Derivatives and Risk Management
MBAB 5P44 – MBA

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A derivative is a contract whose return depends on the price movements of some underlying assets. There are three main families of derivative contracts: options, futures, and swaps. They all have the ability to reduce risk; thus, are widely used for hedging purposes.

This course covers basic topics on options, futures, and swaps. We focus on their 1) market organization, 2) hedging properties, and 3) evaluation principles. For each product, we report on its market rules and specificities, and study some of its associated hedging strategies. Models for derivatives pricing are based on the no-arbitrage assumption. We discuss the most popular: the Black-Scholes-Merton model for options pricing and the cost-of-carry model for forward/futures pricing.

The following textbooks are required. The first one will be used frequently during the term; the others offer useful alternate explanations, and are listed by increasing order of difficulty:


All Figures and Tables referred to in the text are taken from Chance and Brooks (2007). Monitoring the news on financial derivatives is recommended as an important complement to classroom work. Selected articles from business newspapers such as The Financial Times, The Globe and Mail, The National Post, and The Wall Street Journal will be discussed in class.
The grading policy is based on: four assignments each worth 5%; two midterm exams each worth 30%; a quiz worth 10%; and one presentation worth 10%. The midterm exams and the quiz are cumulative.

The assignments will improve your skills and prepare you for the exams. For each assignment, you will be given selected problems to solve, articles to comment on, and (possibly) empirical and numerical experiments to implement. The final requirement, due at the end of the term, consists of a short presentation (20 minutes for each group) on a complex financial derivative of your choice.
1 Introduction

1.1 Derivatives and Hedging

A *derivative* is a contract whose performance depends on the price movement of some underlying assets. An underlying may be either a *financial* asset such as a stock or a *real* asset such as a commodity. The underlying assets we consider here are usually traded on *cash* or *spot markets*, which, in turn, strongly impact derivatives markets.

**Exercise 1**: Give a definition of an *asset*. Provide examples of *financial assets* and *real assets*.

Options trading on organized markets goes back to the 18th century in the Netherlands. Dutch horticulturists used to hold put options on Tulip. They would lock in the Tulip sell price before the harvest period, thereby hedging against a potential decrease on the Tulip spot price.
Derivatives have the ability to reduce risk; thus, are widely used for hedging purposes.

**Example 2:** The spot foreign exchange rate for the US dollar is 0.69056 Euros. Your company agrees to pay a bank 63,694 Euros in 3 months in exchange for 100,000 US dollars. This is a foreign currency *forward* contract. No cash is exchanged up front. Give the underlying asset, the maturity date, and the forward rate (for the US dollar). Compare to the spot forward rate. Conclude.

**Example 3:** A contract promises its holder $1000 if no earthquake hits Tokyo during the next year. This contract is known as a *digital option*, and it’s used here as insurance against earthquake damage. How much would you pay for it?

The *return* (in %) is a numerical measure of investment performance, and the *risk* is the uncertainty of future returns. There are various types of risk, including *market risk* and *credit risk*. Market risk is associated to the
movement of asset prices, and credit risk to the failure of a counterparty to fulfill his obligations.

Among investment opportunities that have the same expected return, a *risk-averse* investor would prefer the one that has the lowest risk, while a *risk-neutral* investor would be indifferent, as long as the expected return remains constant. A risk-averse investor wouldn’t take on additional risk unless he expects a large enough *risk premium*.

Hedging is relevant because investors are usually risk-averse, pessimistic, and they support market imperfections, including transaction costs, information asymmetries, and taxes.

There are three main subfamilies of derivative contracts, namely, *options*, *futures*, and *swaps*. All can be designed to hedge risk. If one party is hedging, then the other is speculating!

**Exercise 4:** Why is hedging relevant for a company? (stability, convexity...)
1.2 Options

An option is a right (privilege), but not an obligation. A call option is a contract that gives its holder the right, but not the obligation, to buy an underlying asset at a specified future maturity date for a known strike price. A put option is a contract that gives its holder the right, but not the obligation, to sell an underlying asset at a specified future maturity date for a known strike price.

Options are traded both on exchanges and over the counter.

The option holder is also called the buyer or the long party. The option buyer benefits from a privilege; therefore, pays for it usually up front. The option signer is also called the writer, the seller, or the short party.

The up-front payment made by the buyer to the seller is the option price, also known as the option premium.

Example 5: The ABC October 30 call option is quoted at $0.5, the ABC October 30 put option is quoted at
$1.5, while the ABC spot price is quoted at $28. The call option is said to be *out of the money* since its immediate exercise (if possible) has no value. Conversely, the put option is *in the money*. Suppose the ABC spot price rises to $31 on October 16, which is the option maturity date. Then, the call holder is better of exercising his right and making a profit of $1. The call expires in the money, while the put expires out of the money.

### 1.3 Forward and Futures Contracts

Under a *forward contract*, one party agree to buy and the other to sell an underlying asset at a specified future *maturity date* for an agreed-upon *forward price*. A forward contract sounds like an option, but the two are fundamentally different. While a forward contract is an obligation for both parties, an option is a right for its holder. In addition, an option assumes an up-front payment by the buyer to the seller, while a forward contract does not.
Forward contracts are traded over the counter, and are written on foreign currencies, bonds, stocks, stock indexes, and commodities.

A *futures contract* is otherwise similar to a forward contract except that it is traded on an exchange and is subject to a *daily settlement*. At the end of each trading day, the futures contract is closed and rewritten at the new closing *settlement price*. If the difference to the previous settlement price is positive, it is subtracted from the short-trader account and added to the long-trader account. Else, the reverse is done. The daily-settlement system protects both parties against counterparty risk.

Few traders hold their positions on a futures contract until the maturity date; thus, deliver or take delivery of the underlying asset. They usually reverse their positions and exit the market before the futures contract maturity.
1.4 Swaps and Other Derivatives

A swap commits two parties to exchange cash flows at specified settlement dates, up to a given maturity date.

Example 6: A company agrees to pay its bank semi-annual interest over 20 years, based on a fixed nominal interest rate of 5.5% (per year), and to receive interest payments based on a floating interest rate, say, the 6-month LIBOR rate. Both interest payments assume a principal amount of $1 million. No payment is exchanged up front.

Several swaps are traded over the counter, namely, interest rate swaps, currency swaps, equity swaps, and the more recently introduced and controversial credit default swaps.

There are several combinations of derivative contracts, such as options on swaps or swaptions, options on futures contracts, and options embedded in bonds, among others.
1.5 Arbitrage

Suppose an investor enters the market without initial wealth, initiates a riskless investment strategy, and leaves the market at a certain future date with a strictly positive riskless profit. This is a *free lunch* or an *arbitrage opportunity*.

A *short selling* of a financial asset consists of borrowing the asset from a broker for an immediate sell, with the promise to return it later with its generated revenues. Short selling is usually required to initiate arbitrage strategies.

Models for pricing derivatives are built to preclude against arbitrage opportunities. They are said to be *arbitrage free*. On the other hand, financial markets may allow for sporadic arbitrage opportunities; however, in efficient markets, they will quickly disappear since arbitrageurs will profit immediately and keep prices in line.

**Example 7:** Two assets $A_1$ and $A_2$ are traded in a one-period market model as follows.
[Please discuss Figure 1.2 on page 10.]

Which asset is over- and which is under-priced? Give an arbitrage opportunity.

Researchers have worked hard to characterize arbitrage-free market models, and have established a strong relationship between the no-arbitrage property and the so-called martingale property of the underlying-asset price, discounted at the risk-free interest rate. The law of one price, which characterizes no-arbitrage market models, stipulates that two portfolios with identical cash-flow streams necessarily have the same present value.

Market participants figure out their investment strategies in part by comparing theoretical values to market prices. One should buy undervalued assets, and sell over-valued assets. Theoretical values of financial assets are computed using the present value principle, or more elaborated methods such as the risk-neutral evaluation principle, which plays a central role for pricing derivatives.
In all cases, *discount* and *compound* factors are used to move cash flows backward and forward in time.

**Exercise 8:** The *nominal interest rate* is fixed at \( r_{\text{nom}} = 4\% \) (per year), and interest is compounded semi-annually. Give the *semi-annual interest rate* \( r_{\frac{1}{2}} \) (in % per six months), which is relevant to compound interest, the compound factor \( c_{0,1} \) over one year. Compute the equivalent *annual interest rate* \( r_1 \) (in % per year), which would apply if interest were compounded annually. This is the *effective interest rate*. Compute again the compound factor \( c_{0,1} \). Compute the equivalent *monthly interest rate* \( r_{\frac{1}{12}} \) (in % per month) and *quarterly interest rate* \( r_{\frac{1}{4}} \) (in % per quarter). Compute again the compound factor \( c_{0,1} \). Recognize that \( r_{\frac{1}{2}}, r_1, r_{\frac{1}{12}} \) and \( r_{\frac{1}{4}} \) are all equivalent rates. The *continuously compounded interest rate* \( r_c \) (in % per year) is a nominal rate, which would apply if interest were compounded at each second (or even a fraction of a second). Compute again the compound factor \( c_{0,1} \).

Recall that:

\[
\left(1 + \frac{a}{n}\right)^n \rightarrow e^a, \quad \text{when } n \rightarrow \infty,
\]

where \( a \in \mathbb{R} \) and \( n \in \mathbb{N} \).
1.6 The Role of Derivatives Markets

Hedgers use derivatives to reduce risk; however, their counterparts, called speculators, use them to increase risk. This is the dark side of derivatives, as they are sometimes considered legalized gambling tools.

Next, derivatives provide investors with significant and valuable information about spot markets. For example, the convergence of the forward price to the spot price at maturity provides useful information to market participants. In addition, derivative markets offer some operational advantages. Transaction costs are often lower and short-selling positions are usually easier than in cash markets. Economic and financial crises have had great impact on derivatives markets. Examples include the major stock-market crash in 1987, and the recent credit-risk crisis. Consequently, derivatives markets suffered; however, investors’ confidence and growth were usually back.

In sum, derivative markets contribute to making financial markets more complete and efficient.
1.7 Assignment

Read the chapter, and give precise and concise definitions for the keywords. This part is to be prepared for the midterm exam, but should not be handed in as part of the assignment.

Answer questions: 7, 11, 13 and 15 on page 19.
2 Structure of Options Markets

2.1 The Risk of an Option Position

Consider a call option on a stock that pays no dividend over the option’s life.

The stock price at maturity is indicated by \( S_T \) and the option strike price by \( X \).

If \( S_T > X \), then the call-option payoff is \( Y_T = S_T - K \). Else if \( S_T \leq K \), then the option payoff is null. In sum, the payoff for the call-option holder at maturity resumes to:

\[
Y_T = \max (0, S_T - K).
\]

The payoff for the call-option holder, as a function of the terminal stock price, is as follows.

[Please plot the graph of a call payoff function.]
The call-option seller, who is in charge to pay the promised cash flow, may fail to fulfill his obligation. In addition, the payoff for the call-option buyer is unbounded, which increases further the seller’s credit risk.

This is also the case for a put option, even though its payoff function is bounded.

[Please plot the graph of a put payoff function.]

Both parties support market risk in both options exchanges and over-the-counter options markets. While long parties support some credit risk in over-the-counter options markets, they are protected in options exchanges by margin systems.

2.2 Organized versus Over-the-Counter Options Markets: Pros and Cons

An exchange is a legal corporate entity that is organized for trading securities. The Chicago Board of Trade, the
first large derivatives exchange known worldwide, was created in 1973. Futures contracts have been traded first, followed by *standardized options*. Since then, derivatives exchanges have been created and promoted all over the world.

[Please comment on Table 2.1 on page 30.]

Meanwhile, over-the-counter markets had been moderately growing up till the early 1980s, when corporations started hedging against interest- and exchange-rate risk. Over-the-counter markets are now larger and deeper.

In the following table, an advantage for an options market is a disadvantage for the other.
<table>
<thead>
<tr>
<th>Over-the-Counter Markets</th>
<th>Organized Markets</th>
</tr>
</thead>
<tbody>
<tr>
<td>1– Great design flexibility</td>
<td>4– Very low counterparty risk</td>
</tr>
<tr>
<td>2– Easy market access</td>
<td>5– High liquidity</td>
</tr>
<tr>
<td>3– Total confidentiality</td>
<td>6– Low transaction costs</td>
</tr>
<tr>
<td></td>
<td>7– Price effectiveness</td>
</tr>
</tbody>
</table>

Dealing over the counter offers great flexibility to fix the option underlying asset, strike price, and maturity, among other parameters. In addition, more flexibility was recently added for revising minimal trading-volume levels.

The flexibility offered in over-the-counter options markets is effective for designing specific hedging strategies. These are usually conducted by corporations and institutional investors in total privacy as they are highly strategic and valuable.
Options buyers support some credit risk in over-the-counter markets since options sellers may fail to fulfill their obligations. In an organized market, however, long parties benefit from a protective *margin system*, mastered by a *clearing house*. Conversely, both parties support market risk whether they trade on exchange-listed options or not.

Standardized options, traded in organized markets, are liquid. A market intervenant may invert his position at any time before the maturity date, offset his risk, and leave the market. Unfortunately, leaving the market prior maturity is not allowed in over-the-counter markets.

Liquid markets provide investors with effective and valuable signals, including prices and volumes.

[Please comment on the statistics of the *Bank for International Settlements*].
2.3 Standardized Options

Exchanges have made options as marketable as stocks by means of standards. Firstly, the exchange specifies the underlying assets to be used. Options tend to be written on large firms; however, options on small firms may be more useful for hedging purposes.

To initiate an exchange-listed option, the decision whether to consider an underlying asset or not belongs to the exchange but not to the underlying firm.

An option class refers to a given underlying asset, and an option serie to the same class and same strike price. Next, the exchange specifies the size of any traded option contract. For example, on the Chicago Board of Options Exchange (CBOE), an option on a stock gives exposure to 100 stocks. The option size is adjusted upward for a stock split or a stock dividend. Meanwhile, the option strike price is revised downward consistently.
For example, in the case of a two-for-one stock split, any old option contract is exchanged for two new option contracts, and, accordingly, the strike price is divided by two. In sum, the payoff to the option holder must remain the same before and after the split.

Exercise 9: Consider the case of a 15% stock dividend. The option size is revised upward, and the strike price is revised downward. Explain.

In addition, exercise prices are also standardized by the exchange to maintain the most attractive options, which are usually signed at-the-money and nearly at-the-money options:

\[ X = S_0 \quad \text{and} \quad X \approx S_0, \]

where \( S_0 \) is the current stock price, and \( X \) is the option exercise price.

Exercise prices are not adjusted when a cash-dividend is paid. Traders must accordingly revise options prices,
downward for call options and upward for put options. Along the same lines, options maturities are standardized. For example, in the CBOE, each stock is assigned to one of the following three expiration cycles:

1. January, April, July, and October;

2. February, May, August, and November;

3. March, June, September, and December.

For a given stock, traded maturities are: the current month that defines the current cycle, the next month in the next cycle, and the next two months in the stock cycle.

**Example 10:** IBM is assigned to the January cycle. The current date is January 20, 2010. Traded maturities of options on IBM are thus: January, February, April, and July.
Finally, exchanges put constraints on options positions and exercising to prevent against market manipulations. A position on an option class on one side of the market is limited to a certain maximum, say, 50,000 stocks. This is a *position limit*.

Unlike a *European option*, which can be exercised only at maturity, an *American option* gives its holder the additional right of early exercise. Most exchange-listed options in the United States allow for early exercise.

The number of option contracts one can exercise on a given trading day may also be limited. This is an *exercise limit*.

Options standards can be seen as limitations and constraints; nevertheless, they add liquidity. Many large institutional investors, however, left options exchanges to trade over the counter, looking for more flexibility. Meanwhile, exchanges have introduced flexibility via some options contracts. Examples include FLEX and LEAPS, both traded on the CBOE. FLEX options allow for various exercise prices, while LEAPS options allow for long maturities.
2.4 The Trading Process

Trading within an exchange is mainly maintained by market makers. Electronic trading systems, however, have been increasing in popularity.

A market maker is responsible for matching options buyers with options sellers. If there is a mismatch, the market maker may complete the trade. Depending on how the trading information is disclosed and shared, the market maker may be called a specialist. To survive, the market maker profits by buying at one price and selling at a higher price. The bid price is the highest price the market maker is willing to pay for an option, and the ask price is the lowest price he is willing to accept for an option. The bid-ask spread, which is the difference between the two prices quoted by a market maker, can be seen as a transaction cost for market participants.

Market makers act in various ways:
1. *Scalpers* profit from bid-ask spreads and tend to close their positions in the very short term;

2. *Position traders* may hold positions to keep the trading process going;

3. *Spreaders* look for quasi-arbitrage opportunities in an attempt to earn small profits at a very low risk.

A *floor broker* is a member of the exchange who executes *trading orders* on behalf of non-member brokerage firms. He earns a flat salary or a commission on each order. Market makers and floor brokers are members of the exchange. Each membership is referred to as a *seat*.

To trade on exchange-listed options, one must establish an account with a *brokerage firm*, which employs or is in business with a floor broker.
A *board broker* is an employee of the exchange who is in charge to introduce trading orders into the computer system; thereby keeping the market maker informed.

An *electronic-trading system* allows for trading from electronic terminals located anywhere. The pertinent information, including bid and ask prices, is vehiculed and processed in the system to keep the trading process going. For example, EUREX, located in Frankfurt, Germany, is a fully automated exchange.

There are several *off-floor options traders*, including large institutional as well as individual investors.

While options on individual stocks may lead to deliver the underlying asset at the exercise date, options on stock indexes obey a *cash settlement* procedure.

**Example 11:** Consider an American index call option, which has a multiple of 100. The current level of the
index is $I_t = 1500$, and the exercise price is $X = 1450$. If the call-option is exercised at $t$, the holder get paid:

$$100 \times (I_t - X) = 500 \text{ dollars in cash.}$$

Quotations are reported in business newspapers and on exchanges Web sites.

[Please read the Microsoft price quotation on page 38].

An investor can place several types of orders on an exchange. While a market order instructs to obtain the best available market price, a limit order specifies a maximum price for buying and a minimum price for selling. They can be either good-till-canceled or day orders. Limit orders are executed as soon as possible in order of priority.

**Example 12:** Suppose you issue a limit order, good-till-cancelled, to buy a call option at a maximum price of $3$, while the bid-ask spread is quoted by the market maker 2.75–3.25. Obviously, your order cannot be completed.
The board broker will input it into the computer system till the ask price, now at $3.25, is revised downward.

A *stop order* specifies a price, which is lower than the current price, at which the broker has to sell for the best available price.

Some trading orders are so large that they cannot be achieved in total for the same price. An *all-or-none order* informs to execute the order in total, even at different prices. On the other hand, an *all-or-none, same-price order* instructs to execute the order in total at a unique price, should it be cancelled.

### 2.5 The Clearing House

A *clearing house*, formally known as an *Options Clearing Corporation*, is an organization that guarantees short parties' performance. Their members are called *clearing firms*. 
The protecting system works as follows. The option seller is constrained to deposit a *margin amount* in a *margin account* held by his broker. The margin amount per option is typically a fraction of the underlying asset price, which depends on whether the option is *covered* or *not* and out or in the money. Similarly, the seller’s broker is himself constrained to deposit a margin amount in a margin account held by his clearing firm within the clearing house.

Now, if the shares are not delivered by an option seller when the option is exercised by an option buyer, the clearing firm of the seller’s broker will appeal to the seller’s broker, and the seller’s broker will appeal to the seller.

Liquidity is the main concern of exchanges. An option trader may offset his position at any time he wishes by issuing an *offsetting order*, which simply consists on inverting his position at the best available market price.

[Please discuss Figure 2.2 on page 35.]
2.6 Transaction Costs

Trading options entails various *transaction costs*. The bid-ask spread is a significant transaction cost for options traders. Floor trading and clearing fees are included in the broker’s commission, and are generally expressed as a fixed amount plus a per-contract charge. Internet rates are about $20 plus $1.25 per contract from major *discount brokers*. They charge less than *full-service brokers*, but usually provide fewer service (valuable information and advice). Margins induce an opportunity cost.

2.7 Types of Options

Options are written on various underlying assets, including stocks, indexes, currencies, bonds, and commodities; however, options on major stocks and indexes are the most widely traded.
Standard options are called *vanilla options*. Several others, with more complex payoff functions, are called *exotic options*. For example, the payoff of an *Asian option* is based on the average price of the underlying asset price along its path, and the one of a *lookback option* is based on extreme prices. On the other hand, options on swaps and futures are actively traded. Options may also be embedded in securities. For example, government bonds often contain *embedded call and put options*, and corporate bonds often contain *embedded call and conversion options*.

An *executive stock option* is a call option written by a corporation and held by its executives. These options are used as incentives for managers. They have to outperform in an attempt to push upward the corporation’s market price at higher levels, and in turn to push their call options deeply in the money.

A *real option* is an option that is embedded in an investment project. Examples are *options to expand*, temporary *shut down*, *abandon*, or *sell* an already started investment project.
2.8 Assignment

Read the chapter, and give precise and concise definitions for the keywords listed on page 46. This part is to be prepared for the midterm exam, but should not be handed in as part of the assignment.

Answer questions no 2, 4, 7, 13, and 17 on page 47.
3 Principles of Options Pricing

3.1 Basic Notation and Concepts

Throughout this material, we assume that arbitrage opportunities (if any) are quickly eliminated by investors. The following notation is used from now on:

1. $S_t$: the stock price at time $t \in [0, T]$, where $t = 0$ refers to the current date and $t = T$ to the option maturity;

2. $X$: the option strike price;

3. $r$: the (effective) risk-free rate (in % per year);

4. $v(S_0, X, T)$ and $V(S_0, X, T)$: a European option value and the associated American option value, respectively, seen as functions of the current underlying asset price, strike price, and maturity;
Lowercase and uppercase letters are used to distinguish between European and American options, respectively. For example, $c$ refers to the premium of a European call option, and $C$ to the premium of an American call option.

The exercise value of an American call option, $C^e(S_0, X, T) = \max(0, S_0 - X)$, is known, and the holding value, $C^h(S_0, X, T)$, is computed based on the future potentialities of the contract. The value of the American call option is:

$$C = \max\left( C^e, C^h \right),$$

with the convention that $C^h(S_T, X, 0) = 0$ and $C(S_T, X, 0) = C^e(S_T, X, 0) = \max(0, S_T - X)$, for all $S_T$.

The exercise value of an American put option, $P^e(S_0, X, T) = \max(0, X - S_0)$, is known, and the holding value, $P^h(S_0, X, T)$, is computed based on the future potentialities of the contract. The value of the American put option is:

$$P = \max\left( P^e, P^h \right),$$
with the convention that $P^h(S_T, X, 0) = 0$ and $P(S_T, X, 0) = \max(0, X - S_T)$, for all $S_T$.

The *time value* of an American option, called also the *speculative value*, is defined as the difference between its value and its exercise value, that is, $V - V^e \geq 0$.

The *risk-free rate* is the return earned on a risk-free investment, which is associated to T-bills for short maturities. They are traded in over-the-counter markets, and quoted by special dealers for various maturities up to one year.

Instead of earning interest, T-bills are quoted *at discount*, that is, they are traded at lower prices than their principal amount, set here at $100. For a given maturity, the *discount* (in $ per $100 of principal) is:

\[
\text{Discount} = \text{Principal amount} \times \text{Discount rate} \times \text{Time to maturity},
\]
where the principal amount is $b_T = $100, the discount rate $r_d$ is expressed in % per year, and the time to maturity is expressed as a fraction of a year, with the convention that one year equals 360 days.

The current T-bill price for a principal amount of $100 comes:

\[
b_0 = \text{Principal} - \text{Discount} = \text{Principal} \times (1 - r_d \times \text{Time to maturity}).
\]

Suppose that the investment horizon is 7 days, the bid discount rate is quoted at 4.45% (per year), and the ask discount rate is quoted 4.37% (per year). The average discount rate, 4.41% (per year), is an approximation of the discount rate. The T-bill price is then:

\[
b_0 = 100 \times \left(1 - 4.41\% \times \frac{7}{360}\right) = $99.91425.
\]
The *periodic return* (in % per 7 days) is:

\[
    r_7 = \frac{b_T - b_0}{b_0} = \frac{100 - 99.91425}{99.91425} = 0.0858\%.
\]

Compounding interest annually at the (effective) risk-free rate \( r = r_{365} \) (in % per year) is equivalent to compounding interest each 7 days at the periodic rate \( r_7 \) (in % per 7 days). Indeed, their associated *compound factors* are identical.

The risk-free rate (in % per year) is then:

\[
    r = (1 + r_7)^\frac{365}{7} - 1 = 4.57\% \text{ (per year)}.
\]
3.2 Bounds for Call Options

A call option is a right for its holder; therefore, must generally pay for it up front. The value of the call option has an obvious lower bound:

$$C(S_0, X, T) \geq c(S_0, X, T) \geq 0.$$ 

Since exercising early is not mandatory, the exercise value of an American call option, called also the intrinsic value, is a lower bound:

$$C(S_0, X, T) \geq \max(0, S_0 - X).$$

[Please explain Figure 3.1 on page 59.]

Exercise 13: Check that this lower bound holds for the American call options in Table 3.1 on page 57.
An upper bound for a call option is simply the current stock price, that is,

$$c(S_0, X, T) \leq C(S_0, X, T) \leq S_0.$$  

Since a call option is a vehicle to purchase the stock, one cannot pay more for it than for the stock. To make sure that this upper bound holds, suppose the opposite, that is,

$$C(S_0, X, T) > S_0.$$  

If this were the case, the call option would be over- and the stock under-priced. An arbitrage strategy is as follows: sell the call option, buy the stock, and make a profit. In the worst-case scenario, whenever the call option is exercised ahead in time, deliver the stock and get paid. This is an arbitrage opportunity. Since a free lunch is supposed to disappear very quickly, the assumed strict inequality cannot hold for long. Thus, its opposite holds durably:

$$C(S_0, X, T) \leq S_0.$$
Exercise 14: Check that this upper bound holds for the American call options in Table 3.1 on page 57.

At expiry, the value of a call option is:

\[ \text{max} (0, S_T - X). \]

[Please explain Figure 3.2 on page 60.]

The value of an American call option is an increasing function of its time to maturity:

\[ C(S_0, X, T_1) \leq C(S_0, X, T_2), \quad \text{for} \ T_1 \leq T_2. \]

The two call options are American. The call option \((S_0, X, T_2)\) may be exercised early whenever \((S_0, X, T_1)\) is exercised.

Exercise 15: Check this inequality for the American call options in Table 3.1 on page 57.
**Exercise 16:** Assume the opposite to hold. Give an arbitrage opportunity, and draw conclusions.

[Please explain Figure 3.4 on page 62.]

This property also holds for European call options.

The value of a call option is a decreasing function of its exercise price:

\[ c(S_0, X_1, T) \geq c(S_0, X_2, T), \quad \text{for } X_1 \leq X_2. \]

The payment promised by the call option \((S_0, X_1, T)\) to its holder is greater than the one promised by \((S_0, X_2, T)\) in all scenarios. The cost for the call-option holder follows the same rule.

**Exercise 17:** Check this inequality for the American call options in Table 3.1 on page 57.

**Exercise 18:** Assume the opposite to hold. Give an arbitrage opportunity, and draw conclusions.
For European call options, one has:

\[
0 \leq c(S_0, X_1, T) - c(S_0, X_2, T) \leq \frac{X_2 - X_1}{(1 + r)^T}
\]

\[
\leq X_2 - X_1, \quad \text{for } X_1 \leq X_2.
\]

The lower bound for the difference in the prices of the two call options has already been established.

**Exercise 19:** To establish the upper bound, suppose the opposite. Give an arbitrage strategy, and draw conclusions.

For American call options, one has:

\[
0 \leq C(S_0, X_1, T) - C(S_0, X_2, T) \leq X_2 - X_1.
\]

**Exercise 20:** Check the upper bound for DCRB European call options in Table 3.4 on page 57 (done in Table 3.4 on page 65).
A lower bound for the European call option \((S_0, X, T)\) is:

\[
c (S_0, X, T) \geq \max \left( 0, S_0 - \frac{X}{(1 + r)^T} \right).
\]

**Exercise 21:** Suppose the opposite to hold. Give an arbitrage opportunity, and draw conclusions.

[Please explain Figure 3.5 on page 66.]

We now consider a European call option on a foreign currency. At time \(t\), one unit of the foreign currency is equivalent to \(S_t\) units of the local currency. The risk-free foreign interest rate is \(\rho\). In this context, a lower bound for the European call option \((S_0, X, T)\) is:

\[
c (S_0, X, T) \geq \max \left( 0, \frac{S_0}{(1 + \rho)^T} - \frac{X}{(1 + r)^T} \right).
\]

**Exercise 22:** Suppose the opposite to hold. Give an arbitrage opportunity, and draw conclusions.
The value of a call option is an increasing function of the risk-free interest rate and of the stock-return volatility.

Interest rates have multiple impacts on the value of a call option. In one hand, higher interest rates reduce present values. This has a negative impact on the value of the call option. On the other hand, the underlying stock price increases more when interest rates are high than it does when interest rates are low. This has a positive impact on the value of the call option. In this context, the positive effect is stronger.

Under high stock-return volatility, the stock price is more likely to reach greater highs and lows. This has a positive impact on the value of the call option, since only high prices are accounted for.
3.3 Optimal Exercise of American Call Options

An American option costs at least as much as its European counterpart with the same parameters since it gives its holder the additional right of early exercise. This additional cost is null for American call options on no-dividend-paying stocks.

Exercising a call option gives:

\[ C^e(S_0, X, T) \leq C(S_0, X, T) = S_0 - X \leq S_0 - \frac{X}{(1 + r)^T} \leq c \leq C^h, \]

which in turn implies:

\[ C^e(S_0, X, T) \leq C^h(S_0, X, T). \]

The conclusion is that it is never optimal to exercise an American call option early when the underlying stock does not pay dividends.
If the stock pays a cash-dividend amount, the stock price tends to fall by the same amount on the ex-dividend date. The value of the call option is revised downward accordingly. The holder of the call option can avoid such a loss by exercising his right just before the ex-dividend date. Exercising early may be optimal for American call options on dividend-paying stocks.

3.4 Bounds for Put Options

A put option is a right for its holder; therefore, must pay for it generally up front.

The value of the put option has an obvious lower bound:

\[ p(S_0, X, T) \geq 0. \]

A lower bound for an American put option is its exercise value, that is,

\[ P(S_0, X, T) \geq \max(0, X - S_0). \]
The value of a European put option has the following upper bound:

\[ p(S_0, X, T) \leq \frac{X}{(1 + r)^T}. \]

Since an American put option can be exercised early, its value has the following upper bound:

\[ P(S_0, X, T) \leq X. \]

**Exercise 23:** For each contract, suppose the opposite to hold. Give an arbitrage strategy, and draw conclusions.

The value of an American put option is an increasing function of its time to maturity:

\[ P(S_0, X, T_1) \leq P(S_0, X, T_2), \text{ for } T_1 \leq T_2. \]
The two put options are American. The put option \((S_0, X, T_2)\) may be exercised early whenever \((S_0, X, T_1)\) is exercised.

**Exercise 24:** Check this property for DCRB American put options in Table 3.1 on page 57.

**Exercise 25:** Suppose the opposite to hold. Give an arbitrage opportunity, and draw conclusions.

[Please explain Figure 3.8 and Figure 3.9 on page 74.]

This property usually holds for European put options.

The value of a put option is an increasing function of its exercise price:

\[ p(S_0, X_1, T) \leq p(S_0, X_2, T), \quad \text{for } X_1 \leq X_2. \]

The payment promised by the put option \((S_0, X_1, T)\) to its holder is lower than the one promised by \((S_0, X_2, T)\)
in all scenarios. The cost for the option holder follows the same rule.

**Exercise 26:** Check this inequality for the DCRB American put options in Table 3.1 on page 57.

**Exercise 27:** Suppose the opposite to hold. Give an arbitrage opportunity, and draw conclusions.

The difference in value of two European put options with the same characteristics except for their strike prices verifies:

\[ p(S_0, X_2, T) - p(S_0, X_1, T) \leq \frac{X_2 - X_1}{(1 + r)^T}. \]

The difference in value of two American put options with the same characteristics except for their strike prices verifies:

\[ P(S_0, X_2, T) - P(S_0, X_1, T) \leq X_2 - X_1. \]
**Exercise 28:** Check the inequality for the DCRB American put options in Table 3.1 on page 57 (done in Table 3.9 on page 77).

**Exercise 29:** Suppose the opposite to hold. Give an arbitrage opportunity, and draw conclusions.

A lower bound for a European put option is:

\[
p(S_0, X, T) \geq \max \left( 0, \frac{X}{(1 + r)^T} - S_0 \right).
\]

**Exercise 30:** Suppose the opposite to hold. Give an arbitrage opportunity, and draw conclusions.

[Please explain Figure 3.10 on page 78.]

**Exercise:** Show that a European put option on a foreign currency verifies:

\[
p(S_0, X, T) \geq \max \left( 0, \frac{X}{(1 + r)^T} - \frac{S_0}{(1 + \rho)^T} \right),
\]
where $S_0$ is the current foreign exchange rate and $\rho$ the risk-free foreign interest rate.

The value of a put option is a decreasing function of the risk-free interest rate and an increasing function of the stock-return volatility. The reasons for this are similar to those evoked for call options.

An American put option is exercised early if, and only if, the stock price is under a certain threshold.

[Explain again Figure 3.8 on page 74.]

### 3.5 Call-Put Parity

The put-call parity states that a stock plus a European put is equivalent to a European call plus some risk-free bounds:

$$ p + S_0 = c + \frac{X}{(1 + r)^T}. $$
No-arbitrage implies a strong relationship between European call and put options.

**Exercise 31:** An equality is equivalent to two inequalities. For each one, suppose the opposite to hold. Give an arbitrage opportunity, and draw conclusions.

The call-put parity for European currency options is:

\[ p + \frac{S_0}{(1 + \rho)^T} = c + \frac{X}{(1 + r)^T}. \]

### 3.6 Assignment

Read the chapter, and give precise and concise definitions for the keywords listed on page 86. This part is to be prepared for the midterm exam, but should not be handed in as part of the assignment.

Answer questions no 1, 2, 6, 8, 9, 12, 14, 16, 20, and 24.
4 Binomial Model

4.1 The One-Period Binomial Tree

We consider a market for a riskless saving account and a risky stock. Trading activities take place only at the current time $t_0 = 0$ and at horizon $t_1 = T$, all positions are then closed at the horizon. No trading is allowed in between.

In addition, the stock price is assumed to move from its current level $S_0$ according to a one-period binomial tree.

$$p \quad S_{1}^{\text{up}} = uS_0$$

$$S_0$$

$$1 - p \quad S_{1}^{\text{down}} = dS_0$$

$t_0 = 0 \quad t_1 = T$
The stock price can rise by a factor $u$ or drop by a factor $d$, where $u > d$.

The probabilities $p$ and $1-p$ define the *physical probability measure* $P$, under which investors evaluate likelihoods and take decisions.

The parameters $u$ and $d$ can be seen as volatility parameters. The greater is $u - d$, the greater the volatility of the stock return.

**Example 32:** Assume that the stock price is currently quoted at $100, and can either increase by 25% or decrease by 20%. The factors $u$ and $d$ are:

$$u = 1.25 \quad \text{and} \quad d = 0.8.$$  

The one-period binomial tree for the stock is as follows.

[Please explain Figure 4.1 on page 94.]
In this case, the one-period binomial tree is:

\[ S_{1}^{\text{up}} = uS_0 = 1.25 \times 100 = 125 \]

\[ S_{1}^{\text{down}} = dS_0 = 0.8 \times 100 = 80 \]

\[ t_0 = 0 \quad t_1 = T \]

The price can move upward from $100 to $125 or downward from $100 to $80.

Trading is allowed at the current time \( t_0 \), and all positions are closed at the horizon \( t_1 \).

The (periodic) risk-free interest rate is indicated by \( r \), and is expressed in % over the time period \([0, T]\). It can be inferred from the discount rate on T-bills that have almost the same maturity.
The binomial tree is arbitrage free if, and only if, the following property holds:

\[ d < 1 + r < u. \]

Indeed, in the case of an upward movement \( S_1 = uS_0 \), the rate of return on the stock \( u - 1 \) (in % per period) must exceed the risk-free interest rate \( r \) (in % per period). Otherwise, the stock would be overpriced, and an arbitrage opportunity would appear. In addition, the risk-free interest rate \( r \) must exceed the rate of return on the stock under of a downward movement \( d - 1 \).

We consider the European call option \((S_0, X, T)\).

[Please explain Figure 4.1 on page 94.]

**Example 32 (continued):** The risk-free interest rate is \( r = 7\% \) (per period). Is the model arbitrage free?

The binomial tree is simple but viable. It is used here to go through the fundamentals of options pricing.
The goal now is to derive a formula for the value of the European call option \( c_0 \) in the one-period binomial tree, as a function of the current stock price \( S_0 \), the option strike price \( X \), the volatility parameters \( u \) and \( d \), and the risk-free interest rate \( r \). As usual, we use the no-arbitrage principle.

Next, we set up a general pricing formula that holds not only within the binomial tree but also in all arbitrage-free models.

### 4.2 Pricing Formula for a One-Period Binomial Tree

Consider a European call option on a stock with a maturity date \( T \) and a strike price \( X \).

The so-called *hedge portfolio* consists of \( h \) shares of stock and a single signed call option with an initial value of:

\[
H_0 = hS_0 - C_0,
\]
and a terminal value of:

\[ H_1 = hS_1 - C_1. \]

For a specific level of \( h \), the hedge portfolio is riskless, and, by the no-arbitrage principle, must earn the risk-free rate. A formula for the call-option value can therefore be derived.

The value of the hedge portfolio moves along the binomial tree as follows:

\[ H_1^{\text{up}} = hS_1^{\text{up}} - C_1^{\text{up}} \]

\[ H_1^{\text{down}} = hS_1^{\text{down}} - C_1^{\text{down}} \]

The hedge portfolio is riskless if, and only if,

\[ H_1^{\text{up}} = H_1^{\text{down}}. \]
Solving for $h$ gives:

$$h = \frac{C_{1}^{up} - C_{1}^{down}}{S_{1}^{up} - S_{1}^{down}},$$

which depends on the known parameters $S_0$, $X$, $u$, and $d$.

Given the *hedge parameter* $h$, the hedge portfolio, as a riskless investment, should earn the risk-free rate:

$$H_0 = hS_0 - C_0 = \frac{H_1}{1 + r} = \frac{H_{1}^{up}}{1 + r} = \frac{H_{1}^{down}}{1 + r}.$$

Solving for $C_0$ gives:

$$C_0 = \frac{p^*C_{1}^{up} + (1 - p^*)C_{1}^{down}}{1 + r},$$

where

$$p^* = \frac{1 + r - d}{u - d} \in (0, 1).$$
The probabilities $p^*$ and $1 - p^*$ define the so-called *risk-neutral probability measure* $P^*$, which is not related in any way to the physical probability measure $P$.

In sum, the value of the European call option can be expressed as a weighted average (an expectation) of its promised cash flow that is discounted at the risk-free interest rate:

$$c_0 = E^* \left[ \frac{c_1}{1 + r} \right],$$

where $E^* [.]$ is the expectation sign under the risk-neutral probability measure $P^*$, $c_0$ is the option value at time $t_0$, and $c_1$ is the option value at time $t_1$.

The pricing formula discounts the risky cash flow of the option by the risk-free interest rate as if investors were risk neutral, while they are not. This was done via a major correction. We move from the real world, seen under the physical probability measure $P$, to a risk-neutral world, seen under the risk-neutral probability measure $P^*$.

**Example 32 (continued):** Consider a European call option on the previously mentioned stock with a strike price of $X = \$100$. See page 96.
1. Compute the risk-neutral probabilities for upward and downward movements.

2. Draw the one-period binomial tree for the call option.

3. Compute the hedge ratio.

4. Use the formula to compute the value of the call option.

5. Check that the hedge portfolio earns the risk-free interest rate.

6. Compute in two different ways the value of its associated European put option.

The risk-neutral probabilities are:

\[ p^* = \frac{1 + r - d}{u - d} = \frac{1 + 7\% - 0.8}{1.25 - 0.8} = 0.6 \text{ and} \]

\[ 1 - p^* = 0.4. \]
The one-period binomial tree for the stock, the call option, and the hedge portfolio is:

\[
S_0 = 100 \\
C_0 = \frac{0.6 \times 25 + 0.4 \times 0}{1 + 7\%} = 14.02 \\
h = \frac{25 - 0}{125 - 80} = 0.556 \\
H_0 = 0.56 \times 100 - 14.02 = 42.02 \\
t_0 = 0
\]

and

\[
S_{1\text{up}} = 125 \\
C_{1\text{up}} = 25 \\
H_{1\text{up}} = 0.56 \times 125 - 25 = 45 \\
S_{1\text{down}} = dS_0 = 0.8 \times 100 = 80 \\
C_{1\text{down}} = 0 \\
H_{1\text{down}} = 0.56 \times 80 - 0 = 45 \\
t_0 = 0 \\
t_1 = T
\]

The *hedge portfolio*, which is a riskless investment, is equivalent to a long position on the stock and a short position on the call option:

\[
H_0 = hS_0 - C_0.
\]
Equivalently, one has:
\[ C_0 = hS_0 - H_0. \]

Thus, the call option is equivalent to a long position on the stock and a short position on T-bills. More precisely, the call option is equivalent to holding \( h \) shares of stock and borrowing \( H_0 \) (in \$) at the risk-free interest rate.

In other words, the call option can be replicated (duplicated) by a hedging strategy based on the underlying stock and the saving account:
\[ C_1 = hS_1 - H_1. \]

If the signer decides to fully replicate the option, he would insure the promised payment to the holder at maturity. A market in which one can fully replicate derivatives is called a complete market. The binomial tree is an example.
The value of the associated put option is rather obtained through the binomial tree or by the put-call parity:

\[ p_0 = c_0 + \frac{X}{1 + r} - S_0 \]
\[ = $7.48. \]

**Example 32 (continued):** Show how the call option can be fully replicated. Do the same work for the associated European put option.

### 4.3 No-Arbitrage Pricing

The extension of the pricing formula is straightforward as long as the model is arbitrage free.

Consider an American option on a stock with a strike price of $X$ and a maturity date of $T$. We use the following notation:
1. $S_t$ : the stock price at time $t$;

2. $V_t(s)$ : the option value at time $t$, seen as a function of the stock price $S_t = s$;

3. $V^h_t(s)$ : the option holding value at time $t$, seen as a function of the stock price $S_t = s$;

4. $V^e_t(s)$ : the option exercise value at time $t$, seen as a function of the stock price $S_t = s$.

The holding value summarizes for all the future potentialities of the option. The notation proposed here is consistent with the one introduced in the previous chapters, but is more general. For an American call option, for example, one has $V_t = C_t$ and $V^e_t = \max(0, S_t - X)$. At each decision date, the option holder has to decide
whether it is optimal to exercise its right or not. The option value becomes:

\[ V_t(s) = \max(V^e_t(s), V^h_t(s)), \quad \text{for all } t \text{ and all } s, \]

with the convention that \( V^h_T(s) = 0 \) or \( V^e_T(s) = V^e_T(s) \), for all \( s \).

A European option can be seen as a special case with:

\[ V^e_t(s) = 0, \quad \text{for all } t < T \text{ and all } s. \]

The holding value of an option at time \( t_n \) can be expressed as a conditional expectation of its future potentialities discounted back at the risk-free rate. The no-arbitrage pricing formula is:

\[ V^h_n(s) = E^* \left[ \frac{V_{n+1}^h(S_{t_{n+1}})}{1 + r} \left| S_{t_n} = s \right. \right], \]

where \( V^h_n(s) \) is the option holding value at the current time \( t_n \) when \( S_{t_n} = s \), \( E^* \left[ . \left| S_{t_n} = s \right. \right] \) is the conditional
expectation symbol taken under the risk-neutral probability measure $P^*$, $V_{n+1}(S_{t_{n+1}})$ is the future value of the option at the next decision date $t_{n+1}$ when the stock price is $S_{t_{n+1}}$, and $r$ is the periodic risk-free rate.

From the perspective of an investor at time $t_n$, the stock price $S_{t_{n+1}}$ is random. The pricing formula consists of computing a weighted average on all feasible future values of the option, discounted at the risk-free rate, the weights being given by the risk-neutral probabilities.

**Example 32 (continued):** Consider a two-period binomial tree for the stock. Compute the value of the European call option. Compute in two ways the value of its associated European put option. The two-period binomial tree for the stock and the call option are as follows. See Figure 4.4 on page 104. Again, the hedge portfolio
can be used to replicate the call option.

\[
\begin{align*}
S_{0} &= 100 \\
c_{0} &= 17.69 \\
S_{1}^{u} &= 125 \\
c_{1}^{u} &= 31.54 \\
S_{1}^{d} &= 80 \\
c_{1}^{d} &= 0 \\
S_{2}^{uu} &= 156.25 \\
c_{2}^{uu} &= 56.25 \\
S_{2}^{ud} &= S_{2}^{du} = 100 \\
c_{2}^{ud} &= C_{2}^{du} = 0 \\
S_{2}^{dd} &= 64 \\
c_{2}^{dd} &= 0 \\
t_{0} &= 0 \\
t_{1} &= t_{2} = T
\end{align*}
\]

### 4.4 Dividends and Early Exercise

Unlike American put options, American call options cannot be exercised optimally before maturity. If the stock pays dividends over the option’s life, however, an American call option can be exercised early just before ex-dividend dates.
Consider an American option on a dividend-paying stock. Let $D_1, \ldots, D_p$ be the dividends, and $\tau_1, \ldots, \tau_p$ their associated ex-dividend dates over the option’s life $[0, T]$. Pricing an option on a dividend-paying stock requires the following adjustments for the binomial tree.

1. Compute the PV of all dividends to be paid over the option’s life as of the present time $t_0 = 0$, that is,

$$D = \sum_{i=1}^{p} D_i e^{-r\tau_i},$$

where $r$ is the continuous risk-free rate;

2. Adjust downward the current stock price by the PV of all dividends:

$$S_0^* = S_0 - D;$$

3. Create the binomial tree that starts from $S_0^*$ using the up and down factors as usual;
4. At each node of the binomial tree, add to the stock price the PV of all future dividends to be paid over the option’s life, including the one that is about to go ex-dividend (if any);

5. Move backward through the binomial tree, and compute the option values as usual from maturity to the start.

This method applies for European as well as American options.

**Example 33:** Consider a stock that promises a dividend of $2 in 90 days. The stock is now traded at $50. The risk-free rate is a continuously compounded rate, expressed in % per year. The parameters of the binomial
tree are:

\[
\begin{align*}
\Delta t &= 24 \text{ days} \\
&= 0.065756 \text{ (years)}, \\
r &= 9\% \text{ (per year)}, \\
u &= 1.108015, \\
d &= 0.902515, \\
p^* &= \frac{e^{r\Delta t} - d}{u - d} = 0.503262.
\end{align*}
\]

The PV of all dividends over the option’s life is:

\[
D = D_1 e^{-r\tau_1} = \$1.96.
\]
The binomial tree in step 3 is as follows:

```
  80.23
   72.40
    65.35
     65.35
      58.98
       58.98
        53.23
         53.23
          53.23
           48.04
            48.04
             48.04
              48.04
               43.36
                43.36
                 43.36
                  43.36
                   39.13
                    39.13
                     39.13
                      35.32
                       35.32
                        35.32
                         31.87
                          28.77
```
The adjusted binomial tree in step 4 is as follows:

Consider an American call option on this stock starting at parity and maturing in 120 days. The binomial tree
The value of this American call option is $4.48. This option will be optimally exercised if the stock price will record an “up-up-up” movement from the start.

### 4.5 Assignment

Read the chapter, and give precise and concise definitions for the keywords listed on page 86. This part is to be
prepared for the midterm exam, but should not be handed in as part of the assignment.

Answer questions no 2, 4, 12, 13, and 20.

**Exercise 34:** Read and comment on the business article entitled: Dow jones sells index business to CME.
5 The Black and Scholes Model

5.1 Introduction

Black and Scholes introduced a frictionless market for a non-dividend paying stock (the risky asset) and a saving account (the riskless asset) in which trading takes place continuously to time horizon $T$.

The stock price process $\{S\}$ is assumed to follow a long-normal process characterized by:

$$S_T = S_t e^{R},$$

where $R$ is the continuously compounded rate of return on the stock over the period $[t, T]$. Under the physical probability measure $P$, the rate of return on the stock over $[t, T]$ follows a normal distribution in the form:

$$R = \left( \alpha - \frac{\sigma^2}{2} \right) (T - t) + \sigma \sqrt{T - t} Z,$$
where $\alpha$ is a real parameter known as the *instantaneous rate of return on the stock* (in % per year), $\sigma$ is a positive parameter known as the *volatility of the stock log-returns* (in % per year), $T - t$ is the investment’s remaining time to maturity (in years), and $Z$ is a random variable that follows the standard normal distribution $\mathcal{N}(0, 1)$ with a mean value of 0 and a standard deviation of 1. The stock-price process is said to be *lognormal* since the natural logarithm of any future stock price $S_T$, given the current stock price $S_t$, follows the normal distribution:

$$\mathcal{N}\left(\ln(S_t) + \left(\alpha - \frac{\sigma^2}{2}\right)(T - t), \sigma\sqrt{T - t}\right).$$

**Example 35:** The rate of return on a stock (in % per year) follows the normal distribution $\mathcal{N}(5, 4\%)$ with a mean value of 5% and a standard deviation of 4%. Draw the curve of its density function $n(.)$. Explain the meaning of its distribution function $N(.)$. Compute the probability that the stock will record a loss during the upcoming year $P(\mathcal{N}(5, 4\%) \leq 0)$. Use Excel. Unfortunately, such a manipulation is not so easy to achieve
in practice since the parameter $\alpha$ is typically unknown, and is hard to estimate from observed data.

**Example 36:** Compute the probability that a standard normal distribution $\mathcal{N}(0, 1)$ records values between its mean value $\pm 1$, $\pm 2$, and $\pm 3$ standard deviations. Use Excel and Table 5.1 on page 135.

Consider a European call option on the risky asset whose current value in the Black and Scholes model is indicated by $c$. The hedge portfolio, as for the binomial tree, is riskless for:

$$h = \Delta = \lim_{\Delta S \rightarrow 0} \frac{\Delta c}{\Delta S} = \frac{\partial c}{\partial S}$$

if it is rebalanced continuously. Therefore, the PV principle can be extended to the so-called *risk-neutral evaluation principle* in the Black and Scholes model, which turns out to be arbitrage free. Under the risk-neutral
probability measure $P^*$, the rate of return on the stock $R^*$ is random, and follows the normal distribution:

$$R^* = \left( r - \frac{\sigma^2}{2} \right) (T - t) + \sigma \sqrt{T - t} Z^*, \quad$$

where $r$ is the continuously compounded risk-free rate (in % per year), and $Z^*$ is a random variable that follows the normal distribution $\mathcal{N}(0, 1)$.

In sum, the rate of return on the stock behaves similarly under $P$ and $P^*$, except that the instantaneous rate of return on the stock $\alpha$ in the real world is exchanged by the risk-free rate $r$ in the risk-neutral world, where investors act as if they were risk neutral.

## 5.2 The Black and Scholes Formula for European Options

Risk-neutral evaluation holds in the Black and Scholes market, for it is arbitrage free. The value of a European
call option is:

\[ c_t = E^* \left[ e^{-r(T-t)} \max(0, S_T - X) \mid S_t \right] \]
\[ = E^* \left[ e^{-r(T-t)} (S_T - X) I (S_T > X) \mid S_t \right] \]
\[ = E^* \left[ e^{-r(T-t)} S_T I (S_T > X) \mid S_t \right] - E^* \left[ e^{-r(T-t)} X I (S_T > X) \mid S_t \right], \]

where the indicator function \( I \) is defined as follows:

\[ I (S_T > X) = \begin{cases} 1, & \text{if } S_T > X \\ 0, & \text{if } S_T \leq X \end{cases} \]

The call-option value consists of two terms. The first (second) term is the conditional expectation of the discounted stock price (strike price) given that the option expires in the money.

Computing this conditional expectation is straightforward using integration theory and calculus. The final result is the Black and Scholes formula for a European call option on a non-dividend paying stock:

\[ c_t = N (d_1) S_t - X e^{-r(T-t)} N (d_2), \]
where

\[
d_1 = \frac{\ln \left( \frac{S_t}{X} \right) + \left( r + \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}} \quad \text{and}
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t}.
\]

The Black and Scholes formula for a European put option on a non-dividend paying stock is

\[
p_t = X e^{-r(T-t)} N (-d_2) - S_t N (-d_1).
\]

**Example 37:** Compute the value of a European call option. The initial stock price is \( S_0 = \$125.94 \), the strike price is \( X = \$125 \), the effective risk-free interest rate is \( r = 4.56\% \) (per year), the volatility is \( \sigma = 83\% \) (per year), and the maturity is \( T = 0.0959 \) (in years). Use the Black and Scholes formula and Excel. See Table 5.2 on page 136. Do a sensitivity analysis of \( c_0 \), as a function of \( S_0, X, \sigma, \) and \( r \).
5.3 Sensitivity Analysis

The value of a European call option \( c = c_0 \) at the start is an increasing function of the stock price \( S_0 = S \), the volatility \( \sigma \), and the risk-free rate \( r \), and a decreasing function of the strike price \( X \).

The Delta coefficient, indicated by \( \Delta \), is the sensitivity of \( c \) to small variations in \( S_0 \), all other parameters being fixed:

\[
\Delta = \frac{\partial c}{\partial S} = \lim_{\Delta S_0 \to 0} \frac{\Delta c}{\Delta S} = N(d_1) \in [0, 1]
\]

\[
\approx \frac{\Delta c}{\Delta S}, \text{ for small } \Delta S.
\]

For small changes in \( S \), the first-order Taylor approximation of \( c \) is:

\[
c^{\text{new}} \approx c^{\text{old}} + \Delta \left( S^{\text{new}} - S^{\text{old}} \right).
\]
The Gamma coefficient, indicated by $\Gamma$, is the sensitivity of the $\Delta$ coefficient to small variations in $S$, all other parameters being fixed:

\[
\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-d_1^2/2}}{S\sigma \sqrt{2\pi T}} > 0, \text{ for all } S_0.
\]

For small changes in $S$, the second-order Taylor approximation of $c$ is:

\[
c_{\text{new}} \approx c_{\text{old}} + \Delta (S_{\text{new}} - S_{\text{old}}) + \frac{1}{2} \Gamma (S_{\text{new}} - S_{\text{old}})^2.
\]

[Please explain Figure 5.6 on page 142.]

**Example 37 (continued):** The coefficient Delta is $\Delta = 0.5692$. Approximate the new call price if the new stock price were quoted at $127$. 

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Example 37 (continued): The coefficient Gamma is 0.0123. Approximate the new call value if the new stock price were quoted at $127. Check that the second-order approximation is more precise than the first-order approximation.

The coefficients $\Delta$ and $\Gamma$ are always positive; hence, the curve of the call option is not only an increasing function, but also convex.

[Please explain Figure 5.6 on page 142.]

The Vega function, indicated by $\vartheta$, is the sensitivity of $C_e$ to small variations in $\sigma$, all other parameters being fixed:

$$
\vartheta = \frac{\partial C_e}{\partial \sigma}
= \lim_{\Delta \sigma \to 0} \frac{\Delta C_e}{\Delta \sigma}
= \frac{S_0 \sqrt{T} e^{-d_1^2/2}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}}
> 0,
$$

for all $\sigma$. 

\[91\]
Example 37 (continued): The coefficient Vega is 15.32. If the volatility changes by 1%, the call value would change by $0.1532$ in the same direction. The call value is highly sensitive to the volatility of stock returns.

The Theta function, indicated by $\Theta$, is the sensitivity of $C_e$ to small variations in the current time $t$, all other parameters being fixed:

$$\Theta = \frac{\partial C_e}{\partial t} = -\frac{\partial C_e}{\partial (T - t)}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta C_e}{\Delta t}$$

$$= -\frac{S_0 \sqrt{T - t} e^{-d_1^2/2}}{2 \sqrt{2\pi (T - t)}} - \frac{r_c X e^{-r_c(T-t)} N(d_2)}{2 \sqrt{2\pi (T - t)}}$$

for all $T$.

The Theta function is always negative.
5.4 Dividends

In the case of a known discrete dividend $D_1$ at an ex-dividend date $t_1$, one has to adjust the stock price downward by the discounted value of the dividend:

$$S_0^* = S_0 - D_1 e^{-r c t_1},$$

and again apply the Black and Scholes formula with $S_0^*$. Now, if the dividends are paid continuously at a rate $\delta_c$ (in % per year), the stock price must be adjusted downward as follows:

$$S_0^* = S_0 e^{-\delta_c T}.$$

Example 37 (continued): Compute the call value if 1– a dividend of $2 is paid in 21 days, or 2– the dividends are paid continuously at a rate of 4% (per year).
5.5 The Volatility of Stock Returns

In the Black and Scholes model, there are two methods for computing the volatility of stock returns $\sigma$.

The first approach, known as the historical method, consists of 1– selecting a time step $\Delta t$, 2– computing the continuous rates of return on the stock $R_{c1}^c, \ldots, R_{cN}^c$ realized on successive time intervals of length $\Delta t$, and 3– estimating the volatility $\sigma$ as follows:

$$\hat{\sigma} = \frac{1}{\sqrt{\Delta t}} \sqrt{\frac{\sum_{n=1}^{N} (R_{cn}^c - \overline{R}_c^c)^2}{N - 1}}, \text{ (per year).}$$

Example 37 (continued): Table 5.6 on page 156 shows $N = 59$ realized daily stock returns. For example, the discrete rate of return on the stock on day 1 is:

$$\frac{S_2 - S_1}{S_1} = R_1^d$$

$$= 7.87\% \quad (\text{on day 1}),$$
and the continuous rate of return on the stock on day 1 verifies:

\[ R_1^c = \ln \left( 1 + R_1^d \right) = \ln \left( \frac{S_2}{S_1} \right), \]

\[ = 7.57\% \text{ (on day 1).} \]

Since the sum \( \sum_{n=1}^{59} \left( R_n^c - \bar{R}^c \right)^2 = 0.182817 \), the volatility estimate is:

\[ \hat{\sigma} = \sqrt{250 \times \frac{0.182817}{58}} \]

\[ = 0.8877 \text{ (per year)} \]

The second approach, known as the implied method, consists of approximating the volatility by solving the non-linear equation for \( \sigma \):

\[ C_e^{BS} (\sigma) = C_e^{\text{market}}. \]

The implied volatility acts as if the market behaved exactly as the Black and Scholes model, while it does not.

[Please explain Figure 5.19 on page 161.]
A more general approach consists of minimizing the model error:

$$\min_{\sigma} \sum_{n=1}^{N} \left( v_{n}^{\text{BS}} - v_{n}^{\text{market}} \right)^2,$$

where the European options are selected to be the most traded contracts.

### 5.6 The Black and Scholes Formula for European Put Options

The Black and Scholes market is arbitrage free. So, given the fair value for a call option, we can deduce the fair value of a put option by using the put-call parity.

The Black and Scholes formula for a European put option is:

$$P_e = X e^{-r c T} N (-d_2) - S_0 N (-d_1).$$
Example 37 (continued): The value of the associated put option is \( P = $12.08 \). 

[See Table 5.8 on page 165 for a sensitivity analysis.]

As for a European call option, a sensibility analysis can be done for a European put option.

For a European put option, the Delta coefficient \( \Delta \) is negative. The \( \Delta \) hedging strategy, which allows one to replicate the put option, starts by taking a short position on \(|\Delta|\) shares of stock and saving at the risk-free rate. Next, when the stock price moves, compute the new coefficient \( \Delta \), adjust the quantity of shares sold short, and save or withdraw money at the risk-free rate consequently.

### 5.7 Relationship to the Binomial Model

The binomial tree can be calibrated to the Black and Scholes model by selecting its coefficients as follows:

\[
u = e^{\sigma \sqrt{\Delta t}} \quad \text{and} \quad d = \frac{1}{u},\]

\( 97 \)
where $\sigma$ is the volatility of stock returns (per year), $T$ the horizon (in years), and $N$ the number of time steps, and $\Delta t = \frac{T}{N}$ the time step (in years).

The binomial tree remains arbitrage free if, and only if, 

$$d < 1 + r < u.$$ 

Selecting the coefficients in that way ensures the convergence of the binomial tree to the Black and Scholes model, when the number of time steps $N$ converges to infinity.

5.8 Assignment

Read the chapter, and give precise and concise definitions for the keywords listed on page 172. This part is to be prepared for the second midterm exam, but should not be handed in as part of the assignment.

Answer questions no 1, 2, 6, 7, and 12.
6 Selected Option Strategies

6.1 Introduction

Combinations of stocks and options are so diverse that any investor can find a strategy that fits his risk preference, and market forecast.

A bull market is a market in which stock prices go up, and a bear market is one in which stock prices go down. A bullish investor believes that a bull market is coming, and a bearish investor believes that a bear market is coming.

We focus on strategies based on European call and put options combined with their underlying stocks. The profit associated to a given portfolio strategy is defined as the difference in the value of the portfolio over the option’s life:

\[ \Pi = V_T - V_0, \]
$V_t$ is the value of the portfolio at time $t$.

The strategy is then analyzed through its profit seen as a function of the (unknown) terminal stock price, indicated by $S_T$.

### 6.2 Basic Strategies

The profit from a long position on the stock is:

$$\Pi(S_T) = S_T - S_0.$$  

The price level $S_T^*$ is the terminal stock price, such that $\Pi(S_T^*) = 0$. The following pairs $(S_T, \Pi(S_T))$ belong to the profit line:

$$(0, -S_0) \quad \text{and} \quad (S_T^* = S_0, 0).$$

The profit from a short position on the stock is simply the opposite of the profit from the long position.

Please explain Figure 6.1 and Figure 6.2 on page 186.
Holding a stock is a bullish strategy, which has an unlimited potential gain and limited potential loss. Selling the stock short is a bearish strategy, which has a limited potential gain and unlimited potential loss.

The profit from a long position on a European call option is:

\[ \Pi(S_T) = \max(0, S_T - X) - C_e. \]

The profit function can be expressed as:

\[ \Pi(S_T) = \begin{cases} -C_e, & \text{for } S_T \leq X \\ S_T - X - C_e, & \text{for } S_T > X \end{cases}, \]

where \( S_T = X \) is a breakeven point. The following pairs belong to the piecewise-linear profit curve:

\((0, -C_e), (X, -C_e), \) and \((S_T^* = X + C_e, 0)\).

A short position on a European call option is also known as an *uncovered call* or a *naked call*.

Please explain Figure 6.3 on page 188 and Figure 6.6 on page 192.
A long position on a European call is a bullish strategy, which has an unlimited potential gain and limited potential loss. Conversely, a short position on a European call option is a bearish strategy, which has a limited potential gain and unlimited potential loss. The profit from a long position on a European put option is:

$$ \Pi(S_T) = \max(0, X - S_T) - P_e. $$

The profit function can be expressed as:

$$ \Pi(S_T) = \begin{cases} X - S_T - P_e, & \text{for } S_T \leq X \\ -P_e, & \text{for } S_T > X \end{cases}. $$

Again, $S_T = X$ is a breakeven point. The following pairs belong to the piecewise-linear profit curve:

$$ (0, X - P_e), \quad (S_T^* = X - P_e, 0), \quad \text{and} \quad (X, -P_e). $$

The profit from a short position on the European put option is simply the opposite of the profit from the long position.

[ Please explain Figures 6.9, 6.12 and 6.15 on pages 194, 197, and 199. ]

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A long position on a European put option is a bearish strategy, which has a limited potential loss and a larger but limited potential gain. Conversely, a short position on a European put option is a bullish strategy, which has a limited potential gain and a larger but limited potential loss.

6.3 More Complex Strategies

A covered call consists of a short position on a call option and a long position on the underlying stock. A similar but somewhat different strategy is the so-called \( \Delta \) hedging strategy. A short position on a call option and a long position on \( \Delta \) shares of stock define a riskless strategy. The covered call is expected to be a bit riskier, but is still a safe strategy.

The profit from a covered call is:

\[
\Pi(S_T) = - \left( \max(0, S_T - X) - C_e \right) + S_T - S_0.
\]
The profit function can be expressed as:

\[
\Pi(S_T) = \begin{cases} 
S_T - (S_0 - C_e), & \text{for } S_T \leq X \\
X - (S_0 - C_e), & \text{for } S_T > X
\end{cases},
\]

where \( S_T = X \) is a breakeven point. The following pairs belong to the piecewise-linear profit curve:

\[
(0, -(S_0 - C_e)), \quad (S_T^* = S_0 - C_e, 0), \quad \text{and} \quad (X, X - (S_0 - C_e)),
\]

where \( S_0 - C_e \geq 0 \), \( S_0 - C_e \leq X \), and \( X - (S_0 - C_e) \geq 0 \).

[Please explain Figure 6.16 on page 201.]

A covered call is a bullish strategy, which has a limited potential loss and a limited and small potential gain. A *protective put* consists of a long position on a put option and a long position on its underlying stock. The profit from a covered call is:

\[
\Pi(S_T) = (\max(0, X - S_T) - P_e) + S_T - S_0.
\]
The profit function can be expressed as:

$$\Pi(S_T) = \begin{cases} 
X - S_0 - P_e, & \text{for } S_T < X \\
S_T - S_0 - P_e, & \text{for } S_T \geq X
\end{cases},$$

where $S_T = X$ is a breakeven point.

The following pairs belong to the piecewise-linear profit curve:

$$\left(0, X - S_0 - P_e\right), \left(X, X - S_0 - P_e\right), \text{ and } \left(\hat{S}_T = S_0 + P_e, 0\right),$$

where $X - S_0 - P_e$ is supposed to be negative, and $X \leq \hat{S}_T$.

[Please explain Figure 6.19 on page 205.]

A protective put is a bullish strategy, which has an unlimited potential gain and a limited and small potential loss. The protective put acts as an insurance policy on the stock.
The put-call parity is:

\[ C_e (S_0, X, T) = S_0 + P_e (S_0, X, T) - \frac{X}{(1 + r)^T}. \]

The right-hand expression defines a portfolio, called a synthetic call, that is equivalent to a European call option.

Suppose that the call option is overpriced. The put-call parity suggests the actual call option to be sold, and the synthetic call option to be bought. This strategy is known as a conversion. A reverse conversion consists of selling the actual put option and buying the synthetic put option. This strategy is motivated when the actual put option is underpriced.

### 6.4 Advanced Option Strategies

A spread is a portfolio consisting of a long position on one option and a short position on another option. Options
under *money spreads* differ only by their strike prices, and by their maturity dates under *calendar spreads*.

A *call bull spread* is a money spread on call options that differ only by their strike prices. The long call has the lower strike price $X_1 < X_2$. The profit from a call bull spread is:

$$
\Pi(S_T) = \max(0, S_T - X_1) - C_1 - \\
\left(\max(0, S_T - X_2) - C_2\right),
$$

where $C_1 > C_2$ are the call-option prices associated to $X_1 < X_2$. The profit function from a call bull spread is:

$$
\Pi(S_T) = \left\{ \begin{array}{ll}
-(C_1 - C_2), & \text{for } S_T \leq X_1 < X_2 \\
S_T - X_1 - (C_1 - C_2), & \text{for } X_1 < S_T \leq X_2 \\
X_2 - X_1 - (C_1 - C_2) & \text{for } X_1 < X_2 < S_T
\end{array} \right.
$$

where $X_1$ and $X_2$ are two breakeven points.

The following pairs belong to the piecewise-linear profit curve:

$$(0, -(C_1 - C_2)),$$  
$$(X_1, -(C_1 - C_2)),$$  
$$(S_T^* = X_1 + (C_1 - C_2), 0),$$  
and

$$(X_2, X_2 - X_1 - (C_1 - C_2)).$$
where $C_1 - C_2 > 0$, $X_1 < X_1 + (C_1 - C_2) < X_2$, and $X_2 - X_1 - (C_1 - C_2) > 0$.

[Please explain Figure 7.1 on page 222.]

A call bull spread is a bullish strategy that has a limited and small gain and loss.

A *put bear spread* is a money spread that consists of a long position on the put option with the higher exercise price $X_2$ and a short position on the put option with the lower exercise price $X_1$.

The profit function from a call bull spread is:

$$
\Pi (S_T) = \begin{cases} 
X_2 - X_1 - (P_2 - P_1), & \text{for } S_T \leq X_1 < X_2 \\
-S_T + X_2 - (P_2 - P_1), & \text{for } X_1 < S_T \leq X_2 \\
-(P_2 - P_1), & \text{for } X_1 < X_2 < S_T 
\end{cases}
$$

where $X_1$ and $X_2$ are two breakeven points.
The following pairs belong to the piecewise-linear profit curve:

\[(0, X_2 - X_1 - (P_2 - P_1)),\]
\[(X_1, X_2 - X_1 - (P_2 - P_1)),\]
\[(S_T^* = X_2 - (P_2 - P_1), 0),\] and
\[(X_2, -(P_2 - P_1)),\]

where \(X_2 - X_1 - (P_2 - P_1) \geq 0, X_1 \leq X_2 - (P_2 - P_1) \leq X_2,\) and \(P_2 - P_1 \geq 0.\)

[Please explain Figure 7.3 on page 225.]

The put bear spread is a bearish strategy that has a limited and small gain and loss.

### 6.5 Assignment

Read the chapter, and give precise and concise definitions for the keywords listed on page 215. This part is to be prepared for the second midterm exam, but should not be handed in as part of the assignment.

Answer questions no 1, 3, 5, 9, and 10 on pages 215 and 216.
7 The Structure of Forward and Futures Markets

7.1 The Development of Forward and Futures Markets

A forward contract commits two parties one to buy and the other to sell an underlying asset at a known future maturity date (delivery date) for an agreed-upon forward price (delivery price). No payment is exchanged up front.

Example 38: Several every-day transactions can be interpreted as forward contracts. Examples include pizza delivery and car rentals.

A forward contract is fundamentally different from an option. The former commits the long party to buying the underlying asset, while the latter does not. Forward
contracts are traded over the counter; therefore, forward-market participants are subject to both market and credit risk.

If the forward price rises (falls), the long party records a gain (loss) since he committed to buying the underlying asset at a lower (higher) price. On the other hand, the short party records a loss (gain). The cumulative loss recorded by one party at maturity can be so great that he can fail to fulfill his obligation.

A futures contract is identical to a forward contract except that it is traded on an exchange and is subject to a daily settlement of gains and losses. Unlike a forward contract, which is settled at maturity, a futures contract is settled at the end of each trading day, thereby keeping the loss and the default probability at a very low level.

Since investors can invert their positions at any time in a liquid market, futures contracts are usually (but not always) settled in cash prior to maturity.
While forward contracts go back to the beginnings of commerce, futures contracts are more recent. Futures contracts on agricultural products have been traded since 1898, when the Chicago Mercantile Exchange was created. The exchange defines a set of terms and conditions for each futures contract such as, size, quotation unit, minimum price fluctuation, grade, trading hours, delivery terms, daily price limits, and delivery procedures. Financial futures contracts started to be traded later, in the early 1970s, when Western economies allowed foreign exchange rates to fluctuate. The International Monetary Market was created in 1972 to trade futures contracts on foreign currencies. Later on, in the mid-1970s, interest-rate futures contracts started to be traded. Examples include the GNMA, T-bill, T-notes, and T-bond futures contracts.

**Example 39:** The T-bond futures contract, traded on the CBOT, calls for the delivery of $100,000 T-bonds with a minimum remaining life of 15 years at the first delivery date. This contract contains several *embedded options*
held by the short trader, such as the choosing and timing options. The short trader can deliver, at his discretion, any bond of a set of eligible long-term T-bonds. The seller is expected to act in an optimal way, and deliver the cheapest T-bond. The short trader also has the right to deliver the underlying asset within a month, following specific delivery rules.

Stock index futures contracts were the next to be traded, followed by options on futures. For example, a call option on a futures contract gives its holder the right to take a long position on the underlying futures contract at option maturity for the agreed-upon option strike price. The option holder will exercise his right if, and only if, the futures price at option maturity is higher than the option strike price.

7.2 The Trading Process

A futures exchange is a corporate entity organized to trade futures contracts. The owners of the exchange are
known as the *full members*. Each one of them owns a *seat*.

[Please explain Figure 8.1 on page 264.]

Several market participants are involved in trading a futures contract: the *buyer* (*seller*), the *buyer’s* (*seller’s*) *broker*, and the *commission broker* of the *buyer’s* (*seller’s*) *broker*. The latter is a *full* or a *partial member* of the exchange, depending on whether he owns or leases a seat.

[Please explain the top of Figure 8.2 on page 267.]

### 7.3 The Daily Settlement System

The *clearinghouse* guarantees the performance of each participant in a futures exchange. The oldest one in North America, associated to the Chicago Board of Trade, was created in 1920. The owners of a clearinghouse are known as the *clearing firms*. 
To ensure the performance of an investor, a futures contract is subject to a daily settlement managed by the clearinghouse.

[Please explain the bottom of Figure 8.2 on page 267.]

First of all, when a futures contract is issued, each party is invited to open a margin account with his broker, and to deposit an initial margin. The broker is invited to do the same with his clearing firm, which in turn, does the same with the clearinghouse. Next, the trading process continues untill the day’s end. The settlement price is then registered based on the last futures price. To make it simple, define the settlement price within a trading day as the last registered futures price. Finally, the futures contract is marked to the market at the end of the trading day: it is closed and reopened at the new settlement price. The difference between the current and the last settlement prices, if positive, is subtracted from the seller’s margin account, and added to the buyer’s margin account. If the difference is negative, the opposite is
done. When a margin account falls below a certain level, known as the *maintenance margin*, the investor receives a *margin call* to raise the margin balance to the initial margin level. The daily settlement system drastically reduces counterparty risk, since losses, which are monitored every day, remain limited.

**Example 39 (continued):** The detail is in Table 8.2 on page 269. Consider a short position on the CBOT T-bond futures contract, which was taken on 08/01 at the quoted futures price of 97-27. The futures price was at $(97+27/32)$ per $100$ of principal (of the underlying T-bond), which corresponded to $97,843.75$ per $100,000$ of principal. The initial margin was $2,500$, and the maintenance margin $2,000$. The seller then deposited $2,500$ into his margin account.

The futures price fell to $97,406.25$ at the end of the first trading day. This was the new settlement price, the old one being $97,843.75$. The seller recorded a gain, and a buyer recorded a loss. The (absolute) difference
between the two settlement prices, that is, $437.50, was subtracted from the buyer’s margin account, and added to the seller’s margin account. The balance of the seller’s margin account was then at $2,937.50.

The futures price rose to $97,781.25 at the end of the second trading day. This was the new settlement price, the old one having been $97,406.25. The seller recorded a loss, and the buyer a gain. The (absolute) difference between the two settlement prices, that is, $375.50, was subtracted from the seller’s margin account, and added to the buyer’s margin account. The balance of the seller’s margin account then was at $2,562.50.

On 08/08, the balance of the seller’s margin account (at $1250) fell under the maintenance margin ($2000) just after the contract was marked to the market. A margin call of $1250 was issued the same day, and honoured the next trading day, on 08/11.

On 08/18, the last trading day, the settlement price fell from $100,781.25 to $100,500.00, and the seller then
recorde a gain of $281.25. The latter closed his position by inversion, and withdrew the remaining balance of $3,031.25.

### 7.4 Assignment

Read the chapter, and give precise and concise definitions for the keywords listed on page 279. This part is to be prepared for the second midterm exam, but should not be handed in as part of Assignment 2.

Answer questions no 1, 2, 3, 15, 16.
8 Pricing Forwards, Futures, and Options on Futures

8.1 Principles of Carry-Arbitrage Pricing

A *forward contract* commits two parties one to buy and the other to sell an underlying asset at a known future maturity date, for an agreed-upon forward price. No payment is exchanged up front. The maturity date is also known as the *delivery date*, and the forward price as the *delivery price*. The underlying asset may be a stock, stock index, foreign currency, bond, or commodity.

A *futures contract* is identical to a forward contract except that it is traded on an exchange, subject to a margin system, and a *daily settlement* of gains and losses.

An important issue is that the forward (futures) price is different from the forward (futures) value. For example,
you may commit to buy a share of stock in three months for the forward price of $50; however, the value of your position is zero at the start since no up-front payment is exchanged.

The value of both positions on a forward contract is zero at the start. This is also the case for a futures contract at the start and end of each trading day, as soon as the contract is marked to market.

We use the following notation:

1. $F(t, T)$: the price of a forward contract starting at $t$ and maturing at $T$;

2. $V_t(0, T)$: the value at $t$ of a long position on a forward contract starting at $0$ and maturing at $T$;

3. $f_t(T)$: the price at $t$ of a futures contract maturing at $T$. 

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4. \( f_{t-1}(T) \): the settlement price on the day prior to \( t \) of a futures contract maturing at \( T \);

5. \( v_t(T) \): the value at \( t \) of a long position on a futures contract maturing at \( T \).

### 8.2 Pricing Forward Contracts on Stocks

Consider a forward contract on a single stock, which pays no dividends over the contract’s life. Prices and values are computed assuming that the market is frictionless and arbitrage-free. The convergence principle is:

\[
F(T, T) = \lim_{t \to T} F(t, T) = S_T,
\]

where \( S_T \) is the spot price at \( T \) of the stock.

The valuation of a forward contract is based on the property:

\[
V_0(0, T) = 0.
\]
So, a forward contract cannot be considered an asset or a liability at the start. The forward price \( F(0, T) \) is negotiated at 0, such that \( V_0(0, T) = 0 \). The value of a long position on the forward contract at \( T \) is:

\[
V_T(0, T) = S_T - F(0, T),
\]

which can be either positive or negative. Thus, looking ahead, the forward contract can be considered an asset or a liability. The value of a short position on a forward (futures) contract is simply the opposite of that of a long position.

Now, consider a portfolio consisting of a long position on the underlying asset and a short position on the forward contract. Its value at \( T \) is:

\[
S_T - V_T(0, T) = F(0, T),
\]

which is known in advance. Since this portfolio bears no risk at maturity, its value before maturity at \( t \) is:

\[
S_t - V_t(0, T) = \frac{F(0, T)}{(1 + r)^{T-t}},
\]
where $r$ is the effective risk-free interest rate (in % per year), and $T - t$ the remaining time to maturity (in years). This results in the following equation:

$$V_t (0, T) = S_t - \frac{F (0, T)}{(1 + r)^{T-t}}.$$

Solving this equation at time 0 for $F (0, T)$ gives the well-known result:

$$F (0, T) = S_0 (1 + r)^T.$$

**Example 40:** You bought a forward contract maturing in 45 days for a price of $100, and have held it for 20 days. The risk-free rate is 10% (per year). Now, the spot price of the underlying stock is $102. Give the stock price that would have been observed at the start, and compute the current value of the forward contract.

For a futures contract, the convergence property holds:

$$f_T (T) = \lim_{t \to T} f_t (T) = S_T.$$
The value of a long position on a futures contract before being marked to market is:

\[ v_t(T) = f_t(T) - f_{t-1}(T). \]

Since the gain/loss under a futures contract is settled every day, the value of both positions switches to zero as soon as the contract is marked to market.

Ignoring the margin system, and assuming a constant risk-free rate, the futures price will be equal to the forward price.

### 8.3 Pricing Forward Contracts on Stock Indices

If the stock pays sure dividends \( D_1, \ldots, D_J \) at \( t_1, \ldots, t_J \) during the contract’s life, the carry-arbitrage evaluation equation at \( t \) becomes:

\[ S_t - V_t(0, T) = D_{t,T} + \frac{F(0, T)}{(1 + r)^{T-t}}, \]
where

\[ D_{t,T} = \sum_{t_j > t} \frac{D_j}{(1 + r)^{t_j}}, \]

is the present value at \( t \) of all remaining dividends from \( t \) to \( T \). The present value at time 0 of all dividends \( D_{0,T} \) is indicated by \( D_0 \). This results in:

\[ F(0, T) = (S_0 - D_0)(1 + r)^T, \]

**Example 41:** The current stock price is at $100, and the risk-free rate at 6%. The stock pays a dividend of $5 in 20 days. Compute the forward price with a maturity of 45 days.

A stock index is a weighted combination of single stocks, each paying dividends on a regular basis, as if the stock index were paying dividends in continuous time. Assuming that the dividends are paid continuously at a known fixed rate \( \delta_c \), one has:

\[ D_0 = S_0 \left(1 - e^{-\delta_c T}\right). \]
The forward price is then:

\[ F(0, T) = S_0 e^{(r_c - \delta_c)T}, \]

where \( r_c \) is the continuous-time risk-free interest rate (in % per year).

**Example 42:** A stock index is at 50, and its dividend rate at 6%. Compute the futures price when the continuous-time risk-free rate is at 8% and the time to maturity is 60 days. Compute the futures price again when the risk-free rate is at 5%.

For a futures contract on a stock index, one can observe the futures price \( f_0(T) \) and solve for the dividend rate \( \delta_c \):

\[ f_0(T) = S_0 e^{(r_c - \delta_c)T}. \]

The solution is:

\[ \delta_c = r_c - \frac{1}{T} \ln \left( \frac{f_0(T)}{S_0} \right). \]
Example 43: Consider a futures contract on a stock index with a time to maturity of 60 days. The futures price is at 50.16, the stock index at 50, and the continuous-time risk-free rate at 8%. Compute the constant dividend rate of this stock index.

8.4 Pricing Forward Contracts on Foreign Currencies

Consider again the portfolio of a long position on the underlying asset and a short position on the forward contract. Interpret the spot exchange rate $S_0$ as the price at time 0 in local money of one unit of the foreign currency. For this portfolio to be riskless at maturity, it must contain one unit of the foreign currency. Less then one unit is required prior to maturity, since holding a foreign currency earns interest. More precisely, the portfolio must contain:

$$
\frac{1}{(1 + \rho)^{T-t}}
$$
units of the foreign currency at time $t < T$, where $\rho$ is the foreign risk-free rate (in % per year).

The carry-arbitrage evaluation equation becomes

$$\frac{S_t}{(1 + \rho)^{T-t}} - V_t (0, T') = \frac{F (0, T)}{(1 + r)^{T-t}}.$$

Solving for $F (0, T)$ at time 0 gives:

$$F (0, T) = \frac{(1 + r)^T}{(1 + \rho)^T} S_0.$$

**Example 44:** The spot exchange rate for US dollars was 0.7908 euros. The US interest rate was 5.84%, while the euro interest rate was 3.59%. The time to expiration was 90 days. Compute the forward exchange rate for dollars.

From the perspective of a European investor, it may be interesting to convert euros to dollars to invest at a higher interest rate. When time comes to convert back dollars to euros at the agreed-upon forward exchange rate, a loss is recorded. This is the well known *interest-rate parity.*
8.5 Pricing Forward Contracts on Commodities

Holding a commodity induces a storage cost $s$ per unit, assumed to be known and paid at maturity. Consider again the portfolio of a long position on the commodity and a short position on the forward contract. The carry-arbitrage evaluation equation at $t$ becomes:

$$S_t - V_t(0, T) = -s + F(0, T)\frac{(1 + r)^{T-t}}{(1 + r)^{T-t}}.$$ 

Solving for $F(0, T)$ at time 0 gives:

$$F(0, T) = S_0(1 + r)^T + s.$$
8.6 Futures Prices and Expected Future Spot Prices

Depending on the benefit/cost of carry, the futures price may be higher or lower than the spot price:

\[
f_0(T) : \begin{cases} 
> S_0, & \text{then the market is called Contango} \\
< S_0, & \text{then the market is called Backwardation}
\end{cases}
\]

On the other hand, futures prices give insights into future spot prices. Empirical investigations, however, show that futures prices are generally biased estimations of expected future spot prices:

\[
f_0(T) : \begin{cases} 
> E[S_T], & \text{then the market is called Normal Contango} \\
< E[S_T], & \text{then the market is called Normal Backwardation}
\end{cases}
\]

where the expectation is computed under the physical probability measure.
8.7 Call-Put-Futures Parity

Consider a portfolio of a long position on a European call option and a short position on a European put option. Both have the same underlying asset, exercise price, and maturity. The value of this portfolio at $T$ is:

$$\max(0, S_T - X) - \max(0, X - S_T) = S_T - X = S_T - F(0, T) + F(0, T) - X.$$ 

This results in the call-put-futures parity:

$$C_e - P_e = \frac{F(0, T) - X}{(1 + r)^T}.$$ 

**Example 45:** On May 14, the S&P 500 index was at 1337.80, and the June futures price on the S&P 500 index at 1339.30. The June 1340 call option on this stock index was at 40, and the June 1340 put option at 39. The maturity date was June 18, and the risk-free rate at 4.56% (per year). Check the call-put-futures parity. Conclude. See the example on page 310.


8.8 Pricing Options on Futures

A call (put) option on a futures contract gives its holder the right (but not the obligation) to take a long (short) position in the underlying futures contract at the option maturity.

The call (put) option is exercised at maturity if, and only if, its strike price is lower (higher) than the futures price. The option is then settled in cash. While the call-option holder is paid the difference between the futures price and the option strike price, the put-option holder is paid the opposite amount when the option expires in the money.

Black (1976) developed a variation of the Black and Scholes formula (1973) for pricing options on futures. The futures contract in the Black model plays the same role as the stock in the Black and Scholes model.
To alleviate the notation, the futures price at the option start \( f_0 (T_2) \) is indicated by \( f_0 \), where \( T_2 \) is the maturity of the futures contract. Black's formula is as follows:

\[
C_e = e^{-r_c T_1} \left[ f_0 N (d_1) - X N (d_2) \right],
\]

and

\[
P_e = e^{-r_c T_1} \left[ X N (-d_2) - f_0 N (-d_1) \right],
\]

with

\[
d_1 = \frac{\ln (f_0/X) + \sigma^2 T_1/2}{\sigma \sqrt{T_1}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T_1},
\]

where \( T_1 \) (prior to \( T_2 \)) is the option maturity, \( X \) the option strike price, \( r_c \) is the risk-free rate, and \( \sigma \) is the volatility associated to the futures contract.

### 8.9 Assignment

Read the chapter, and give precise and concise definitions for the keywords. This part is to be prepared for
the exams, but should not be handed in as part of the assignment.

Answer questions no 10, 11, 14, 16, 19.
9  Forward and Futures Hedging

9.1  Is Hedging Relevant?

Hedging is a trading activity designed to reduce risk. We have been discussing hedging strategies, and giving examples such as covered calls and protective puts. Now, we focus on activities based on forward/futures contracts and their underlying assets.

Mainly, a hedger should take opposite positions in the spot and forward/futures markets. Indeed, the underlying-asset spot and forward/futures prices move in the same direction, so that a loss (gain) recorded in the spot market is at least partially offset by a gain (loss) in the forward/futures market.

Hedging activities reduce the possibility of loss, but also of gain. They have no impact on the average of the profit function, but reduce its fluctuation. In addition,
the value of a company is independent of any financial decision, including hedging (Modigliani and Miller)! Is hedging relevant?

The answer is obviously, yes. The result by Modigliani and Miller is true only in a frictionless market. Real activities involve various frictions, including transaction costs, information asymmetries, and taxes.

Firms hedge to save taxes, improve credit terms, and reduce bankruptcy costs. Also, hedging a sporadic transaction may be extremely effective.

### 9.2 Hedging Design

A *short hedge* consists of a short position in the forward/futures market, possibly coupled with a long position in the spot market. A *long hedge* consists of a long position in the forward/futures market, possibly coupled with a short position in the spot market.
The profit from a short position in the futures market and a long position in the spot market is:

\[
\Pi_t = S_t - S_0 - (f_t - f_0) \\
= S_t - f_t - (S_0 - f_0) \\
= b_t - b_0 \\
= \Delta b,
\]

where \(b_t = S_t - f_t\) is the basis at time \(t\).

The profit turns out to be the change in the basis. In this context, analysing the risk is equivalent to analysing the basis.

[Please explain Figure 11.1 on page 360.]

For a hedge to completely eliminate risk, there must be a perfect match between the hedging and the underlying asset, maturity, and size. This is a perfect hedge. A perfect hedge can be achieved only by using a forward contract since the underlying asset, maturity, and size can be selected by the hedger at his discretion. Hedging
activities based on forward contracts, however, are often undesirable because forward markets are illiquid, and involve large transaction costs.

Hedging activities based on futures contracts are viable alternatives. Hedging strategies with futures contracts also have their disadvantages.

First of all, the hedging asset may not be traded on the futures markets; instead, a related underlying asset may be used. Suppose one holds a corporate bond, and wants to hedge against a rise in its yield to maturity. Futures contracts on corporate bonds are not traded; however, futures contracts on T-bonds can be used. Next, the hedging maturity may not be known in advance, or may not match the trading maturities. The maturity of the futures contract should be subsequent to the hedging maturity, so that the hedger can leave the market at the hedging maturity, just before the delivery month. This works for a short-maturity hedge, but is less viable for longer hedging maturities. Indeed, the longer the maturity of a futures
contract, and the lower its liquidity. For a long-maturity hedge, hedgers often roll forward the hedge. Finally, the hedge size should be selected with care. The hedge ratio is the number of futures contracts used to hedge a given exposure in the spot market.

To start with, assume that the size of the exposure and of the futures contract is one. The profit from a long position on one asset to be hedged and \( N_f \) futures contracts is:

\[
\Pi_t = S_t - S_0 + N_f (f_t - f_0) = \Delta S + N_f \Delta f.
\]

The profit function is risky. Here, the asset to be hedged is not necessarily the asset underlying the futures contract. The optimal hedge ratio \( N_f^* \) is the one that minimizes the variance of the profit function, which is:

\[
\text{Var}[\Pi] = \text{Var}[\Delta S] + \text{Var}[N_f \Delta f] + 2 \text{Cov}[\Delta S, N_f \Delta f] = N_f^2 \text{Var}[\Delta f] + 2N_f \text{Cov}[\Delta S, \Delta f] + \text{Var}[\Delta S].
\]
The variance of the profit function can be seen as a function of the unknown parameter $n_f$. Check that the profit is a convex function that reaches its minimum at:

$$N_f^* = \frac{-\text{Cov} [\Delta S, \Delta f]}{\text{Var} [\Delta f]}$$

$$= -\rho [\Delta S, \Delta f] \frac{\sigma [\Delta S]}{\sigma [\Delta f]},$$

where $\rho [\Delta S, \Delta f]$ is the correlation between $\Delta S$ and $\Delta f$, $\sigma [\Delta S]$ the standard deviation of $\Delta S$, and $\sigma [\Delta f]$ the standard deviation of $\Delta f$. These coefficients are estimated using their sample counterparts. This formula may be expressed in different ways, depending on the asset to be hedged and the asset underlying the futures contract.

Now, given the size of the futures contract, the number of futures contracts required to hedge a given exposure is obtained by computing the quantity:

$$N_f^* \times \frac{\text{Size of the exposure}}{\text{Size of the futures contract}},$$

and rounding it to the first whole number.
Example 46: A company knows it will buy 1 million gallons of jet fuel in three months. The standard deviation of the three-month change in the price of jet oil is 0.032. The company chooses to hedge by buying futures contracts on heating oil. The standard deviation of the change in the futures price is 0.04. The correlation between the three-month change in the price of jet fuel and the three-month change in the futures price is 0.8. Compute the optimal hedge ratio and the number of futures contracts to be used. Each futures contract on heating oil is on 42,000 gallons.

The correlation \( \rho \) between two variables measures their linear association, and indicates how much their realisations lie along the same line. The correlation \( \rho \) belongs to the interval \([-1, 1]\), and must be interpreted as follows:

\[
\rho = \begin{cases} 
\text{around } +1 \rightarrow \text{strong linear increasing relationship} \\
\text{around } 0 \rightarrow \text{no linear relationship} \\
\text{around } -1 \rightarrow \text{strong linear decreasing relationship}
\end{cases}
\]
The coefficient $\rho [\Delta S, \Delta f]$ is expected to be positive since the asset underlying the futures contract should be closely related to the asset to be hedged. A negative sign of $N_f^*$ shows that the hedge is achieved by a short position in the futures market in case of a long position in the spot market. A long hedge is required in case of a short position in the spot market.

**Example 47:** Discuss examples for $\rho$ around 1, 0, and $-1$.

### 9.3 Hedging T-Bonds and Stock Portfolios

Consider a position on a T-bond to be hedged by an opposite position on a T-bond futures contract. In this context, a rise (fall) in the term-structure of interest rates
induces a fall (rise) in the bond’s price. The current T-bond’s price is:

\[ B = \sum_{n=1}^{N} \frac{CF_n}{(1 + y)^{t_n}}, \]

where \( y \) is the bond’s yield to maturity (effective rate in \( \% \) per year), and \( CF_n \) (in $) the cash flow made at time \( t_n \) (in years). One has:

\[
\frac{\partial B}{\partial y} = \sum_{n=1}^{N} \frac{CF_n}{(1 + y)^{t_n-1}} \times (-t_n)
\]

\[
= -\frac{1}{1 + y} \sum_{n=1}^{N} \frac{CF_n}{(1 + y)^{t_n}} \times t_n
\]

\[
= -\frac{B}{1 + y} \sum_{n=1}^{N} \frac{CF_n / (1 + y)^{t_n}}{B} \times t_n
\]

\[
= -\frac{B}{1 + y} \sum_{n=1}^{N} \omega_n t_n,
\]
where the weights $\omega_n$, for $n = 1, \ldots, N$, are non-negative and verify

$$\sum_{n=1}^{N} \omega_n = 1.$$ 

The duration of the bond, defined as:

$$D = \sum_{n=1}^{N} \omega_n t_n,$$

is a weighted average of the time to each cash-flow payment.

Finally, for a small change in $y$, one has:

$$\frac{\partial B}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta B}{\Delta y} \approx \frac{\Delta B}{\Delta y} = \frac{B}{1 + y} D.$$ 

Said in other words, one has:

$$\frac{\Delta B / B}{\Delta y} \approx - \frac{D}{1 + y} = MD_B.$$
where $MD_B$ is known as the \textit{modified duration} of the bond.

Along the same lines, we define the modified duration of the T-bond futures contract as:

$$MD_f = \frac{\Delta f / f}{\Delta y},$$

where $\Delta y$ is the change in the yield to maturity of the T-bond to be hedged.

The optimal hedge ratio is:

$$N_f^* = -\frac{\text{Cov} [\Delta B, \Delta f]}{\text{Var} [\Delta f]} = -\frac{\text{Cov} [\Delta B / B, \Delta f / f] \times B \times f}{\text{Var} [\Delta f / f] \times f^2} = -\frac{B \text{Cov} [MD_B \Delta y, MD_f \Delta y]}{f \text{Var} [MD_f \Delta y]} = -\frac{B}{f} \times \frac{MD_B}{MD_f}.$$

This is the \textit{price-sensitivity hedge ratio}. 
Example 48: A portfolio manager holds $1 million face value of a T-bond. The bond is currently priced 101 per $100 par value, and has a modified duration of 7.83 (years). The manager plans to sell the bond in one month to meet an obligation, and is concerned by an increase in interest rates. A short position on the T-bond futures contract maturing in four months is used. Its quoted price is at 70 16/32, and implied modified duration at 7.2 (years). Give the price-sensitivity hedge ratio. See Figure 11.7 on page 376.

Now, consider a position on a stock portfolio to be hedged by an opposite position on a stock index futures. Assume that the stock portfolio is not well diversified, while the stock index underlying the futures contract is diversified enough to be considered as a market proxy.
The optimal hedge ratio is:

\[
- \frac{\text{Cov}[\Delta S, \Delta f]}{\text{Var}[\Delta f]} = - \frac{\text{Cov}[\Delta S/S, \Delta f/f] \times S \times f}{\text{Var}[\Delta f/f] \times f^2} \\
= - \frac{S}{f} \times \frac{\text{Cov}[\Delta S/S - r, \Delta f/f - r]}{\text{Var}[\Delta f/f - r]} \\
\approx - \frac{S}{f} \times \frac{\text{Cov}[\Delta S/S - r, \Delta I/I - r]}{\text{Var}[\Delta I/I - r]} \\
= - \frac{S}{f} \times \beta,
\]

where \( r \) is the risk-free rate, \( I \) the index underlying the futures contract, and \( \beta \) is the beta of the portfolio. The latter parameter indicates the sensitivity of the excess return on the asset to be hedged to a change in the excess return on the market portfolio.

**Example 49:** A company wishes to hedge a portfolio worth $2,100,000 over the next three months using the S&P 500 index futures contract maturing in four months. The S&P 500 index is quoted 900, and has a multiplier of 250. The beta of the portfolio is 1.5. Give the optimal hedge ratio. How can the company reduce the beta of the portfolio to 0.75?
9.4 Assignment

Read the chapter, and give precise and concise definitions for the keywords. This part is to be prepared for the exams, but should not be handed in as part of the assignment.

Exercise (worth an extra 5% in the second midterm exam): Consider the variance of the profit from a long position in the spot market and a short position in the futures market, seen as a function of \( N_f \in \mathbb{R} \):

\[
\text{Var} [\Pi] = f \left( N_f \right).
\]

The parameters \( \rho [\Delta S, \Delta f] \), \( \sigma [\Delta S] \), and \( \sigma [\Delta f] \) are fixed at 0.6, 0.2, and 0.1, respectively. Draw the curve of \( \text{Var}[\Pi] \). Describe the shape of the curve. Locate the optimal hedge ratio, and check its consistency with the formula. Answer the same questions, for \( \rho [\Delta S, \Delta f] \in \{0.8, 0.6, 0.4, -0.4, -0.6, -0.8\} \). Interpret your result.
10 Vanilla Interest-Rate Swaps

10.1 Design

A swap commits two parties to exchanging series of payments or assets in the future, in accordance with specified rules and schedule. There are swaps on interest rates, exchange rates, equities, and commodities.

Swaps are traded in over-the-counter markets, and designed in several ways to match clients’ needs. In the following, we focus on swaps on interest rates, which are the most-traded swap contracts.

A vanilla interest-rate swap commits two parties to exchanging interest payments on a regular basis. The swap payer commits to paying interest based on a fixed rate, known as the swap rate, and the swap receiver to paying interest based on a floating rate. Only the net payments are exchanged. No payment is made up front. The swap
rate is negotiated so that the value of both parties is null at the start.

A vanilla swap is characterized by its *starting date*, *settlement dates* (including its maturity), *tenor* or *settlement period*, *notional principal*, *floating rate*, and *fixed rate*.

**Example 50:** A firm enters a swap as a payer with the following parameters: a notional principal of $50 million, settlement period of 3 months, and maturity of one year. The floating rate is the 3-month LIBOR rate, and the fixed rate is 7.5% (per year). The swap is *settled in arrears*, that is, the interest payment at time $t_n$ is based on the floating rate observed at time $t_{n-1}$, which covers the period $[t_{n-1}, t_n]$. The 3-month LIBOR rate and the swap rate are nominal rates, both expressed in % per year. They assume here that the interest is compounded quarterly. Draw a figure for the cash-flow stream, from the perspective of the swap payer.

A swap payer can exchange a floating-rate loan for a fixed-rate loan, and thus hedge against interest-rate fluctuations.
Example 50 (continued): Explain Table 12.1 on page 410.

10.2 Pricing

The position of a swap payer can be seen as a portfolio consisting of a long position on a floating-rate bond and a short position on a fixed-rate bond, both with the same maturity, settlement dates, and principal as those of the swap. The floating-rate bond pays coupons at the floating-rate of the swap, and the fixed-rate bond pays coupons at the swap rate.

The discount factor for the horizon $t_n$, $B_0(t_n)$, is the price at time 0 of $1$ to be received at time $t_n$. This is the price at time 0 of a zero-coupon bond maturing at time $t_n$.

In the LIBOR market, the discount factors are related to the term-structure of interest rates as follows:

$$B_0(t_n) = \frac{1}{1 + L_0(t_n) \times t_n}, \quad \text{for } n = 1, \ldots, N,$$
where \( L_0(t_n) \) (in % per year) is the rate of return associated to the zero-coupon bond maturing at \( t_n \). In the following, we assume that the LIBOR term-structure of interest rates is given, as are the discount factors.

The value of the swap at the start is:

\[
v_0 = 1 - \sum_{n=1}^{N} R \times \Delta t \times B_0(t_n) + B_0(t_N)
\]

\[= 0\]

where \( R \) is the swap rate (in % per year), \( t_1, \ldots, t_N \) are the settlement dates (including the swap maturity), and \( \Delta t \) is the tenor. Solving this equation for \( R \) gives:

\[
R = \frac{1}{\Delta t} \times \frac{1 - B_0(T)}{\sum_{n=1}^{N} B_0(t_n)}
\]

**Example 51:** Explain Table 12.2.