WHAT PRACTITIONERS NEED TO KNOW . . .

. . . About Time Diversification

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Suppose you plan to purchase a new home in three months, at which time you will be required to pay $100,000 in cash. Assuming you have the necessary funds, would you be more inclined to invest these funds in a riskless asset such as a Treasury bill or in a risky asset such as an S&P 500 index fund?

Now consider a second question. Suppose you plan to purchase a new home *10 years* from now, and that you currently have $100,000 to apply toward the purchase of this home. How would you invest these funds, given the choice between a riskless investment and a risky investment?

The only difference between these two scenarios is the length of your investment horizon. In the first case, you have a three-month horizon; in the second case, your investment horizon equals 10 years. If you are a typical investor, you would probably select the riskless investment for the three-month horizon and the risky investment for the 10-year horizon.

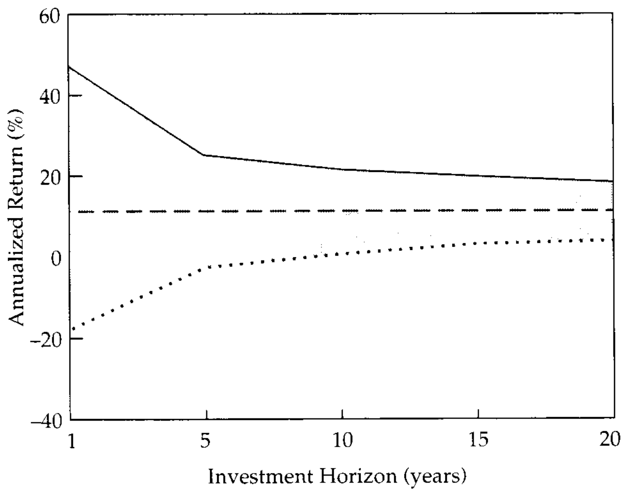
You might rationalize your choice as follows. Even though you expect stocks to generate a higher return over the long term, by investing in Treasury bills you are certain to have the requisite funds to satisfy your down payment three months from now. If you were to invest in stocks, there is a significant chance you could lose part of your savings, with little opportunity to recoup this loss, and be unable to meet the down payment requirement. But over a 10-year investment horizon, favorable short-term stock returns are likely to offset poor short-term stock returns; it is thus more likely that stocks will realize a return close to their expected return.

The Argument for Time Diversification

The notion that above-average returns tend to offset below-average returns over long horizons is called *time diversification*. Specifically, if returns are independent from one year to the next, the standard deviation of annualized returns diminishes with time. The distribution of annualized returns consequently converges as the investment horizon increases.

Figure A shows a 95% confidence interval of annualized returns as a function of investment horizon, assuming that the expected return is 10% and the standard deviation of returns equals 15%. These confidence intervals are based on the assumption that the returns are lognormally distributed; thus the standard deviation measures the dispersion of the logarithms of one plus the returns. It is apparent from Figure A that the distribution of annualized returns converges as the investment horizons lengthens.

Figure A. Annualized Returns



It might also be of interest to focus on the notion of time diversification from the perspective of *losing* money. We can determine the likelihood of a negative return by measuring the difference in standard deviation units between a 0% return and the expected return. Again, if we assume that the S&P's expected return equals 10% and its standard deviation equals 15%, the expected return is 0.64 standard deviation above a 0% return, given a one-year horizon. This value corresponds to a 26% probability that the S&P 500 will generate a negative return in any one year.

Given a 10-year horizon, however, the annualized expected return is 2.01 standard deviations above an annualized return of 0.0%. There is only a 2.2% chance that the S&P 500 will produce a negative return, on average, over 10 years.[1] This does not imply that it is just as improbable to lose money in any *one* of these 10 years; it merely reflects the tendency of above-average returns to cancel out below-average returns.

Time Diversification Refuted

Several prominent financial economists, most notably Paul Samuelson, have argued that the notion of time diversification is specious for the following reason.[2]

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Although it is true that the annualized dispersion of returns converges toward the expected return with the passage of time, the dispersion of terminal wealth also diverges from the expected terminal wealth as the investment horizon expands.

This result implies that, although you are less likely to lose money over a long horizon than over a short horizon, the magnitude of your potential loss increases with the duration of your investment horizon. According to the critics of time diversification, if you elect the riskless alternative when you are faced with a three-month horizon, you should also elect the riskless investment when your horizon equals 10 years, or 20 years or, indeed, any duration.

This criticism applies to cross-sectional diversification as well as to temporal diversification. Suppose you have an opportunity to invest $10,000 in a risky venture, and you decline this opportunity because you think it is too risky. Would you be less averse to investing in 10 independent ventures, each of which has the same risk as the venture you declined and each of which requires a $10,000 investment?

You are clearly less likely to lose money by investing in 10 equally risky but independent ventures than by investing in just one of these ventures. The amount you could conceivably lose, however, is 10 times as great as your exposure in a single venture.

Now consider a third choice. Suppose you are offered a chance to invest a total of $10,000 in 10 independent but equally risky ventures. In this case you would invest only $1000 in each of the 10 risky ventures. This investment opportunity diversifies your risk across the 10 ventures without increasing your total exposure. You might still choose not to invest, but your opposition to it should be less intense than it was to the first two alternatives.

Perhaps you are unpersuaded by these arguments. You reason as follows. Although it is true that the dispersion of terminal wealth increases with the passage of time or with the number of risky opportunities, the expected wealth of the risky venture also increases. The dispersion of wealth thus expands around a growing mean as the horizon lengthens or as the number of independent risky ventures increases.

Consider again the choice of investing in an S&P 500 index fund versus a riskless asset. Suppose the riskless asset has a certain 3% annual return compared with the S&P's 10% expected return and 15% standard deviation. Table 1 compares the dispersion of terminal wealth of the S&P 500 with the certain terminal wealth of the riskless investment.

After one year, the terminal wealth of an initial $100,000 investment in the S&P index fund ranges from $81,980 to $147,596 given a confidence interval of 95%, while the riskless investment grows with certainty to $103,000. After 10 years, the spread in the S&P investment's terminal wealth expands from $65,616 to $554,829, but it surrounds a higher expected wealth. Thus the lower boundary of the 95% confidence interval is greater than the initial investment. If the investment horizon is extended to 20 years, the lower boundary of the 95% confidence interval actually exceeds the terminal wealth of the riskless investment.

Although this line of reasoning might strike you as a credible challenge to the critics of time diversification, in the limit it fails to resurrect the validity of time diversification.[3] Even though it is true that the lower boundary of a 95% confidence interval of the S&P investment exceeds the terminal wealth of the riskless investment after 20 years, the lower boundary of a 99% confidence interval falls below the riskless investment, and the lower boundary of a 99.9% confidence interval is even worse. The growing improbability of a loss is offset by the increasing magnitude of potential losses.

It is an indisputable mathematical fact that if you prefer a riskless asset to a risky asset given a three-month horizon, you should also prefer a riskless asset to a risky asset given a 10-year horizon, assuming the following conditions are satisfied:

1. Your risk aversion is invariant to changes in your wealth.
2. You believe that risky returns are random.
3. Your future wealth depends only on investment results.

Risk aversion implies that the satisfaction you derive from increments to your wealth is not linearly related to increases in your wealth. Rather, your satisfaction increases at a decreasing rate as your wealth increases. You thus derive more satisfaction when your wealth grows from $100,000 to $150,000 than you do when it grows from $150,000 to $200,000. It also follows that a decrease in your wealth conveys more disutility

Table 1. Risky versus Riskless Terminal Wealth

	S&P 500 95% Confidence Interval		
	Lower Boundary	Upper Boundary	Riskless Asset Terminal Wealth
1 Year	$ 81,980	$ 147,596	$103,000
5 Years	83,456	310,792	115,927
10 Years	102,367	657,196	134,392
15 Years	133,776	1,304,376	155,797
20 Years	180,651	2,565,345	180,611

than the utility that comes from an equal increase in your wealth.[4]

The financial literature commonly assumes that the typical investor has a utility function equal to the logarithm of wealth. Based on this assumption, I will demonstrate numerically why it is that your investment horizon is irrelevant to your choice of a riskless versus a risky asset.

Suppose you have $100.00. This $100.00 conveys 4.60517 units of utility [ln(100.00) = 4.60517]. Now consider an investment opportunity that has a 50% chance of a ⅓ gain and a 50% chance of a ¼ loss. A $100.00 investment in this risky venture has an expected terminal wealth equal to $104.17, but it too conveys 4.60517 units of utility [50% × ln(133.33) + 50% × ln(125.00) = 4.60517]. Therefore, if your utility function is defined by the logarithm of wealth, you should be indifferent between holding onto your $100.00 or investing it in this risky venture. In this example, $100.00 is the certainty equivalent of the risky venture because it conveys the same utility as the riskless venture.

Now suppose you are offered an opportunity to invest in this risky venture over two periods, and the same odds prevail. Your initial $100.00 investment can either increase by ⅓ with a 50% probability in each of the two periods or it can decrease by ¼ with a 50% probability in each of the two periods. Over two periods, the expected terminal wealth increases to $108.51, but the utility of the investment opportunity remains the same. You should thus remain indifferent between keeping your $100.00 and investing it over two independent periods.

The same mathematical truth prevails irrespective of the investment horizon. The expected utility of the risky venture will always equal 4.60517, implying that you derive no additional satisfaction by diversifying your risk across time. This result holds even though the standard deviation of returns increases approximately with the square root of time, while the expected terminal wealth increases almost linearly with time.

Table 2 shows the possible outcomes of this investment opportunity after one, two and three periods, along with the expected wealth and expected utility after each period. The possible wealth values are computed by linking all possible sequences of return. Expected wealth equals the probability-weighted sum of each possible outcome, while expected utility equals the probability-weighted sum of the logarithm of each possible wealth outcome.

This result does not require that you have a log wealth utility function. Suppose, instead, that your utility function is defined by minus the reciprocal of wealth. This utility function implies greater risk aversion than a log wealth utility function. You would thus prefer to hold onto your $100.00 given the opportunity to invest in a risky venture that has an equal chance of increasing by ⅓ or decreasing by ¼. You would, however, be indifferent between a certain $100.00 and a risky venture that offers an equal chance of increasing by ⅓ or decreasing by ⅕.

Table 3 shows that the expected utility of this risky venture remains constant as a function of investment horizon, even though the expected terminal wealth grows at a faster pace than it does in the previous

Table 2. Utility = ln(Wealth)

		Distribution of Wealth After		
	Starting Wealth	One Period	Two Periods	Three Periods
				1/8 × 237.04
			1/4 × 177.78	
				1/8 × 133.33
		1/2 × 133.33		
				1/8 × 133.33
			1/4 × 100.00	
				1/8 × 75.00
	100.00			
				1/8 × 133.33
			1/4 × 100.00	
				1/8 × 75.00
		1/2 × 75.00		
				1/8 × 75.00
			1/4 × 56.25	
				1/8 × 42.19
Expected Wealth	100.00	104.17	108.51	113.03
Expected Utility	4.60517	4.60517	4.60517	4.60517

Table 3. Utility = − 1/Wealth

		Distribution of Wealth After		
	Starting Wealth	One Period	Two Periods	Three Periods
				1/8 × 237.04
			1/4 × 177.78	
				1/8 × 142.22
		1/2 × 133.33		
				1/8 × 142.22
			1/4 × 106.67	
				1/8 × 85.33
	100.00			
				1/8 × 142.22
			1/4 × 106.67	
				1/8 × 85.33
		1/2 × 80.00		
				1/8 × 85.33
			1/4 × 64.00	
				1/8 × 51.20
Expected Wealth	100.00	106.67	113.78	121.36
Expected Utility	−0.01000	−0.01000	−0.01000	−0.01000

example. Again, time diversification would not induce you to favor the risky venture over a multiperiod horizon if you did not prefer it for a single-period horizon.

Time Diversification Resurrected

Now that you have been exposed to the incontrovertible truth that time does not diversify risk, would you truly invest the same in your youth as you would in your retirement? There are several valid reasons why you might still condition your risk posture on your investment horizon, even though you accept the mathematical truth about time diversification.

First, you may not believe that risky asset returns are random. Perhaps investment returns follow a mean-reverting pattern. If returns revert to their mean, then the dispersion of terminal wealth increases at a slower rate than implied by a lognormal distribution (the distribution that results from random returns). If you are more averse to risk than the degree of risk aversion implicit in a log wealth utility function, then a mean-reverting process will lead you to favor risky assets over a long horizon, even if you are indifferent between a riskless and a risky asset over a short horizon.[5]

Suppose, for example, that returns are not random. Instead, the risky venture in Table 3 has a 60% chance of reversing direction and, therefore, only a 40% chance of repeating its prior return. Table 4 reveals that expected utility rises from −0.010 over a single period to −0.00988 over two periods and to −0.00978 over three periods. Thus, if you believe in mean reversion and you are more risk averse than a log wealth investor, you would rationally increase your exposure to risk as your investment horizon expands.

This result does not apply, however, to investors who have a log wealth utility function. These investors would not be induced to accept more risk over longer horizons, even if they believed in mean reversion.

Second, you might believe that the extremely bad outcomes required to justify the irrelevancy of time diversification would result from events or conditions that would have equally dire consequences for the so-called riskless asset, especially if you measure wealth in consumption units.

Third, even if you believe that returns are random, you might still choose to accept more risk over longer horizons than over shorter horizons because you have more discretion to adjust your consumption and work habits.[6] If a risky investment performs poorly at the beginning of a short horizon, there is not much you can do to compensate for this loss in wealth. If a risky investment performs poorly at the beginning of a long horizon, however, you can postpone consumption or work harder to achieve your financial goals. The argument against time diversification assumes implicitly that your terminal wealth depends only on investment performance.

Fourth, you may have a discontinuous utility function. Consider, for example, a situation in which you require a minimum level of wealth to maintain a certain standard of living. Your lifestyle might change drastically if you penetrate this threshold, but further reductions in wealth are less meaningful. You might be more likely to penetrate the threshold given a risky investment over a short horizon than you would be if you invested in the same risky asset over the long run.

Moreover, even if you are not confronted with a real

Table 4. Utility = − 1/Wealth with Mean Reversion

	Starting Wealth	Distribution of Wealth After: One Period	Two Periods	Three Periods
				0.08 × 237.04
			0.20 × 177.78	
				0.12 × 142.22
		1/2 × 133.33		
				0.18 × 142.22
			0.30 × 106.67	
				0.12 × 85.33
	100.00			
				0.12 × 142.22
			0.30 × 106.67	
				0.18 × 85.33
		1/2 × 80.00		
				0.12 × 85.33
			0.20 × 64.00	
				0.08 × 51.20
Expected Wealth	100.00	106.67	112.36	118.63
Expected Utility	−0.0100	−0.0100	−0.00988	−0.00978

threshold, you might still behave as though you have a discontinuous utility function. Perhaps we can only process a finite set of possible outcomes, or maybe human nature leads us to ignore terrible outcomes that are extremely remote. Only the passage of time will reveal whether or not such behavior is prudent.

Finally, you are irrational. This does not mean you are a bad person. It simply implies that you behave inconsistently.

Footnotes

1. For a review of the relation between probability estimation and the dispersion of returns, see M. Kritzman, "What Practitioners Need to Know About Uncertainty," *Financial Analysts Journal*, March/April 1991.
2. For example, see P. Samuelson, "Risk and Uncertainty: A Fallacy of Large Numbers," *Scientia*, April/May 1963; P. Samuelson, "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, August 1969; and Z. Bodie, A. Kane and A. Marcus, *Investments* (Homewood, IL: Irwin, 1989), 222–26.
3. I posed this argument in a letter to Paul Samuelson on December 2, 1991. In his reply, he eloquently and convincingly disabused me of the notion that a rising mean overcomes the increase in dispersion.
4. For a review of utility theory, see M. Kritzman, "What Practitioners Need to Know About Utility," *Financial Analysts Journal*, May/June 1992.
5. Samuelson addresses this result in P. Samuelson, "Longrun Risk Tolerance When Equity Returns are Mean Reverting: Pseudoparadoxes and Vindication of 'Businessman's Risk'," in W. Brainard, W. Nordhaus and H. Watts, eds., *Macroeconomics, Finance and Economic Policy: Essays in Honor of James Tobin* (Cambridge, MA: MIT Press, 1991).
6. This idea is attributed to Zvi Bodie and William Samuelson, both of Boston University.