Real Exchange Rate Persistence in DGE Sticky-Price Models: An Analytical Characterization

Hafedh Bouakez*
International Department, Bank of Canada

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Abstract

This paper assesses analytically the ability of dynamic general-equilibrium sticky-price models to generate persistent real exchange rate fluctuations. It develops a tractable general-equilibrium model with Calvo-type price stickiness. The model has a closed-form solution and the persistence of the real exchange rate is explicitly characterized. The paper shows that real exchange rate persistence is pinned down by the probability of not changing prices. This result suggests that standard sticky-price models are unable to generate endogenous persistence.

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1. Introduction

Examining real exchange rate movements within dynamic general-equilibrium (DGE) sticky-price models has been a major development in the international finance literature over the past decade. This approach has become popular because it reformulates the intuitive arguments of Keynesian theory within a neoclassical micro-founded intertemporal framework. The first generation of DGE sticky-price models of exchange rate determination includes those by Obstfeld and Rogoff (1995) and Betts and Devereux (1996). In both studies, a closed-form solution of the model is made possible by assuming that prices are held fixed for only one period. Because such short-lived price rigidity is obviously insufficient to generate persistent effects of monetary shocks on the real exchange rate, subsequent studies that build on Obstfeld and Rogoff's work endeavored to allow for richer mechanisms of price setting. For example, Bergin and Feenstra (2001) and Chari, Kehoe, and McGrattan (2002) assume staggered price contracts à la Taylor, while Kollmann (2001) introduces a Calvo-type price setting. These pricing mechanisms are certainly more plausible than the one-period price stickiness, but this gain in realism came at the cost of precluding analytical solutions, so that numerical methods were needed to solve the model.[1]

Does price staggering generate persistent real exchange rates, once incorporated in a DGE open-economy framework? This question has been recently investigated by Chari, Kehoe, and McGrattan (2002). Using a simulation-based approach, they show that, unless one assumes an implausibly long duration of price contracts, standard DGE sticky-price models fail to match the persistence found in real exchange rate data.

This paper adopts an analytical approach rather than a simulation-based approach to assess the ability of sticky-price models to generate persistent real exchange rates. The paper develops a tractable DGE two-country sticky-price model that can be solved explicitly. The model embeds the standard features found in a typical DGE sticky-price model, yet is simple enough to yield a closed-form solution. The type of price stickiness introduced follows Calvo (1983) where each firm has a constant probability of changing its price in every period.

The paper shows analytically that real exchange rate persistence is pinned down by the probability of not changing prices. More precisely, the real exchange rate response to a monetary shock dies out *exactly* at the same rate at which prices adjust. An interpretation of this result is that the effects of monetary shocks on the real exchange rate do not last

[1] Bergin and Feenstra (2001) do find an analytical solution to their model in the special case where price contracts last exactly two periods. For longer durations, however, their model cannot be solved analytically.

beyond the average duration of price contracts. Thus, price staggering does not generate endogenous real exchange rate persistence. Using Taylor's terminology, this means that the *contract multiplier* is exactly equal to 1. This finding parallels the *persistence problem* emphasized by Chari, Kehoe, and McGrattan (2000) who show that, in a closed-economy set-up, staggered price contracts do not generate endogenous output persistence.

The rest of the paper is organized as follows. Section 2 presents the theoretical model. Section 3 analyzes the model's dynamics. Section 4 performs a robustness analysis. Section 5 concludes.

2. The Model

The model consists of two countries, each characterized by (i) a representative infinitely lived household, (ii) a representative final-good producer, (iii) a continuum of intermediate-good producers indexed by $i \in [0, 1]$, and (iv) a government. A fraction n (respectively, $1 - n$) of intermediate-good producers are located in the home (foreign) country. Intermediate goods are differentiated and are used to produce the final good in both countries. The final good is used exclusively for consumption and is not tradable between the two countries. Following Chari, Kehoe, and McGrattan (2002), I assume that, in each period t, the world economy experiences one of finitely many events s_t. The history of events realized up to period t (or the state of nature in period t) is denoted by $s^t = (s_0, s_1, ..., s_t)$. The probability, as of period 0, of any particular state s^t is denoted by $\omega(s^t)$. The initial state, s^0, is given.

2.1 Households

The representative household in the home country has the following lifetime utility function:

$$U_0 = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \omega(s^t) u(c(s^t), m(s^t), l(s^t)),$$

where β is the subjective discount factor ($0 < \beta < 1$), and u is the instantaneous utility function. Households derive utility from consumption (c), from holding real money balances (m), and from leisure ($1 - l$).[2] The instantaneous utility function is assumed to be

$$u(c(s^t), m(s^t), l(s^t)) = v(c(s^t), l(s^t)) + \gamma \log(m(s^t)), \tag{1}$$

[2] In each period, the household's total endowment of time is normalized to unity.

where the function v satisfies $v_c > 0, v_l < 0, v_{cc} < 0$, and $v_{ll} > 0$, $m = M/P$, M is the nominal money stock, P is the aggregate price index, and γ is a positive parameter.

Financial markets are assumed to be complete. That is, there exists a complete set of state-contingent bonds, which are assumed to be denominated in domestic currency. Let $B(s^{t+1})$ denote the home household's holdings of a bond purchased in period t that pays one unit of the home currency in period $t+1$ if state s^{t+1} occurs and 0 otherwise. The price of this bond in units of the home currency is denoted by $Q(s^{t+1}|s^t)$.

The representative household carries $M(s^{t-1})$ units of home currency and a portfolio of state-contingent nominal bonds into period t. After the realization of the event s_t, the household receives $B(s^t)$ additional units of home currency. It also receives a lump-sum transfer, $T(s^t)$, from the government and dividends, $D(i, s^t)$, from each intermediate-good producer $i \in [0, n]$. The household sells $l(i, s^t)$ units of labor to each intermediate-good producer $i \in [0, n]$ at the nominal wage, $W(s^t)$. The household's income in period t is allocated to consumption, money holdings and the purchase of nominal bonds. The representative household's budget constraint in period t is

$$\begin{aligned} P(s^t)c(s^t) + \sum_{s^{t+1}} Q(s^{t+1}|s^t)B(s^{t+1}) + M(s^t) \\ \leq B(s^t) + M(s^{t-1}) + W(s^t)l(s^t) + D(s^t) + T(s^t), \end{aligned} \tag{2}$$

where $l(s^t) = \int_0^n l(i, s^t)di$ is the household's total labor supply, and $D(s^t) = \int_0^n D(i, s^t)di$ are total dividends.

The representative household in the foreign country has the following budget constraint:

$$\begin{aligned} P^*(s^t)c^*(s^t) + \sum_{s^{t+1}} Q(s^{t+1}|s^t)B^*(s^{t+1})/e(s^t) + M^*(s^t) \\ \leq B^*(s^t)/e(s^t) + M^*(s^{t-1}) + W^*(s^t)l^*(s^t) + D^*(s^t) + T^*(s^t), \end{aligned} \tag{3}$$

where the asterisk denotes variables in the foreign country, and e_t is the nominal exchange rate, defined as the price of one unit of the foreign currency in terms of the home currency.

The representative household maximizes its lifetime utility subject to its budget constraint (2). The first-order necessary conditions for this problem are

$$v_l(s^t) + w(s^t)v_c(s^t) = 0, \tag{4}$$

$$\gamma/m(s^t) - v_c(s^t) + \beta \sum_{s^{t+1}} \omega(s^{t+1}|s^t)v_c(s^{t+1})/\pi(s^{t+1}) = 0, \tag{5}$$

$$Q(s^{t+1}|s^t)v_c(s^t) - \beta\omega(s^{t+1}|s^t)v_c(s^{t+1})/\pi(s^{t+1}) = 0, \tag{6}$$

where $w(s^t)$ is the real wage, $\omega(s^{t+1}|s^t) = \omega(s^{t+1})/\omega(s^t)$ is the conditional probability of s^{t+1} given s^t, and $\pi(s^{t+1}) = P(s^{t+1})/P(s^t)$ is the gross inflation rate between t and $t+1$.

The foreign household's problem implies an analogous set of first-order conditions. Equation (6) and its foreign counterpart imply the following risk-sharing condition:

$$q(s^t) = \varsigma v_c^*(s^t)/v_c(s^t), \tag{7}$$

where $q = eP^*/P$ is the real exchange rate, and ς is a constant reflecting initial wealth differences.

From this point on, I will use the operator E_t to denote the expected value of variables dated $\tau \geq t$ conditional on the current state s^t. More specifically, for a given variable x, $E_t x(s^\tau) = \sum_{s^\tau} \omega(s^\tau|s^t)x(s^\tau)$. Moreover, for ease of presentation, I will slightly abuse the notation by writing $x_t \equiv x(s^t)$.[3]

2.2 The final-good producer

Final-good producers are perfectly competitive. They use the differentiated intermediate goods from both countries to produce a single country-specific perishable commodity using the following CES technology:

$$y_t = \left[\int_0^n y_{ht}(i)^{(\theta-1)/\theta} di + \int_n^1 y_{ft}(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)}, \tag{8}$$

where $y_{ht}(i)$ (respectively, $y_{ft}(i)$) is the input of intermediate good i produced in the home (foreign) country, and $\theta > 1$ is the elasticity of substitution between different goods. As $\theta \to \infty$, goods become perfect substitutes in production. It is assumed that exports are invoiced in the currency of the importing country. This assumption, often called local currency pricing, was introduced by Betts and Devereux (1996, 2000) into Obstfeld and Rogoff's (1995) model to characterize pricing-to-market behavior by monopolistic firms. Pricing-to-market is the ability of a monopoly to set different prices in the home and foreign countries by somehow segmenting the market. Typically, this price discrimination leads to the violation of the law of one price among traded goods, and ultimately to a departure from the purchasing power parity. It is clear, though, that such behavior is possible only if there are economic and/or institutional constraints that prevent consumers from taking advantage of international arbitrage opportunities in the goods market. Empirically, studies by Knetter (1989, 1993), Engel (1993), and Engel and Rogers (1996) seem to provide strong

[3] The first-order condition (5), therefore, becomes $\gamma/m_t - v_{ct} + \beta E_t (v_{ct+1}/\pi_{t+1}) = 0$.

evidence in favor of pricing-to-market, as departures from purchasing power parity were found to reflect mainly the failure of the law of one price between traded goods, rather than the presence of non-traded goods. Under the assumption of local currency pricing, the final-good producer solves the following problem:

$$\underset{\{y_{ht}(i),y_{ft}(i)\}}{Max} \qquad P_t y_t - \int_0^n P_{ht}(i) y_{ht}(i) di + \int_n^1 P_{ft}(i) y_{ft}(i) di,$$

subject to (8), where $P_{ht}(i)$ (respectively, $P_{ft}(i)$) is the price of intermediate good i produced in the home (foreign) country. The solution of this problem yields the input demand of good i:

$$y_{jt}(i) = (P_{jt}(i)/P_t)^{-\theta} y_t, \tag{9}$$

where $j = h$ for $i \in [0, n]$ and $j = f$ for $i \in]n, 1]$, and the elasticity of demand is equal to θ. The zero-profit condition implies that the aggregate price index is given by

$$P_t = \left[\int_0^n P_{ht}(i)^{1-\theta} di + \int_n^1 P_{ft}(i)^{1-\theta} di \right]^{1/(1-\theta)}. \tag{10}$$

Let P_{ht} and P_{ft} denote, respectively, the price indexes of home and foreign intermediate goods sold in the home country.[4] Hence, the aggregate price index can be written as

$$P_t = \left[n P_{ht}{}^{1-\theta} + (1-n) P_{ft}{}^{1-\theta} \right]^{1/(1-\theta)}. \tag{11}$$

The problem of the representative foreign final good producer is described in an analogous manner.

2.3 The intermediate-good producer

The representative firm i in the home country produces its differentiated good using the following technology:

$$y_t(i) \equiv y_{ht}(i) + y_{ht}^*(i) = F(k_t(i), h_t(i)),$$

where $k_t(i)$ and $h_t(i)$ are, respectively, capital and labor input used in the production of the intermediate good i, and F is a concave production function with constant returns to

[4] More precisely, P_{ht} and P_{ft} are defined as follows:
$P_{ht} \equiv \left[\frac{1}{n} \int_0^n P_{ht}^{1-\theta}(i) di \right]^{1/(1-\theta)}$ and $P_{ft} \equiv \left[\frac{1}{1-n} \int_n^1 P_{ft}^{1-\theta}(i) di \right]^{1/(1-\theta)}$.

scale.[5] Denoting by r_t the real rental rate of capital, the representative final-good producer solves the following problem:

$$\underset{\{k_t(i),h_t(i)\}}{Min} \quad r_t k_t(i) + w_t h_t(i),$$

subject to $F(k_t(i), h_t(i)) = 1$. First-order conditions are

$$r_t = mc_t F_{kt}, \tag{12}$$

$$w_t = mc_t F_{ht}, \tag{13}$$

where the real marginal cost, mc_t, is the Lagrange multiplier associated with the constraint. Using equations (12) and (13), and the fact that the function F is homogenous of degree 1, it is straightforward to show that the total real cost for firm i is equal to $mc_t(y_{ht}(i)+y^*_{ht}(i))$.

Intermediate-good producers are monopolistically competitive. Each firm faces a downward-sloping demand curve for its differentiated good in each country. Firm i chooses its (nominal) prices, $P_h(i)$ and $P^*_h(i)$, taking as given the aggregate demand and the price level in each country. Nominal prices are assumed to be sticky. Price stickiness is modeled *à la* Calvo (1983). That is, each period, some firms are randomly selected to set new prices for the home and foreign markets. The probability of being selected in any particular period is constant and is equal to $1 - \varphi$.

Let $\tilde{P}_{ht}$ and $\tilde{P}^*_{ht}$ denote the optimal prices set by a typical firm at period t in the home and foreign countries, respectively. It is not necessary to index $\tilde{P}_{ht}$ and $\tilde{P}^*_{ht}$ by firm, because all of the firms that change their prices at a given time choose the same price. The total domestic and foreign demands facing this firm at time τ for $\tau \geq t$ are $\tilde{y}_{h\tau} = (\tilde{P}_{ht}/P_\tau)^{-\theta} y_\tau$ and $\tilde{y}^*_{h\tau} = (\tilde{P}^*_{ht}/P_\tau)^{-\theta} y^*_\tau$, respectively. The probability that $\tilde{P}_{ht}$ and $\tilde{P}^*_{ht}$ "survive" at least until period τ, for $\tau \geq t$, is $\varphi^{\tau-t}$. Thus, the intermediate-good producer chooses $\tilde{P}_{ht}$ and $\tilde{P}^*_{ht}$ to maximize

$$E_t \sum_{\tau=t}^{\infty} (\varphi\beta)^{\tau-t} \Lambda_{t,\tau} \left[\tilde{P}_{ht}\tilde{y}_{h\tau} + e_\tau \tilde{P}^*_{ht}\tilde{y}^*_{h\tau} - MC_\tau(\tilde{y}_{h\tau} + \tilde{y}^*_{h\tau})\right],$$

where $\Lambda_{t,\tau}$ is the marginal utility of a dollar earned at time τ relative to its marginal utility at time t, and $MC_t = mc_t P_t$ is the nominal marginal cost. First-order conditions for this

[5] Labor-market clearing requires that $\int_0^n h_t(i)di = nl_t$.

problem are

$$\tilde{P}_{ht} = \frac{\theta}{\theta-1}\frac{E_t\sum_{\tau=t}^{\infty}(\varphi\beta)^{\tau-t}\Lambda_{t,\tau}MC_\tau\tilde{y}_{h\tau}}{E_t\sum_{\tau=t}^{\infty}(\varphi\beta)^{\tau-t}\Lambda_{t,\tau}\tilde{y}_{h\tau}}, \tag{14}$$

$$\tilde{P}_{ht}^{*} = \frac{\theta}{\theta-1}\frac{E_t\sum_{\tau=t}^{\infty}(\varphi\beta)^{\tau-t}\Lambda_{t,\tau}MC_\tau\tilde{y}_{h\tau}^{*}}{E_t\sum_{\tau=t}^{\infty}(\varphi\beta)^{\tau-t}\Lambda_{t,\tau}e_\tau\tilde{y}_{h\tau}^{*}}. \tag{15}$$

Assuming that price changes are independent across firms, the law of large numbers implies that $1-\varphi$ is also the proportion of firms that set a new price each period. The proportion of firms that set a new price at time τ and have not changed it as of time t (for $\tau \leq t$), is given by the probability that a time-τ price is still in effect in period t. It is easy to show that this probability is $\varphi^{t-\tau}(1-\varphi)$. It follows that P_{ht} and P_{ht}^{*} can be written respectively as

$$P_{ht} = \left((1-\varphi)\sum_{\tau=-\infty}^{t}\varphi^{t-\tau}\tilde{P}_{h\tau}^{1-\theta}\right)^{\frac{1}{1-\theta}}, \tag{16}$$

$$P_{ht}^{*} = \left((1-\varphi)\sum_{\tau=-\infty}^{t}\varphi^{t-\tau}\tilde{P}_{h\tau}^{*1-\theta}\right)^{\frac{1}{1-\theta}}. \tag{17}$$

2.4 The government

The government represents both the fiscal and monetary authorities in each country. There is no government spending or investment. Each period, the government makes lump-sum transfers to households. Transfers are financed by printing additional money in each period. Thus, the government budget constraint in the home country is

$$T_t = M_t - M_{t-1}. \tag{18}$$

Money is supplied exogenously by the government according to $M_t = \mu_t M_{t-1}$, where μ_t is the gross rate of money growth. In real terms, this process implies

$$m_t\pi_t = \mu_t m_{t-1}. \tag{19}$$

The rate of money growth, μ_t, is assumed to be a non-autocorrelated shock that follows the process

$$\ln\mu_t = \ln\mu + \epsilon_t, \tag{20}$$

where μ is the steady-state rate of money growth, and ϵ_t is a normally distributed zero-mean disturbance.

3. Analytical Results

To solve the model, I follow the usual strategy of considering an approximate solution in the neighborhood of the steady state. I do so by log-linearizing the equilibrium conditions around a zero-shock initial steady state in which all variables are constant. The steady state corresponds to a symmetric flexible-price equilibrium. From the log-linearized version of the model, it is easy to show that the real exchange rate (expressed as a percentage deviation from its steady-state value) is fully determined by the following three-equation system (see Appendix A for the derivation):

$$\hat{m}_t - \hat{m}_t^* = \hat{m}_{t-1} - \hat{m}_{t-1}^* - (\hat{\pi}_t - \hat{\pi}_t^*) + \hat{\mu}_t - \hat{\mu}_t^*, \tag{21}$$

$$E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^*\right) = \frac{1}{\beta}(\hat{\pi}_t - \hat{\pi}_t^*) - \frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\hat{q}_t, \tag{22}$$

$$E_t\hat{q}_{t+1} = \frac{1}{\beta}\hat{q}_t - E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^*\right) - \frac{1-\beta}{\beta}(\hat{m}_t - \hat{m}_t^*), \tag{23}$$

where the circumflex denotes the percentage deviation of a variable from its steady-state value $[\hat{x}_t = (x_t - x)/x]$. The log-linearized model (21)–(23) can be written as

$$E_t\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}z_t, \tag{24}$$

where $\mathbf{x}_t = (\hat{m}_{t-1} - \hat{m}_{t-1}^*, \hat{\pi}_t - \hat{\pi}_t^*, \hat{q}_t)'$, $z_t = \hat{\mu}_t - \hat{\mu}_t^*$, and $\mathbf{A}$ and $\mathbf{B}$ are respectively, a 3×3 matrix and a 3×1 vector given by

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{\beta} & -\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta} \\ -\frac{1-\beta}{\beta} & -1 & \frac{1-\varphi\beta(1-\varphi)}{\varphi\beta} \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1-\beta}{\beta} \end{bmatrix}$$

Blanchard and Kahn (1980) derive the necessary and sufficient condition for the existence of a unique stable solution to linear dynamic rational-expectation models similar to (24). This condition requires that the number of eigenvalues of the matrix $\mathbf{A}$ outside the unit circle be equal to the number of non-predetermined variables. This saddle-point property guarantees the existence of unique initial values for the non-predetermined variables such that the model always converges to a stationary equilibrium following a shock.[6] Of the three variables included in the vector $\mathbf{x}_t$, one is predetermined ($\hat{m}_{t-1} - \hat{m}^*_{t-1}$) and two are forward-looking ($\hat{\pi}_t - \hat{\pi}^*_t$ and $\hat{q}_t$). Hence, for the model (24) to possess a unique stable solution, it must have one stable and two unstable roots.

Proposition 1 *The eigenvalues of the matrix* $\mathbf{A}$ *are* φ, $1/\beta$, *and* $1/\varphi\beta$.

Proof.

$$det(\mathbf{A} - \delta\mathbf{I}) = \begin{vmatrix} 1-\delta & -1 & 0 \\ 0 & \frac{1}{\beta}-\delta & -\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta} \\ -\frac{1-\beta}{\beta} & -1 & \frac{1-\varphi\beta(1-\varphi)}{\varphi\beta}-\delta \end{vmatrix}$$

$$= (1-\delta)\begin{vmatrix} \frac{1}{\beta}-\delta & -\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta} \\ -1 & \frac{1-\varphi\beta(1-\varphi)}{\varphi\beta}-\delta \end{vmatrix} + \begin{vmatrix} 0 & -\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta} \\ -\frac{1-\beta}{\beta} & \frac{1-\varphi\beta(1-\varphi)}{\varphi\beta}-\delta \end{vmatrix}$$

$$= (1-\delta)\left[\left(\frac{1}{\beta}-\delta\right)\left(\frac{1-\varphi\beta(1-\varphi)}{\varphi\beta}-\delta\right) - \left(\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\right)\right] - \left(\frac{1-\beta}{\beta}\right)\left(\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\right)$$

$$= \left(\delta^2 - \frac{1+\varphi^2\beta}{\varphi\beta}\delta + \frac{1}{\beta}\right)\left(\frac{1}{\beta}-\delta\right)$$

$$= (\delta-\varphi)\left(\delta - \frac{1}{\varphi\beta}\right)\left(\frac{1}{\beta}-\delta\right).$$

The roots of the characteristic equation $det(\mathbf{A} - \delta\mathbf{I}) = \mathbf{0}$ are $\delta = \varphi$, $\delta = 1/\beta$, or $\delta = 1/\varphi\beta$, which are therefore the eigenvalues of $\mathbf{A}$. ■

The matrix $\mathbf{A}$ has two eigenvalues outside the unit circle ($1/\beta$, and $1/\varphi\beta$) and one eigenvalue of modulus less than 1 (φ). Thus, a unique and stable solution to (24) always exists for any sensible values of the parameters φ and β. The magnitude of the stable eigenvalue (φ) characterizes the speed of convergence of the model's variables to their steady-state values, following a structural perturbation. In particular, the following result holds:

Proposition 2 *The deviation of the real exchange rate from its steady-state value evolves according to the following process:*

$$\hat{q}_t = \varphi\hat{q}_{t-1} + \varphi z_t.$$

[6] The Blanchard-Kahn condition rules out the possibility of explosive solutions (which arise in the case where the number of eigenvalues greater than 1 exceed the number of non-predetermined variables) and sunspot equilibria (which arise in the case where the number of eigenvalues greater than 1 falls short of the number of non-predetermined variables).

Proof.

Equation (21) implies that

$$E_t\left(\hat{\pi}_{t+1}-\hat{\pi}^*_{t+1}\right)=\hat{m}_t-\hat{m}^*_t-E_t(\hat{m}_{t+1}-\hat{m}^*_{t+1}). \tag{25}$$

Using this equation to substitute for $E_t\left(\hat{\pi}_{t+1}-\hat{\pi}^*_{t+1}\right)$ in equation (23), gives

$$E_t\hat{q}_{t+1}=\frac{1}{\beta}\hat{q}_t+E_t(\hat{m}_{t+1}-\hat{m}^*_{t+1})-\frac{1}{\beta}\left(\hat{m}_t-\hat{m}^*_t\right).$$

The solution to this first-order stochastic difference equation is

$$\hat{q}_t=\hat{m}_t-\hat{m}^*_t. \tag{26}$$

Substituting for $\hat{\pi}_t-\hat{\pi}^*_t$ and $E_t\left(\hat{\pi}_{t+1}-\hat{\pi}^*_{t+1}\right)$ in (22) using (21), (25), and (26), yields the following second-order stochastic difference equation:

$$E_t\hat{q}_{t+1}-\left(\frac{1+\beta}{\beta}+\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\right)\hat{q}_t+\frac{1}{\beta}\hat{q}_{t-1}=-\frac{1}{\beta}z_t. \tag{27}$$

Using standard techniques to solve (27), yields

$$\hat{q}_t=a\hat{q}_{t-1}+az_t,$$

where a is the root of absolute value less than 1 that solves the characteristic equation: $a^2-\left(\frac{1+\beta}{\beta}+\frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\right)a+\frac{1}{\beta}=0$. It is easy to show that this root is

$$a=\varphi. \blacksquare$$

Proposition 2 shows that the parameter φ characterizes both the magnitude and the persistence of the real exchange rate response to a relative money growth shock. The higher φ is, the stronger the shock's initial effect and the more persistent the real exchange rate response will be. More importantly, proposition 2 states that φ is precisely the first-order autocorrelation coefficient of the real exchange rate.[7] Hence, real exchange rate persistence is always pinned down by the probability of not changing prices, so that there is no scope for endogenous persistence in this model. Therefore, one cannot replicate the persistence found in real exchange rate movements using a short period of price stickiness. This result parallels the so-called persistence problem raised by Chari, Kehoe, and McGrattan (2000)

[7] In fact, it is easy to show that φ is also the first-order autocorrelation coefficient of the inflation differential $(\hat{\pi}_t-\hat{\pi}^*_t)$ and the relative real money stock $(\hat{m}_t-\hat{m}^*_t)$.

who show that, in a closed-economy set-up, standard DGE models fail to generate output effects of monetary shocks beyond the exogenous length of price contracts, even if contracts are set in a staggered fashion. Chari, Kehoe, and McGrattan conclude that the contract multiplier in standard DGE models is essentially 1.[8] The present study shows that the persistence problem extends to open-economy models. By demonstrating that real exchange rate deviations from the steady state decay at a rate equal to the (exogenous) probability of not changing prices, proposition 2 implies that the contract multiplier in this model is exactly equal to 1.

4. Robustness

In this section, I assess the sensitivity of my results regarding real exchange rate persistence to alternative specifications of the utility function and the money-growth shock process.

4.1 The utility function

The virtue of the functional form (1) is that it yields a closed-form solution to the model. However, the logarithmic specification (with respect to real money balances) has an important drawback: it implies an interest elasticity of money demand equal to unity. This is obviously a counterfactual feature, as empirical estimates of such elasticity range from 0.05 in Mankiw and Summers (1986) to 0.39 in Chari, Kehoe, and McGrattan (2000). To allow for plausible values of the interest elasticity of money demand, I consider the following instantaneous utility function:

$$u(c_t, l_t, m_t) = v(c_t, l_t) + \frac{\gamma}{1-\eta} m_t^{1-\eta},$$

where the parameter $\eta > 0$ can be interpreted as the inverse of the interest elasticity of money demand. In the general case where η is different from unity, the eigenvalues of the matrix $\mathbf{A}$ cannot be computed analytically and must, therefore, be evaluated numerically. Table 1 shows the "stable" eigenvalue of the matrix $\mathbf{A}$ (which is also the first-order autocorrelation of the model's variables) computed for different values of η ranging from 1

[8] Chari, Kehoe, and McGrattan (2000) define the contract multiplier as the ratio of the half-life of output to one-half the length of exogenous stickiness. In their model, price rigidity is introduced via Taylor-type staggered contracts, thereby implying that prices fully adjust after the duration of price contracts. With Calvo-type rigidity, however, the aggregate price level, never fully adjusts (by construction). An appropriate definition of the contract multiplier in this case (which shall be used in this paper) would be the ratio of the half-life of a given variable to the half-life of price stickiness.

to 20 (implying an interest elasticity of money demand ranging from 1 to 0.05). In this experiment, the parameters β and φ are set to 0.99 and 0.75, respectively.

Table 1. The Stable Eigenvalue of A for Different Values of the Parameter η ($\beta = 0.99$ and $\varphi = 0.75$)

η	Stable eigenvalue
1	0.75
2	0.7459
3	0.7420
4	0.7383
5	0.7347
6	0.7313
7	0.7281
8	0.7249
9	0.7219
10	0.7190
11	0.7162
12	0.7134
13	0.7108
14	0.7082
15	0.7052
16	0.7032
17	0.7008
18	0.6985
19	0.6962
20	0.6940

Table 1 shows that a plausible parameterization of η does not overturn the lack of persistence generated by the model. In fact, the magnitude of the stable root proves to be a decreasing function of η, meaning that the maximum level of real exchange rate persistence is obtained when $\eta = 1$.

4.2 The shock process

As shown earlier, with non-autocorrelated money-growth shocks, the first-order autocorrelation of the real exchange rate is explicitly determined and is equal to the stable eigenvalue of the matrix **A**. When those shocks are serially correlated, however, the first-order autocorrelation of the real exchange rate is no longer pinned down by the magnitude of the stable eigenvalue. To assess the sensitivity of the results to the persistence of money-growth shocks, I assume that the rate of money growth, μ_t, follows a first-order autoregressive process given by

$$\ln \mu_t = (1-\rho)\ln \mu + \rho \ln \mu_{t-1} + \epsilon_t,$$

where ρ is strictly bounded between -1 and 1, and μ and ϵ_t are defined as in (20). The first-order autocorrelation coefficient, ρ, is assumed to be the same for both countries. Under this assumption, the relative shock, z_t, follows a first-order autoregressive process given by

$$z_t = \rho z_{t-1} + \epsilon_t - \epsilon_t^*. \tag{28}$$

In this case, the deviation of the real exchange rate from its steady-state value evolves according to

$$\hat{q}_t = \varphi \hat{q}_{t-1} + \phi z_t + \omega z_{t-1},$$

where $\phi = \frac{\varphi(1-\beta\rho+\beta^2\rho-\varphi\beta^2\rho)}{(1-\beta\rho)(1-\varphi\beta\rho)}$, and $\omega = -\frac{\varphi\beta\rho}{1-\beta\rho}$.

Proof.

From equation (21), I obtain

$$E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^*\right) = \hat{m}_t - \hat{m}_t^* - E_t(\hat{m}_{t+1} - \hat{m}_{t+1}^*) + \rho z_t. \tag{29}$$

Using this equation to substitute for $E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^*\right)$ in (23), gives

$$E_t\hat{q}_{t+1} = \frac{1}{\beta}\hat{q}_t + E_t(\hat{m}_{t+1} - \hat{m}_{t+1}^*) - \frac{1}{\beta}\left(\hat{m}_t - \hat{m}_t^*\right) - \rho z_t.$$

The solution to this equation is

$$\begin{aligned} \hat{q}_t &= \hat{m}_t - \hat{m}_t^* + \beta\rho\sum_{i=0}^{\infty}\beta^i z_{t+i} \\ &= \hat{m}_t - \hat{m}_t^* + \frac{\beta\rho}{1-\beta\rho}z_t. \end{aligned} \tag{30}$$

Using (21), (29), and (30) to substitute for $\hat{\pi}_t - \hat{\pi}_t^*$ and $E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^*\right)$ in (22), yields

$$E_t\hat{q}_{t+1} - \left(\frac{1+\beta}{\beta} + \frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\right)\hat{q}_t + \frac{1}{\beta}\hat{q}_{t-1} = -\left(\frac{1-\beta\rho}{\beta} + \frac{\rho(1-\beta\rho+\beta)}{1-\beta\rho}\right)z_t + \frac{\rho}{1-\beta\rho}z_{t-1}.$$

The solution to this equation is

$$\hat{q}_t = \varphi\hat{q}_{t-1} + \varphi\beta\left(\frac{1-\beta\rho}{\beta} + \frac{\rho(1-\beta\rho+\beta)}{1-\beta\rho}\right)\sum_{i=0}^{\infty}(\varphi\beta)^i E_t z_{t+i} - \varphi\beta\frac{\rho}{1-\beta\rho}\sum_{i=0}^{\infty}(\varphi\beta)^i E_t z_{t-1+i}. \tag{31}$$

Finally, noting that $\sum_{i=0}^{\infty}(\varphi\beta)^i E_t z_{t-1+i} = z_{t-1} + \sum_{i=1}^{\infty}(\varphi\beta)^i E_t z_{t+i}$, and using (28), equation (31) can be written as

$$\hat{q}_t = \varphi\hat{q}_{t-1} + \frac{\varphi(1-\beta\rho+\beta^2\rho-\varphi\beta^2\rho)}{(1-\beta\rho)(1-\varphi\beta\rho)}z_t - \frac{\varphi\beta\rho}{1-\beta\rho}z_{t-1}. \blacksquare$$

The parameter φ still characterizes the volatility and the persistence of the real exchange rate, but it is no longer interpreted as a first-order autocorrelation coefficient. Table 2 shows the serial correlation of the real exchange rate for different values of the parameter ρ.[9] The parameters β and φ are calibrated as in subsection 4.1. The table shows that even when the relative money-growth shock is highly autocorrelated, real exchange rate persistence is virtually unaffected, and therefore the contract multiplier remains essentially 1.

Table 2. First-Order Autocorrelation of the Real Exchange Rate for Different Values of the Parameter ρ ($\beta = 0.99$ and $\varphi = 0.75$)

ρ	Autocorrelation
0	0.75
0.1	0.7505
0.2	0.7510
0.3	0.7515
0.4	0.7521
0.5	0.7528
0.6	0.7535
0.7	0.7541
0.8	0.7545
0.9	0.7540

5. Conclusion

This paper has constructed a tractable DGE two-country sticky-price model whose dynamics can be characterized analytically. Real exchange rate fluctuations generated by the model

[9] Because money-growth shocks are usually found to be positively autocorrelated in the data, I only focus on positive values of ρ.

do not last beyond the average duration of price contracts. More precisely, the contract multiplier delivered by the model is shown to be exactly equal to 1. This outcome is reminiscent of the persistence problem found by Chari, Kehoe, and McGrattan (2000) in a closed-economy set-up.

The failure of sticky-price models to generate persistent real exchange rates suggests that price rigidities might not be the only source driving real exchange rate movements. A similar conclusion is reached by Ng (2003) who uses a VAR-based approach to empirically assess the importance of sticky prices in explaining real exchange rate fluctuations. Her results show that U.S. sticky-price shocks have short half-lives and cannot account for the observed real exchange rate persistence, despite the fact that they explain most of real exchange rate variability.

In view of these results, future research on real exchange rate determination should focus on developing models where real shocks, such as technology and fiscal shocks, affect the real exchange rate in non-trivial ways. Such shocks are likely to be highly persistent and thus, are plausible candidates to account for real exchange rate persistence.

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Appendix A: Derivation of Equations (21)–(23)

Derivation of equation (21)

Equation (19) is approximated as

$$\hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t + \hat{\mu}_t. \tag{A.2}$$

Taking the difference between (A.2) and its foreign counterpart results in

$$\hat{m}_t - \hat{m}_t^* = \hat{m}_{t-1} - \hat{m}_{t-1}^* - (\hat{\pi}_t - \hat{\pi}_t^*) + \hat{\mu}_t - \hat{\mu}_t^*,$$

which is equation (21) in the main text.

Derivation of equation (22)

Dividing both sides of equation (14) by P_t and using the fact that $MC_\tau / P_t = \left(\Pi_{k=t+1}^{\tau} \pi_k\right) mc_\tau$, I obtain

$$\tilde{p}_{ht} = \frac{\theta}{\theta - 1} \frac{E_t \sum_{\tau=t}^{\infty} (\varphi\beta)^{\tau - t} \Lambda_{t,\tau} \left(\Pi_{k=t+1}^{\tau} \pi_k\right) mc_\tau \tilde{y}_{h\tau}}{E_t \sum_{\tau=t}^{\infty} (\varphi\beta)^{\tau - t} \Lambda_{t,\tau} \tilde{y}_{h\tau}},$$

where $\tilde{p}_{ht} = \tilde{P}_{ht}/P_t$. This equation can be approximated as

$$\hat{\tilde{p}}_{ht} = (1 - \varphi\beta)\, E_t \sum_{\tau=t}^{\infty} (\varphi\beta)^{\tau - t} \left(\hat{mc}_\tau + \sum\nolimits_{k=t+1}^{\tau} \hat{\pi}_k\right),$$

which can be rewritten in the following recursive form:

$$\hat{\tilde{p}}_{ht} - \varphi\beta E_t \hat{\tilde{p}}_{ht+1} = (1 - \varphi\beta)\hat{mc}_t + \varphi\beta E_t \hat{\pi}_{t+1}. \tag{A.3}$$

Similarly, dividing both sides of equation (15) by P_t^* and using the fact that $e_\tau P_t^*/P_t = \left(\Pi_{k=t+1}^{\tau} \pi_k\right) \left(\Pi_{k=t+1}^{\tau} \pi_k^*\right)^{-1} q_\tau$, yields

$$\tilde{p}_{ht}^* = \frac{\theta}{\theta - 1} \frac{E_t \sum_{\tau=t}^{\infty} (\varphi\beta)^{\tau - t} \Lambda_{t,\tau} \left(\Pi_{k=t+1}^{\tau} \pi_k\right) mc_\tau \tilde{y}_{h\tau}^*}{E_t \sum_{\tau=t}^{\infty} (\varphi\beta)^{\tau - t} \Lambda_{t,\tau} \left(\Pi_{k=t+1}^{\tau} \pi_k\right) \left(\Pi_{k=t+1}^{\tau} \pi_k^*\right)^{-1} q_\tau \tilde{y}_{h\tau}^*}, \tag{A.4}$$

where $\tilde{p}_{ht}^* = \tilde{P}_{ht}^*/P_t^*$. Following the same steps involved in obtaining equation (A.3), it is easy to show that the approximation of equation (A.4) can be written as

$$\hat{\tilde{p}}_{ht}^* - \varphi\beta E_t \hat{\tilde{p}}_{ht+1}^* = (1 - \varphi\beta) \left(\hat{mc}_t - \hat{q}_t\right) + \varphi\beta E_t \hat{\pi}_{t+1}^*. \tag{A.5}$$

By analogy to (A.3) and (A.5), the pricing decisions by the foreign monopolistic firm are approximated by

$$\hat{\tilde{p}}_{ft} - \varphi\beta E_t \hat{\tilde{p}}_{ft+1} = (1-\varphi\beta)\left(\hat{mc}_t^* + \hat{q}_t\right) + \varphi\beta E_t \hat{\pi}_{t+1}, \tag{A.6}$$

and

$$\hat{\tilde{p}}_{ft}^* - \varphi\beta E_t \hat{\tilde{p}}_{ft+1}^* = (1-\varphi\beta)\hat{mc}_t^* + \varphi\beta E_t \hat{\pi}_{t+1}^*, \tag{A.7}$$

where $\tilde{p}_{ft} = \tilde{P}_{ft}/P_t$ and $\tilde{p}_{ft}^* = \tilde{P}_{ft}^*/P_t^*$.
Using equations (11), (16), and its foreign counterpart for P_{ft}, I obtain

$$P_t = (1-\varphi)\sum_{\tau=-\infty}^{t} \varphi^{t-\tau}\left[n\tilde{P}_{h\tau}^{1-\theta} + (1-n)\tilde{P}_{f\tau}^{1-\theta}\right]. \tag{A.8}$$

Dividing both sides of equation (A.8) by P_t results in

$$1 = (1-\varphi)\sum_{\tau=-\infty}^{t} \varphi^{t-\tau}\left[n\tilde{p}_{h\tau}\left(\Pi_{k=\tau+1}^{t}\pi_k\right)^{-1} + (1-n)\tilde{p}_{f\tau}\left(\Pi_{k=\tau+1}^{t}\pi_k\right)^{-1}\right].$$

The linearization of this equation yields

$$0 = (1-\varphi)\sum_{\tau=-\infty}^{t} \varphi^{t-\tau}\left[n\left(\hat{\tilde{p}}_{h\tau} - \sum\nolimits_{k=\tau+1}^{t}\hat{\pi}_k\right) + (1-n)\left(\hat{\tilde{p}}_{f\tau} - \sum\nolimits_{k=\tau+1}^{t}\hat{\pi}_k\right)\right],$$

or

$$\hat{\pi}_t = (1-\varphi)\sum_{\tau=-\infty}^{t} \varphi^{t-\tau-1}\left[n\left(\hat{\tilde{p}}_{h\tau} - \sum\nolimits_{k=\tau+1}^{t-1}\hat{\pi}_k\right) + (1-n)\left(\hat{\tilde{p}}_{f\tau} - \sum\nolimits_{k=\tau+1}^{t-1}\hat{\pi}_k\right)\right].$$

Subtracting $\varphi\hat{\pi}_t$ from both sides of this equation gives

$$\hat{\pi}_t = \frac{1-\varphi}{\varphi}\left[n\hat{\tilde{p}}_{ht} + (1-n)\hat{\tilde{p}}_{ft}\right]. \tag{A.9}$$

The foreign counterpart of equation (A.9) is

$$\hat{\pi}_t^* = \frac{1-\varphi}{\varphi}\left[n\hat{\tilde{p}}_{ht}^* + (1-n)\hat{\tilde{p}}_{ft}^*\right]. \tag{A.10}$$

Substituting (A.3) and (A.6) into (A.9) and rearranging, I obtain

$$E_t\hat{\pi}_{t+1} = \frac{1}{\beta}\hat{\pi}_t - \frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\left[n\hat{mc}_t + (1-n)(\hat{mc}_t^* + \hat{q}_t)\right]. \tag{A.11}$$

Similarly, equation (A.10), with (A.5) and (A.7) substituted in for $\hat{\tilde{p}}^*_{ht}$ and $\hat{\tilde{p}}^*_{ft}$, becomes

$$E_t \hat{\pi}^*_{t+1} = \frac{1}{\beta}\hat{\pi}^*_t - \frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\left[n(\hat{mc}_t - \hat{q}_t) + (1-n)\hat{mc}^*_t\right]. \tag{A.12}$$

Finally, subtracting (A.12) from (A.11) yields

$$E_t\left(\hat{\pi}_{t+1} - \hat{\pi}^*_{t+1}\right) = \frac{1}{\beta}\left(\hat{\pi}_t - \hat{\pi}^*_t\right) - \frac{(1-\varphi)(1-\varphi\beta)}{\varphi\beta}\hat{q}_t,$$

which is equation (22) in the main text.

Derivation of equation (23)

Linearizing the first-order condition (5) and its foreign counterpart (with (7) substituted in for $\hat{v}^*_{ct}$ and $E_t\hat{v}^*_{ct+1}$) yields, respectively,

$$\hat{m}_t = \frac{\beta}{1-\beta}(E_t\hat{v}_{ct+1} - E_t\hat{\pi}_{t+1}) - \frac{1}{1-\beta}\hat{v}_{ct}, \tag{A.13}$$

and

$$\hat{m}^*_t = \frac{\beta}{1-\beta}(E\hat{v}_{ct+1} + E_t\hat{q}_{t+1} - E_t\hat{\pi}^*_{t+1}) - \frac{1}{1-\beta}(\hat{v}_{ct} + \hat{q}_t). \tag{A.14}$$

Subtracting (A.13) from (A.14) and rearranging, I obtain

$$E_t\hat{q}_{t+1} = \frac{1}{\beta}\hat{q}_t - E_t\left(\hat{\pi}_{t+1} - \hat{\pi}^*_{t+1}\right) - \frac{1-\beta}{\beta}\left(\hat{m}_t - \hat{m}^*_t\right),$$

which is equation (23) in the main text.