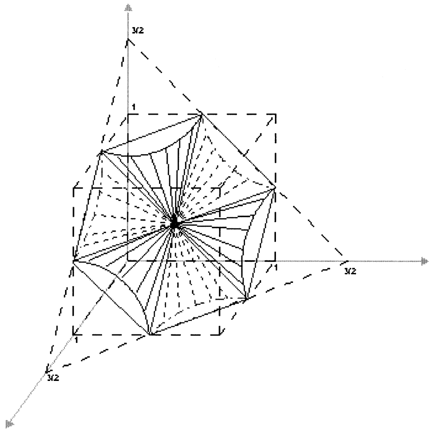

Figure 2. $\mathcal{E}_\mu$

Thus, we see for the simpler case where $n = 3$ and $m = 2$ that all balanced allocations are not equitable and *vice versa;* also, there exist allocations other than the egalitarian one (where $\omega^1 = \omega^2 = (1/2, 1/2, 1/2)$) which are both balanced and equitable. This result is a typical one, as is established in the following proposition.

Proposition 3 *For $n > 2$, not all balanced allocations are equitable, and not all equitable allocations are balanced; moreover, there exist allocations, other than the egalitarian one, which are both balanced and equitable.*

Proof.
i) Let $\omega = (\omega^1, \omega^2)$ be an allocation of 3 goods between 2 recipients. One can always write $\omega^1 = \frac{1}{2}(e+u)$ and $\omega^2 = \frac{1}{2}(e-u)$ where $e = (1,1,1)$ and $u = \omega^1 - \omega^2$. Now consider the following corresponding allocation of 3 goods between $m \geq 3$ recipients: $\omega' = (\omega'^1, \omega'^2, ..., \omega'^m)$ where $\omega'^1 = \frac{1}{m}(e+u)$, $\omega'^2 = \frac{1}{m}(e-u)$ and $\omega'^i = \frac{1}{m}e$ for $i > 2$. One checks immediately that ω' is balanced if and only if $\mu \cdot u = 0$, i.e. if and only if ω is balanced. Similarly, ω' is equitable if and only if $u \in \Gamma_3$, i.e. if and only if ω is equitable. Thus, the proposition being true for $n = 3$ and $m = 2$ (showed in the above text), it remains true for $n = 3$ and $m \geq 2$.
ii) Finally let $\omega = (\omega^1, \omega^2, ..., \omega^m)$ be an allocation of $n = 3$ goods between $m \geq 2$ recipients and consider the following corresponding allocation of $n' > 3$ between m recipients: $\omega' = (\omega'^1, \omega'^2, ..., \omega'^m)$ such that $\omega'^i_j = \omega^i_j$ for all i and for $j = 1, 2$ or 3, and $\omega'^i_j = \frac{1}{m}$ otherwise. Again, it is immediately checked that ω' is balanced if and only if ω is; similarly, ω' is equitable if and only if ω is equitable. The property being established for $n = 3$ (and $m \geq 2$) in i), it generalizes to $n \geq 3$, completing the proof. □

Before introducing the impact of the divisibility constraints of the goods on the existence of balanced and equitable allocations, we wish to point out that the concepts of balancedness and equitableness introduced above are consistent in the following sense. Let $\omega \in \Omega$ be an allocation and let $A \subseteq \{1, ..., m\}$ be a subset of recipients. For every $j \in \{1,...,n\}$ and every $i \in A$, define $\omega_j^{Ai} = \frac{\omega_j^i}{\sum_{i\in A}\omega_j^i}$ and $\mu_j^A = \varsigma.\mu_j \sum_{i\in A}\omega_j^i$, where $\varsigma = \frac{n}{\sum_{i\in A}\mu.\omega^i}$ such that $\sum_{j=1}^n \mu_j^A = n$; μ_j^A can be thought of as being the expected valuation of good j which was allocated to the subset A of recipients under allocation ω. Thus, ω^A is the allocation obtained by restricting allocation $\omega \in \Omega$ to the recipients of A as well as to the total amount of goods actually received by these recipients under ω. We show that ω^A is balanced (resp. equitable) if ω is balanced (resp. equitable):

Proposition 4 *For any $A \subseteq \{1,...,m\}$ and $\omega \in \Omega$:*

1. $\omega \in \mathcal{B}_\mu \implies \omega^A \in \mathcal{B}_{\mu^A}$
2. $\omega \in \mathcal{E}_\mu \implies \omega^A \in \mathcal{E}_{\mu^A}$

Proof. Let $A \subseteq \{1,...,m\}$ and let $\omega \in \Omega$. Let $i, i' \in A$, it follows immediately from the definition of ω^A and μ^A that $\frac{\mu^A.\omega^{Ai}}{\mu^A.\omega^{Ai'}} = \frac{\mu.\omega^i}{\mu.\omega^{i'}}$, proving proviso 1. Now, denote $\Lambda_n^A \equiv ch(\mu_1^A e^1, ..., \mu_n^A e^n)$ and $H^A(\mu_j^A e^j) \equiv \mu_j^A(\omega_j^{Ai} - \omega_j^{Ai'}) = \varsigma\mu_j(\omega_j^i - \omega_j^{i'}) = \varsigma H(\mu_j e^j)$. The equality: $\frac{Area(\Lambda_n^{A+}(\omega^{Ai}-\omega^{Ai'}))}{Area(\Lambda_n^A)} = \frac{Area(\Lambda_n^{+}(\omega^{i}-\omega^{i'}))}{Area(\Lambda_n)}$ (and proviso 2 of the proposition) follows from noticing that the formula in Theorem 1 is homogenous of degree zero in $H(\mu_j e^j)$. □

We next impose divisibility constraints on the goods, rendering the problem more challenging.

4 Divisibility constraints

As mentioned above, we reserve our attention to cases where the egalitarian allocation (which is both balanced and equitable, of course) is unattainable due to divisibility constraints imposed on the goods to be divided. More specifically, we derive conditions on these divisibility constraints for balanced and equitable allocations to exist.

4.1 Conditions for $\mathcal{B}_\mu \neq \emptyset$

The existence of balanced allocations depends not only on the divisibility of the goods but also on the vector of relative prices, μ, that the arbitrator associates with these goods. In this paper, we do not attempt to solve this general (and tricky) problem as it typically pertains to operations research rather than economics. We restrict our analysis to the case where each good is divisible in a number of shares of equal size, i.e. for every imperfectly divisible good, there exists a positive integer, p, representing the number of pieces – of size $\frac{1}{p}$ – into which this good can be divided, we then say that this good is *p-divisible*. Also, in order to make the presentation clearer, we consider the "median" case where the arbitrator values all the goods equally, that is to say, the case where $\mu = (1,...,1) \in \mathbb{R}^n$. The qualitative results we obtain readily generalize to any vector μ.

Since the egalitarian allocation is assumed to be unattainable, the arbitrator must take into account the divisibility constraints imposed on the goods. Assume that the n goods to be distributed comprise n_1 p_1-divisible goods, n_2 p_2-divisible goods,..., n_s p_s-divisible goods (by convention $1 \leqslant p_1 < p_2 ... < p_s$), and the remaining ones being perfectly divisible. Consider the vector $\tilde{N} = (\tilde{n}_{p_1}, \tilde{n}_{p_2}, ..., \tilde{n}_{p_s}, n)$ where $\tilde{n}_{p_t} = n_{p_1} + ... + n_{p_t}$, for all $t = 1, ..., s$.

When the allocation is a balanced one, each individual receives a bundle of "value" $\frac{n}{m}$. This value can uniquely be divided in the following manner:

$$\frac{n}{m} = \frac{f_1}{p_1} + \frac{f_2}{p_2} + ... + \frac{f_s}{p_s} + \varepsilon \tag{2}$$

$$\frac{1}{p_t} > \sum_{t'=t+1}^{s} \frac{f_{t'}}{p_{t'}} + \varepsilon \quad \text{for all } t = 1, ..., s \tag{3}$$

where $\varepsilon \geqslant 0$ and $f_1, ..., f_s$ are integers. An elementary computation yields $f_t = E\left(\left(\frac{n}{m} - \left(\frac{f_1}{p_1} + ... + \frac{f_{t-1}}{p_{t-1}}\right)\right) p_t\right)$ for all $t = 1, ..., s$, where $E(\cdot)$ returns the integer part of any real number.

Finally, let $mf_t = c_t p_t + r_t$, with c_t and r_t respectively the result and the remainder in the Euclidean division of mf_t by p_t. The following proposition provides a necessary condition for the existence of a balanced allocation. It consists in comparing the vector $\tilde{N}$ with another vector $\tilde{A} = (\tilde{a}_{p_1}, ..., \tilde{a}_{p_s}, n)$ such that $\tilde{a}_{p_t} = c_1 + ... + c_t + E(\frac{r_1}{p_1} + ... + \frac{r_t}{pt})$ for all $t = 1, ..., s$.

Proposition 5 *A necessary condition on the divisibility constraint of the goods for balanced allocations to exist is:* $\tilde{A} \geq \tilde{N}$.

Proof. The construction of vector $\tilde{A}$ amounts to taking the maximum number of p_1-divisible goods, making sure that the aggregate value of no individual's bundle exceeds $\frac{n}{m}$. If mf_1 is not a multiple of p_1, the number c_1 leaves r_1 fractions of size $\frac{1}{p_1}$ not allocated. Then, the maximum number of p_2-divisible goods allowed by formula (2) is c_2, or $c_2 + 1$ if this extra unit allows to partially fill in the gaps obtained in the previous steps, i.e. if $E(\frac{r_1}{p_1} + \frac{r_2}{p_2}) \geqslant 1$. And so on.

Notice how these compensations are performed solely on the basis of global quantity considerations rather than on fractions among recipients. Therefore, vector $\tilde{A}$ gives an upper bound for this type of compensations. Hence, if $\tilde{A} \geq \tilde{N}$ is not true, there exists a rank t such that $\tilde{a}_t < \tilde{n}_t$ and $\tilde{a}_{t'} \geqslant \tilde{n}_{t'}$ for all $t' < t$, meaning that there is at least one p_t-divisible good too many compared to formula (2) and, according to (3), it must be that the aggregate share of a recipient exceeds $\frac{n}{m}$. □

The above condition is only a necessary one (even if it proves to be sufficient in some cases, for instance when p_t is a multiple of $p_{t'}$ for any $t' < t$). Nevertheless, it provides valuable indication as to how weak the divisibility requirements are for a balanced allocation to exist. In particular, if some of the goods are "divisible enough" ($p_s \geqslant m$), this necessary condition requires that only one good be perfectly divisible. This statement can be completed by the following sufficient condition.

Proposition 6 *A balanced allocation will exist if at least one of the goods is perfectly divisible and if the other goods satisfy the divisibility constraint* $p_t \geqslant \frac{m-1}{n-\tilde{n}_{p_t}}$ *for all* $t = 1, ..., s$.

Proof. We proceed by allocating the goods in an increasing order of divisibility. In order to obtain a balanced allocation, one only needs for the allocation procedure to unravel for the $n-1$ least divisible goods without the cumulative value of a bundle exceeding $\frac{n}{m}$ for any recipient; the nth good (which is perfectly divisible) then allows us to complete each bundle up to the desired value of $\frac{n}{m}$.

This only requires that all the recipients have not been given a bundle of value exceeding $\left(\frac{n}{m} - \frac{1}{p_t}\right)$ when allocating the last fraction (of size $\frac{1}{p_t}$) of the last p_t-divisible good. This will certainly be the case if the total value of what has already been distributed, i.e. $\tilde{n}_{p_t} - \frac{1}{p_t}$, is less than $\left(\frac{n}{m} - \frac{1}{p_t}\right) m$. □

What is most instructive about these last two propositions is that balanced allocations will exist provided some mild divisibility constraints are satisfied. In most cases, only one good is required to be perfectly divisible.

4.2 Conditions for $\mathcal{E}_\mu \neq \emptyset$

The existence of equitable allocations is also directly related to how finely divisible are the goods to be allocated. Recall that an allocation $\omega \in \Omega$ is equitable if and only if $(\omega^i - \omega^{i'}) \in \Gamma_n$ for all $i, i' = 1, ..., m$. A notable special case of equitableness yielding tractable computations is that in which all the $(\omega^i - \omega^{i'})$ are collinear; i.e. $\exists u \in \Gamma_n, \exists h_1, ..., h_m \in \mathbb{R}$ s.t. $\sum_{i=1}^m h_i = 0$ and $\omega^i = \bar{\omega} + h_i u$, where $\bar{\omega}$ stands for the egalitarian bundle $\bar{\omega} = (\frac{1}{m}, ..., \frac{1}{m}) \in \mathbb{R}^n$. We next give a sufficient condition on the divisibility structure of the goods to allow for such a special allocation to exist.

Proposition 7 *A solution,* $\omega \in \Omega$*, in which the differences in bundles,* $(\omega^i - \omega^{i'})$*, are collinear for all* $i, i' = 1, ..., m$ *exists if at least* $(n-1)$ *of these goods are perfectly divisible and the last one is* p*-divisible with* $p \geq \frac{m}{n}$.

Proof. In Appendix 6.3. □

The intuition behind this proposition is fairly straightforward; if there are "enough" goods to go about (i.e. $n \geqslant m$), one can always rectify the inequity induced by the allocation of the one good that is not perfectly divisible. However, if there are more recipients than goods, there is a divisibility requirement that the good which is not perfectly divisible must fulfill: one must be able to allocate it in shares that do not exceed $\frac{n}{m}$.

We now turn to a more general result.

Proposition 8 *An equitable allocation will exist in general*[8] *if at least* $\frac{m}{2}$ *goods are perfectly divisible. Also, an allocation which is both balanced and equitable will exist in general if there are at least* $\frac{m}{2}+1$ *goods are perfectly divisible.*

[8] Provided some conditions on the goods which are not perfectly divisible are satisfied.

Proof. Let q be the number of perfectly divisible goods, the set of possible allocations, Ω, allows for $q(m-1)$ degrees of freedom.

Also, the existence of an equitable solution demands that the $\frac{m(m-1)}{2}$ vectors of differences, $(\omega^i - \omega^{i'})$ for $i > i'$, be elements of Γ_n. Moreover, since Γ_n is diffeomorphic to $\mathbb{R}^{n-1} \setminus \{0\}$, each of these vectors takes up one degree of freedom.

Hence, an equitable allocation will exist in general if $(m-1)q \geqslant \frac{m(m-1)}{2}$; i.e. if $q \geq \frac{m}{2}$.

The existence of an allocation which is not only equitable but balanced as well requires the $(m-1)$ additional equations $\sum_{j=1}^{n} \mu_j \omega_j^i = \frac{n}{m}$ for $i = 1, ..., m-1$ (the equation then holds for $i = m$ by feasibility). Therefore, such an allocation will exist in general if $(m-1)q \geqslant \frac{m(m-1)}{2} + (m-1)$; i.e. if $q \geq \frac{m}{2} + 1$. □

Although rather vague, this last proposition clearly indicates that the conditions of existence of equitable allocations are much more demanding (in the number of perfectly divisible goods) than that of balanced allocations.

5 Conclusion

In this paper we have investigated a fair division situation in which the arbitrator has no information about the preferences of the recipients and where no preference-revealing procedure can be implemented. Furthermore, the goods to be allocated are non-marketable, non-exchangeable (for exogenous reasons: technical, legal or ethical ones) and are subject to divisibility constraints.

Such informational situations in which the arbitrator knows nothing about the recipients' preferences or claims often occur in practice. The first main case is a situation in which it may be impossible to discriminate between claims, when the urgency of the situation is such that priority statuses are rendered indistinguishable. Another case of such ignorance could occur when the demand procedure is not strategyproof. Practical cases where this lack of information is coupled with the above heavy technical constraints bearing on the goods to be divided are easy to find.

The model presented here allowed us to derive two types of results: substantial results and formal ones. The main substantial lesson is that balanced allocations retain some of their appeal even though they are not adapted when a problem with such informational restriction is not repeated often. Balanced allocations are not nearly as satisfying on equity grounds as equitable ones, yet this does not mean that the arbitrator can necessarily do better than suggesting a balanced allocation; equitable allocations may not always exist because of the divisibility constraints on the goods to be distributed and, when they do exist, actually finding equitable allocations may prove to be a (very) difficult task despite the formula allowing us to easily compute the ratio $\frac{Area(\Lambda_n^+)}{Area(\Lambda_n)}$ (Theorem 1). This is why we suggest that the arbitrator restrict her attention to balanced allocations (as they are much easier to find and exist more often than equitable ones and also provide a safeguard against high probabilities of envy occurring) and make an extra effort to single out an allocation that fares much better on equity grounds than the threshold α, which turns out to be considerably less than our equity norm of 50%.

The formal results offered by our investigation concern the remarkable property of simplices shown in Section 2.2: Theorem 1 and Corollary 1. Many calculations implied by our reasoning, such as the ones required to define equitable allocations or to discriminate amongst balanced allocations, are greatly facilitated by this property. Moreover, beyond the use made here, the prominence of simplices in many areas of economics suggests that this property may have interesting applications in different contexts. For these reasons, this geometrical property, which to our knowledge never appeared in the economic literature (nor the mathematical one), could have an interest of its own.[9]

The main shortcomings of our approach lie in the simplifying assumptions imposed: the linearity of the preferences, the divisibility of the goods in shares of equal size. Most of all, one could object that requiring the vectors of marginal utilities to be uniformly distributed on the simplex Λ_n is arbitrary. Nevertheless, the qualitative results of this paper are robust to changes within a large range of normalizing surfaces (as discussed in Appendix 6.4).

6 Appendix

6.1 Proof of Theorem 1

Theorem 1 is an application of a more general result. We first introduce some notation before stating this more general theorem:

Let $n \in \mathbb{N}$, and let $r \in \{2, ..., n+1\}$. For any set $A \subset \mathbb{R}^n$, the expressions $clo(A)$ and $ch(A)$ denote the closure and the convex hull of A, respectively. Let $x \in \mathbb{R}^n$ and $K \in \mathbb{R}$, define $H = \{q \in \mathbb{R}^n | q \cdot x - K = 0\}$ to be a hyperplane of $\mathbb{R}^n$. Accordingly, for any $q \in \mathbb{R}^n$, define $H(q) = q \cdot x - K$. Let $s_1, s_2, ..., s_r$ be r affinely independent points and let Λ_r be the $(r-1)$-dimensional simplex having the s_k's as vertices. The hyperplane H divides $\mathbb{R}^n$ into two half-spaces, F^+ and F^-, and the simplex Λ_r into two subsets: $\Lambda_r^+ = \{q \in \Lambda_r | H(q) \geqslant 0\} \subset F^+$ and $\Lambda_r^- = \{q \in \Lambda_r | H(q) \leqslant 0\} \subset F^-$(see Fig. 3).

We also use the following notational convention. If $A \subset \mathbb{R}^n$ is a set of dimension q, we denote by $Area_q$ the function associating with A its q-dimensional volume $Area_q(A)$. Throughout the paper, we have implicitly used $Area$ in lieu of $Area_{n-1}$.

Theorem 2 *If H is not parallel to any edge of Λ_r (i.e. $H(s_j) \neq H(s_{j'})$ for all $j \neq j'$), then:*

$$\frac{Area_{r-1}(\Lambda_r^+)}{Area_{r-1}(\Lambda_r)} = \sum_{j \in J} \prod_{j' \neq j} \frac{H(s_j)}{H(s_j) - H(s_{j'})}$$

where $J = \{j \mid H(s_j) > 0\}$.

[9] For example, another application of our formal results could be the following, assuming that individuals are described by a vector of functionings. The usual problem of the functionings approach lies in the relative weighing of these functionings. By considering a family of possible vectors in the simplex, one can ask whether this makes a given agent i better off than another given agent i'. Thus, thanks to our results, one may give second thoughts to using the expected weighing vector and may instead look for an equitable one.We are grateful to Marc Fleurbaey for this comment.

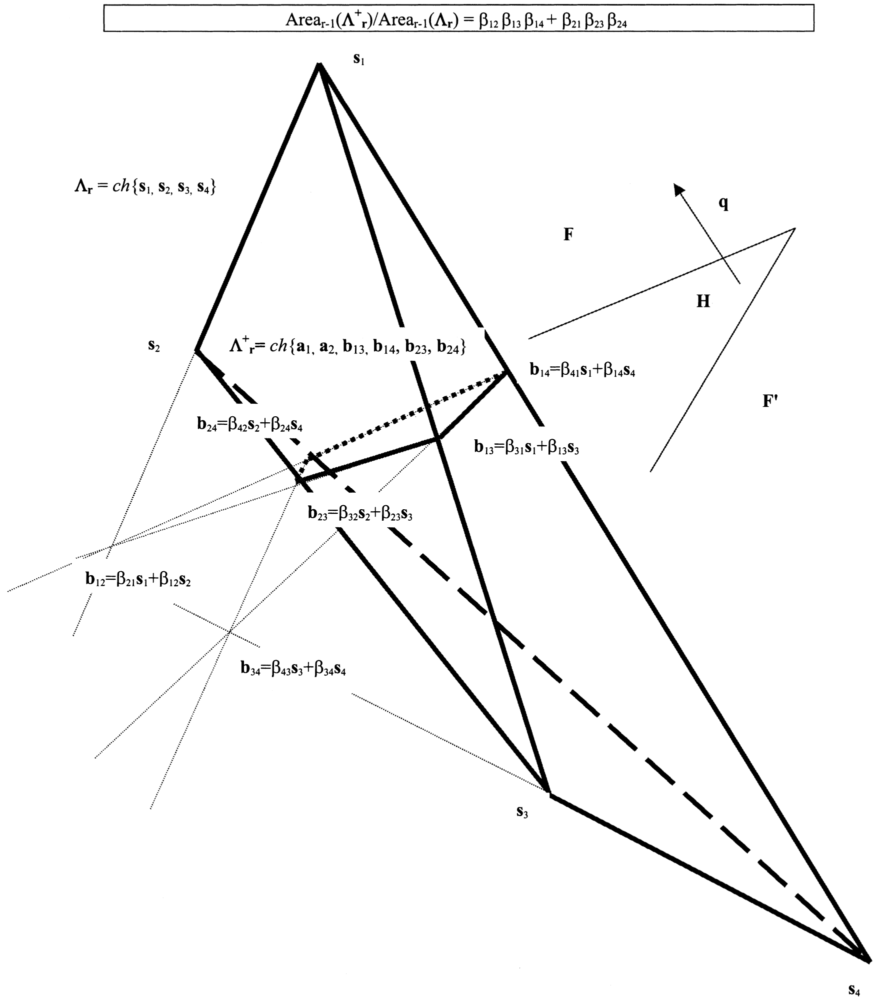

Figure 3

The proof of the theorem requires establishing a lemma. We denote by Θ_{hk} (or, equivalently Θ_{kh}) the line passing through s_h and s_k. Given that the hyperplane H is assumed not to be parallel to any edge of Λ_r, we have $H(s_k) \neq H(s_h)$ for every k and h, and the line Θ_{hk} intersects H at a point that we denote b_{hk} (or, equivalently, b_{kh}). We define β_{kh} and β_{hk} such that $b_{hk} = b_{kh} = \beta_{kh} s_h + \beta_{hk} s_k$ with β_{kh} $+\beta_{hk} = 1$. It follows immediately from these notations that $\beta_{kh} = \frac{H(s_k)}{H(s_k)-H(s_h)}$.

We relabel the vertices of Λ_r that are located in F^+ by $j \in J^+ = \{1, 2, ..., p\}$, such that: for any j and $j' \in J^+$, where $j < j\prime$, we have $\beta_{jj'} \geqslant 1$ (i.e. $s_{j'} \in [s_j, b_{jj'}]$), and for any $i \notin J^+$, $1 \geqslant \beta_{ji} \geqslant 0$ (i.e. $b_{ji} \in [s_j, s_i]$). This labeling of the p vertices of Λ_r located in F^+ is clearly unique.

Finally, for any $j \in J^+$, we define $T_j = ch\left(s_j, (b_{jk})_{k \neq j}\right)$. And, for any $j \in J^+$, we define $R_j = clo(T_j \backslash R_{j+1})$, where $R_{p+1} = \emptyset$, by convention. Then:

Lemma 1 *$R_j \subset T_{j-1}$ and $R_1 = \Lambda_r^+$.*

Proof. First of all, notice that for any $j, j' \in J^+$ with $j < j'$ and for any $k \notin \{j, ..., p\}$ we have $b_{j'k} \in [b_{jk}, b_{jj'}]$. Indeed, these three points, $b_{j'k}$, b_{jk} and $b_{jj'}$ lie on one same line since they belong to the intersection of the hyperplane H and the two-dimensional face of Λ_r generated by the triplet $(s_j, s_{j'}, s_k)$. One can therefore write $b_{j'k} = \beta b_{jk} + (1-\beta) b_{jj'}$, where $\beta = \frac{\beta_{jj'}-1}{\beta_{jj'}-1+\beta_{kj}}$; notice that $1 \geqslant \beta \geqslant 0$ (i.e. $b_{j'k} \in [b_{jk}, b_{jj'}]$).

We now establish the following equality by backward induction:

$$\begin{aligned} \forall j \in J^+, \quad R_j &= clo(T_j \diagdown R_{j+1}) \\ &= ch\left((s_{j'})_{j' \in \{j,\dots,p\}}, (b_{j'k})_{j' \in \{j,\dots,p\}, k \notin \{j,\dots,p\}}\right). \end{aligned} \tag{4}$$

This equality is obviously satisfied for $j = p$ since, by definition, $R_{p+1} = \emptyset$ and thus $R_p = T_p = ch\left(s_p, (b_{pk})_{k \neq p}\right)$.

Now assume that equality (4) is true for $j+1, ..., p$. Then $b_{jj'} \in R_{j+1} \cap T_j$ for any $j \in \{j+1, ..., p\}$. The labeling we used ensures that $s_{j'} \in [s_j, b_{jj'}]$ for any $j' \in \{j+1, ..., p\}$; and we previously noticed that $b_{j'k} \in [b_{jk}, b_{jj'}]$. Moreover, from (4), $R_{j+1} = ch(s_{j'}, b_{j'k})$ for any $j' \in \{j+1, ..., p\}$ and any $k \notin \{j+1, ..., p\}$; therefore $R_{j+1} \subset T_j = ch\left(s_j, (b_{jk})_{k \neq j}\right)$ and R_j verifies (and thus establishes) equality (4). Finally, when $j = 1$, formula (4) implies that $R_1 = ch\left((s_{j'})_{j' \in \{1,\dots,p\}}, (b_{j'k})_{j' \in \{1,\dots,p\}, k \notin \{1,\dots,p\}}\right)$, i.e. $R_1 = \Lambda_r^+$. □

Let Z be a q-dimensional simplex, s is a vertex of Z and if Z^s denotes the $(q-1)$-dimensional face opposite to s; then $Area_q(Z) = \frac{1}{q} Area_{q-1}(Z^s) \times d(s, Z^s)$ where $d(s, Z^s)$ is the Euclidean distance from s to the linear manifold containing Z^s.

We can now establish Theorem 2.

Proof. $T_j = ch\left\{s_j, (b_{jk})_{k \neq j}\right\}$ is an $(r-1)$-dimensional simplex (obvious). We denote by $T_j^{b_{jr}}$ the face of T_j opposite to $b_{jr}$and $\Lambda_r^{s_r}$ the face of Λ_r that is opposite to s_r. From our notations:

$$\frac{d(b_{jr}, T_j^{b_{jr}})}{d(s_r, \Lambda_r^{s_r})} = |\beta_{jr}| \Longrightarrow \frac{Area_{r-1}(T_j)}{Area_{r-1}(\Lambda_r)} = |\beta_{jr}| \times \frac{Area_{r-2}(T_j^{b_{jr}})}{Area_{r-2}(\Lambda_r^{s_r})}$$

Denoting by $T_j^{b_{jr}, b_{j(r-1)}}$ and $\Lambda_r^{s_r, s_{r-1}}$ the faces opposite to $b_{j(r-1)}$ and s_{r-1}in $T_j^{b_{jr}}$ and $\Lambda_r^{s_r}$ respectively, a similar argument yields:

$$\frac{Area_{r-1}(T_j)}{Area_{r-1}(\Lambda_r)} = |\beta_{jr}| \times \left|\beta_{j(r-1)}\right| \times \frac{Area_{r-3}(T_j^{b_{jr}, b_{j(r-1)}})}{Area_{r-3}(\Lambda_r^{s_r, s_{r-1}})}$$

and so on. Finally,

$$\frac{Area_{r-1}(T_j)}{Area_{r-1}(\Lambda_r)} = \prod_{i \neq j} |\beta_{ji}| \Longrightarrow \frac{Area_{r-1}(T_j)}{Area_{r-1}(\Lambda_r)} = (-1)^{j-1} \prod_{i \neq j} \beta_{ji}$$

due to the signs of the β_{ji}'s.

From Lemma 1, $Area(\Lambda_r^+) = Area(T_1) - Area(T_2) + ... + (-1)^{p-1} Area(T_p)$. Hence

$$\frac{Area_{r-1}(\Lambda_r^+)}{Area_{r-1}(\Lambda_r)} = \sum_{j \in J^+} \prod_{i \neq j} \beta_{ji}$$

where $\beta_{ji} = \frac{H(s_j)}{H(s_j) - H(s_i)}$ for all i, j; one easily notices that if $H(s_j) = 0$ (i.e. $s_j \in H$), then $\prod_{i \neq j} \beta_{ji} = 0$, allowing us to restrict the above formula to the vertices s_j such that $H(s_j) > 0$. □

6.2 Proof of Proposition 1

By rescaling units one can transform the simplex $\Lambda_n = ch(s_1, ..., s_n)$, where $s_k = \mu_k e_k$ for all k, into the unit simplex $\Delta_n = ch(e_1, ..., e_n)$. Without loss of generality, it is therefore sufficient to prove that Proposition 1 holds on Δ_n. I.e., we only need to prove the following proposition.

Proposition 9 *The set Γ_n of vectors u such that the hyperplane $H = \{q \in \mathbb{R}^n | q \cdot u = 0\}$ divides Δ_n into two surfaces of equal area is diffeomorphic to $\mathbb{R}^{n-1} \setminus \{0\}$.*

Proof.
a) Let Ψ^+ and $\Psi^- = 1 - \Psi^+$ be functions defined on $\mathbb{R}^n$ by: $\forall u \in \mathbb{R}^n \setminus \{0\}$ $\Psi^+(u) = \frac{Area(\Delta_n^+(u))}{Area(\Delta_n)}$ and $\Psi^-(u) = \frac{Area(\Delta_n^-(u))}{Area(\Delta_n)}$ respectively, where $\Delta_n^+(u)$ and $\Delta_n^-(u)$ are the two subsets of Δ_n delimited by the hyperplane $H = \{q \in \mathbb{R}^n | q \cdot u = 0\}$.

Thanks to Corollary 1, an alternative definition of Ψ^+ is:

- $\forall u \in \mathbb{R}^n \setminus \{0\}$ such that $u_i \neq u_j, \forall i \neq j$:

$$\Psi^+(u) = \sum_{j \in \{j | u_j > 0\}} \prod_{i \neq j} \frac{u_j}{u_j - u_i}$$

- $\forall u \in \mathbb{R}^n \setminus \{0\}$ such that $u_i = u_j$ for some $i \neq j$:

$$\Psi^+(u) = \lim_{u' \to u} \left(\sum_{j \in \{j | u'_j > 0\}} \prod_{i \neq j} \frac{u'_j}{u'_j - u'_i} \right)$$

where $u'_i \neq u'_j$ for all $i \neq j$.

Both functions, Ψ^+ and Ψ^-, belong to $\mathcal{C}^\infty$ (the class of infinitely differentiable functions).
b) Denote $e = (1, 1, ..., 1) \in \mathbb{R}^n$ and $H_e = \{q \in \mathbb{R}^n \setminus \{0\} \mid q \cdot e = 0\}$. Let $a \in H_e$ and let $\lambda, \lambda' \in \mathbb{R}_{++}$, denote $u = \lambda a + (1-\lambda)e$ and $u' = \lambda' a + (1-\lambda')e$. Then, if $\lambda' > \lambda$, for all $q \in \Delta_n$ such that $q \cdot u \leqslant 0$, then $q \cdot u' \leqslant 0 \Longleftrightarrow \Delta_n^-(u) \subset \Delta_n^-(u')$. Therefore, for any $a \in H_e$ such that $\Psi^+(a) \geqslant \frac{1}{2}$, there exists a unique λ_a such that $\Psi^+(\lambda_a a + (1-\lambda_a)e) = \frac{1}{2}$ (and $\lambda_a \geqslant 1$). By symmetry, when $\Psi^-(a) \geqslant \frac{1}{2}$ there exists a unique λ_a such that $\Psi^+(\lambda_a a + (1-\lambda_a)e) = \frac{1}{2}$ (and $1 \geqslant \lambda_a > 0$). For any $a \in H_e$, we denote $\Phi(a) = \lambda_a a + (1-\lambda_a)e$.

We have just defined a bijection $\Phi : H_e \longrightarrow \Gamma_n$ mapping any $a \in H_e$ to a vector $u = \Phi(a)$ such that $H = \{q \in \mathbb{R}^n | q \cdot u = 0\}$ divides Δ_n into two surfaces of equal area, i.e. such that $u \in \Gamma_n$.
c) We still need to check that this bijection is indeed a diffeomorphism. It suffices to show that Φ and its inverse, Φ^{-1}, both belong to $\mathcal{C}^\infty$.

Let $a \in H_e$ and let $u = \Phi(a) = \lambda_a a + (1-\lambda_a)e$. We know that Ψ^+ belongs to $\mathcal{C}^\infty$. Therefore, defining $\Psi^+(\lambda, a) = \Psi^+(\lambda a + (1-\lambda)e)$ it is easy to check that $\frac{\partial \Psi^+}{\partial \lambda} \neq 0$, by construction. Moreover, since $\Psi^+(\lambda_a a + (1-\lambda_a)e) = \frac{1}{2}$, it follows from the implicit function theorem that the function $\lambda_a : H_e \longrightarrow \mathbb{R}$ also belongs to $\mathcal{C}^\infty$. Therefore Φ and Φ^{-1} both belong to $\mathcal{C}^\infty$. □

6.3 Proof of Proposition 7

Assume that good 1 is the imperfectly divisible one. Let $(\omega_1^1, \omega_1^2, ..., \omega_1^m)$ be any possible division of good 1. A direct application of Corollary 1 yields that the vectors $(u_1, -u_1, 0, ..., 0)$ and $(u_1, -u_1(2^{\frac{1}{n-1}} - 1), -u_1(2^{\frac{1}{n-1}} - 1), ..., -u_1(2^{\frac{1}{n-1}} - 1))$ belong to Γ_n. Moreover, for any values of n and m greater than one, $-1 \leqslant \inf\left(-\frac{1}{n-1}, -\frac{1}{m-1}\right) \leqslant -(2^{\frac{1}{n-1}} - 1)$. Consequently, a continuity argument ensures that there exists a vector, $u \in \Gamma_n$, such that:

$$u_1 = \bar{u} = \sup_h (u_h) = 1 \quad \text{and} \quad \underline{u} = \inf_h (u_h) = \inf\left(-\frac{1}{n-1}, -\frac{1}{m-1}\right).$$

We now show that there exist $h_1, ..., h_m \in \mathbb{R}$ such that $\omega^i = \left(\frac{1}{m}, ..., \frac{1}{m}\right) + h_i u$ for all i, and $\sum_{i=1}^m h_i = 0$. Indeed, choosing $h_i = \omega_1^i - \frac{1}{m}$ for all i, we verify immediately that:

- $\sum_{i=1}^m h_i = 0$;
- the distribution of good 1 is feasible,

To complete the proof, one must check that:

- $\sum_{i=1}^m \omega_j^i = 1$ for every good $j > 1$ (immediate),
- $0 \leqslant \omega_j^i \leqslant 1$ for all $i = 1, ..., m$ and all $j = 2, ..., n$; this condition translates into:

$$\sup\left(-\frac{1}{m}\frac{1}{\bar{u}}, \frac{m-1}{m}\frac{1}{\underline{u}}\right) \leqslant h_i \leqslant \inf\left(\frac{m-1}{m}\frac{1}{\bar{u}}, -\frac{1}{m}\frac{1}{\underline{u}}\right) \quad \text{for all } i, \quad (5)$$

for which we distinguish the following two cases. If $n \geqslant m$, then $-\frac{\bar{u}}{\underline{u}} = n-1 \geqslant m-1$, therefore expression (5) can also be written as $-\frac{1}{m} \leqslant h_i \leqslant \frac{m-1}{m}$ for all i. The left-hand-side inequality of the latter expression is always verified; as for the right-hand-side inequality, it amounts to $\omega_1^i \leqslant 1$, which is also always true. Now, if $m > n$, then $-\frac{\bar{u}}{\underline{u}} = n-1 < m-1$, therefore expression (5) can also be written as $-\frac{1}{m} \leqslant h_i \leqslant \frac{n-1}{m}$ for all i. The left-hand-side inequality of the latter expression is, again, always verified; as for the right-hand-side inequality, it is equivalent to $\omega_1^i \leqslant \frac{n}{m}$ for all i completing the proof.[10]

6.4 Choosing the normalizing surface

Due to the absence of information about the preferences of the recipients, the arbitrator is compelled to assume that the random process generating their vectors of marginal utilities is equiprobable. But the choice of the surface on which this uniform density is considered has critical consequences on the final decision of the arbitrator. The reason is that it itself defines the probability distribution of the directions of $\mathbb{R}_+^n$ that the marginal utility vectors, λ^i, will take. Throughout the paper, we chose the simplex Λ_n to be the normalizing surface for convenience: it is simpler to analyze and to present, computations are made easier (especially thanks to Theorem 1) and, finally, it allows one to easily take into account the vector of relative "prices", μ, which the arbitrator associates with the goods to be allocated.

Deliberately choosing a particular framework because it makes the analysis easier can be a quite questionable procedure. Indeed, one may object that the choice of a simplex as the normalizing surface is unjustified on the basis that it is tantamount to considering a peculiar probability distribution of directions of $\mathbb{R}_+^n$: for any cone with its origin at 0, of angle $d\alpha$ and direction u, the surface (and hence the probability) that it defines on Λ_n increases as u gets further away from the direction $(1/\mu_1, ..., 1/\mu_2)$.[11] Our answer to this point is twofold.

First of all, we claim that no other shape of the normalizing surface is more natural or less arbitrary except in the special case where all goods are valued identically by the arbitrator (i.e. when $\mu = (1, 1, ..., 1)$), in which one may be compelled to choose S_n as the normalizing surface; where S_n is defined as the portion of the sphere centered in 0 (with radius 1) included in $\mathbb{R}_+^n$. The reason is that, in this case, what is really at work is an equiprobable distribution of "directions" in $\mathbb{R}_+^n$ around the diagonal of the positive orthant. And, the surface that associates with any cone of origin 0, of axis in the "direction" u and of "angle" $d\alpha$, a surface $d\sigma$ which only depends on $d\alpha$ (and hence is independent of u) is a sphere centered at 0.[12] Yet, apart from this latter case, and as soon as we assume that the arbitrator's reasoning works effectively on marginal utilities (and not on direction of $\mathbb{R}_+^n$), it becomes

[10] Note that this results still holds even if the imperfectly divisible good (good 1) is not divisible in shares of equal size, as long as its largest indivisible share is no greater than $\frac{n}{m}$.

[11] Indeed, the interpretation of this property is not obvious and seems even counterintuitive in the case of the unit simplex (i.e. Δ_n corresponding to $\mu = (1, ..., 1)$).

[12] This argument was pointed out to us by Serge-Christophe Kolm.

necessary to accept the artificiality introduced by the choice of any normalizing surface (because no one surface is "natural").

Our second and main argument in favor of the simplex is the following: as long as our qualitative results are robust to the choice of the normalizing surface, we might as well consider the most convenient one. First, one should note that all the theoretical properties established in this paper hinge on the convexity of the cone originating from 0 and based on the surface of normalization. They will therefore remain valid for any surface such that this cone is convex. In particular, balanced allocations remain only α-equitable, where α is the same as in the case of the simplex, because it only depends on the number of goods, n, and not on the shape of the probability density function.[13]

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[13] The numerical results will certainly depend on the surface chosen by the arbitrator. Yet, a software of our creation (based on Theorem 1) allows us to compute the probabilities of envy occuring for any such admissible surface. The reader may contact the authors for more information on this tool.