This document contains the scanned handwritten results left by Professor Serge Tardif upon his death on September 26, 1998. These results consist of statements with proofs, but with little or no motivation. At Professor Tardif's request, Bellavance and van Eeden used these results to produce the article *A nonparametric procedure for the analysis of balanced crossover designs* published in *The Canadian Journal of Statistics.* Tardif's results are for the case where p is even and ≥ 4.

The document also contains some lengthy proofs that are not included in the Appendix of the article. Reference to an equation number in the proofs presented in this document refer to the corresponding equation number in the paper.

Dispositif expérimental pour M pair :

- on se donne un carré latin équilibré $d = ((d(j,k)))_{\substack{1\le j\le M\\1\le k\le M}}$

- on choisit au hasard et avec remise n permutations $\pi_1, \dots, \pi_n \in \mathcal{R}_M$

- à la $i^{ème}$ réplique, $1\le i\le n$, on utilise le carré latin $d_i = \pi_i \circ d$

(c.-à-d. on utilise le carré latin d dont on a permuté les symboles selon π_i)

Désignons par b le carré latin dont les lignes sont les permutations inverses de d
— b_i ——— d_i, $1\le i\le n$

et posons $s_i = \pi_i^{-1}$

on a $b_i = b\circ\pi_i^{-1} = b\circ s_i = \begin{bmatrix} b(1,s_{i1}) & \dots & b(1,s_{iM}) \\ \vdots & \ddots & \vdots \\ b(M,s_{i1}) & \dots & b(M,s_{iM}) \end{bmatrix}$.

Considérons

$$\hat\theta_{i\ell} = \alpha\sum_{j=1}^{M} Y_{ij\,b(j,s_{i\ell})} + \beta\sum_{\{j:\,b(j,s_{i\ell})\ne M\}} Y_{ij,\,b(j,s_{i\ell})+1}\quad, 1\le \ell\le M,$$

supposons que $H_0^\theta : \theta_1 = \dots = \theta_M = 0$ soit exacte ;
π_i ayant été choisie au hasard, s_i peut être considérée comme ayant été choisie au hasard et ceci induit la distribution de permutation :

$$P\Big[(\hat\theta_{i1},\dots,\hat\theta_{iM}) = \Big(\alpha\sum_j Y_{ij\,b(j,s_{i1})} + \beta\sum_{j\neq} Y_{ij,\,b(j,s_{i1})+1}, \dots, \alpha\sum_j Y_{ij\,b(j,s_{iM})} + \beta\sum_{j\neq} Y_{ij,\,b(j,s_{iM})+1}\Big)\Big] = \frac{1}{M!},$$

$s_i \in \mathcal{R}_M$.

manifestement, $(\hat\theta_{i1},\dots,\hat\theta_{iM}) \overset{\mathcal{L}_P}{=} (\hat\theta_{is_1},\dots,\hat\theta_{is_M})\quad \forall\, s = (s_1,\dots,s_M)\in\mathcal{R}_M$

marginalement

$$\mathbb{P}\left[\hat{\theta}_{i\ell} = \alpha \sum_{j=1}^{q} Y_{ij b \ell(j,h)} + \beta \sum_{j \in b(j,h) \neq M} Y_{ij b(j,h)+1}\right] = \frac{1}{M}, \quad 1 \le h \le M, \text{ pour chaque } \ell = 1, \dots, M$$

Puisque $\hat{\theta}_{i1}, \dots, \hat{\theta}_{iM}$ sont interchangeables sous $\mathbb{P}$ (qui n'a de sens que si H_0^B est exacte) et que $\sum_{\ell=1}^{M} \hat{\theta}_{i\ell} = 0$, on a :

$$E_{\mathbb{P}}(\hat{\theta}_{i\ell}) = 0 \qquad \text{et} \quad \mathrm{cov}_{\mathbb{P}}(\hat{\theta}_{i\ell}, \hat{\theta}_{i\ell'}) = \frac{M}{M-1}\left(\delta_{\ell\ell'} - \frac{1}{M}\right) \mathrm{var}_{\mathbb{P}}(\hat{\theta}_{i1}), \quad 1 \le \ell, \ell' \le M.$$

Or, d'une part, $\mathrm{var}_{\mathbb{P}}(\hat{\theta}_{i\ell}) = E_{\mathbb{P}}(\hat{\theta}_{i\ell}^2) = \frac{1}{M}\sum_{h=1}^{M}\left\{\alpha \sum_j Y_{ijb\ell(j,h)} + \beta \sum_{\neq M} Y_{ij b(j,h)+1}\right\}^2$ et, d'autre part,

$$\sum_{\ell=1}^{M} \hat{\theta}_{i\ell}^2 = \sum_{\ell=1}^{M}\left\{\alpha \sum_j Y_{ijb\ell(j,h_{i\ell})} + \beta \sum_{\neq M} Y_{ijb(j,h_{i\ell})+1}\right\}^2 = \sum_{h=1}^{M}\left\{\alpha \sum_j Y_{ijb(j,h)} + \beta \sum_{\neq M} Y_{ij b(j,h)+1}\right\}^2.$$

On peut donc écrire $\mathrm{var}_{\mathbb{P}}(\hat{\theta}_{i\ell}) = \frac{1}{M}\sum_{\ell=1}^{M} \hat{\theta}_{i\ell}^2$ et, en définitive, on a :

$$E_{\mathbb{P}}(\hat{\theta}_{i\ell}) = 0 \qquad \text{et} \quad \mathrm{cov}_{\mathbb{P}}(\hat{\theta}_{i\ell}, \hat{\theta}_{i\ell'}) = \frac{1}{M-1}\left(\delta_{\ell\ell'} - \frac{1}{M}\right) \sum_{\ell''=1}^{M} \hat{\theta}_{i\ell''}^2, \quad 1 \le \ell, \ell' \le M$$

Considérons maintenant le vecteur $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_{M-1})$, où $\hat{\theta}_\ell = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_{i\ell}$, $1 \le \ell \le M$

on a :

$$E_{\mathbb{P}}(\hat{\theta}_\ell) = 0 \qquad \text{et} \quad \mathrm{cov}_{\mathbb{P}}(\hat{\theta}_\ell, \hat{\theta}_{\ell'}) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{cov}_{\mathbb{P}}(\hat{\theta}_{i\ell}, \hat{\theta}_{i\ell'}) = \frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{M-1}\sum_{\ell''=1}^{M} \hat{\theta}_{i\ell''}^2 \left(\delta_{\ell\ell'} - \frac{1}{M}\right)$$

d'où

$$E_{\mathbb{P}}(\hat{\theta}) = 0 \qquad \text{et} \quad \mathrm{cov}_{\mathbb{P}}(\hat{\theta}) = \frac{1}{n^2}\sum_{i=1}^{n} \left(\frac{1}{M-1}\right) \sum_{\ell''=1}^{M} \hat{\theta}_{i\ell''}^2 \left(I_{M-1} - \frac{1}{M} 1_{M-1} 1'_{M-1}\right)$$

$$\Rightarrow \left[\mathrm{cov}_{\mathbb{P}}(\hat{\theta})\right]^{-1} = \frac{1}{\frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{M-1}\sum_{\ell''=1}^{M} \hat{\theta}_{i\ell''}^2} \left(I_{M-1} + 1_{M-1} 1'_{M-1}\right)$$

alors $Q_0 = [\hat{\theta} - E_{\mathbb{P}}(\hat{\theta})]' [\mathrm{cov}_{\mathbb{P}}(\hat{\theta})]^{-1} [\hat{\theta} - E_{\mathbb{P}}(\hat{\theta})]$

$$= \frac{1}{\frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{M-1}\sum_{\ell=1}^{M} \hat{\theta}_{i\ell}^2} \hat{\theta}' (I_{M-1} + 1_{M-1} 1'_{M-1}) \hat{\theta} \quad = \frac{1}{\frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{M-1}\sum_{\ell=1}^{M} \hat{\theta}_{i\ell}^2} \left\{\sum_{\ell=1}^{M-1} \hat{\theta}_\ell^2 + \Big(\underbrace{\sum_{\ell=1}^{M-1} \hat{\theta}_\ell}_{= -\hat{\theta}_M}\Big)^2\right\}$$

c.-à-d.

$$Q_\theta = (M-1)n \times \frac{\sum_{\ell=1}^{M} \hat{\theta}_\ell^2}{\frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=1}^{M}\hat{\theta}_{i\ell}^2}$$

est la statistique du test de permutation (paramétrique) à considérer.

Trouvons sa version asymptotique inconditionnelle.

Puisque les variables $\sum_{\ell=1}^{M}\hat{\theta}_{i\ell}^2$, $i=1,2,\ldots$, sont i.i.d., le dénominateur de Q_θ converge en probabilité (sous H_0^θ) vers $E\left(\sum_{\ell=1}^{M}\hat{\theta}_{1\ell}^2\right) = \sum_{\ell=1}^{M}E(\hat{\theta}_{1\ell}^2)$.

Trouvons donc $E(\hat{\theta}_{1\ell}^2) = \text{var}(\hat{\theta}_{1\ell})$ (car, sous H_0^θ, $E(\hat{\theta}_{1\ell}) = 0$).

A cet effet, laissons tomber l'indice $i=1$, remplaçons le nombre $r_{1\ell}$ par s et désignons par $j(s)$ l'indice tel que $l(j(s), s) = M$.

On a $\text{var}(\hat{\theta}_{1\ell}) = \text{var}\left\{\alpha \sum_j Y_{j\,l(j,s)} + \beta \sum_{j\neq j(s)} Y_{j,\,l(j,s)+1}\right\}$

$$= \underbrace{\alpha^2 \text{var}\left\{\sum_j Y_{j\,l(j,s)}\right\}}_{①} + \underbrace{2\alpha\beta\,\text{cov}\left\{\sum_j Y_{j\,l(j,s)}, \sum_{j\neq j(s)} Y_{j,\,l(j,s)+1}\right\}}_{②} + \underbrace{\beta^2 \text{var}\left\{\sum_{j\neq j(s)} Y_{j,\,l(j,s)+1}\right\}}_{③}$$

Si on se rappelle que, peu importe j, j', k et k', $\text{cov}(Y_{jk}, Y_{j'k'}) = \left(\delta_{jj'} - \frac{1}{M}\right)\left(\delta_{kk'} - \frac{1}{M}\right)\sigma^2(1-\rho_1-\rho_2+\rho_3)$

$$① = \text{var}\left\{\sum_j Y_{j\,l(j,s)}\right\} = \sum_j \text{var}(Y_{j\,l(j,s)}) + \sum_j\sum_{j'\neq j} \text{cov}(Y_{j\,l(j,s)}, Y_{j'\,l(j',s)})$$

$$= M\frac{(M-1)^2}{M^2}\sigma^2(1-\rho_1-\rho_2+\rho_3) + M(M-1)\frac{1}{M^2}\sigma^2(1-\rho_1-\rho_2+\rho_3)$$

$$= (M-1)\sigma^2(1-\rho_1-\rho_2+\rho_3)$$

$$② = \text{cov}\left\{\sum_j Y_{j\,l(j,s)}, \sum_{j'\neq j(s)} Y_{j',\,l(j',s)+1}\right\}$$

$$= \text{cov}\left\{Y_{j(s)M}, \sum_{j'\neq j(s)} Y_{j',\,l(j',s)+1}\right\} + \text{cov}\left\{\sum_{j\neq j(s)} Y_{j\,l(j,s)}, \sum_{j'\neq j(s)} Y_{j',\,l(j',s)+1}\right\}$$

$$= \sum_{j' \neq j(s)} cov(Y_{j(s)M}, Y_{j', b(j',s)+1}) + \sum_{j \neq j(s)} cov(Y_{j\, b(j,s)}, Y_{j, b(j,s)+1}) + \sum_{j \neq j(s)} \sum_{\substack{j' \neq j(s) \\ j' \neq j}} cov(Y_{j\, b(j,s)}, Y_{j', b(j',s)+1})$$

$$= C_1 + C_2 + C_3 \text{ , disons}$$

pour C_1 : on remarque que $\{ b(j',s)+1 : j' \neq j(s) \} = \{ 1+1, 2+1, \ldots, (M-1)+1 \} = \{ 2, 3, \ldots, M \}$

$$\text{ainsi } C_1 = 1 \times \frac{-(M-1)}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3) + (M-2) \times \frac{1}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$= -\frac{1}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

pour C_2 :

$$C_2 = (M-1) \times \frac{-(M-1)}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$= -\frac{(M-1)^2}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

pour C_3 : on remarque que $\{ b(j,s) : j \neq j(s) \} = \{ 1, 2, \ldots, M-1 \}$

$\{ b(j',s)+1 : j' \neq j(s) \text{ et } j' \neq j \} = \{ 2, 3, \ldots, M \} - \{ b(j,s)+1 \}$

- si j est tel que $b(j,s) = 1$:

$$\sum_{\substack{j' \neq j(s) \\ j' \neq j}} cov(Y_{j1}, Y_{j', b(j',s)+1}) = (M-2) \times \frac{1}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

- si j est tel que $b(j,s) = 2$ ou 3 ou … ou $(M-1)$ mais fixe :

$$\sum_{\substack{j' \neq j(s) \\ j' \neq j}} cov(Y_{j\, b(j,s)}, Y_{j', b(j',s)+1}) = 1 \times \frac{-(M-1)}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3) + (M-3) \times \frac{1}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$= -\frac{2}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$\text{d'où } C_3 = 1 \times \frac{M-2}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3) + (M-2) \times \frac{-2}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$= -\frac{(M-2)}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$\text{en définitive } ② = -\frac{1}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3) - \frac{(M-1)^2}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3) - \frac{(M-2)}{M^2} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$= -\frac{(M-1)}{M} \sigma^2 (1-\rho_1-\rho_2+\rho_3)$$

$$③ = \operatorname{var}\Big\{ \sum_{j\neq j(k)} Y_{j,k_j,n+1} \Big\} = \sum_{j\neq j(k)} \operatorname{var}(Y_{j,k_j,n+1}) + \sum_{j\neq j(k)} \sum_{\substack{j'\neq j(k)\\ j'\neq j}} \operatorname{cov}(Y_{j,k_j,n+1}, Y_{j',k_{j'},n+1})$$

$$= (M-1) \times \frac{(M-1)^2}{M^2} \sigma^2(1-\rho_1-\rho_2+\rho_3) + (M-1)(M-2) \times \frac{1}{M^2} \sigma^2(1-\rho_1-\rho_2+\rho_3)$$

$$= \frac{(M-1)(M^2-M-1)}{M^2} \sigma^2(1-\rho_1-\rho_2+\rho_3)$$

Par conséquent,

$$\operatorname{Var}(\hat\theta_{i\ell}) = \Big\{ \alpha^2(M-1) - 2\alpha\beta \frac{(M-1)}{M} + \beta^2 \frac{(M-1)(M^2-M-1)}{M^2} \Big\} \sigma^2(1-\rho_1-\rho_2+\rho_3)$$

$$= \frac{M-1}{M^2} \Big\{ M^2\alpha^2 - 2M\alpha\beta + \beta^2 + (M^2-M-2)\beta^2 \Big\} \sigma^2(1-\rho_1-\rho_2+\rho_3)$$

$$= \frac{M-1}{M^2} \Big\{ [M\alpha-\beta]^2 + (M^2-M-2)\beta^2 \Big\} \sigma^2(1-\rho_1-\rho_2+\rho_3).$$

On en conclut que :

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=1}^{M} \hat\theta_{i\ell}^2 \xrightarrow{P} \sum_{\ell=1}^{M} E(\hat\theta_{i\ell}^2) = \frac{M-1}{M} \Big\{ [M\alpha-\beta]^2 + (M^2-M-2)\beta^2 \Big\} \sigma^2(1-\rho_1-\rho_2+\rho_3)$$

et la version asymptotique de Q_θ est :

$$\frac{Mn}{[M\alpha-\beta]^2 + (M^2-M-2)\beta^2} \times \frac{\sum_{\ell=1}^{M} \hat\theta_\ell^2}{\sigma^2(1-\rho_1-\rho_2+\rho_3)}.$$

1° Si $\theta = \lambda$, on a $\alpha = \dfrac{1}{M^2-M-2}$ et $\beta = \dfrac{M}{M^2-M-2}$

et la version asymptotique de Q_λ est : $\dfrac{(M^2-M-2)\,n}{M} \times \dfrac{\sum_{\ell=1}^{M} \hat\lambda_\ell^2}{\sigma^2(1-\rho_1-\rho_2+\rho_3)}$.

2° Si $\theta = \tau$, on a $\alpha = \dfrac{M^2-M-1}{(M^2-M-2)M}$ et $\beta = \dfrac{1}{M^2-M-2}$

et la version asymptotique de Q_τ est : $\dfrac{M(M^2-M-2)}{M^2-M-1}\, n \times \dfrac{\sum_{\ell=1}^{M} \hat\tau_\ell^2}{\sigma^2(1-\rho_1-\rho_2+\rho_3)}$.

Pour chaque $1 \le i \le n$,

$$(\hat{\theta}_{i1}, \dots, \hat{\theta}_{iM})' \text{ a pour } E = (\theta_1, \dots, \theta_M)' \text{ et cov} = \underbrace{\frac{1}{M}\{(M\alpha-\beta)^2+(M^2-M-2)\beta^2\}\sigma^2(1-\rho_1-\rho_2+\rho_3)}_{=c,\ \text{disons}} ((\delta_{\ell\ell'} - \frac{1}{M}))$$

(voir feuille ⑥)

ou encore

$$\frac{1}{\sqrt{c}}(\hat{\theta}_{i1}, \dots, \hat{\theta}_{iM})' \text{ a pour } E = \frac{1}{\sqrt{c}}(\theta_1, \dots, \theta_M)' \text{ et cov} = ((\delta_{\ell\ell'} - \frac{1}{M}))$$

par le TCL multivarié,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{c}} (\hat{\theta}_{i1}, \dots, \hat{\theta}_{iM})' \text{ est asympt. } N\left(\sqrt{\frac{n}{c}}(\theta_1, \dots, \theta_M)', ((\delta_{\ell\ell'} - \frac{1}{M}))\right)$$

et, sous K_n^{θ} : $\theta_\ell = \frac{1}{\sqrt{n}}\mu_\ell$, $1 \le \ell \le M$, où $(\mu_1, \dots, \mu_M)'$ est un vecteur fixe tel que $\sum_{\ell=1}^{M} \mu_\ell = 0$

$$\frac{1}{\sqrt{nc}} \sum_{i=1}^{n} (\hat{\theta}_{i1}, \dots, \hat{\theta}_{iM})' \text{ est asympt. } N\left(\frac{1}{\sqrt{c}}(\mu_1, \dots, \mu_M)', ((\delta_{\ell\ell'} - \frac{1}{M}))\right);$$

par le théorème I.4.1 de Hájek et Šidák (1967), il s'ensuit que :

$$\sum_{\ell=1}^{M} \left\{ \frac{1}{\sqrt{nc}} \sum_{i=1}^{n} \hat{\theta}_{i\ell} \right\}^2 \text{ est asympt. } \chi^2_{M-1}\left(\frac{1}{c}\sum_{\ell=1}^{M}\mu_\ell^2\right)$$

$$\left[\!\left[\sum_{\ell=1}^{M} \left\{ \sqrt{\frac{n}{c}} \cdot \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_{i\ell} \right\}^2 = \frac{n}{c} \sum_{\ell=1}^{M} \hat{\theta}_\ell^2 = \text{version asympt. de } Q_\theta. \right.\right.$$

On en conclut finalement que :

1° sous K_n^{λ}, Q_λ est asympt. $\chi^2_{M-1}\left(\frac{M^2-M-2}{M\sigma^2(1-\rho_1-\rho_2+\rho_3)} \sum_{\ell=1}^{M} \mu_\ell^2\right)$

2° sous K_n^{τ}, Q_τ est asympt. $\chi^2_{M-1}\left(\frac{M(M^2-M-2)}{(M^2-M-1)\sigma^2(1-\rho_1-\rho_2+\rho_3)} \sum_{\ell=1}^{M} \mu_\ell^2\right)$.

Puisque $\mathcal{F}_\lambda = \dfrac{\frac{M^2-M-2}{M} \cdot n \cdot \sum_{\ell=1}^{M} \hat{\lambda}_\ell^2 \,/\, (M-1)}{S_{\text{rés.}} / \{[n(M-1)-2]\}(M-1)}$, la version asympt. de $(M-1)\mathcal{F}_\lambda$ est $\dfrac{(M^2-M-2)\, n \sum_{\ell=1}^{M} \hat{\lambda}_\ell^2}{M\sigma^2(1-\rho_1-\rho_2+\rho_3)}$,

c.-à-d. la version asympt. de Q_λ. Donc $(M-1)\mathcal{F}_\lambda$ et Q_λ sont asymptotiquement de même loi.

Puisque $\hat{F}_{\ell} = \dfrac{\dfrac{M(M^2-M-2)}{M^2-M-1} \cdot n \cdot \sum_{\ell=1}^{M} \hat{r}_{\ell}^2 \Big/ (M-1)}{S_{rés.} / [n(M-1)-2](M-1)}$, la version asympt. de $(M-1)\hat{F}_{\ell}$ est

$\dfrac{M(M^2-M-2)\, n \sum_{\ell=1}^{M} \hat{r}_{\ell}^2}{(M^2-M-1)\,\sigma^2(1-\rho_1-\rho_2+\rho_3)}$, c.-à-d. la version asympt. de Q_{ℓ}. Donc $(M-1)\hat{F}_{\ell}$ et Q_{ℓ}

sont asymptotiquement de même loi.

Dispositif expérimental pour p pair :

- on se donne un carré latin équilibré $d = ((d(j,k)))_{\substack{1\le j\le p\\ 1\le k\le p}}$
- on choisit au hasard et avec remise n permutations $\pi_1, \dots, \pi_n \in \mathcal{R}_p$
- à la $i^{\text{ème}}$ réplique, $1\le i\le n$, on utilise le carré latin $d_i = \pi_i \circ d$

(c.-à-d., on utilise le carré latin dont on a permuté les symboles selon π_i)

Désignons par b le carré latin dont les lignes sont les permutations inverses de celles de d — b_i —————— d_i, $(1\le i\le n)$

et posons $s_i = \pi_i^{-1}$

$$\text{on a } b_i = b\circ \pi_i^{-1} = \begin{bmatrix} b(1,s_{i1}) & \cdots & b(1,s_{ip}) \\ \vdots & \ddots & \vdots \\ b(p,s_{i1}) & \cdots & b(p,s_{ip}) \end{bmatrix}.$$

Considérons
$$\hat{\xi}_{il} = c_{31}\sum_{j=1}^{p} Y_{ij\,b(j,s_{il})} + c_{32}\sum_{\{j:\, b(j,s_{il})\neq p\}} Y_{ij,\,b(j,s_{il})+1}, \quad 1\le l\le p,$$
$$= \xi_l + c_{31}\sum_{j=1}^{p}\eta_{ij\,b(j,s_{il})} + c_{32}\sum_{\{j:\, b(j,s_{il})\neq p\}} \eta_{ij,\,b(j,s_{il})+1}, \quad 1\le l\le p,$$
$$= \xi_l + \omega_{il} \quad \text{, disons,} \qquad , 1\le l\le p.$$

Puisque π_i a été choisie au hasard, on peut considérer que s_i a été choisie au hasard et ceci induit la distribution de permutation :

$$\mathbb{P}\left[(\omega_{i1},\dots,\omega_{ip}) = \left(c_{31}\sum_j \eta_{ij\,b(j,s_{i1})} + c_{32}\sum_{j^*}\eta_{ij\,b(j,s_{i1})+1}, \dots, c_{31}\sum_j \eta_{ij\,b(j,s_{ip})} + c_{32}\sum_{j^*}\eta_{ij\,b(j,s_{ip})+1}\right)\right] = \frac{1}{p!},$$
$$s_i\in\mathcal{R}_p,$$

ce qui entraîne manifestement que $\mathcal{L}_p(\omega_{i1},\dots,\omega_{ip}) = \mathcal{L}_p(\omega_{i s_1},\dots,\omega_{i s_p})$, pour tout $(s_1,\dots,s_p)\in\mathcal{R}_p$. Puisque les répliques sont indépendantes, on est donc en pré-

Sens du modèle : $\hat{\xi}_{i\ell} = \xi_\ell + \omega_{i\ell}$, $1 \le \ell \le p$ et $1 \le i \le n$,

où $(\omega_{i1}, \dots, \omega_{ip})$ sont des vecteurs indépendants à composantes interchangeables selon la distribution de permutation.

Du fait que $cov(\eta_{ijk}, \eta_{ij'k'}) = (\delta_{jj'} - \frac{1}{p})(\delta_{kk'} - \frac{1}{p})\theta$, où $\theta = \sigma^2(1 - \rho_1 - \rho_2 + \rho_3)$,

on obtient $cov(\hat{\xi}_{i\ell}, \hat{\xi}_{i\ell'}) = \frac{1}{p}\{(pc_{\xi 1} - c_{\xi 2})^2 + (p^2 - p - 2)c_{\xi 2}^2\}\theta(\delta_{\ell\ell'} - \frac{1}{p})$, $1 \le \ell, \ell' \le p$.

À noter que

$$\mathcal{F}_\xi = \frac{\dfrac{p}{\{(pc_{\xi 1} - c_{\xi 2})^2 + (p^2 - p - 2)c_{\xi 2}^2\}} \sum_{\ell=1}^{p} \hat{\xi}_\ell^2 \Big/ (p-1)}{S_{rés.} / [n(p-1) - 2](p-1)},$$

où $\hat{\xi}_\ell = \frac{1}{n}\sum_{i=1}^{n} \hat{\xi}_{i\ell}$, $1 \le \ell \le p$, et puisque le dénominateur est un estimateur convergent de θ, la version asymptotique de $(p-1)\mathcal{F}_\xi$ est :

$$\frac{p}{\{(pc_{\xi 1} - c_{\xi 2})^2 + (p^2 - p - 2)c_{\xi 2}^2\}\theta} \sum_{\ell=1}^{p} \hat{\xi}_\ell^2 .$$

①

m crépuin $s \{ d_i(j,k) \}_{1 \le j \le 2m, 1 \le k \le m}$ plans de Williams, $1 \le i \le n$

$$X_{ijk} = \mu + \alpha_{ij} + \pi_{ik} + \tau_{d_i(j,k)} + \lambda_{d_i(j,k-1)} + \varepsilon_{ijk} \quad , 1 \le i \le n \; ; 1 \le j \le 2m \; ; 1 \le k \le m$$

Contraintes : $\sum_i \sum_j \alpha_{ij} = 0$; $\sum_k \pi_{ik} = 0 \; \forall i$; $\sum_{l=1}^{m} \tau_l = 0$; $\sum_{l=1}^{m} \lambda_l = 0$

minimisons $\sum_{i=1}^{n} \sum_{j=1}^{2m} \sum_{k=1}^{m} \left(X_{ijk} - \mu - \alpha_{ij} - \pi_{ik} - \tau_{d_i(j,k)} - \lambda_{d_i(j,k-1)} \right)^2$

1° $\frac{\partial}{\partial \mu} = 0 \iff \sum_i \sum_j \sum_k \left(X_{ijk} - \mu - \alpha_{ij} - \pi_{ik} - \tau_{d_i(j,k)} - \lambda_{d_i(j,k-1)} \right) = 0$

$\iff \hat{\mu} = \sum_i \sum_j \sum_k X_{ijk} = \bar{X}_{...}$

2° $\frac{\partial}{\partial \alpha_{ij}} = 0 \iff \sum_k \left(X_{ijk} - \mu - \alpha_{ij} - \pi_{ik} - \tau_{d_i(j,k)} - \lambda_{d_i(j,k-1)} \right) = 0$

$\iff \hat{\alpha}_{ij} = \bar{X}_{ij.} - \bar{X}_{...} + \frac{1}{m} \hat{\lambda}_{d_i(j,m)}$

3° $\frac{\partial}{\partial \pi_{ik}} = 0 \iff \sum_j \left(X_{ijk} - \mu - \alpha_{ij} - \pi_{ik} - \tau_{d_i(j,k)} - \lambda_{d_i(j,k-1)} \right) = 0$

$\iff \hat{\pi}_{ik} = \bar{X}_{i.k} - \bar{X}_{...} - \frac{1}{2m} \sum_j \hat{\alpha}_{ij} = \bar{X}_{i.k} - \bar{X}_{...} - (\bar{X}_{i..} - \bar{X}_{...}) = \bar{X}_{i.k} - \bar{X}_{i..}$

4° $\frac{\partial}{\partial \tau_l} = 0 \iff \sum_i \sum_{(j,k) : d_i(j,k) = l} \left(X_{ijk} - \mu - \alpha_{ij} - \pi_{ik} - \tau_l - \lambda_{d_i(j,k-1)} \right) = 0$

~~$\iff \hat{\tau}_l = \frac{1}{2nm} T_l - \bar{X}_{...}$~~

$\iff \sum_i \sum_j \left(X_{ijk_i(j,l)} - \mu - \alpha_{ij} - \pi_{ik_i(j,l)} - \tau_l - \lambda_{d_i(j, k_i(j,l)-1)} \right) = 0$

$\iff \hat{\tau}_l = \frac{1}{2nm} T_l - \bar{X}_{...} - \frac{1}{2nm} \underbrace{\sum_i \sum_j \hat{\lambda}_{d_i(j, k_i(j,l)-1)}}$

$2n \sum_{l' \ne l}^{n} \hat{\lambda}_{l'}$ car le trait. l est précédé 2 fois par tous les autres trait. (sauf lui-même)

$= -2n \hat{\lambda}_l$

$\iff \hat{\tau}_l = \frac{1}{2nm} T_l - \bar{X}_{...} + \frac{1}{m} \hat{\lambda}_l$

5° $\frac{\partial}{\partial \lambda_\ell} = 0 \Leftrightarrow \sum_i \sum\sum_{(j,k):\, d_{i(j,k-1)}=\ell} (Y_{ijk} - \mu - \alpha_{ij} - \pi_{ik} - \tau_{d_i(j,k)} - \lambda_\ell) = 0$

or $\{(j,k) : d_{i(j,k-1)} = \ell\} = \{(j,k) : \kappa_{i(j,\ell)} = k-1\} = \{(j,k) : k = \kappa_{i(j,\ell)+1}\}$

$= \{(j, \kappa_{i(j,\ell)+1}) : j \neq j_1(\ell) \text{ et } j_2(\ell)\}$ où $j_1(\ell)$ et $j_2(\ell)$ t.q. $\kappa_i(j_1(\ell),\ell) = \kappa_i(j_2(\ell),\ell) = M$

$\Leftrightarrow \sum_i \sum_{j \neq j_1(\ell),\, \neq j_2(\ell)} (Y_{i j \kappa_{i(j,\ell)+1}} - \mu - \alpha_{ij} - \pi_{i \kappa_{i(j,\ell)+1}} - \tau_{d_i(j, \kappa_{i(j,\ell)+1})} - \lambda_\ell) = 0$

$\Leftrightarrow \hat{\lambda}_\ell = \frac{L_\ell}{2n(m-1)} - \bar{X}_{\cdot\cdot\cdot} - \frac{1}{2n(m-1)} \sum_i \sum_{j \neq j_1(\ell),\, \neq j_2(\ell)} \hat{\alpha}_{ij} - \frac{1}{2n(m-1)} \sum_i 2 \sum_{k=2}^{m} \pi_{ik} - \frac{1}{2n(m-1)} \sum_i 2 \sum_{\ell' \neq \ell} \tau_{\ell'}$

car chaque traitement est suivi 2 fois par tous les autres tract. (sauf lui-même)

$= \frac{L_\ell}{2n(m-1)} - \bar{X}_{\cdot\cdot\cdot} + \frac{1}{2n(m-1)} \sum_i (\hat{\alpha}_{i j_1(\ell)} + \hat{\alpha}_{i j_2(\ell)}) + \frac{1}{n(m-1)} \sum_i \hat{\pi}_{i1} + \frac{1}{m-1} \hat{\tau}_\ell$

$= \frac{L_\ell}{2n(m-1)} - \bar{X}_{\cdot\cdot\cdot} + \frac{1}{2n(m-1)} \sum_i \{ \bar{X}_{i j_1(\ell)\cdot} - \bar{X}_{\cdot\cdot\cdot} + \frac{1}{m} \hat{\lambda}_{d_i(j_1(\ell),m)} + \bar{X}_{i j_2(\ell)\cdot} - \bar{X}_{\cdot\cdot\cdot} + \frac{1}{m} \hat{\lambda}_{d_i(j_2(\ell),m)} \}$

(avec $\hat{\lambda}_{d_i(j_1(\ell),m)} = \hat{\lambda}_\ell$ et $\hat{\lambda}_{d_i(j_2(\ell),m)} = \hat{\lambda}_\ell$)

$+ \frac{1}{n(m-1)} \sum_i \{ \bar{X}_{i\cdot 1} - \bar{X}_{i\cdot\cdot} \} + \frac{1}{m-1} \hat{\tau}_\ell$

$\Leftrightarrow \hat{\lambda}_\ell = \frac{m L_\ell}{2n(m^2-m-1)} + \frac{m}{m^2-m-1} \times \frac{\bar{X}_{\cdot j_1(\ell)\cdot} + \bar{X}_{\cdot j_2(\ell)\cdot}}{2} + \frac{m}{m^2-m-1} \bar{X}_{\cdot\cdot 1} - \frac{m(m+1)}{m^2-m-1} \bar{X}_{\cdot\cdot\cdot} + \frac{m}{m^2-m-1} \hat{\tau}_\ell$

si on substitue à $\hat{\tau}_\ell$ sa valeur $\frac{1}{2n}T_\ell - \bar{X}_{\cdot\cdot\cdot} + \frac{1}{m}\hat{\lambda}_\ell$, on obtient

$(m^2-m-2)\hat{\lambda}_\ell = \frac{m L_\ell}{2n} + \frac{T_\ell}{2n} + m \times \frac{\bar{X}_{\cdot j_1(\ell)\cdot} + \bar{X}_{\cdot j_2(\ell)\cdot}}{2} + m \bar{X}_{\cdot\cdot 1} - m(m+2)\bar{X}_{\cdot\cdot\cdot}$

enfin, puisque $(m^2-m-2)\hat{\tau}_\ell = \frac{(m^2-m-2)T_\ell}{2nm} - (m^2-m-2)\bar{X}_{\cdot\cdot\cdot} + \frac{(m^2-m-2)}{m}\hat{\lambda}_\ell$, on obtient

$(m^2-m-2)\hat{\tau}_\ell = \frac{(m^2-m-1)}{m} \times \frac{T_\ell}{2n} + \frac{L_\ell}{2n} + \frac{\bar{X}_{\cdot j_1(\ell)\cdot} + \bar{X}_{\cdot j_2(\ell)\cdot}}{2} + \bar{X}_{\cdot\cdot 1} - m^2 \bar{X}_{\cdot\cdot\cdot}$

posons $\tilde{T}_\ell = \sum_{i=1}^{n}\sum_{j=1}^{2m} Y_{ijk_i(j,\ell)} = \sum_{i=1}^{n}\sum_{j=1}^{2m}\{X_{ijk_i(j,\ell)} - \bar{X}_{ij.} - \bar{X}_{i.k_i(j,\ell)} + \bar{X}_{i..}\}$

$= T_\ell - 2nm\bar{X}_{...} - \underbrace{\sum_i 2\sum_{k=1}^{m} X_{i.k}}_{-2nm\bar{X}_{...}} + 2nm\bar{X}_{...}$

donc $\dfrac{\tilde{T}_\ell}{2n} = \dfrac{T_\ell}{2n} - m\bar{X}_{...}$

posons $\tilde{L}_\ell = \sum_{i=1}^{n}\sum_{\substack{j\neq j_1(\ell)\\ \neq j_2(\ell)}} Y_{ij,k_i(j,\ell)+1} = \sum_i \sum_{\substack{j\neq j_1(\ell)\\ \neq j_2(\ell)}}\{X_{ij,k_i(j,\ell)+1} - \bar{X}_{ij.} - \bar{X}_{i.,k_i(j,\ell)+1} + \bar{X}_{i..}\}$

$= L_\ell - \underbrace{n\sum_{\substack{j\neq j_1(\ell)\\ \neq j_2(\ell)}}\bar{X}_{.j.}}_{2nm\bar{X}_{...} - \{\bar{X}_{.j_1(\ell).} + \bar{X}_{.j_2(\ell).}\}} - \underbrace{\sum_i 2\sum_{k=2}^{m}\bar{X}_{i.k}}_{2n\sum_{k=2}^{m}\bar{X}_{..k} = 2n(m\bar{X}_{...} - \bar{X}_{..1})} + 2n(m-1)\bar{X}_{...}$

donc $\dfrac{\tilde{L}_\ell}{2n} = \dfrac{L_\ell}{2n} + \dfrac{\{\bar{X}_{.j_1(\ell).} + \bar{X}_{.j_2(\ell).}\}}{2} + \bar{X}_{..1} - (m+1)\bar{X}_{...}$

Ainsi $\dfrac{m\tilde{L}_\ell}{2n} + \dfrac{\tilde{T}_\ell}{2n} = \dfrac{mL_\ell}{2n} + m\dfrac{\{\quad\}}{2} + m\bar{X}_{..1} - m(m+1)\bar{X}_{...} + \dfrac{T_\ell}{2n} - m\bar{X}_{...}$

$= \dfrac{mL_\ell}{2n} + \dfrac{T_\ell}{2n} + m\dfrac{\{\quad\}}{2} + m\bar{X}_{..1} - m(m+2)\bar{X}_{...} = (m^2-m-2)\hat{\lambda}_\ell$

et $\dfrac{m^2-m-1}{m}\dfrac{\tilde{T}_\ell}{2n} + \dfrac{\tilde{L}_\ell}{2n} = \dfrac{m^2-m-1}{m}\dfrac{T_\ell}{2n} - (m^2-m-1)\bar{X}_{...} + \dfrac{L_\ell}{2n} + \dfrac{\{\quad\}}{2} + \bar{X}_{..1} - (m+1)\bar{X}_{...}$

$= \dfrac{m^2-m-1}{m}\dfrac{T_\ell}{2n} + \dfrac{L_\ell}{2n} + \dfrac{\{\quad\}}{2} + \bar{X}_{..1} - m^2\bar{X}_{...} = (m^2-m-2)\hat{\tau}_\ell$

DONC

$$(m^2-m-2)\hat{\lambda}_\ell = \frac{\tilde{T}_\ell}{2n} + \frac{m\tilde{L}_\ell}{2n}, \quad 1\le \ell\le m$$

$$(m^2-m-2)\hat{\tau}_\ell = \frac{(m^2-m-1)}{m}\frac{\tilde{T}_\ell}{2n} + \frac{\tilde{L}_\ell}{2n}, \quad 1\le \ell\le m$$

	SC	d.l.	
inter-sujets	$m \sum_{i=1}^{n} \sum_{j=1}^{2m} (\bar{X}_{ij.} - \bar{X}_{...})^2$	$2mn-1$	
intra-sujets	$\sum_{i=1}^{n} \sum_{j=1}^{2m} \sum_{k=1}^{m} (X_{ijk} - \bar{X}_{ij.})^2$	$2m(m-1)n$	
périodes	$2m \sum_{i=1}^{n} \sum_{k=1}^{m} (\bar{X}_{i.k} - \bar{X}_{...})^2$	$(m-1)n$	
traitements (non ajustés)	$2mn \sum_{\ell=1}^{m} \tilde{\tau}_\ell^2$ (1)	$(m-1)$	$\sigma^2(1-\rho_1-\rho_2+\rho_3) + \frac{2mn}{m-1} \sum_{\ell=1}^{m} (\tau_\ell - \frac{1}{m}\lambda_\ell)^2$
arr-effets (ajustés)	$\frac{2(m^2-m-2)n}{m} \sum_{\ell=1}^{m} \hat{\lambda}_\ell^2$ (2)	$(m-1)$	$\sigma^2(1-\rho_1-\rho_2+\rho_3) + \frac{2(m^2-m-2)n}{m(m-1)} \sum_{\ell=1}^{m} \lambda_\ell^2$
ou			
arr-effets (non ajustés)	$\frac{2(m^2-m-1)n}{m} \sum_{\ell=1}^{m} \tilde{\lambda}_\ell^2$ (3)	$(m-1)$	$\sigma^2(1-\rho_1-\rho_2+\rho_3) + \frac{2(m^2-m-1)n}{m(m-1)} \sum_{\ell=1}^{m} (\lambda_\ell - \frac{m}{m^2-m-1}\tau_\ell)^2$
traitements (ajustés)	$\frac{2m(m^2-m-2)n}{(m^2-m-1)} \sum_{\ell=1}^{m} \hat{\tau}_\ell^2$ (4)	$(m-1)$	$\sigma^2(1-\rho_1-\rho_2+\rho_3) + \frac{2m(m^2-m-2)n}{(m^2-m-1)(m-1)} \sum_{\ell=1}^{m} \tau_\ell^2$
résiduelle	$\sum_{i=1}^{n} \sum_{j=1}^{2m} \sum_{k=1}^{m} y_{ijk}^2 - \begin{cases} (1)+(2) \\ \text{ou} \\ (3)+(4) \end{cases}$	$(m-1)[(2m-1)n-2]$	$\sigma^2(1-\rho_1-\rho_2+\rho_3)$
totale	$\sum_{i=1}^{n} \sum_{j=1}^{2m} \sum_{k=1}^{m} (X_{ijk} - \bar{X}_{...})^2$	$2m^2n-1$	

PROOF OF THE COVARIANCE STRUCTURE OF η_i=$\{\eta_{ijk} : 1 \le j \le mp, 1 \le k \le p\},\ i = 1, \ldots, n$.

Recall that

$$\operatorname{cov}(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \begin{cases} \sigma^2 & \text{if } j = j' \quad \textit{and} \quad k = k' \\ \rho_1\sigma^2 & \text{if } j = j' \quad \textit{and} \quad k \ne k' \\ \rho_2\sigma^2 & \text{if } j \ne j' \quad \textit{and} \quad k = k' \\ \rho_3\sigma^2 & \text{if } j \ne j' \quad \textit{and} \quad k \ne k' \end{cases}$$

and $\eta_{ijk} = \varepsilon_{ijk} - \varepsilon_{ij\cdot} - \varepsilon_{i\cdot k} + \varepsilon_{i\cdot\cdot}$. Then,

$$\begin{aligned} Cov(\eta_{ijk}, \eta_{ij'k'}) &= Cov(\varepsilon_{ijk} - \varepsilon_{ij\cdot} - \varepsilon_{i\cdot k} + \varepsilon_{i\cdot\cdot}\ ,\ \varepsilon_{ij'k'} - \varepsilon_{ij'\cdot} - \varepsilon_{i\cdot k'} + \varepsilon_{i\cdot\cdot}) \\ &= Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) - Cov(\varepsilon_{ijk}, \varepsilon_{ij'\cdot}) - Cov(\varepsilon_{ijk}, \varepsilon_{i\cdot k'}) + Cov(\varepsilon_{ijk}, \varepsilon_{i\cdot\cdot}) \\ &\quad - Cov(\varepsilon_{ij\cdot}, \varepsilon_{ij'k'}) + Cov(\varepsilon_{ij\cdot}, \varepsilon_{ij'\cdot}) + Cov(\varepsilon_{ij\cdot}, \varepsilon_{i\cdot k'}) - Cov(\varepsilon_{ij\cdot}, \varepsilon_{i\cdot\cdot}) \\ &\quad - Cov(\varepsilon_{i\cdot k}, \varepsilon_{ij'k'}) + Cov(\varepsilon_{i\cdot k}, \varepsilon_{ij'\cdot}) + Cov(\varepsilon_{i\cdot k}, \varepsilon_{i\cdot k'}) - Cov(\varepsilon_{i\cdot k}, \varepsilon_{i\cdot\cdot}) \\ &\quad + Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{ij'k'}) - Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{ij'\cdot}) - Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{i\cdot k'}) + Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{i\cdot\cdot}). \end{aligned}$$

Now, we have for $i = 1, \ldots, n$,

$$Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \sigma^2(1 - \rho_1 - \rho_2)\delta_{jj'}\delta_{kk'} + \rho_1\sigma^2\delta_{jj'} + \rho_2\sigma^2\delta_{kk'} + \rho_3\sigma^2(\delta_{jj'} - 1)(\delta_{kk'} - 1);$$

$$\begin{aligned} Cov(\varepsilon_{ijk}, \varepsilon_{ij'\cdot}) &= \frac{1}{p}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) \\ &= \frac{1}{p}[\sigma^2(1 - \rho_1 - \rho_2)\delta_{jj'} + p\rho_1\sigma^2\delta_{jj'} + \rho_2\sigma^2 + \rho_3\sigma^2(\delta_{jj'} - 1)(1 - p)] \\ &= C_{jj'},\ \text{say}; \end{aligned}$$

$$\begin{aligned} Cov(\varepsilon_{ijk}, \varepsilon_{i\cdot k'}) &= \frac{1}{mp}\sum_{j'=1}^{mp} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) \\ &= \frac{1}{mp}[\sigma^2(1 - \rho_1 - \rho_2)\delta_{kk'} + \rho_1\sigma^2 + mp\rho_2\sigma^2\delta_{kk'} + \rho_3\sigma^2(1 - mp)(\delta_{kk'} - 1)] \\ &= C^*_{kk'},\ \text{say}; \end{aligned}$$

$$Cov(\varepsilon_{ijk}, \varepsilon_{i\cdot\cdot}) = \frac{1}{mp^2}\sum_{j'=1}^{mp}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'})$$

$$
\begin{aligned}
&= \frac{1}{mp^2}[\sigma^2(1-\rho_1-\rho_2)+p\rho_1\sigma^2+mp\rho_2\sigma^2+\rho_3\sigma^2(1-mp)(1-p)] \\
&= C, \text{ say;} \\
Cov(\varepsilon_{ij\cdot}, \varepsilon_{ij'k'}) &= \frac{1}{p}\sum_{k=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = C_{jj'}; \\
Cov(\varepsilon_{ij\cdot}, \varepsilon_{ij'\cdot}) &= \frac{1}{p^2}\sum_{k=1}^{p}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{p}\sum_{k=1}^{p} C_{jj'} = C_{jj'}; \\
Cov(\varepsilon_{ij\cdot}, \varepsilon_{i\cdot k'}) &= \frac{1}{mp^2}\sum_{k=1}^{p}\sum_{j'=1}^{mp} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{p}\sum_{k=1}^{p} C^*_{kk'} = C; \\
Cov(\varepsilon_{ij\cdot}, \varepsilon_{i\cdot\cdot}) &= \frac{1}{mp^3}\sum_{k=1}^{p}\sum_{j'=1}^{mp}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{p}\sum_{k=1}^{p} C = C; \\
Cov(\varepsilon_{i\cdot k}, \varepsilon_{ij'k'}) &= \frac{1}{mp}\sum_{j=1}^{mp} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = C^*_{kk'}; \\
Cov(\varepsilon_{i\cdot k}, \varepsilon_{ij'\cdot}) &= \frac{1}{mp^2}\sum_{j=1}^{mp}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = C; \\
Cov(\varepsilon_{i\cdot k}, \varepsilon_{i\cdot k'}) &= \frac{1}{m^2p^2}\sum_{j=1}^{mp}\sum_{j'=1}^{mp} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{mp}\sum_{j=1}^{mp} C^*_{kk'} = C^*_{kk'}; \\
Cov(\varepsilon_{i\cdot k}, \varepsilon_{i\cdot\cdot}) &= \frac{1}{m^2p^3}\sum_{j=1}^{mp}\sum_{j'=1}^{mp}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{mp}\sum_{j=1}^{mp} C = C; \\
Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{ij'k'}) &= \frac{1}{mp^2}\sum_{j=1}^{mp}\sum_{k=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = C; \\
Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{ij'\cdot}) &= \frac{1}{mp^3}\sum_{j=1}^{mp}\sum_{k=1}^{p}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{p}\sum_{k'=1}^{p} C = C; \\
Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{i\cdot k'}) &= \frac{1}{m^2p^3}\sum_{j=1}^{mp}\sum_{k=1}^{p}\sum_{j'=1}^{mp} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{mp}\sum_{j'=1}^{mp} C = C;
\end{aligned}
$$

$$Cov(\varepsilon_{i\cdot\cdot}, \varepsilon_{i\cdot\cdot}) = \frac{1}{m^2p^4}\sum_{j=1}^{mp}\sum_{k=1}^{p}\sum_{j'=1}^{mp}\sum_{k'=1}^{p} Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) = \frac{1}{mp^2}\sum_{j=1}^{mp}\sum_{k=1}^{p} C = C.$$

Therefore,

$$\begin{aligned}
Cov(\eta_{ijk}, \eta_{ij'k'}) &= Cov(\varepsilon_{ijk}, \varepsilon_{ij'k'}) - C_{jj'} - C^*_{kk'} + C \\
&= \sigma^2(1-\rho_1-\rho_2)\delta_{jj'}\delta_{kk'} + \rho_1\sigma^2\delta_{jj'} + \rho_2\sigma^2\delta_{kk'} + \rho_3\sigma^2(\delta_{jj'}-1)(\delta_{kk'}-1) \\
&\quad -\frac{1}{p}\sigma^2(1-\rho_1-\rho_2)\delta_{jj'} - \rho_1\sigma^2\delta_{jj'} - \frac{1}{p}\rho_2\sigma^2 - \frac{1}{p}\rho_3\sigma^2(\delta_{jj'}-1)(1-p) \\
&\quad -\frac{1}{mp}\sigma^2(1-\rho_1-\rho_2)\delta_{kk'} - \frac{1}{mp}\rho_1\sigma^2 - \rho_2\sigma^2\delta_{kk'} \\
&\quad -\frac{1}{mp}\rho_3\sigma^2(1-mp)(\delta_{kk'}-1) + \frac{1}{mp^2}\sigma^2(1-\rho_1-\rho_2) + \frac{1}{mp}\rho_1\sigma^2 \\
&\quad +\frac{1}{p}\rho_2\sigma^2 + \frac{1}{mp^2}\rho_3\sigma^2(1-mp)(1-p) \\
&= \sigma^2(1-\rho_1-\rho_2)[\delta_{jj'}\delta_{kk'} - \frac{1}{p}\delta_{jj'} - \frac{1}{mp}\delta_{kk'} + \frac{1}{mp^2}] \\
&\quad +\rho_3\sigma^2[\delta_{jj'}\delta_{kk'} - \frac{1}{p}\delta_{jj'} - \frac{1}{mp}\delta_{kk'} + \frac{1}{mp^2}] \\
&= \sigma^2(1-\rho_1-\rho_2+\rho_3)(\delta_{jj'} - \frac{1}{mp})(\delta_{kk'} - \frac{1}{p}).
\end{aligned}$$

□

Proof of Lemma 2.

First we have,

$$\begin{aligned}
Cov(T^*_{il}, T^*_{il'}) &= Cov(\sum_{j=1}^{mp}\eta_{ijb_i(j,l)}, \sum_{j'=1}^{mp}\eta_{ij'b_i(j',l')}) \\
&= \sum_{j=1}^{mp}\sum_{j'=1}^{mp}(\delta_{jj'} - \frac{1}{mp})(\delta_{b_i(j,l)b_i(j',l')} - \frac{1}{p})\gamma \\
&= \sum_{j=1}^{mp}(\delta_{jj} - \frac{1}{mp})(\delta_{b_i(j,l)b_i(j,l')} - \frac{1}{p})\gamma
\end{aligned}$$

$$+ \sum_{j'=1}^{mp}\sum_{j\neq j'}^{mp}(\delta_{jj'} - \frac{1}{mp})(\delta_{b_i(j,l)b_i(j',l')} - \frac{1}{p})\gamma \ .$$

We now consider separately, in (A) and (B) below, the cases when $l = l'$ and $l \neq l'$.

(A) If $l = l'$,

$$\begin{aligned} Cov(T^*_{il}, T^*_{il'}) &= mp[(1-\frac{1}{mp})(1-\frac{1}{p})\gamma] + \sum_{j'=1}^{mp}\sum_{j\neq j'}^{mp}(-\frac{1}{mp})(\delta_{b_i(j,l)b_i(j',l)} - \frac{1}{p})\gamma \\ &= mp[(1-\frac{1}{mp})(1-\frac{1}{p})\gamma] + mp[(m-1)(-\frac{1}{mp})(1-\frac{1}{p})\gamma] \\ &\quad +mp[m(p-1)(-\frac{1}{mp})(-\frac{1}{p})\gamma] = m(p-1)\gamma \ . \end{aligned}$$

(B) If $l \neq l'$,

$$\begin{aligned} Cov(T^*_{il}, T^*_{il'}) &= \sum_{j=1}^{mp}(\delta_{jj} - \frac{1}{mp})(\delta_{b_i(j,l)b_i(j,l')} - \frac{1}{p})\gamma \\ &\quad + \sum_{j=1}^{mp}\sum_{j'\neq j}^{mp}(\delta_{jj'} - \frac{1}{mp})(\delta_{b_i(j,l)b_i(j',l')} - \frac{1}{p})\gamma \\ &= mp[(1-\frac{1}{mp})(-\frac{1}{p})\gamma] + mp[m(-\frac{1}{mp})(1-\frac{1}{p})\gamma] \\ &\quad +mp[(mp-m-1)(-\frac{1}{mp})(-\frac{1}{p})\gamma] \\ &= mp\gamma[\frac{(mp-1)}{mp}(-\frac{1}{p}) - \frac{m}{mp}(1-\frac{1}{p}) + \frac{1}{mp^2}(mp-m-1)] \\ &= \gamma[-m + \frac{1}{p} - m + \frac{m}{p} + m - \frac{m}{p} - \frac{1}{p}] = -m\gamma \ . \end{aligned}$$

Therefore, from (A) and (B) we have

$$Cov(T^*_{il}, T^*_{il'}) = mp(\delta_{ll'} - \frac{1}{p})\gamma \ .$$

The second covariance to find in Lemma 2 is

$$\begin{aligned} Cov(L^*_{il}, L^*_{il'}) &= Cov(\sum_{\{j|1\leq b_{i(j,l)}\leq p-1\}} \eta_{ijb_i(j,l)+1}, \sum_{\{j'|1\leq b_{i(j',l')}\leq p-1\}} \eta_{ij'b_i(j',l')+1}) \\ &= \sum_{\{j|1\leq b_{i(j,l)}\leq p-1\}}\ \sum_{\{j'|1\leq b_{i(j',l')}\leq p-1\}} (\delta_{jj'} - \frac{1}{mp})(\delta_{b_i(j,l)+1,b_i(j',l')+1} - \frac{1}{p})\gamma \end{aligned}$$

$$= \sum_{\{j|1\le b_{i(j,l)}\le p-1\}} (\delta_{jj} - \frac{1}{mp})(\delta_{b_i(j,l)+1, b_i(j,l')+1} - \frac{1}{p})\gamma$$

$$+ \sum_{\{j|1\le b_{i(j,l)}\le p-1\}} \sum_{\{j'\ne j|1\le b_{i(j',l')}\le p-1\}} (\delta_{jj'} - \frac{1}{mp})(\delta_{b_i(j,l)+1, b_i(j',l')+1} - \frac{1}{p})\gamma \, .$$

We now consider separately, in (C) and (D) below, the cases when $l = l'$ and $l \neq l'$.

(C) If $l = l'$,

$$\begin{aligned} Cov(L^*_{il}, L^*_{il'}) &= m(p-1)[(1-\frac{1}{mp})(1-\frac{1}{p})\gamma] + m(p-1)[(m-1)(-\frac{1}{mp})(1-\frac{1}{p})\gamma] \\ &\quad + m(p-1)[m(p-2)(-\frac{1}{mp})(-\frac{1}{p})\gamma] \\ &= m(p-1)\gamma[1 - \frac{1}{p} - \frac{1}{mp} + \frac{1}{mp^2} - \frac{1}{p} + \frac{1}{p^2} \\ &\quad + \frac{1}{mp} - \frac{1}{mp^2} + \frac{1}{p} - \frac{2}{p^2}] \\ &= m(p-1)\gamma[1 - \frac{1}{p} - \frac{1}{p^2}] = m(p-1)\gamma[\frac{p^2-p-1}{p^2}] \, . \end{aligned}$$

(D) If $l \neq l'$,

$$\begin{aligned} Cov(L^*_{il}, L^*_{il'}) &= m(p-2)[(1-\frac{1}{mp})(-\frac{1}{p})\gamma] + m(p-1)m[(-\frac{1}{mp})(1-\frac{1}{p})]\gamma \\ &\quad + [m^2(p-1)^2 - m(p-2) - m^2(p-1)][(-\frac{1}{mp})(-\frac{1}{p})\gamma] \\ &= m\gamma[-1 + \frac{1}{mp} + \frac{2}{p} - \frac{2}{mp^2} - 1 + \frac{1}{p} + \frac{1}{p} - \frac{1}{p^2} \\ &\quad + \frac{p^2 - 5p + 6 + p - 2 + mp^2 - 3mp + 2m - p^2 + 3p - 2}{mp^2}] \\ &= m\gamma[-1 + \frac{1}{p} + \frac{1}{p^2}] = -m\gamma[\frac{p^2-p-1}{p^2}] \, . \end{aligned}$$

Therefore, from (C) and (D) we have

$$\begin{aligned} Cov(L^*_{il}, L^*_{il'}) &= mp(\delta_{ll'} - \frac{1}{p})\gamma(\frac{p^2-p-1}{p^2}) \\ &= m(\frac{p^2-p-1}{p})(\delta_{ll'} - \frac{1}{p})\gamma \, . \end{aligned}$$

The third covariance in Lemma 2 is

$$
\begin{aligned}
Cov(T^*_{il}, L^*_{il'}) &= Cov(\sum_{j=1}^{mp} \eta_{ijb_i(j,l)}, \sum_{\{j'|1\le b_{i(j',l')}\le p-1\}} \eta_{ij'b_i(j',l')+1}) \\
&= \sum_{j=1}^{mp} \sum_{\{j'|1\le b_{i(j',l')}\le p-1\}} (\delta_{jj'} - \frac{1}{mp})(\delta_{b_i(j,l),b_i(j',l')+1} - \frac{1}{p})\gamma \\
&= \sum_{j=1}^{mp} (\delta_{jj} - \frac{1}{mp})(\delta_{b_i(j,l),b_i(j,l')+1} - \frac{1}{p})\gamma \\
&\quad + \sum_{j=1}^{mp} \sum_{\{j'\neq j|1\le b_{i(j',l')}\le p-1\}} (\delta_{jj'} - \frac{1}{mp})(\delta_{b_i(j,l),b_i(j',l')+1} - \frac{1}{p})\gamma \ .
\end{aligned}
$$

We now consider separately, in (E) and (F) below, the cases when $l = l'$ and $l \neq l'$.

(E) If $l = l'$,

$$
\begin{aligned}
Cov(T^*_{il}, L^*_{il'}) &= m(p-1)(1-\frac{1}{mp})(-\frac{1}{p})\gamma + m^2(p-1)(-\frac{1}{mp})(1-\frac{1}{p})\gamma \\
&\quad + m(p-1)(mp-m-1)(-\frac{1}{mp})(-\frac{1}{p})\gamma \\
&= m(p-1)\gamma[\frac{1}{mp^2} - \frac{1}{p} - \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{mp^2}] \\
&= -m\frac{(p-1)}{p}\gamma = -m(1-\frac{1}{p})\gamma \ .
\end{aligned}
$$

(F) If $l \neq l'$,

$$
\begin{aligned}
Cov(T^*_{il}, L^*_{il'}) &= m(1-\frac{1}{mp})(1-\frac{1}{p})\gamma + m(p-2)(1-\frac{1}{mp})(-\frac{1}{p})\gamma \\
&\quad + m[m(p-1)-1](-\frac{1}{mp})(1-\frac{1}{p})\gamma \\
&\quad + [m(p-1)[mp-m-1]+m](-\frac{1}{mp})(-\frac{1}{p})\gamma \\
&= m\gamma[1 - \frac{1}{mp} - \frac{1}{p} + \frac{1}{mp^2} - 1 + \frac{1}{mp} + \frac{2}{p} - \frac{2}{mp^2} \\
&\quad -1 + \frac{1}{p} + \frac{1}{p} - \frac{1}{p^2} + \frac{1}{mp} - \frac{1}{mp^2}
\end{aligned}
$$

$$+1 - \frac{1}{p} - \frac{1}{mp} - \frac{1}{p} + \frac{1}{p^2} + \frac{2}{mp^2}] = m\gamma\frac{1}{p} \ .$$

Therefore, from (E) and (F) we have

$$Cov(T^*_{il}, L^*_{il'}) \quad = \quad -m(\delta_{ll'} - \frac{1}{p})\gamma.$$

□

The ANOVA Table.

The derivation of the sums of squares of the ANOVA table for balanced crossover designs will follow the theory of the general linear model presented in Chapter 8 of Searle S.R. (1987), *Linear Models for Unbalanced Data*. New York: John Wiley & Sons.

In matrix notation, let $X_{(i,j)} = (X_{ij1}, \ldots, X_{ijp})'$ be the vector of the doubly-centered observations from the jth experimental unit within the ith replication,
$X_{(i)} = (X^{'}_{(i,1)}, \ldots, X^{'}_{(i,mp)})'$ the vector of mp^2 doubly-centered observations from the ith replication, and $X = (X^{'}_{(1)}, \ldots, X^{'}_{(n)})'$ the vector of all nmp^2 doubly-centered observations. Then, the linear model (4) considered for crossover designs can be written in the form $X = Z\beta + \eta$. The vector of parameters $\beta = (\tau \mid \lambda)'$ is decomposed in the vector of treatment effects $\tau = (\tau_1, \ldots, \tau_p)^{'}$ and the vector of carryover effects $\lambda = (\lambda_1, \ldots, \lambda_p)^{'}$. The design matrix is $Z = (D_\tau \mid D_\lambda + \frac{1}{p}D^p_\lambda)$, where D_τ and D_λ are the design matrices associated with τ and λ and D^p_λ is a matrix where the rows are formed by vectors with $p-1$ zeros and a single one. The single one for row ijk is in the column corresponding to the treatment which the jth experimental unit within the ith replication received in period p.

Partitioning the total sum of squares:

The total sum of squares is given by $SS_{total} = X'X$. The reduction in the total sum of squares due to fitting the model (4) is denoted $R(\beta) = R(\tau, \lambda) = X'Z\hat{\beta}$ where $\hat{\beta}$ is the vector of least squares estimates of β (Searle 1987, page 258). Therefore, the residual sum of squares is given by $SS_{res} = X'X - R(\beta)$. The model sum of squares $R(\beta)$ can be further partioned into the following sequential sum of squares (Searle 1987, page 273): the unadjusted treatment effects $R(\tau)$ and the carryover effects adjusted for treatment effects $R(\lambda \mid \tau) = R(\tau, \lambda) - R(\tau)$; or in the alternative sequence, the unadjusted carryover effects $R(\lambda)$ and the treatment effects adjusted for carryover effects $R(\tau \mid \lambda) = R(\tau, \lambda) - R(\lambda)$. Therefore, $R(\beta) = R(\tau) + R(\lambda \mid \tau) = R(\lambda) + R(\tau \mid \lambda)$.
The adjusted sum of squares used to construct the F-statistics for testing the hypotheses H^τ_0 and H^λ_0 are respectively $SS_\tau = R(\tau \mid \lambda)$ and $SS_\lambda = R(\lambda \mid \tau)$. To find explicitly these sum of squares we need $R(\tau)$, $R(\lambda)$ and $R(\tau, \lambda)$. First we have,

$$\begin{aligned} R(\tau, \lambda) &= X'Z\hat{\beta} = X'D_\tau\hat{\tau} + X'(D_\lambda + \frac{1}{p}D^p_\lambda)\hat{\lambda} \\ &= X'D_\tau\hat{\tau} + X'D_\lambda\hat{\lambda} + \frac{1}{p}X'D^p_\lambda\hat{\lambda} = \sum_{l=1}^{p} T_l\hat{\tau}_l + \sum_{l=1}^{p} L_l\hat{\lambda}_l + \sum_{l=1}^{p} F_l\hat{\lambda}_l \\ &= \sum_{l=1}^{p} T_l\hat{\tau}_l + \sum_{l=1}^{p} L_l\hat{\lambda}_l \quad \text{because } F_l = 0 \ . \end{aligned}$$

Also, the unadjusted sum of squares for the treatment effects is given by

$$R(\tau) = X'D_\tau \tilde{\tau} = \sum_{l=1}^{p} T_l \tilde{\tau}_l = nmp \sum_{l=1}^{p} \tilde{\tau}_l{}^2 \qquad by \qquad (12)$$

and the unadjusted sum of squares for the carryover effects is given by

$$\begin{aligned} R(\lambda) &= X'D_\lambda \tilde{\lambda} + \frac{1}{p} X'D_\lambda^p \tilde{\lambda} = \sum_{l=1}^{p} L_l \tilde{\lambda}_l + \sum_{l=1}^{p} F_l \tilde{\lambda}_l = \sum_{l=1}^{p} L_l \tilde{\lambda}_l \quad \text{because } F_l = 0 \\ &= \frac{nm(p^2-p-1)}{p} \sum_{l=1}^{p} \tilde{\lambda}_l{}^2 \qquad by \qquad (15)\ . \end{aligned}$$

Now, the adjusted sum of squares due to the carryover effects after the treatment effects have been included in the model is given by

$$\begin{aligned} R(\lambda \mid \tau) &= R(\tau, \lambda) - R(\tau) = \sum_{l=1}^{p} T_l \hat{\tau}_l + \sum_{l=1}^{p} L_l \hat{\lambda}_l - \sum_{l=1}^{p} T_l \tilde{\tau}_l \\ &= \sum_{l=1}^{p} T_l (\hat{\tau}_l - \tilde{\tau}_l) + \sum_{l=1}^{p} L_l \hat{\lambda}_l = \frac{1}{p} \sum_{l=1}^{p} T_l \hat{\lambda}_l + \sum_{l=1}^{p} L_l \hat{\lambda}_l \qquad by \qquad (13) \\ &= \frac{1}{p} \sum_{l=1}^{p} \hat{\lambda}_l [T_l + pL_l] = \frac{nm(p^2-p-2)}{p} \sum_{l=1}^{p} \hat{\lambda}_l{}^2 \qquad by \qquad (14)\ . \end{aligned}$$

Finally, the adjusted sum of squares due to the treatment effects after the carryover effects have been included in the model is given by

$$\begin{aligned} R(\tau \mid \lambda) &= R(\tau, \lambda) - R(\lambda) = \sum_{l=1}^{p} T_l \hat{\tau}_l + \sum_{l=1}^{p} L_l \hat{\lambda}_l - \sum_{l=1}^{p} L_l \tilde{\lambda}_l \\ &= \sum_{l=1}^{p} T_l \hat{\tau}_l + \sum_{l=1}^{p} L_l (\hat{\lambda}_l - \tilde{\lambda}_l) = \sum_{l=1}^{p} T_l \hat{\tau}_l + \sum_{l=1}^{p} L_l (\frac{p}{p^2-p-1}) \hat{\tau}_l \qquad by \qquad (16) \\ &= \sum_{l=1}^{p} \hat{\tau}_l [T_l + \frac{p}{(p^2-p-1)} L_l] = \frac{p}{(p^2-p-1)} \sum_{l=1}^{p} \hat{\tau}_l [\frac{(p^2-p-1)}{p} T_l + L_l] \\ &= \frac{nmp(p^2-p-2)}{(p^2-p-1)} \sum_{l=1}^{p} \hat{\tau}_l{}^2 \qquad by \qquad (17). \end{aligned}$$

To complete the ANOVA table, we need to find the expectation of the mean squares under model (4). We first find the expectation of T_l and L_l under model (4):

$$E(T_l) = \sum \sum \sum_{\{i,j,k \mid d_i(j,k)=l\}} (\tau_{d_i(j,k)} + \lambda_{d_i(j,k-1)} + \frac{1}{p} \lambda_{d_i(j,p)})$$

$$= \quad nmp\tau_l + nm \sum_{l'=1, l' \neq l}^{p} \lambda_{l'} + \frac{nm}{p} \sum_{l'=1}^{p} \lambda_{l'}$$

$$= \quad nmp\tau_l + nm \sum_{l'=1}^{p} \lambda_{l'} - nm\lambda_l = nmp\tau_l - nm\lambda_l$$

and

$$E(L_l) \quad = \quad \sum\sum \sum_{\{i,j,k|d_i(j,k-1)=l\}} (\tau_{d_i(j,k)} + \lambda_{d_i(j,k-1)} + \frac{1}{p}\lambda_{d_i(j,p)})$$

$$= \quad -nm\tau_l + nm(p-1)\lambda_l + \frac{nm}{p} \sum_{l'=1, l' \neq l}^{p} \lambda_{l'}$$

$$= \quad -nm\tau_l + nm(p-1)\lambda_l + \frac{nm}{p} \sum_{l'=1}^{p} \lambda_{l'} - \frac{nm}{p}\lambda_l$$

$$= \quad -nm\tau_l + nm(p-1-\frac{1}{p})\lambda_l.$$

Then, from (14) we have

$$E(\hat{\lambda}_l) \quad = \quad \frac{1}{nm(p^2-p-2)}(nmp\tau_l - nm\lambda_l)$$

$$+\frac{p}{nm(p^2-p-2)}[-nm\tau_l + nm(p-1-\frac{1}{p})\lambda_l]$$

$$= \quad \lambda_l \ ;$$

from (17) we have

$$E(\hat{\tau}_l) \quad = \quad \frac{(p^2-p-1)}{nmp(p^2-p-2)}(nmp\tau_l - nm\lambda_l)$$

$$+\frac{1}{nm(p^2-p-2)}[-nm\tau_l + nm(p-1-\frac{1}{p})\lambda_l] = \tau_l \ ;$$

from (15) we have

$$E(\tilde{\lambda}_l) \quad = \quad \frac{p}{nm(p^2-p-1)}[-nm\tau_l + nm(p-1-\frac{1}{p})\lambda_l] = \lambda_l - \frac{p}{(p^2-p-1)}\tau_l \ ;$$

and from (12) we have

$$E(\tilde{\tau}_l) \quad = \quad \frac{1}{nmp}[nmp\tau_l - nm\lambda_l] = \tau_l - \frac{1}{p}\lambda_l \ .$$

Now, to find the expectation of the mean squares for the unadjusted treatment effects, note that

$$Var(\tilde{\tau}_l) \quad = \quad Var(\frac{1}{nmp}\sum_{i=1}^{n} T_{il}) = \frac{1}{nm^2p^2}Var(T^*_{il})$$

$$= \frac{1}{nmp}(1-\frac{1}{p})\gamma \quad \text{by Lemma 2.}$$

Thus,

$$E(\tilde{\tau}_l{}^2) = Var(\tilde{\tau}_l) + [E(\tilde{\tau}_l)]^2 = \frac{1}{nmp}(1-\frac{1}{p})\gamma + (\tau_l - \frac{1}{p}\lambda_l)^2$$

and therefore

$$E[\frac{R(\tau_l)}{(p-1)}] = \frac{nmp}{(p-1)}\sum_{l=1}^{p} E(\tilde{\tau}_l{}^2) = \gamma + \frac{nmp}{(p-1)}\sum_{l=1}^{p}(\tau_l - \frac{1}{p}\lambda_l)^2 \ .$$

To find the expectation of the mean squares for the adjusted carryover effects, note that

$$\begin{aligned}
Var(\hat{\lambda}_l) &= Var[\frac{1}{nm(p^2-p-2)}T_l + \frac{p}{nm(p^2-p-2)}L_l] \\
&= Var[\frac{1}{nm(p^2-p-2)}\sum_{i=1}^{n} T^*_{il} + \frac{p}{nm(p^2-p-2)}\sum_{i=1}^{n} L^*_{il}] \\
&= Var\{\frac{1}{nm}\sum_{i=1}^{n}[\frac{1}{(p^2-p-2)}T^*_{il} + \frac{p}{(p^2-p-2)}L^*_{il}]\} \\
&= \frac{1}{nm^2}Var[\frac{1}{(p^2-p-2)}T^*_{il} + \frac{p}{(p^2-p-2)}L^*_{il}] \\
&= \frac{1}{nm^2}[\frac{1}{(p^2-p-2)^2}Var(T^*_{il}) + \frac{2p}{(p^2-p-2)^2}Cov(T^*_{il}, L^*_{il}) \\
&\quad + \frac{p^2}{(p^2-p-2)^2}Var(L^*_{il})] \\
&= \frac{1}{nm^2}[\frac{1}{(p^2-p-2)^2}mp(1-\frac{1}{p})\gamma - \frac{2pm}{(p^2-p-2)^2}(1-\frac{1}{p})\gamma \\
&\quad + \frac{p^2}{(p^2-p-2)^2}m\frac{(p^2-p-1)}{p}(1-\frac{1}{p})\gamma] \quad \text{by Lemma 2} \\
&= \frac{\gamma}{nm(p^2-p-2)^2}[(p-1) - 2(p-1) + (p^2-p-1)(p-1)] \\
&= \frac{\gamma}{nm(p^2-p-2)^2}[(p^2-p-2)(p-1)] = \frac{(p-1)\gamma}{nm(p^2-p-2)}.
\end{aligned}$$

Therefore,

$$E[\frac{R(\lambda \mid \tau)}{(p-1)}] = \frac{nm(p^2-p-2)}{p(p-1)}\sum_{l=1}^{p} E(\hat{\lambda}_l^2)$$

$$= \frac{nm(p^2-p-2)}{p(p-1)} \sum_{l=1}^{p} \{Var(\hat{\lambda}_l) + [E(\hat{\lambda}_l)]^2\}$$

$$= \gamma + \frac{nm(p^2-p-2)}{p(p-1)} \sum_{l=1}^{p} \lambda_l^2 \ .$$

To find the expectation of the mean squares for the unadjusted carryover effects, note that

$$Var(\tilde{\lambda}_l) = Var[\frac{p}{nm(p^2-p-1)} L_l] = Var[\frac{p}{nm(p^2-p-1)} \sum_{i=1}^{n} L_{il}^*$$

$$= \frac{p^2}{n^2m^2(p^2-p-1)^2} \sum_{i=1}^{n} Var(L_{il}^*)$$

$$= \frac{p^2}{nm^2(p^2-p-1)^2}[\frac{m(p^2-p-1)}{p}(1-\frac{1}{p})\gamma] \quad \text{by Lemma 2}$$

$$= \frac{(p-1)\gamma}{nm(p^2-p-1)} \ .$$

Therefore,

$$E[\frac{R(\lambda)}{(p-1)}] = \frac{nm(p^2-p-1)}{p(p-1)} \sum_{l=1}^{p} E(\tilde{\lambda_l^2})$$

$$= \frac{nm(p^2-p-1)}{p(p-1)} \sum_{l=1}^{p} \{Var(\tilde{\lambda}_l) + [E(\tilde{\lambda}_l)]^2\}$$

$$= \gamma + \frac{nm(p^2-p-1)}{p(p-1)} \sum_{l=1}^{p} (\lambda_l - \frac{p}{(p^2-p-1)}\tau_l)^2 \ .$$

To find the expectation of the mean squares for the adjusted treatment effects, note that

$$Var(\hat{\tau}_l) = Var[\frac{(p^2-p-1)}{nmp(p^2-p-2)} T_l + \frac{1}{nm(p^2-p-2)} L_l]$$

$$= Var[\frac{(p^2-p-1)}{nmp(p^2-p-2)} \sum_{i=1}^{n} T_{il}^* + \frac{1}{nm(p^2-p-2)} \sum_{i=1}^{n} L_{il}^*]$$

$$= \frac{1}{nm^2}[\frac{(p^2-p-1)^2}{p^2(p^2-p-2)^2} Var(T_{il}^*) + \frac{2(p^2-p-1)}{p(p^2-p-2)^2} Cov(T_{il}^*, L_{il}^*)$$

$$+ \frac{1}{(p^2-p-2)^2} Var(L_{il}^*)]$$

$$
\begin{aligned}
&= \frac{1}{nm^2}[\frac{(p^2-p-1)^2}{p^2(p^2-p-2)^2}mp(1-\frac{1}{p})\gamma - \frac{2(p^2-p-1)m}{p(p^2-p-2)^2}(1-\frac{1}{p})\gamma \\
&\quad + \frac{1}{(p^2-p-2)^2}\frac{m(p^2-p-1)}{p}(1-\frac{1}{p})\gamma] \quad \text{by Lemma 2} \\
&= \frac{(p^2-p-1)(p-1)}{nmp^2(p^2-p-2)^2}\gamma[(p^2-p-1)-2+1] \\
&= \frac{(p^2-p-1)(p-1)}{nmp^2(p^2-p-2)}\gamma \ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[\frac{R(\tau \mid \lambda)}{(p-1)}] &= \frac{nmp(p^2-p-2)}{(p-1)(p^2-p-1)}\sum_{l=1}^{p} E(\hat{\tau}_l^2) \\
&= \frac{nmp(p^2-p-2)}{(p-1)(p^2-p-1)}\sum_{l=1}^{p}\{Var(\hat{\tau}_l)+[E(\hat{\tau}_l)]^2\} \\
&= \gamma + \frac{nmp(p^2-p-2)}{(p-1)(p^2-p-1)}\sum_{l=1}^{p}\tau_l^2 \ .
\end{aligned}
$$

Finally, to find the expectation of the residual sum of squares, we first need to find the expectation of $X'X$:

$$
E(X'X) = \sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p} E(X_{ijk}^2) = \sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}\{Var(X_{ijk})+[E(X_{ijk})]^2\} \ .
$$

Recall that,

$$
Var(X_{ijk}) = Var(\eta_{ijk}) = (1-\frac{1}{mp})(1-\frac{1}{p})\gamma
$$

and

$$
\begin{aligned}
E(X_{ijk}) &= \tau_{d_i(j,k)} + \lambda_{d_i(j,k-1)} + \frac{1}{p}\lambda_{d_i(j,p)} \\
[E(X_{ijk})]^2 &= \tau_{d_i(j,k)}^2 + \lambda_{d_i(j,k-1)}^2 + \frac{1}{p^2}\lambda_{d_i(j,p)}^2 \\
&\quad + 2\tau_{d_i(j,k)}\lambda_{d_i(j,k-1)} + \frac{2}{p}\tau_{d_i(j,k)}\lambda_{d_i(j,p)} + \frac{2}{p}\lambda_{d_i(j,k-1)}\lambda_{d_i(j,p)} \ .
\end{aligned}
$$

But because the crossover design considered is balanced, we have

$$
\sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}\tau_{d_i(j,k)}^2 = nmp\sum_{l=1}^{p}\tau_l^2
$$

$$\sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}\lambda^2_{d_i(j,k-1)} = nm(p-1)\sum_{l=1}^{p}\lambda_l^2$$

$$\sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}\frac{1}{p^2}\lambda^2_{d_i(j,p)} = \frac{nm}{p}\sum_{l=1}^{p}\lambda_l^2$$

$$\sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}2\tau_{d_i(j,k)}\lambda_{d_i(j,k-1)} = 2nm\sum_{l=1}^{p}\sum_{l'\neq l}\tau_l\lambda_{l'}$$

$$= 2nm[\sum_{l=1}^{p}\sum_{l'=1}^{p}\tau_l\lambda_{l'} - \sum_{l=1}^{p}\tau_l\lambda_l] = -2nm\sum_{l=1}^{p}\tau_l\lambda_l$$

$$\sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}\frac{2}{p}\tau_{d_i(j,k)}\lambda_{d_i(j,p)} = \frac{2n}{p}\sum_{j=1}^{mp}\lambda_{d_i(j,p)}\sum_{k=1}^{p}\tau_{d_i(j,k)} = 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}\frac{2}{p}\lambda_{d_i(j,k-1)}\lambda_{d_i(j,p)} = \frac{2n}{p}\sum_{j=1}^{mp}\lambda_{d_i(j,p)}[\sum_{k=1}^{p}\lambda_{d_i(j,k)} - \lambda_{d_i(j,p)}]$$

$$= \frac{2n}{p}\sum_{j=1}^{mp}\lambda_{d_i(j,p)}(-\lambda_{d_i(j,p)}) = -\frac{2n}{p}\sum_{j=1}^{mp}\lambda^2_{d_i(j,p)} = -\frac{2nm}{p}\sum_{l=1}^{p}\lambda_l^2\ .$$

Therefore,

$$\begin{aligned}
E(SS_{res}) &= \sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}\{Var(X_{ijk}) + [E(X_{ijk})]^2\} - E[R(\tau)] - E[R(\lambda \mid \tau)] \\
&= nmp^2(1-\frac{1}{mp})(1-\frac{1}{p})\gamma + nmp\sum_{l=1}^{p}\tau_l^2 + nm(p-1)\sum_{l=1}^{p}\lambda_l^2 \\
&\quad + \frac{nm}{p}\sum_{l=1}^{p}\lambda_l^2 - 2nm\sum_{l=1}^{p}\tau_l\lambda_l - \frac{2nm}{p}\sum_{l=1}^{p}\lambda_l^2 \\
&\quad -(p-1)\gamma - nmp\sum_{l=1}^{p}(\tau_l - \frac{1}{p}\lambda_l)^2 \\
&\quad -(p-1)\gamma - \frac{nm}{p}(p^2-p-2)\sum_{l=1}^{p}\lambda_l^2 \\
&= [n(mp-1)(p-1) - 2(p-1)]\gamma + nmp\sum_{l=1}^{p}\tau_l^2
\end{aligned}$$

$$
+\frac{nm}{p}(p^2 - p + 1 - 2 - p^2 + p + 2)\sum_{l=1}^{p}\lambda_l^2
$$

$$
-2nm\sum_{l=1}^{p}\tau_l\lambda_l - nmp\sum_{l=1}^{p}(\tau_l^2 - \frac{2}{p}\tau_l\lambda_l + \frac{1}{p^2}\lambda_l^2)
$$

$$
= \quad (p-1)[n(mp-1)-2]\gamma\ .
$$

Hence, we have the following ANOVA table for balanced crossover designs using n replications of m Latin squares of order $p \geq 3$:

source of variation	sum of squares	degrees of freedom	expectation of mean square under the full model
treatment effects (unadjusted)	$R(\tau) = SS^*_{\boldsymbol{\tau}} = nmp\sum_{l=1}^{p}\tilde{\tau}_l^2$	$p-1$	$\gamma + \frac{nmp}{(p-1)}\sum_{l=1}^{p}(\tau_l - \frac{1}{p}\lambda_l)^2$
carry-over effects (adjusted)	$R(\lambda \mid \tau) = SS_{\boldsymbol{\lambda}} = \frac{nm(p^2-p-2)}{p}\sum_{l=1}^{p}\hat{\lambda}_l^2$	$p-1$	$\gamma + \frac{nm(p^2-p-2)}{p(p-1)}\sum_{l=1}^{p}\lambda_l^2$
or			
carry-over effects (unadjusted)	$R(\lambda) = SS^*_{\boldsymbol{\lambda}} = \frac{nm(p^2-p-1)}{p}\sum_{l=1}^{p}\tilde{\lambda}_l^2$	$p-1$	$\gamma + \frac{nm(p^2-p-1)}{p(p-1)}\sum_{l=1}^{p}(\lambda_l - \frac{p}{(p^2-p-1)}\tau_l)^2$
treatment effects (adjusted)	$R(\tau \mid \lambda) = SS_{\boldsymbol{\tau}} = \frac{nmp(p^2-p-2)}{(p^2-p-1)}\sum_{l=1}^{p}\hat{\tau}_l^2$	$p-1$	$\gamma + \frac{nmp(p^2-p-2)}{(p^2-p-1)(p-1)}\sum_{l=1}^{p}\tau_l^2$
residual	$SS_{res} = \begin{cases} \sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}X_{ijk}^2 - SS^*_{\boldsymbol{\tau}} - SS_{\boldsymbol{\lambda}} \\ \text{or} \\ \sum_{i=1}^{n}\sum_{j=1}^{mp}\sum_{k=1}^{p}X_{ijk}^2 - SS^*_{\boldsymbol{\lambda}} - SS_{\boldsymbol{\tau}} \end{cases}$	$(p-1)[n(mp-1)-2]$	γ