STOCHASTIC NAVIER-STOKES EQUATIONS IN VORTICITY FORM

(ÉQUATIONS DE NAVIER-STOKES STOCHASTIQUES POUR LA VORTICITÉ)

par

XIAOGANG CHEN

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FACULTÉ DES SCIENCES
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Contents

Résumé en français

Prennez note que le sommaire qui suit retrace en français les principaux éléments de contexte de cette étude présentés en anglais au chapitre suivant. On brosse ici également les grandes lignes de la structure du texte.

L'étude de la dynamique des fluides fascine l'oeil et l'esprit, qu'il s'agisse de l'écoulement laminaire d'un long fleuve tranquille ou de celui, turbulent, que soulève en cascade le fond irrégulier d'une rivière trop inclinée. Faire un historique complet de la mécanique des fluides serait présomptueux ici et d'autres que nous l'ont fait avec autorité (par exemple, McComb [65] et Chorin et Marsden [20]). Nous en extrayons les quelques informations qui suivent et nous y renvoyons le lecteur intéressé d'en savoir davantage.

Contentons nous ici de rappeler que c'est Newton qui, en 1657, a le premier décrit mathématiquement la viscosité, la définissant comme étant la variation de la quantité de mouvement des particules d'un fluide en écoulement laminaire par unité de surface. Cette variation est physiquement interprétée comme résultant de la dissipation d'énergie qu'entraîne l'interaction entre éléments du fluide macroscopiquement distants.

Ce n'est que près de cent ans plus tard, en 1755, qu'Euler décrit avec rigueur et exactitude l'écoulement d'un fluide parfait, c'est-à-dire, un fluide dont la viscosité est nulle, dont les forces internes d'interaction sont isotropes et pour lequel l'écoulement s'effectue sans dissipation d'énergie thermique. Les hypothèses de travail d'Euler étaient que l'écoulement est laminaire et en phase unique. Par exemple, si le milieu est un tuyau, il n'y a donc pas d'air dans le tuyau et la description du flot vaut partout sauf trop près des bords, là où pourrait sévir la turbulence.

L'équation de Navier-Stokes apparaît pour la première fois dans le cours d'hydrodynamique de Navier (1822), qui marque le début d'un siècle et demi de foisonnement scientifique autour de l'utilisation de cette équation en mécanique des fluides. L'objectif poursuivi en la dérivant est de décrire complètement le mouvement des éléments de volume d'un fluide qui peut être visqueux ou non, compressible ou non, turbulent ou non.

Rappellons ici que l'incompressibilité est définie comme la transmission intégrale des pressions externes sur le fluide, par le biais de l'équilibre des rapports de force par aire de section. Citons quelques exemples ici. Sont incompressibles tous les liquides, localement; les gaz à vitesse hypersonique (plus de MACH 8) et tous les fluides en phase fortement visqueuse. Sont compressibles les gaz s'écoulant à des vitesses plus modérées et les liquides s'écoulant dans des tuyaus longs à pression variable comme des pipelines.

On peut maintenant énoncer la fameuse équation de Navier-Stokes, comme suit.

Considérons $\rho = \rho(x,t)$ la densité (masse volumique), $p = p(x,t)$ la pression (volumique), $f = f(x,t)$ la densité (volumique) du champ de force externe (supposé connu), $T = T(x,t)$ la température (connue elle aussi) et ν la viscosité (celle dite dynamique, qu'on supposera connue également mais qu'il ne faut pas confondre avec la viscosité cinématique, qui est donnée par le rapport ν/ρ). La formulation exacte nécessite également deux autres fonctions $\Phi = \Phi(\rho, p, T)$ et $\lambda = \lambda(\rho, p, T)$, qui sont bien connues mais dont on ne détaillera pas la forme ici.

On recherche des solutions $u = u(x,t)$ aux trois équations suivantes, munies de conditions aux frontières appropriées. Ici u représente la vitesse du flot sous considération. Tout d'abord l'équation de Navier-Stokes (quand $\nu = 0$ on l'appelle équation d'Euler)

$$\rho\Big(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\Big) - \nu \triangle u - (3\lambda + \nu)\nabla div(u) + \nabla p = f \tag{0.0.1}$$

résulte de la conservation des bilans d'énergie et de force par unité de surface, comme on le verra au chapitre 3. C'est là que sera expliquée plus en détail l'interprétation exacte des opérateurs de gradient ∇ et de Laplace $\triangle$.

La seconde équation, appelée équation de continuité,

$$\frac{\partial \rho}{\partial t} + div(\rho u) = 0 \tag{0.0.2}$$

est une conséquence directe de la conservation de la quantité de mouvement, via le théorème de Gauss liant intégrales de surfaces et de volumes induits. Finalement, un bilan thermodynamique fournit la troisième équation

$$\Phi(\rho, p, T) = 0\,. \tag{0.0.3}$$

La plus remarquable qualité de (0.0.1) est sans doute la vaste étendue de son domaine de validité aux yeux des ingénieurs. En effet, au moins jusqu'en 1990 lorsqu'écrit McComb [65], il n'existe aucun contrexemple expérimental d'écoulement gazeux turbulent remettant en cause sa validité en bas de MACH 15, soit quinze fois la vitesse du son! Cette versalité d'application explique l'abondance tant des méthodes développées au fil des ans pour la résoudre et l'analyser, que de la littérature s'y rapportant.

Pour un mathématicien, la validité de (0.0.1) passe d'abord par une définition précise de ce qu'on entend par solution de cette équation, puis par la démonstration rigoureuse de son existence et, idéalement, de son unicité. Dans tous les manuels mathématiques de facture classique sur le sujet, même les plus récents ([20], [26], [72]), on ne s'attarde toujours à la démonstration de l'existence d'une telle solution, que dans le cas restreint des fluides homogènes et incompressibles, autrement dit, lorsque les paramètres ρ, p et T sont des constantes connues. Les éléments de base des preuves proposées dans ces ouvrages, remontent aux travaux de Jean Leray [60]. Il existe bien sûr maintenant des méthodes déterministes beaucoup plus puissantes, dues en particulier aux travaux de Pierre-Louis Lions [61] en analyse fonctionnelle, mais elles seraient trop complexes à décrire ici et l'approche que nous adoptons est résolument probabiliste dans sa facture.

Nous verrons au chapitre 3 comment les méthodes stochastiques utilisées dans la présente thèse, permettent d'aborder et de résoudre le problème de l'existence et de la caractérisation de la solution dans certains cas où l'hypothèse d'incompressibilité est invalide. On aura cependant toujours l'hypothèse d'homogénéité et on stipulera donc pour simplifier que $\rho = 1$ est vérifiée partout. Soulignons qu'alors l'équation (0.0.2) devient tout simplement $div(u) = 0$, qui doit être vérifiée en tout point à l'intérieur du domaine de validité de l'équation (0.0.1).

La littérature sur les équations de Navier-Stokes abonde, comme en fait foi une brève recherche sur le site http://www.ams.org/mathscinet/search , qui décompte plus de 3,800 articles sur le sujet, dont plus de 100 actuellement en revue critique. Afin de préserver une taille raisonable à une introduction déjà longue, nous ne traiterons qu'au premier chapitre la revue de la littérature récente (surtout au cours des deux dernières années) et on s'y astreindra aux ouvrages entourant les constructions stochastiques de systèmes de particules ayant pour limites des solutions à (0.0.1). On excluera même le cas des équations classiques forcées aléatoirement, où seule f est aléatoire, puisque les articles qui s'y rapportent utilisent généralement des méthodes déterministes de construction et que, dans le cadre des constructions qui nous occuperont ici, cette subtilité n'apporte pas vraiment d'éléments nouveaux au traitement mathématique.

La présente thèse s'attardera donc sur une seule et unique approche, celle de la recherche de solutions dites statistiques, telles qu'introduites par Wiener dans [76], deux ans seulement avant que Kolmogorov publie sa célèbre théorie de la turbulence. L'idée de Wiener, délaissée pendant plusieurs décennies, fut reprise indépendamment semble-t-il par Bleher et Vishik [9] et par Foias [37] qui produirent les premiers résultats rigoureux en ce sens. Aujourd'hui, leur présentation demeure moderne puisqu'elle repose sur la théorie des équations différentielles stochastiques, dont nous rappellerons l'essentiel pour nos fins au chapitre 2.

L'objectif de notre étude est d'évaluer le comportement asymptotique de systèmes de particules en interaction qui ont comme limite macroscopique un flot d'écoulement fluide solution de l'équation de Navier-Stokes classique, tant dans le cas réaliste à trois dimensions (3D) que dans le cas d'un flot laminaire idéalisé sur deux dimensions (2D). Le mouvement du nuage de particules est décrit par une famille d'équations stochastiques nonlinéaires de type McKean-Vlasov, incorporant à la fois la représentation de la vorticité du nuage et la présence de cohérence dans le mouvement du fluide par le biais d'un champs de force attractif sous la forme d'un environnement aléatoire.

Le texte est organisé comme suit. Au premier chapitre, on rappelle le contexte historique décrit plus haut puis on fait une revue de la littérature tant en physique qu'en mathématique, concernant le schème de Wiener. Le second chapitre regroupe les outils de base qui nous serviront subséquemment afin d'exprimer nos équations stochastiques avec rigueur. Au troisième chapitre, on fait la revue des principales contributions des vingt dernières années sur l'équation de Navier-Stokes stochastique, tout en attirant l'attention sur les grandes difficultés techniques rencontrées par les auteurs impliqués. Les chapitres 4 et 5 contiennent nos contributions originales dans les cas bi- et tri-dimensionnels, respectivement.

La dérivation des principaux résultats se fait par le biais de la formulation mathématique du mouvement du nuage limite contenant une infinité de particules, sous la forme d'un processus à valeurs prises dans un espace de mesures de Borel et dont les trajectoires s'avèrent être continues. Nous démontrons en particulier pour la première fois, pour le système fini et les systèmes voisins existants déjà dans la littérature, l'existence de conditions aisément vérifiables qui garantissent l'absence de collision en temps fini entre particules.

Chapter 1

Historical introduction

The study of fluid dynamics fascinates the eye and the mind, whether it be the observation of the smooth, laminary flow of a long and quiet river, or the turbulent cascades caused by the irregular bottom of a babbling brook. The amount of scientific litterature on the subject is enormous and goes back a long way. A complete historical review of fluid mechanics would take us too far afield and our betters have written with authority on the subject (for instance, McComb [65] as well as Chorin and Marsden [20]). Let us glean the relevant information there for our purpose and send the interested reader back to these sources in order to find out more.

Let us first recall that Newton is the one who, in 1657, first described mathematically the concept of viscosity, defining it as the variation per unit of cross-section surface, of the momentum of particles in a fluid with laminar flow. This variation is physically interpreted as the result energy dissipation resulting from the interaction of macroscopically distant fluid elements.

Almost a hundred years later, in 1755, Euler described exactly and with full rigour the flow a perfect fluid, that is, a fluid with no viscosity, isotropic particle to particle interaction forces and isothermic flow (no thermal energy generated by its movement). Euler's working hypotheses were that the flow is laminary and in a single physical phase. For example, if the flow is that of water through a cylindrical pipe, this description prevails away from the walls of the pipe provided there is no air in it. If either condition fails, some turbulence will occur.

The Navier-Stokes equations appear for the first time in a course on hydrodynamics written by Navier in 1822, a date which marks the beginning of a tremendous amount of scientific activity which continues to this day, in fluid mechanics. The equations describe completely the motion of the elements of volume of a fluid which may well be viscous or not, compressible or not, turbulent or not.

Recall here that incompressibility is defined as the integral transmission of the external pressures on the fluid to its movement. A few examples are in order here. Instances of incompressible fluids are : all liquids, locally; gases travelling at hypersonic speed (more than MACH 8, eight times the speed of sound) and all strongly viscous liquids. Instances of compressible fluids are : all gases flowing at more moderate speeds and liquids travelling through long pipes with irregular pressure areas, like pipelines.

The Navier-Stokes equations can be formulated as follows.

Consider $\rho = \rho(x,t)$ the density (mass per volume) of the fluid, $p = p(x,t)$ the pressure (per unit volume) it exerts, $f = f(x,t)$ the density (per unit volume) of the external force field (which we assume known), $T = T(x,t)$ the temperature (also known) and ν the viscosity. (We refer here to the intrinsic, dynamic viscosity, which we assume known for the fluid under observation, which must not be confused with the kinematic viscosity, given by the ratio ν/ρ.) The exact formulation also requires two other functions $\Phi = \Phi(\rho, p, T)$ and $\lambda = \lambda(\rho, p, T)$, which are well-known to physicists and which we will not detail further here, for reasons that will be clear in a moment.

We are looking for solutions $u = u(x,t)$ to the following three equations, equipped with appropriate boundary conditions. Here u represents the speed of the flow under study. To begin with, the Navier-Stokes equation itself (when $\nu = 0$ we talk about Euler's equation)

$$\rho\Big(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\Big) - \nu \triangle u - (3\lambda + \nu)\nabla div(u) + \nabla p = f \tag{1.0.1}$$

follows from the conservation of energy and force per unit area of the cross-section of the flow. The second one, called the continuity equation,

$$\frac{\partial \rho}{\partial t} + div(\rho u) = 0 \tag{1.0.2}$$

is a direct consequence of the conservation of momentum and Gauss's theorem linking surface integrals and induced volumes. Finally, a thermodynamic conservation law yields the third equation

$$\Phi(\rho, p, T) = 0\,. \tag{1.0.3}$$

The most remarquable property of (1.0.1) is undoubtedly the vast range of its validity in the eyes of engineers. Indeed, at least as late as 1990 when McComb [65] was writing, there is not a single experimental counterexample of a turbulent gas flow putting (1.0.1) in question at speeds below MACH 15! This versatility explains the abundance of methods developped over the years to solve it and analyze the behavior of its solutions, as well as that of the litterature pertaining to it.

For a mathematician, the validity of (1.0.1) requires a precise definition of what is meant by a solution and a rigourous demonstration of its existence and, ideally, of its unicity. In all the classic books on the subject, even the more recent ones like [20], [26] or even [72], the proof of existence is invariably produced in the restricted case of homogeneous and incompressible fluids, that is when parameters ρ, p and T are known constants. Basically, all the proposed proofs in these books go back to the essential work of Jean Leray [60]. Of course there are now far more powerful deterministic methods, especially those of Fields medallist Pierre-Louis Lions [61] which rely on new functional analytic structures, but they are too complex to describe succintly here and the approach we have adopted is resolutely probabilistic in nature.

We shall see in chapter 3 how the stochastic methods used in this thesis, allow us to define the solution and prove its existence and unicity in some cases. **We will require both incompressibility and homogeneity at all times and therefore assume that $\rho = 1$ everywhere for the sake of simplicity in writing**. This condition entails that equation (1.0.2) becomes simply $div(u) = 0$, which must be satisfied at all interior points of the domain where equation (1.0.1) is required to hold.

As we mentionned above, the litterature on Navier-Stokes equations is abundant. Anyone who needs convincing of this may do a straightforward search on website http://www.ams.org/mathscinet/search in order to find over 3,800 scientific papers on the subject, more than a hundred of which are so recent so as to be currently under review. Keeping this introduction within reason demands that we treat here only the most recent litterature surrounding the stochastic constructions of those particle systems having solutions to (1.0.1) as their limits. We will even exclude the case of the classical equations with random forcing, where only f is stochastic, since the papers on this subject tend to use deterministic methods that will not be considered here and since this additionnal subtlety would not add any new element to our mathematical treatment.

From hereon, our interest lies in turbulent flows, which are unstable and chaotic. It is now known to be unrealistic to try to model a single trajectory in that case for more than a short span of time. Instead, one considers collections of solutions known as ensembles, usually best described as measures over the phase space. We therefore concentrate henceforth on one and only one approach, that of the search for these ensembles, first introduced by Wiener in [76] under the name statistical solutions and two years only prior to the publication by Kolmogorov of his celebrated theory of turbulence in [51]. Wiener's ideas, abandonned for several decades, were taken up independently around 1970 by Bleher and Vishik [9] and by Foias [37], who first obtained rigourous results along this line of enquiry. Another major paper along these lines is Bensoussan and Temam [8] in 1973, which extends the previous two while keeping the noise term purely additive (now called linearized stochastic Navier-Stokes equations), a physically unrealistic assumption as it turns out (see Kotelenez [53]). Nevertheless, all these presentations remain mathematically modern and pertinent even today, as they all rely on the still very active field of stochastic differential equations, on which more will be said in the next chapter.

Also in the early seventies, physicists seized upon this scheme of Wiener's and devised discretized versions of it in order to analyze numerically the properties of both inviscid (non viscous) and viscous flows. In 1973, after Chorin observed in [18] that viscosity at a boundary slows down a fluid by creating vortices in it, Chorin and Bernard in [24] devised a deterministic, discrete vortex representation for two dimensional (hereafter 2D) inviscid flows that should converge to the solution of the Euler equation when the number of such vortices tends to infinity. Hald and del Prete first together in [40] and then Hald alone in [41] and [42], proved that convergence under increasingly weak assumptions.

In a series of papers ([4], [5], [6], [7]), Beale and Majda attacked the three dimensional (hereafter 3D) case similarly and we'll have more to say on these attempts in a moment. In parallel to these attempts at proving rigorously the convergence of these various numerical schemes to the solution of the full fledged Navier-Stokes equation (1.0.1), Chorin gained some insight into such phenomena as the rapid stretching of vortices (in [21]), intermittency (the uneven distribution of vortices in space as the flow evolves, in [19]), the apparent fractal dimension of the support of the vorticity (in [22] and [23]) and so on. Chorin summarizes his contributions on the subject in his book [25]. All in all, interesting numerical work but very few rigourous results. This line of study goes on to this day in the physics litterature. See, for instance, in chronological order from 1985 on, [59], [3], [69], [63], [38], [17], [62], [44], [50], [28], [1] and [39], where random forcing via Wiener's scheme appears on occasion but non random forcing dominates. Forcing refers here to the presence of a non zero external force field f in equation (1.0.1).

Rigourous results for the general case of (1.0.1) took longer to appear. The 2D dynamics were formulated in terms of vortex theory in an excellent treatise by Marchioro and Pulvirenti [64] in 1982. As expected, the 3D dynamics offered a lot more resistance. The first spectacular success came in 1985 through a joint effort by three of the best mathematicians of the day. Constantin, Lax and Majda built in [27] a one-dimensional (deterministic) caricature of the 3D vorticity equation, that is, the equation which governs the rotational of the velocity of a Navier-Stokes particle. This was accomplished six years after a physical description of such a caricature appeared in [56]. It is indeed remarkable that the corresponding (stochastic) caricature for the Boltzmann equation, governing the motion of an idealized gas, was obtained by Mark Kac [47] back in 1956.

In 1989, Esposito and Pulvirenti [31] published a detailed analysis of 3D vortex flows upon which more will be said in chapter 3. Many of the key ideas used in this thesis go back to this important paper, which builds upon the work of Beale and Majda quoted above. Rigourous treatment was now being attempted in the more realistic modelling afforded by making the noise term multiplicative. Further progress was achieved by Brzezniak, Capinski and Flandoli [11] in 1992, under some rather severe a priori estimates and then by Capinski and Cutland [13] in 1994, by way of nonstandard analysis. A year later in [14], these same two authors went back to the ideas of Foias [37] and were able to avoid nonstandard analysis altogether, at the cost of some technical conditions on the characteristic functional. Flandoli and his collaborators got rid of some of these conditions by considering solutions in a weaker sense than previously used, in both the 2D [36] and 3D [34] contexts. Alternative treatments can be found somewhat later in [49], [70], [15] and [33]. More recently, these techniques have been used to obtain qualitative results on the regularity of most paths of these stochastic systems, showing them to be (at least microscopically) quite a bit better behaved than the deterministic ones (see for instance Flandoli and Romito [35]), as well as amenable to stability (Caraballo et al. [16]) and even ergodic behavior (Weinan et al. [75]). Even vortex interactions can now be analyzed precisely, as done by He in [43].

Our line of study here follows both the work of Esposito and Pulvirenti [31] quoted above and the construction of systems of interacting particles on random sheets due to Kotelenez in [53], [54], [55] after the original ideas of Walsh [74]. The premise Kotelenez postulates is the necessity for a good system of strongly interacting particles that mirrors the microscopic behavior of the fluid in a physically realistic way, while providing macroscopic limits (in a mathematically rigourous sense) that verify the Navier-Stokes equations. This he accomplishes for these equations in the case where they can be put in their vorticity form described below. Explicit constructions of the paths of particles moving around in an ambiant random medium described by common Brownian sheets, a choice which ensures joint continuity of motion of the medium in time and space, a highly desirable feature from the physical viewpoint. The characterization of both the finite and infinite systems were carried out in detail in 1995 by Kotelenez in [55] for a large class of multi-particle interactions using stochastic integration and independently by Dawson and Vaillancourt in [29] where the martingale problem formulation was used instead, for a closely related family of such systems.

In this stochastic systems approach, the motion of the cloud of particles is described by a family of nonlinear stochastic partial differential equations incorporating both the force fields generated by a large class of physical, multi-particle interactions, and an external force field acting on each particle. The corresponding infinite cloud of particles is obtained as a strong limit of the sequence of weighted empirical measure processes generated by these finite systems and solves a stochastic evolution equation of Navier-Stokes type on the space of Borel signed measures. These strong limits are defined rigorously in chapter 3 below and are refered to in the litterature as mezoscopic limits, in reference to their incorporation of large enough ensembles of particle movements that have retained the particle-to-cloud interaction in their stochastic description while forgetting the much weaker individual particle-to-particle effects that are present at the microscopic level. The stochastic Navier-Stokes equations that are thus obtained arise from a stochastic form of the microscopic vortex model of fluid mechanics mentionned above.

The macroscopic limits of these systems are deterministic (fluid) flows that solve the classical Navier-Stokes equation (1.0.1), in both the realistic three dimensional case (hereafter 3D) and the case of an idealized laminary flow in two dimensions (hereafter 2D). The movement of the cloud of particles is described by a family of nonlinear stochastic McKean-Vlasov equations made explicit in chapter 3. These equations incorporate simultaneaously the representation of the vorticity of motion of the cloud and the presence of coherence in the fluid motion by way of an attractive free force field mathematically formulated as a random environment.

In the following chapter, we will assemble the tools of the trade from the theory of stochastic processes. This will enable us in chapter 3 to review the most recent papers related to this approach (specifically [2], [10], [12], [46], [52], [66] and [67]) in greater detail and with the precision required of a mathematical thesis.

The remainder of the text is organised as follows. Chapter 2 comprises all the tools that we need later on to express our stochastic equations rigourously. Chapter 3 is a critical review of the recent litterature (last twenty years) on stochastic Navier-Stokes equations. Chapters 4 and 5 are new and center on the two and three dimensional cases, respectively.

The reader will do well to remember that throughout this thesis, we only consider the evolution of particles and flows on the whole space and not on some bounded subset of it. No boundary conditions will be postulated nor checked.

Chapter 2

Notation and basics

We use the following metric spaces: E is any Polish space, that is, any topologically complete, separable, metric space; Ω_E, the Polish space of continuous paths $[0, \infty) \to E$ with the topology of uniform convergence on compact sets; $C(E)$, the Banach space of real-valued bounded continuous functions on E with the uniform topology; $\mathcal{M}(E)$, the Polish space of all finite signed measures on the Borel subsets of E with the weak topology, defined by $\mu_n \Rightarrow \mu$ iff $\lim_{n\to\infty} \int \phi \, d\mu_n = \int \phi \, d\mu$, for every $\phi \in C(E)$ (see [30]); $\mathcal{M}^+(E) \subset \mathcal{M}(E)$, its closed subset of all finite positive measures; $C^j(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, the space of j times continuously differentiable functions with bounded first, second, $\ldots, j^{\text{th}}$ derivatives, for $1 \leq j \leq \infty$; $C_k^j(\mathbb{R}^n)$, its subspace of functions with compact support; along the same lines, $C^{i,j} \subset C([0, \infty) \times \mathbb{R}^n)$ the space of j times continuously differentiable functions with bounded first, second, $\ldots, j^{\text{th}}$ derivatives in all coordinates of the second component and i times continuously differentiable functions with bounded first, second, $\ldots, i^{\text{th}}$ derivatives in the first (time) coordinate-component; $L_2(E)$, the Hilbert space of real-valued, square integrable functions, with the scalar product $(\cdot,\cdot)_2$ and corresponding norm $|\cdot|_2$; more generally, $L_p(E)$ will denote the Banach space of real-valued, p-integrable functions, for $p \geq 1$, with the usual Hölder p-norm noted $|\cdot|_p$. The scalar product of vectors in $\mathbb{R}^n$ will be denoted by $\cdot$ or simply by juxtaposition, the vector product by $\times$ resulting in a n-dimensional vector and the tensor product by $\otimes$ resulting in a n by n matrix.

A first result that will be required repeatedly is Gronwall's celebrated inequality. The proof is in Appendix 5 of [32].

Theorem 2.0.1 *Given a Borel measurable function f which is bounded on bounded intervals in $[0,\infty)$ and a measure $\mu \in \mathcal{M}^+([0,\infty))$ such that for some $\epsilon > 0$ there holds $0 \leq f(t) \leq \epsilon + \int_{[0,t)} f(s)\,\mu(ds)$ for all $t \geq 0$, then there follows $f(t) \leq \epsilon \exp \mu[0,t)$ for all $t \geq 0$ as well.*

Various operators will be required throughout this thesis, including in particular the following: ∂_{x_j} (occasionnally written ∂_j) is the derivative with respect to coordinate j acting on $C^1(\mathbb{R}^n)$, $\partial^2_{x_j}$ or ∂^2_j the second order derivative (and so on), $\nabla = (\partial_1, \partial_2, \ldots, \partial_n)^T$ is the gradient operator, $div = \sum_{j=1}^n \partial_j$ and $\Delta = \sum_{j=1}^n \partial^2_j$ the Laplacian on $C^2(\mathbb{R}^n)$. When a function is vector-valued, the derivatives will be interpreted coordinatewise as is usual.

In the specific case of 3D motion ($n = 3$), we will use the curl of vector field $f = (f_1, f_2, f_3)$ defined on some open set in 3D space, the curl itself being an operator defined by $curl\, f := (\partial_2 f_3 - \partial_3 f_2, \partial_3 f_1 - \partial_1 f_3, \partial_1 f_2 - \partial_2 f_1)$. Of course there holds $div\, curl\, f = 0$ whenever f is smooth enough.

Because of the highly nonlinear nature of the Navier-Stokes dynamics, a solution to our equations will at times mean a function but at other times only a distribution, in the sense explained next.

We denote by $\mathcal{S}'$ the Schwartz space of tempered distributions, defined as the topological dual to the space $\mathcal{S}$ of smooth functions on $\mathbb{R}^d$ with rapidly decreasing derivatives of all order at infinity (for more on these spaces, including the definitions of weak and strong derivatives, see chapter 25 of Trèves [73]).

Let $h_n(t) := (\pi^{1/2} 2^n n!)^{-1/2} (-1)^n e^{t^2/2} D_t^n (e^{-t^2})$ for $n = 0, 1, \ldots$ and $t \in \mathbb{R}$, where D_t^n denotes the n^{th} derivative with respect to t. Define the Hermite function of index $n = (n_1, n_2, \ldots, n_d)$ by $h_n(z) := h_{n_1}(z_1) \cdot \ldots \cdot h_{n_d}(z_d)$ for each $z = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d$. The set $\{h_n\}$ forms a complete orthonormal system in $L_2(\mathbb{R}^d)$ and satisfies the upper bound $\sup_{n,z} |h_n(z)| = h_0(0) = \pi^{-d/4} < 1$ due to Szász [71]. The separable Hilbert spaces

$$H^j := \left\{ X \in \mathcal{S}' : ||X||_j^2 = \sum_n (2|n|_1 + d)^j X[h_n]^2 < \infty \right\},$$

with $|n|_1 := \sum_{i=1}^d n_i$, provide the dense continuous inclusions

$$\mathcal{S} \subset H^j \subset H^0 = L_2(\mathbb{R}^d) \subset H^{-j} \subset \mathcal{S}'$$

for any real-valued $j \geq 0$. Remember that the elements of H^j have all their first order partial derivatives valued inside H^{j-1} for every $j \in \mathbb{R}$ and that the inclusion $H^j \subset H^i$ is of Hilbert-Schmidt type as soon as $j - i > d$.

Finally it will be useful at times to remember the continuity of the inclusion $\mathcal{M}^+(\mathbb{R}^d) \subset H^{-d-1}$ with respect to the norm $||\cdot||_{-d-1}$. A proof can be found in [29].

We can now turn to probabilistic matters.

Given a filtration $\{\mathcal{F}_t : t \in T\}$ indexed by some time set $T \subset [0, \infty]$ on a probability space $(\Omega, \mathcal{F}, P)$, we call the quadruplet $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ a filtered probability space. This last is said to be complete when $(\Omega, \mathcal{F}, P)$ itself is complete (we follow [45] for basic definitions in probability theory but any of [32], [48] or [68] will also do) and all P-null sets in $\mathcal{F}$ actually lie in $\mathcal{F}_0$. Hereafter we always assume that our processes are built on such a complete filtered probability space and will refer to it as the (stochastic) basis. We will further assume that right continuity of the filtration holds, that is $\mathcal{F}_{t+} = \mathcal{F}_t$ for all values of t, where we define

$$\mathcal{F}_{t+} := \bigcap_{h>0} \mathcal{F}_{t+h} .$$

We can always select this basis so that countably many standard Brownian motions can be built on it. Again see [45] for details. This will allow us to shorten many of the statements in the sequel.

An $\mathbb{R}^n$-valued stochastic process $\{x_t\}$ defined on this basis is said to be adapted to this filtration if x_t is $\mathcal{F}_t$-measurable for all $t \in T$. Such an adapted process $\{m_t\}$ forms a martingale with respect to filtration $\{\mathcal{F}_t\}$ if each m_t is P-integrable and there holds $E[m_t|\mathcal{F}_s] = m_s$ for every choice of $0 \leq s \leq t < \infty$.

Martingales possess many fine properties. For instance, if an $\mathbb{R}^n$-valued continuous martingale m is square integrable, that is, such that $\int_0^t E|m_t|^2\,dt$ is finite for every t, then there exists a unique adapted process $\langle m\rangle$ with increasing paths called the quadratic variation of m such that $\{|m_t|^2 - \langle m\rangle_t\}$ is also a martingale. Further, given two such continuous square-integrable martingales m and M, there are unique adapted processes $\langle m, M\rangle$ and $\langle\langle m, M\rangle\rangle$ with paths of bounded variation on compact time sets respectively called the mutual variation and tensor mutual variation of m and M, which are respectively such that $\{m_t \cdot M_t - \langle m, M\rangle_t\}$ is once again a real-valued martingale and $\{m_t \otimes M_t - \langle\langle m, M\rangle\rangle_t\}$ is an $n \times n$ matrix-valued martingale (in the obvious sense, i.e., coordinatewise).

Among other interesting properties of martingales is the following, known as the Burkholder-Davis-Gundy inequality (see [45] again) and used later on.

Theorem 2.0.2 *Given is a real-valued continuous martingale m started at $m_0 = 0$. If we write $m_t^* = \sup_{s\le t}|m_s|$ then for all $p > 0$ there are universal constants $0 < c_p < C_p < \infty$ such that*

$$c_p \cdot E\langle m\rangle_t^{\frac{p}{2}} \le E(m_t^*)^p \le C_p \cdot E\langle m\rangle_t^{\frac{p}{2}}\,.$$

One way to construct martingales in particular and processes in general, is through stochastic integration. We consider here processes x which are square integrable and real-valued. Without going into the details of the construction, let it be said here that it is possible to define rigorously on such a basis as above, the application mapping any square integrable process x with continuous paths $\{t \to x_t(\omega) : \omega \in \Omega\}$ to a square integrable martingale with continuous paths noted $\int_0^t x_s\,d\beta_s$ called the stochastic integral with respect to standard Brownian motion β. Up to a null set, it is the only such martingale with mean 0 and quadratic variation equal to $\langle \int_0^\cdot x_s \, d\beta_s \rangle_t = \int_0^t (x_s)^2 \, ds$.

The most important tool for computing with stochastic integrals is Itô's formula, which we state next.

Theorem 2.0.3 *Given are mutually independent standard Brownian motions $\{\beta^i\}$; real-valued continuous processes $\{a^j\}$ with bounded variations on compact time sets, started at $a_0^j = 0$; real-valued continuous square integrable processes $\{c^{ij}\}$; these conditions holding for all $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, n$. Then for every choice of $f \in C^{1,2}$, there holds almost surely, with process $x = (x^1, x^2, \ldots, x^n)$ defined by $x_t^j = x_0^j + a_t^j + \sum_{i=1}^d \int_0^t c_s^{ij} d\beta_s^i$,*

$$f(t, x_t) - f(0, x_0) = \int_0^t \frac{\partial f(s, x_s)}{\partial s} \, ds + \sum_{j=1}^n \int_0^t \frac{\partial f(s, x_s)}{\partial x^j} \, da_s^j$$

$$+ \sum_{i=1}^d \sum_{j=1}^n \int_0^t \frac{\partial f(s, x_s)}{\partial x^j} \, c_s^{ij} \, d\beta_s^i + \frac{1}{2} \sum_{i=1}^d \sum_{j,k=1}^n \int_0^t \frac{\partial^2 f(s, x_s)}{\partial x^j \partial x^k} \, c_s^{ij} c_s^{ik} \, ds \, .$$

For a proof, see [45], [32], [48] or [68], all of which cover more general cases including that where the Brownian motions are dependent, a case we shall require later.

The formulation of physical phenomena such as fluid flows at the microscopic level, leads naturally to stochastic expressions known as stochastic integral equations, which are defined as follows. Consider the expressions (valid for all $j = 1, 2, \ldots, n$)

$$x_t^j = x_0^j + \int_0^t g_s^j ds + \sum_{i=1}^d \int_0^t c_s^{ij} d\beta_s^i \tag{2.0.1}$$

where of course one must require at least that real-valued processes g_j be adapted and integrable, as well as real-valued processes c_{ij} be adapted and square-integrable, for the expression on the right to make sense at all. Once again here, the standard Brownian motions are assumed to be independent.

A (strong) solution to equation (2.0.1) will be any $\mathbb{R}^n$-valued process $x_t = (x_t^1, x_t^2, \ldots, x_t^n)$ with continuous trajectories, built on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, which is $\mathcal{F}_t$-adapted, jointly measurable in (t, ω) and verifies (2.0.1) almost everywhere, jointly in $t \in [0, \infty)$ and $\omega \in \Omega$.

Uniqueness of solution will always mean pathwise uniqueness: given any filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with independent Brownian motions on it and any two solutions x and x' to equation (2.0.1) such that $P(x(0) = x'(0)) = 1$ holds, then $P(x(t) = x'(t) \quad \text{for every } t > 0) = 1$ holds as well.

Theorem 2.0.4 *Assume that all the functions $g^j : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and $c^{ij} : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ are locally bounded and Borel measurable. If on each compact subset $C \subset [0, \infty) \times \mathbb{R}^n$, there is a constant $K_C > 0$ such that there holds on C:*

$$\sum_j |g^j(t, x) - g^j(t, y)| + \sum_{i,j} |c^{ij}(t, x) - c^{ij}(t, y)| \leq K_C |x - y|$$

and for all $T > 0$ there is a constant $L_T > 0$ such that there holds for all $x \in \mathbb{R}^d$ and $t \in [0, T]$:

$$\sum_j x^j \cdot g^j(t, x) + \sum_{i,j} |c^{ij}(t, x)|^2 \leq L_T (1 + |x|^2)$$

then there is a unique strong solution to equation (2.0.1).

The proof may be found at the end of chapter 5 of [32].

Other processes of interest for us will require integration with respect to a collection of Brownian motions with a high level of coherence in their behavior, known as the Brownian sheet.

Once again recall that we are given a basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ on which are built independent sequences $\{\beta_k^i : k \geq 1\}$ of independent real-valued standard Brownian motions, for $i = 1, 2, \ldots, d$.

Recall from the seminal work [74] of Walsh that a standard Brownian sheet w_i on $\mathbb{R}^d$ is simply a continuous version of the Gaussian random field defined by the almost surely convergent series

$$w_i(p,t) = \sum_{k=1}^{\infty} (I_{R(p)}, h_k)_2 \; \beta_k^i(t), \tag{2.0.2}$$

where $\{h_k\}$ is a complete orthonormal system for $L_2(\mathbb{R}^d)$, we denote by $R(p) = R(p_1, p_2, \ldots, p_d)$ the rectangle in $\mathbb{R}^d$ with corners $\{(\pm p_1, \ldots, \pm p_d)\}$ and by $I_{R(p)}$ its indicator function.

The stochastic integral, with respect to such a Brownian sheet, of an $L_2(\mathbb{R}^d)$-valued, $\mathcal{F}_t$-adapted predictable process $\{f_t : t \geq 0\}$ which satisfies $E\int_0^t |f_s|_2^2 ds < \infty$ for each $t > 0$, is the continuous, locally square integrable $\mathcal{F}_t$-martingale given by the almost surely convergent series

$$\int_0^t\!\!\int f_s(y)\, w_i(dy, ds) = \sum_{k=1}^{\infty} \int_0^t (f_s, h_k)_2 \, d\beta_k^i(s)\,.$$

The quadratic variation process for this martingale is given by

$$\langle \int_0^{\cdot}\!\!\int f_s(y)\, w_i(dy, ds) \rangle_t = \int_0^t |f_s|_2^2 \, ds\,.$$

The corresponding stochastic integral martingale measure of f with respect to Brownian sheet w_i is then a well-defined, continuous, worthy martingale measure ([74]). We will henceforth write $w = (w_1, w_2, \ldots, w_d)$ and interpret stochastic integrals with respect to w simply coordinate by coordinate.

The stochastic evolution equations that arise in the next chapter are of the form

$$x_t^j = x_0^j + \int_0^t g_s^j ds + \sum_{i=1}^{d} \int_0^t \int c_s^{ij} dw_s^i \tag{2.0.3}$$

where now the (always adapted and continuous in the sequel) processes c^{ij} need to be $L_2(\mathbb{R}^d)$-valued for this equation to make sense.

We are now ready to move on to the description of the major results of recent years on the solution of stochastic evolution equations of the form (2.0.3), including existence and uniqueness issues, as they pertain to the description of vorticity.

Chapter 3

Recent litterature on stochastic vorticity equations

3.1 Navier-Stokes equation in 3D

3.1.1 Derivation of the vortex form

The Navier-Stokes equations of chapter 1 can be written explicitely for homogeneous and incompressible fluid flows in 3D with no forcing, for all $t > 0$ and x in a given domain,

$$\begin{cases} \frac{\partial u_1}{\partial t} + (u \cdot \nabla)u_1 - \nu \bigtriangleup u_1 = \frac{\partial p}{\partial x_1} \\ \frac{\partial u_2}{\partial t} + (u \cdot \nabla)u_2 - \nu \bigtriangleup u_2 = \frac{\partial p}{\partial x_2} \\ \frac{\partial u_3}{\partial t} + (u \cdot \nabla)u_3 - \nu \bigtriangleup u_3 = \frac{\partial p}{\partial x_3} \\ div(u(x,t)) = 0 \\ u(x,0) \quad \text{some known function or distribution} \end{cases} \tag{3.1.1}$$

where $u = (u_1, u_2, u_3)$ is the fluid velocity, p is the pressure, $x = (x_1, x_2, x_3)$ is the position of the particle and finally we set $(u \cdot \nabla) = \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i}$.

The matters of existence and uniqueness for this system in 3D are still open to a large extent. Existence of strong solutions have been proven only over short time intervals and the only global existence theorems known to date are for weak solutions over regular or periodic domains. The unicity of the weak solution remains open in 3D. We shall therefore assume for the rest of this section that we are dealing with a case of the equation where a global weak solution exists, that is, one where u takes its values in the Schwartz space of distributions and the derivatives are defined accordingly (see [72]). Sufficient conditions for this to occur may be found in [60], [58] and [72].

We will not require to know the exact formulation of any such sufficient conditions, which are quite long to explain with any degree of precision, since our goal here is only to show that the vorticity of any such solution will satisfy the Navier-Stokes equations in their so-called vortex form (also refered to hereafter as the vorticity equations) stated below. The reconstruction of solutions to the Navier-Stokes equations from solutions to the vorticity equations is also formally possible. In 3D Biot-Savart's law, made explicit in the next subsection, allows us to show that the solution to the vorticity equations can be convoluted with a singular kernel function to yield a solution to the original Navier-Stokes equations. The 2D case will be touched upon in the following section and can be handled easily using Poisson's law instead.

It is important to stress that the standard initial conditions for the Navier-Stokes equations (3.1.1) are on the value of the initial velocity and not on that of the initial curl. Only in this last instance do we find a true correspondance between the two set of equations and the reader will do well to bear this in mind. Our purpose in this thesis will be to study the existence and uniqueness of solution to these vorticity equations directly, without reference back to system (3.1.1) inasmuch as the solutions in question are wholly dependent on the type of initial conditions contemplated in each context.

Recall from chapter 2 the definition of operator *curl*:

$$\omega = (\omega_1, \omega_2, \omega_3) = curl\, u = \nabla \times u = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} \tag{3.1.2}$$

or $\omega_1 = (\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3})$; $\omega_2 = (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1})$; $\omega_3 = (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})$. The vorticity of any solution u to system (3.1.1) is precisely ω. Two basic facts are used below, both of which are well-known : for any smooth real-valued function ϕ there holds $div\ (curl\ \phi) = 0$ and $curl\ (\nabla\ \phi) = 0$ in 3D space.

Let us derive the equation satisfied by the vorticity of any solution to the system (3.1.1). We provide a detailed proof here for the 3D case. (Since the 2D case is easily obtained by similar methods, we only state the results in the next section of this chapter.)

Theorem 3.1.1 *(Strong Navier-Stokes equations in vortex form in 3D). If u denotes any smooth (function) solution to the Navier-Stokes equations on an open set in 3D, then there comes*

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - \nu \triangle \omega = (\omega \cdot \nabla)u \ \ \text{or} \ \ \omega \cdot \frac{\partial u}{\partial x} \qquad (3.1.3)$$

for ω, the curl of u.

Proof. The proof will be found in the Appendix at the end of this thesis.□

Next we give the weaker form of (3.1.3) which is the form we will study in detail.

Theorem 3.1.2 *(Weak Navier-Stokes equations in vortex form in 3D). If u denotes any solution (either a smooth function defined on an open set in 3D or a tempered distribution) to the Navier-Stokes equations on the whole domain of definition of u, then for any $\phi \in \mathcal{S}$ there comes*

$$\frac{\partial}{\partial t} \langle \omega, \phi \rangle - \langle \omega, u \cdot \nabla \phi \rangle - \nu \langle \omega, \triangle \phi \rangle = \langle u, \omega \cdot \nabla \phi \rangle \qquad (3.1.4)$$

for ω, the curl of u. When u is a function (respectively, a distribution), then so is ω and the derivatives are to be taken in the usual (resp., weak) sense.

Proof. The proof will be found in the Appendix at the end of this thesis.□

The two terms in each of (3.1.3) and (3.1.4) that involve the gradient operator ∇ display a bilinear form, jointly in (u, ω), that is known to cause the partial differential equations containing them to possess only explosive and non smooth solutions in general. In particular the explosion times are usually related to singularities in the topological description of the dynamics at hand. Because these singularities make a rigorous treatment very difficult, we next give a smooth form of (3.1.3).

3.1.2 Smooth form of Navier-Stokes vortex equations

Since the very specific form of the Navier-Stokes dynamics allow for an explicit representation through the so-called Biot-Savard formula, let us first make explicit the aforementionned singularity in (3.1.3). The smoothing operation will thus become completely transparent.

Remember that the convolution of two integrable (or square-integrable) functions K and Ω over $\mathbb{R}^3$ is another integrable function (see [73]) defined by

$$U(x) = K * \Omega(x) = \int_{\mathbb{R}^3} K(x-y) \cdot \Omega(y) dy \,.$$

A simple computation allows us (at least formally) to recover a solution to the original Navier-Stokes equations (3.1.1) from any solution to the vorticity equations (3.1.3) under the following new formulation, valid at every $t > 0$:

$$\begin{cases} \partial_t \omega(x,t) + (u \cdot \nabla)\omega(x,t) = (\omega(x,t) \cdot \nabla)u(x,t) + \nu \Delta \omega(x,t), \\ \omega(\cdot,t) = \text{curl } (u(\cdot,t)), \\ u(\cdot,t) = [k * \omega](\cdot,t) \\ \omega(x,0) \quad \text{some known function } \omega_0(x), \end{cases} \tag{3.1.5}$$

provided we choose $\omega_0 = \text{curl } (u(\cdot,0))$ and $k(x)$ the matrix-valued kernel defined by

$$k(x) = -\frac{1}{4\pi|x|^3} \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}. \tag{3.1.6}$$

The first and third equations in (3.1.5) together actually imply the second one in this case, as well as the necessary $div(u) = 0$. Because of its importance, this third equation in (3.1.5) is known as the Biot-Savart formula, explicitely stating

$$u(x,t) = [k * \omega](x,t) = \int_{\mathbb{R}^3} k(x-y) \cdot \omega(y,t) dy. \tag{3.1.7}$$

Of course, the Biot-Savard kernel k is singular at the origin and ω needs to be a very nice function in order for formula (3.1.7) to make sense mathematically. For instance if $\omega(\cdot, t)$ has compact support for each fixed t, then $u(\cdot, t)$ is a well-defined integrable function for almost every t (see [73]). Unfortunately, in general the solution to (3.1.5) does not have compact support at any positive time t. There lies the major difficulty in creating a full correspondence between (3.1.5) and (3.1.1).

Because of this difficulty, many authors (starting with Chorin over twenty years ago, see the history in [25]) have suggested altering the Biot-Savard kernel locally in order to get at least approximate results. This we do next.

Here we introduce the mollified kernel $k^\varepsilon(x)$ defined by:

$$k^\varepsilon(x) = \int_{\mathbb{R}^3} k(x-y)\rho_\varepsilon(|y|)dy$$

where the mollifier ρ_ε has the following properties: $\rho_\varepsilon \in C^\infty(\mathbb{R}_+)$ is a positive approximation to the usual Dirac point mass distribution at the origin, such that there holds $k^\varepsilon(x) \longrightarrow k(x)$ as $\varepsilon \longrightarrow 0$ and $\int_{\mathbb{R}^3} \rho_\varepsilon(|y|)dy = 1$. It is further possible to choose it in such a way that there holds as well

$$\max(\sup_x |k^\varepsilon(x)|, \sup_{x,\alpha} |\partial_\alpha k^\varepsilon(x)|, \sup_{x,\alpha,\beta} |\partial_\alpha \partial_\beta k^\varepsilon(x)|) \le c\varepsilon^{-4}, \tag{3.1.8}$$

where c is some positive constant. For explicit examples, see Hald [41] or Beale and Majda [4].

Remark 3.1.3 *k^ε is a matrix-valued function just like k.*

For any $\varepsilon > 0$, the following is our regularized or smooth 3D Navier-Stokes equations in vortex form

$$\begin{cases} \partial_t \omega^\varepsilon(x,t) + (u^\varepsilon \cdot \nabla)\omega^\varepsilon(x,t) = (\omega^\varepsilon(x,t) \cdot \nabla)u^\varepsilon(x,t) + \nu\Delta\omega^\varepsilon(x,t), \\ u^\varepsilon = k^\varepsilon * \omega^\varepsilon, \\ \omega^\varepsilon(x,0) = \omega_0(x). \end{cases} \tag{3.1.9}$$

Matters relating to existence and uniqueness for (3.1.9) will be taken up through the work of Esposito and Pulvirenti on a stochastic version of this equation later on in this chapter.

3.2 Navier-Stokes equations in 2D

The corresponding Navier-Stokes equations for homogeneous and incompressible fluid flows in 2D in the absence of forcing are, for all $t > 0$ and x in a given domain,

$$\begin{cases} \frac{\partial u_1}{\partial t} + (u \cdot \nabla) u_1 - \nu \triangle u_1 = 0 \\ \frac{\partial u_2}{\partial t} + (u \cdot \nabla) u_2 - \nu \triangle u_2 = 0 \\ div(u(x,t)) = 0 \\ u(x,0) \quad \text{some known function or distribution} \end{cases} \tag{3.2.10}$$

In 2D, the vorticity or curl ω of the flow is a real-valued function on $\mathbb{R}^2$, corresponding to any one of the three coordinates of the original 3D vector-valued curl appearing in (3.1.2). It is defined as $\omega := (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})$.

Following the derivation process used in the 3D case, we obtain the 2D vorticity equation, first in strong form

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - \nu \triangle \omega = 0 \tag{3.2.11}$$

and similarly for the weak form

$$\frac{\partial}{\partial t} \langle \omega, \phi \rangle - \langle \omega, u \cdot \nabla \phi \rangle - \nu \langle \omega, \triangle \phi \rangle = 0 \,. \tag{3.2.12}$$

Notice how the pressure term vanishes in 2D, a consequence of the physical fact that the (scalar) vorticity is then conserved along particle paths in 2D.

To get an explicit representation of the vorticity in 2D just introduce the operator $\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$. By virtue of $div\ u = 0$, we have

$$u(x,t) = \int (\nabla^\perp g)(x-y)\omega(y,t)dy. \tag{3.2.13}$$

where $g(r) = -\frac{1}{2\pi} \ln |r|$ is the fundamental solution of the Poisson equation. Some authors refer to equation (3.2.13) as the 2D Biot-Savard formula.

Just like in 3D, the smooth version of the vorticity equations (3.2.11) and (3.2.12) in 2D are obtained by perturbing the kernel g in order to remove the singularity at the origin. This is what we do next so let $0 < \varepsilon \leq 1$ and $g_\varepsilon(|r|) = g(r)$ for $\varepsilon \leq r \leq \frac{1}{\varepsilon}$ and arbitrarily extended to an even $C^2(\mathbb{R}^1)$ function such that $\left|g_\varepsilon^{'}(r)\right| \leq |g^{'}(r)|$, $\left|g_\varepsilon^{''}(r)\right| \leq |g^{''}(r)|$.

Such a filter of smooth approximations is easy to build (for instance, see Leonard [59]).

Set $K_\varepsilon(r) = (\nabla^\perp g_\varepsilon)(r)$, $r \in \mathbb{R}^2$.

Then the (regularized or) smooth Navier-Stokes equations are given by

$$\begin{cases} \frac{\partial \omega_\varepsilon}{\partial t}(x,t) + (u_\varepsilon \cdot \nabla)\omega_\varepsilon(x,t) - \nu\Delta\omega_\varepsilon(x,t) = 0, \\ u_\varepsilon(x,t) = \int K_\varepsilon(x-y)\omega_\varepsilon(y,t)dy, \\ \omega_\varepsilon(x,0) = \omega_0(x). \end{cases} \tag{3.2.14}$$

Note that not only could Marchioro and Pulvirenti give explicit conditions for these equations to possess one and only one solution, they did so even when it was perturbed by independent Brownian motions. Further, Kotelenez was also able to extend Marchioro and Pulvirenti's treatment when the stochastic driving terms involve a random environment, a more realistic description.

These results and the work of Esposito and Pulvirenti on the 3D case are described next.

3.3 Random vortex method for Navier-Stokes Equations

The random vortex method was conceived by Chorin [18] and used to study a slightly viscous flow with boundary condition. This method consists of solving Euler's equations by a vortex method and then sampling Gaussian random variables to model the diffusion equation. This method has been studied by several people. A vortex method for solving Euler's equations can be briefly described as follows. In a vortex method, the initial vorticity field is partionned into a sum of vortex blobs called vortices; and Euler's equations are replaced by a finite set of ordinary differential equations according to which the vortices evolve.

Unlike Euler's equations, in the Navier-Stokes equations one cannot keep track of the paths of the physical vortices by solving a system of ordinary differential equations due to the existence of a highly nonlinear viscosity term. In a random vortex method, each vortex carries a certain weight determined by the initial vorticity field and their collective motion generates at each time step a probability distribution. The velocity field in turn is determined via the distribution while the weights are governed by the Biot-Savart law.

In short, the random vortex method is a method that provides an approximation to the velocity field through the distribution of random vorticies, each of them specified by both its position and it weight.

The random vortex method has been successfully used in the study of several physical phenomena. The early study, by Chorin and Bernard [24], of a vortex method without using vortex blobs (called the point vortex method) shows that method to be unstable in predicting the rollup of nonuniform vortex sheets. In a vortex method, the velocity is determined by integrating the vorticity against a kernel with a singularity at the origin. The above instability is thus due to the coming close together of two vortex blobs.

The solution that allows them the luxury of using these vortex blobs is simply to cut off this non physical singularity. For this purpose, a class of so-called cutoff functions was introduced by Hald and they enabled him to give the first convergence proof of vortex method in [41] for this specific class.

3.4 Results of Marchioro and Pulvirenti in 2D from 1982

Marchoro and Pulvirenti [64] studied a stochastic 2D modified Euler's and Navier-Stokes equations in vortex form with independent Brownian motions driving each vortex.

More specifically they consider the Navier-Stokes equation for a viscous and incompressible fluid in $\mathbb{R}^2$. They show such an equation may be interpreted as a mean field equation for a system of particles, called vortices, interacting via a logarithmic potential upon which a stochastic perturbation is also acting.

For the deterministic Euler's equation they gave a system of $\mathbb{N}$ deterministic vortices of (fixed scalar) intensity $a_1^N, a_2^N, ..., a_N^N$ interacting via a smoothed potential :

$$\begin{cases} r_i^N(t) = r_i(0) + \int_0^t \sum_{j=0}^N a_j^N K_\varepsilon(r_i^N(s) - r_j^N(s))ds \\ r_j(0) = r_j, \quad i = 1, 2, \ldots, N, \end{cases} \tag{3.4.15}$$

Existence and uniqueness of solution to this system of equations pose no difficulty here (for instance, use the Picard iteration method). Writing

$$\omega_t^N = \sum_{j=1}^N a_j^N \delta_{r_j^N(t)} \quad \text{and} \quad \omega_t^N(f) = \sum_{j=1}^N a_j^N f(r_j^N(t)) \quad \text{for all integrable } f$$

for the approximating, discrete measure-valued mapping representing the discrete vortices, they restrict their attention to triangular arrays $\{a_j^N\}$ of values for the intensities of the vorticies inside the restricted set of only one positive and one negative value per line, namely $a_j^N \in \{A^+/N, -A^-/N\}$ for every choice of j and N, given two values $A^+ \geq 0$ and $A^- \geq 0$. They go on to prove the following limit theorem.

Theorem 3.4.1 *(Mean field limit). Consider the Euler case $\nu = 0$ (inviscid flow). Given is some starting value $\omega \in \mathbb{L}_1 \bigcap \mathbb{L}_\infty$ for which we make the choice $A^+ = \int_{\omega>0} \omega$ and $A^- = -\int_{\omega<0} \omega$. Assume that the triangular array of values for the coefficients $\{a_j^N\}$ is restricted as above. Assume also that the sequence of initial discretized measures $\{\omega_0^N\}$ satisfies $lim_{N\to\infty}\omega_0^N(f) = \omega(f)$ for every $f \in C(\mathbb{R}^2)$ whenever the sequence $\varepsilon = \varepsilon(N)$ goes to 0 as N goes to infinity. Then there holds also, at each fixed $t > 0$ and for every $f \in C(\mathbb{R}^2)$, $lim_{N\to\infty}\omega_t^N(f) = \omega_t(f)$, where the limit point ω_t is the solution of the Euler equation in vorticity form (3.2.12) with initial datum ω.*

Similarly for Navier-Stokes equation, they consider the following $\mathbb{N}$ stochastic vortices of intensity $a_1^N, a_2^N, ...a_N^N$, interacting via the same smoothed potential :

$$\begin{cases} r_i^N(t) = r_i(0) + \int_0^t \sum_{j=0}^N a_j^N K_\varepsilon(r_i^N(s) - r_j^N(s))ds + \sqrt{2\nu} b_i(t) \\ r_j(0) = r_j, \quad i = 1, 2, \ldots, N, \end{cases} \tag{3.4.16}$$

where the b_i's are 2-dimensional independent Brownian motions. With ω_t^N defined exactly as above, with the same restrictions on the values of the array $\{a_j^N\}$ they prove the following slightly more difficult result.

Theorem 3.4.2 *(Mean field limit). Here ν is allowed to be non null. Given is some starting value $\omega \in \mathbb{L}_1 \bigcap \mathbb{L}_\infty$ for which we choose $A^+ = \int_{\omega>0} \omega$ and $A^- = -\int_{\omega<0} \omega$. Assume that the triangular array of values for the coefficients $\{a_j^N\}$ is restricted as above. Assume also that the sequence of initial discretized measures $\{\omega_0^N\}$ satisfies $lim_{N\to\infty} \omega_0^N(f) = \omega(f)$ for every $f \in C(\mathbb{R}^2)$ whenever the sequence $\varepsilon = \varepsilon(N)$ goes to 0 as N goes to infinity. Then there holds also, at each fixed $t > 0$ and for every $f \in C(\mathbb{R}^2)$,*

$$lim_{N\to\infty} E[\omega_t^N(f)] = \omega_t(f)\,,$$

where the limit point ω_t is the solution of the Navier-Stokes equations in vorticity form (3.2.12) with initial datum ω.

Note that the authors did not prove the two theorems in the sense of weak convergence of processes, only in the sense of pointwise convergence in law, since the tightness arguments in the paper are for intermediate scaled processes only.

3.5 Results of Esposito and Pulvirenti in 3D from 1989

To find realistic particle models for flows in fluid mechanics is a much more intricate problem in 3D than in 2D. Esposito and Pulvirenti [31] propose one such particle system, based on their so-called stochastic Lagrangian picture of the vortex form of the Navier-Stokes equations in 3D. This picture was arrived at by generalising the approach already developped by Beale and Majda in [6] several years before in the (inviscid) Euler case.

First the authors consider the smoothed 3D version (3.1.9) of the Navier-Stokes equations and claim the following result.

Theorem 3.5.1 *For each $\omega_0 \in L_1$ such that its Fourier transform $\widehat{\omega_0} \in L_1$, one can find some positive time $T^* = T^*(\|\omega_0\|_1, \|\widehat{\omega_0}\|_1) > 0$ such that there exists a unique weak solution $[0,T^*] \ni t \to \omega_t^\varepsilon$ to (3.1.9), that is*

$$\frac{d\omega_t^\varepsilon(f)}{dt} = \nu\omega_t^\varepsilon(\Delta f) + \omega_t^\varepsilon(u_t^\varepsilon \cdot \nabla f) + \omega_t^\varepsilon(\nabla u_t^\varepsilon \cdot f) \tag{3.5.17}$$

for any smooth vector-valued function f. Here ∇u_t^ε is a matrix-valued operator and one must read $(\nabla u_t^\varepsilon)_{\alpha,\beta} = \partial_\alpha u_\beta^\varepsilon, \alpha, \beta = 1, 2, 3$. Furthermore there exists a constant $a > 0$ such that

$$\sup_{t\in[0,T^*]} \{\|u_t^\varepsilon\|_1 + \|\widehat{u_t^\varepsilon}\|_1\} \leq a \tag{3.5.18}$$

The $\lim_{\varepsilon\to 0} \omega_t^\varepsilon = \omega_t$, exist uniformly in $[0,T^*]$ in L_1 sense and ω_t is the weak solution to (3.1.4). Finally there also holds $\lim_{\varepsilon\to 0} u_t^\varepsilon(x) = u_t(x) = k * \omega_t(x)$ uniformly in $x \in \mathbb{R}^3$ and $t \in [0,T^*]$.

Warning! It is our belief that the proof in Esposito and Pulvirenti [31] provides inequality (3.5.18) only for an unbounded sequence of real values a_ϵ and not for a universal bound a as claimed. The universal bound is used by them in a crucial way to prove the last two sentences of Theorem (3.5.1) as well as Theorem (3.5.3) below. To our knowledge, the existence of a solution to (3.1.4) on unbounded domains remains unproved. For this reason our work in 3D will not rely on these questionable statements.

For our purpose in this thesis, the main contribution of Esposito and Pulvirenti [31] is in providing us with the following particle systems, used to approximate $\omega_t^\varepsilon(x)$:

$$\begin{cases} r_i(t) = r_i(0) + \int_0^t \sum_{j=0}^N K^\varepsilon(r_i(s) - r_j(s))\omega_j(s)ds + \sqrt{2\nu}b_i(t), \\ \frac{d\omega_i(t)}{dt} = \omega_i(t) \cdot \sum_{j=0}^N \nabla K^\varepsilon(r_i(t) - r_j(t))\omega_j(t), \\ \omega_j(0) = \omega_j, \quad i = 1, 2, \ldots, N, \end{cases} \tag{3.5.19}$$

where the b_i's are independent 3D Brownian motions.

Remark 3.5.2 *Note that this is clearly an adaptation to 3D of the system (3.4.16). Here $r_i(t)$ and $\omega_i(t)$ are vector-valued random functions. If $k(x)$ is a matrix-valued function, we define $(\nabla k(x))_{\alpha,\beta} = (\nabla k_{\alpha,\beta}(x))_{\alpha,\beta}, x \in \mathbb{R}^3$. So $\nabla k(x)$ is also a matrix-valued function, but now each of its components is a vector.*

For each time t, define the weighted empirical process associated with this system by

$$\omega_t^N(dx) = \sum_{j=1}^N \omega_j(t) \cdot \delta_{r_j(t)}(dx)\,, \tag{3.5.20}$$

so that ω_t^N is a vector-valued signed measure on $\mathbb{R}^3$.

The main result claimed by Esposito and Pulvirenti [31, Theorem 1.1] is

Theorem 3.5.3 *Assume the integrability of* ω_0 *and its Fourier transform* $\widehat{\omega_0}$ *and select a sequence of starting measures* $\omega_0^N(dx) = \sum_{j=1}^N \omega_j \cdot \delta_{r_j(0)}$ *such that it satisfies*

$$\lim_{N\to\infty} \int \omega_0^N(x) f(x) dx = \int \omega_0(x) f(x) dx$$

for any bounded continuous vector-valued function f*. Then there exists* T^* *sufficiently small and* $\varepsilon = \varepsilon(N) \to 0$ *as* $N \to \infty$*, such that*

$$\lim_{N\to\infty} \sup_{t\in[0,T^*]} \sup_{x\in\mathbb{R}^3} |u(x,t) - k^\varepsilon * \omega_t^N(x,t)| = 0,$$

where $u(x,t) = k(x) * \omega(x,t)$ *and* $\omega(x,t)$ *is the weak solution of (3.1.4).*

We shall see in Chapter 5 of this thesis what we were able to achieve towards a rigorous treatment of the problem addressed by the claims of the two would-be theorems stated in the present section.

3.6 Results of Kotelenez in 2D from 1992 on

The smooth Navier-Stokes equations (3.2.14) for the vorticity of a viscous and incompressible fluid in $\mathbb{R}^2$ can be analyzed with more physical realism than in Marchioro and Pulvirenti [64], by viewing it as a macroscopic equation for an underlying microscopic model of randomly moving vortices. This was done by Kotelenez [54]. The N point vortices positions satisfy a stochastic ordinary differential equation on $\mathbb{R}^{2N}$, where the fluctuation forces are state dependent and driven by Brownian sheets. The state dependence is modeled to yield a short correlation length ν between the fluctuation forces of different vortices. The bonus here over [64] is that the energy conservation law of physics is respected by the microscopic level, an important contribution from the point of view of the physics community.

Kotelenez [54] defines the following particle systems which serves as an approximation of (3.2.14). For each particle labelled $i = 1, \ldots, N$ define

$$dr^i(t) = \sum_{j=1}^{N} a_j^N K_\varepsilon(r^i - r^j)dt + \sqrt{2\nu} \int \hat{\Gamma}_\upsilon(r^i, p)w(dp, dt)\,, \tag{3.6.21}$$

where $w(p,t) = (w_1(p,t), w_2(r,t))^T$ is a pair of independent Brownian sheets on $\mathbb{R}^2 \times R_+$ with mean zero and variance $t\lambda(A)$, where A is a Borel set in $\mathbb{R}^2$ with finite Lebesque measure $\lambda(A)$. The Gaussian interaction kernel $\widehat{\Gamma}_\upsilon(r,p)$ will be defined in the next chapter when we discuss the conditions of existence and uniqueness of solution to equation (3.6.21). For the moment let r (which of course depends on the number N of vortices, the level ε of the mollifying kernel and what will turn out to be the range υ of the correlation structure built into the random medium where the particles bathe) be this solution and set $\omega_t^N := \omega_{\varepsilon(N),\upsilon(N)}(t) = \sum_{i=1}^N a_i^N \delta_{r^i(t)}$ where δ_x is the Dirac point measure at $x \in \mathbb{R}^2$. In [54] the following weak convergence result is proved, under slighlty less stringent conditions on the real coefficients $\{a_i^N\}$.

Theorem 3.6.1 *Assume that $E\omega_0^N(\phi) \to \int \omega_0\phi$ holds for every $\phi \in C(\mathbb{R}^2)$ as $N \to \infty$. For simplicity, we assume the conditions of section 3.4 on* *the coefficients* $\{a_i^N\}$. *Then, whatever the choice of sequence* $\epsilon(N) \to 0$ *as* $N \to \infty$, *there is another sequence* $\upsilon(N) \to 0$ *as* $N \to \infty$ *such that, for any* $t > 0$, *there comes* $\lim_{N\to\infty} E\omega_t^N(\phi) = \int \omega_t \phi$ *as* $N \to \infty$ *where* ω_t *is the solution to equation (3.2.14).*

Here again, as in the case of the work of Esposito and Pulvirenti above, some clarifications are required in order to make proper sense of this statement, since the proof provided by Kotelenez relies on the assumption of completeness of some metric space of signed measures which in fact is not complete, as we shall see. In Section 4.2 of this thesis, we were able to provide a rigorous treatment of this problem as well.

Chapter 4

New results on Navier-Stokes equations in 2D

We consider the Navier-Stokes equation of a viscous and incompressible fluid in $\mathbb{R}^2$. We show that such an equation may be interpreted as a mean field equation for a system of particles called vortices, interacting via a logarithmic potential, upon which in addition, a stochastic perturbation is acting. More precisely we prove that the solution of the Navier-Stokes equation may be approximated, in a suitable way, by finite dimensional diffusion processes with a diffusion constant related to the viscosity.

4.1 Introduction

In this section we deal with an incompressible, viscible or inviscible fluid in two dimension and study the connection between the equations governing the motion of such a fluid and the vortex theory and particle systems.

It is well known that an incompressible and viscous two dimensional fluid, under the action of an external conservative field, is described by the following evolution equations

$$\begin{cases} \frac{\partial \omega}{\partial t}(x,t) + (u \cdot \nabla)\omega(x,t) - \nu \Delta \omega(x,t) = 0, \\ \omega(x,t) = \text{curl } u(x,t) = \frac{\partial \omega}{\partial x_1} - \frac{\partial \omega}{\partial x_2}, \\ \nabla \cdot u = 0, \end{cases} \tag{4.1.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $u = (u_1, u_2) \in \mathbb{R}^2$ is the velocities field, $\nu \geq 0$ is the viscosity coefficient, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $\Delta = \nabla \cdot \nabla$ is the Laplace operator.

In the following we shall refer to (4.1.1) as the Navier-Stokes equation (NS for short) when $\nu \geq 0$ and as the Euler equation when $\nu = 0$.

Introduce the operator $\nabla^{\perp} = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$. By virtue of $\nabla \cdot u = 0$, we have

$$u(x,t) = \int (\nabla^{\perp} g)(x-y)\omega(y,t)dy, \tag{4.1.2}$$

where $g(r) = G(\|r\|) = -\frac{1}{2\pi} \ln \|r\|$ is the fundamental solution of the Poisson equation, where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^2$.

There is an extensive literature on the solution of (4.1.1) by the so-called point vortex method. A theoretical model related to the point vortex model has been analyzed by [64].

First we treat a regularized version of the Navier-Stokes equations (3.2.14) of the following form

$$\left\{\begin{array}{rcl} d\chi(t) & = & [\nu\Delta\chi - \chi\nabla\cdot(U_\varepsilon)]\,dt \\ & & \quad -\sqrt{2\nu}\nabla\cdot(\chi\int\widehat{\Gamma}_\delta(\cdot,p)w(dp,dt)), \\ U_\varepsilon(r,t,\chi) & = & \int K_\varepsilon(r-q)\chi(t,dq), \end{array}\right.$$

where $\chi(t)$ is a signed measure in an appropriate space, where $w(p,t)$ is a Brownian sheet process. We show that for any initial condition $\chi(0)$ with finite support, the above equation admits a weak solution, by constructing explicitly the solution using systems of ordinary stochastic differential equations with respect to Brownian sheets. Then it is proven that these solutions can be extended to any "nice" initial condition.

To give a meaning to the above equation, we used basically the same notations as in Kotelenez [54].

Let $g_\varepsilon(r) = G_\varepsilon(\|r\|)$, $0 < \varepsilon \leq 1$ where $G_\varepsilon(s) = g(s)$ for $\varepsilon \leq s \leq \frac{1}{\varepsilon}$ G_ε is $C^2(\mathbb{R}^1)$ with bounded derivatives of order 1 and 2, $G'_\varepsilon(0) = 0$, and for all $s > 0$, $|G'_\varepsilon(s)| \leq |G'(s)|$, $|G''_\varepsilon(s)| \leq |G''(s)|$.

For $r \neq 0$, set $K_\varepsilon(r) = (\nabla^\perp g_\varepsilon)(r), r \in \mathbb{R}^2$. It follows from the assumption $G'_\varepsilon(0) = 0$ that $K_\varepsilon(0) = 0$ makes K_ε continuous on $\mathbb{R}^2$. Moreover, since $G'_\varepsilon(0) = 0$ and G''_ε is bounded by C_ε (say), it follows that $|K_\varepsilon(r) - K_\varepsilon(q)| \leq 2C_\varepsilon|r-q|$, that is K_ε is Lipschitz.

Next define an interactive kernel $\widehat{\Gamma}_\delta(r,p)$ as follows. Let $\delta > 0$ and define correlation functions $\widetilde{\Gamma}_\delta : \mathbb{R}^4 \to \mathbb{R}_+$ to be bounded Borel-measurable functions which are symmetric in $r, p \in \mathbb{R}^2$ such that the following conditions are satisfied:

For any $r, p \in \mathbb{R}^2$:

(C1) $\int \widetilde{\Gamma}^2_\delta(r,p)dp = 1.$

(C2) There are finite positive constants c_δ such that, if we define

$$\rho(r,q) = (c_\delta \|r-q\|) \wedge 1, \tag{4.1.3}$$

we have

$$\int \left[\widetilde{\Gamma}_\delta(r,p) - \widetilde{\Gamma}_\delta(q,p)\right]^2 dp \le c\rho^2(r,q), \tag{4.1.4}$$

where $\wedge$ denotes the minimum of two numbers.

Set

$$\widehat{\Gamma}_\delta(r,p) = \begin{pmatrix} \widetilde{\Gamma}_\delta(r,p) & 0 \\ 0 & \widetilde{\Gamma}_\delta(r,p) \end{pmatrix}.$$

For this smooth model we define the following particle systems that will serve as an approximation of (3.2.14).

$$dr^i(t) = \sum_{j=1}^{N} a_j K_\varepsilon(r^i - r^j)dt + \sqrt{2\nu}\int \hat{\Gamma}_\delta(r^i,p)w(dp,dt), \tag{4.1.5}$$

$i = 1,\ldots,N$, where $w(p,t) = (w_1(p,t), w_2(r,t))^T$ and $w_l(r,t), l = 1,2$ are independent Brownian sheets on $\mathbb{R}^2 \times R_+, l = 1,2,$ with mean zero and variance $t\lambda(A)$, where A is a Borel set in $\mathbb{R}^2$ with finite Lebesgue measure $\lambda(A)$.

Let us now introduce some assumptions. In what follows, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a stochastic basis with right continuous filtration. All stochastic process are assumed to live on Ω and to be $\mathcal{F}_t-$adapted, including all initial conditions in SDE$'$ and SPDE$'$s. Moreover, the processes are assumed to be $P \otimes \lambda$ measurable, where λ is the Lebesgue measure on $[0, \infty)$. To be adapted for $w_l(r,t)$ means that $\displaystyle\int_A w_l(dp,t)$ is adapted for any Borel set $A \subset \mathbb{R}^2$ with $\lambda(A) < \infty$.

Let us assume for the moment that for suitably adapted squared integrable initial conditions, the Itô equations in (4.1.5) have a unique solution $r_N = (r^1(t), r^2(t), \ldots, r^N(t))^\top$.

For the kernel satisfying assumptions C1 and C2, we choose the Gaussian kernel:

$$\widetilde{\Gamma}_\delta(r,p) = \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{\|r-p\|^2}{4\delta}\right). \tag{4.1.6}$$

This kernel clearly satisfies (C1), that is

$$\int \widetilde{\Gamma}^2_\delta(r,p)dp = 1.$$

We also have

$$\int \widetilde{\Gamma}_\delta(r,p)\widetilde{\Gamma}_\delta(q,p)dp = \exp\left(-\frac{\|r-q\|^2}{8\delta}\right). \tag{4.1.7}$$

To see that (4.1.7) holds true, just remark that $\widetilde{\Gamma}_\delta(q,p)/(2\sqrt{2\pi\delta})$ is the density of a bivariate Gaussian vector with mean q and covariance matrix $2\delta I$, where I is the identity matrix. Then (4.1.7) follows from the well-known fact it represents, up to a constant, the density evaluated at r of the sum of two independent Gaussian vectors with respective means $\mu_1 = 0, \mu_2 = q$, covariances $\Sigma_i = 2\delta I$. Since the sum is also Gaussian with mean $\mu = \mu_1 + \mu_2 = q$ and covariance $\Sigma = \Sigma_1 + \Sigma_2 = 4\delta I$, the result follows.

Therefore

$$\int \left[\widetilde{\Gamma}_\delta(r,p) - \widetilde{\Gamma}_\delta(q,p)\right]^2 dp = 2\left(1 - \exp\left(-\frac{\|r-q\|^2}{8\delta}\right)\right) \le 2\left(1 \wedge \frac{\|r-q\|^2}{8\delta}\right).$$

Hence if we set $\rho(r,q) = 1 \wedge \frac{\|r-q\|}{\sqrt{8\delta}} = (c_\delta\|r-q\|) \wedge 1$, we easily verify that conditions (4.1.3) and (4.1.4) are satisfied.

Proposition 4.1.1 *$\rho(r,q)$ is a metric on $\mathbb{R}^2$ and $(\mathbb{R}^2, \rho)$ is a Polish space.*

Proof. First note that $\rho(r,q) = \rho(q,r)$, and $\rho(r,q) = 0$ if and only if $r = q$. Next to prove the triangular inequality, let r, q, p are three points in $\mathbb{R}^2$. We need to prove that $\rho(r,q) \le \rho(r,p) + \rho(p,q)$, that is

$$(c_\delta\|r-q\|) \wedge 1 \le (c_\delta\|r-p\|) \wedge 1 + (c_\delta\|p-q\|) \wedge 1.$$

If either $c_\delta\|r-p\|$ or $c_\delta\|p-q\|$ is greater than 1, it is trivial because the left-hand side is less that or equal to 1.

If both $c_\delta\|r-p\|$ and $c_\delta\|p-q\|$ are smaller than 1, then

$$\begin{aligned} \rho(r,q) &\le c_\delta\|r-q\| \\ &\le c_\delta\|r-p\| + c_\delta\|p-q\| \\ &= \rho(r,p) + \rho(r,q). \end{aligned}$$

Hence the result. Finally, separability and completeness follow from the fact $B_d(p,a) = B_\rho(p, ac_\delta)$ if $a < 1/c_\delta$, where $d(p,q) = \|p-q\|$. $\square$

Assume that $q^1(t)$ and $q^2(t)$ are $\mathbb{R}^2$-valued adapted processes. Then $M_{ij}(t) = \displaystyle\int_0^t \int \widetilde{\Gamma}_\delta(q^i(s),p) w_j(dp,ds)$ are $\mathbb{R}$-valued square integrable continuous martingales and their mutual quadratic variation are given by

$$\langle\langle M_{ik}(t), M_{jl}(t)\rangle\rangle = \int_0^t \int \widetilde{\Gamma}_\delta(q^i(s),p)\widetilde{\Gamma}_\delta(q^j(s),p) dp ds \cdot \delta_{k,l}, \tag{4.1.8}$$

$i,j = 1,2$, $k,l = 1,2$, with $\delta_{1,1} = \delta_{2,2} = 1$ and $\delta_{1,2} = 0$.

Moreover, from (4.1.7) it follows that the correlations between $M_{ik}(t)$ and $M_{jk}(t)$ are negligible if $\|q^1(s) - q^2(s)\|^2 >> \delta$ and that they are observable if $\|q^1(s) - q^2(s)\|^2 \sim \delta$. In other words, δ is the correlation length.

For metric spaces S_1, S_2, let $C(S_1, S_2)$ be the space of continuous function from S_1 into S_2.

Endow $\mathbb{R}^2$ with the metric ρ defined in (4.1.3) and also endow $\mathbb{R}^{2N}$ with $\rho_N(r,q) = \max\limits_{1\le i\le N} \rho(r^i, q^i)$. Set $\|r-q\|_N = \max\limits_{1\le i\le N} \|r^i - q^i\|$. To indicate what metric is used on $\mathbb{R}^2$ and $\mathbb{R}^{2N}$, we will write $(\mathbb{R}^2, \rho)$ and $(\mathbb{R}^{2N}, \rho_N)$, while $(\mathbb{R}^2, \|\cdot\|)$ and $(\mathbb{R}^{2N}, \|\cdot\|_N)$, refer to the usual Euclidean metric.

4.2 The analysis of particle systems and SPDEs

Now we look at the following SPDEs that describe the particle movement

$$\begin{cases} dr^i(t) = \sum_{j=1}^N a_j K_\varepsilon(r^i - r^j)dt + \sqrt{2\nu}\int \hat{\Gamma}_\delta(r^i, p)w(dp, dt), \\ r^i(0) = r_0^i, \ i = 1, 2, \ldots, N. \end{cases} \tag{4.2.9}$$

Lemma 4.2.1 *For every $r_N(0) \in (\mathbb{R}^{2N}, \rho_N)$ $\mathcal{F}_0$-adapted initial condition, (4.2.9) has a unique $\mathcal{F}_t$-adapted solution $r_N(\cdot) \in C([0,\infty); \mathbb{R}^{2N})$ a.s., which is an $\mathbb{R}^{2N}$-valued Markov process.*

For sake of completeness, we fill in the missing details in Kotelenez's proof [54].

Proof. Let $q_{N,l}(\cdot) = (q_l^1(\cdot), \ldots, q_l^N(\cdot))^\top$ be $\mathbb{R}^{2N}$-valued adapted $P \otimes \lambda$ measurable stochastic process, $l = 1, 2$. Set

$$Q_{N,l}(t); = \sum_{i=1}^N a_i \delta_{q_l^i(t)},$$

and

$$\widehat{q}_l^i(t) = q_l^i(0) + \int_0^t \int K_\varepsilon(q_l^i(s) - p)Q_{N,l}(dp, s)ds + \int_0^t \int \widehat{\Gamma}_\delta(q_l^i(s), p)w_i(dp, ds).$$

First, we need some estimations on $\|\widehat{q}_1^i(t) - \widehat{q}_2^i(t)\|^2$. We have

$$\left\|\int_0^t \int K_\varepsilon(q_1^i(s) - p)Q_{N,1}(dp, s)ds - \int_0^t \int K_\varepsilon(q_2^i(s) - p)Q_{N,2}(dp, s)ds\right\|^2$$
$$= \left\|\int_0^t \sum_{j=1}^N a_j K_\varepsilon(q_1^i(s) - q_1^j(s))ds - \int_0^t \sum_{j=1}^N a_j K_\varepsilon(q_2^i(s) - q_2^j(s))ds\right\|^2$$
$$= \left\|\int_0^t \sum_{j=1}^N a_j (K_\varepsilon(q_1^i(s) - q_1^j(s)) - K_\varepsilon(q_2^i(s) - q_2^j(s))ds\right\|^2$$

$$\leq T \left\| \int_0^t (\sum_{j=1}^N a_j (K_\varepsilon(q_1^i(s) - q_1^j(s)) - K_\varepsilon(q_2^i(s) - q_2^j(s)))^2 ds \right\|$$
$$\leq 2T \int_0^t \sum_{j=1}^N a_j^2 (K_\varepsilon(q_1^i(s) - q_1^j(s)) - K_\varepsilon(q_2^i(s) - q_2^j(s)))^2 ds$$
$$\leq 2T \int_0^t \sum_{j=1}^N a_j^2 c_\delta^2 ((q_1^i(s) - q_1^j(s) - q_2^i(s) + q_2^j(s))^2 ds$$
$$\leq 2T \int_0^t c_\delta^2 (\sum_{j=1}^N (a_j^2 \|q_1^i(s) - q_2^i(s)\|^2 + \|q_1^j(s) - q_2^j(s)\|^2) ds$$
$$\leq 2T(c_\delta a)^2 \int_0^t (\sum_{j=1}^N (\|q_1^i(s) - q_2^i(s)\|^2 + \|q_1^j(s) - q_2^j(s)\|^2) ds$$
$$\leq 2T(c_\delta a)^2 N \int_0^t \|q_{N,1}(s) - q_{n,2}(s)\|_N^2 \, ds,$$

and

$$E \sup_{0 \leq t \leq T} \left\| \int_0^t \int \widehat{\Gamma}_\delta(q_1^i(s), p) w(dp, ds) - \int_0^t \int \widehat{\Gamma}_\delta(q_2^i(s), p) w(dp, ds) \right\|^2$$
$$= E \sup_{0 \leq t \leq T} \sum_{j=1}^2 \left\langle \left\langle \int_0^t \int (\widetilde{\Gamma}_\delta(q_1^i(s), p) - \widetilde{\Gamma}_\delta(q_2^i(s), p)) w_j(dp, ds) \right\rangle \right\rangle$$
$$= 2E \sup_{0 \leq t \leq T} \int_0^t \int (\widetilde{\Gamma}_\delta(q_1^i(s), p) - \widetilde{\Gamma}_\delta(q_2^i(s), p))^2 dp ds$$
$$\leq 2cE \sup_{0 \leq t \leq T} \int_0^t \rho^2(q_1^i(s) - q_2^i(s)) ds \leq 2c \int_0^T E\rho^2(q_1^i(s) - q_2^i(s)) ds,$$

where we used inequality (4.1.4).

To complete the proof of existence and uniqueness of solution, it suffices to use a version of the standard contraction method adapted to this kind of stochastic partial differential equations. An example of this can be found in Ethier and Kurtz [32, p. 300]. □

Remark 4.2.2 *Several important papers in the litterature on this subject (e.g. Kotelenez [54], Amirdjanova [2]) contain proofs that rely on an assumption of completeness of some spaces of signed measures which are in fact not complete. In what follows we define the right setting for the true space of solutions.*

For any $\lambda \geq 0$, let $M(\lambda)$ be the set of (non negative) Borel measures μ with $\mu(\mathbb{R}^2) = \lambda$. This space is equipped with the Wasserstein metric defined in the following way:

$$\gamma_m(\mu, \nu) = \left[\inf_{Q \in C(\mu,\nu)} \int_{\mathbb{R}^4} \rho^m(x, y) Q(dx, dy) \right]^{1/m}$$

where $C(\mu, \nu)$ is the set of all joint representation of (μ, ν), that is, for any Borel subset A of $\mathbb{R}^2$, $Q(A \times \mathbb{R}^2) = \mu(A)$ and $Q(\mathbb{R}^2 \times A) = \nu(A)$.

Further let $M_f(\lambda)$ be the subset of measure in $M(\lambda)$ concentrated on finite sets. It follows from [30, Lemma 11.8.4] that $M_f(\lambda)$ is dense in $M(\lambda)$.

Let a^+ and a^- be nonnegative numbers such that $a = a^+ + a^- > 0$.

Let $M(a^+,\, a^-) = M(a^+) \times M(a^-)$. For any $\mu, \eta \in M(a^+,\, a^-)$, define

$$\gamma_m(\mu, \eta) = \gamma_m \left\{ (\mu^+, \mu^-), (\eta^+, \eta^-) \right\} = \left[\gamma_m^m \left(\mu^+, \eta^+ \right) + \gamma_m^m \left(\mu^-, \eta^- \right) \right]^{1/m}.$$

The proof of the following lemma is a consequence of Minkowski's inequality together with a lemma in Dudley [30, p.330].

Lemma 4.2.3 *$\gamma_1(\cdot, \cdot)$ and $\gamma_2(\cdot, \cdot)$ are metrics on $M(a^+,\, a^-)$.*

Therefore $(M(a^+,\, a^-), \gamma_m)$ is a metric space, with topology equivalent to the product topology of the product spaces $M(a^+) \times M(a^-)$.

Remark that since $\rho \leq 1$,

$$\gamma_2^2(\mu, \eta) \leq \gamma_1(\mu, \eta) \leq 2^{1/2} a \gamma_2(\mu, \eta).$$

It follows that γ_1 and γ_2 generate equivalent topologies.

Lemma 4.2.4 *Under metric γ_m, $(M(a^+,\, a^-), \gamma_m)$ is a Polish space and this metric generates on $M(a^+,\, a^-)$ the topology of weak convergence of pairs of positive measures. Moreover $M_f(a^+,\, a^-)$ is dense in $M(a^+,\, a^-)$.*

Recall the Lipschitz seminorm for functions $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by

$$\|f\|_L = \sup_{r \neq q \in \mathbb{R}^2} \frac{|f(r) - f(q)|}{\rho(r, q)}.$$

Note that by hypothesis on G_ε, we have $\|K_\varepsilon\|_L \leq C_\varepsilon$.

If $\mu, \eta \in M(a^+, a^-)$ then, by denoting $\mu^s = \mu^+ - \mu^-$ and $\nu^s = \nu^+ - \nu^-$, we notice that there holds

$$\sup_{\|f\|_L \leq 1} |< \mu^s - \eta^s, f >| \leq \gamma_1(\mu, \eta) \tag{4.2.10}$$

and

$$\sup_{\|f\|_L \leq 1} < \mu^s - \eta^s, f >^2 \leq 2a^2 \gamma_2^2(\mu, \eta). \tag{4.2.11}$$

Throughout the rest of the section, let a, a^+, a^- be fixed.

The following stochastic Navier-Stokes equation on $M = M(a^+,\, a^-)$ will by analyzed next.

A path $t \mapsto \chi(t) \in M$, is called a (weak) solution of the stochastic Navier-Stokes equation if $\chi^s(t) = \chi^+(t) - \chi^-(t)$ satisfies

$$\left\{ \begin{array}{rcl} d\chi^s(t) & = & [\nu \Delta \chi^s - \chi^s \nabla \cdot (U_\varepsilon)]\, dt \\ & & \qquad -\sqrt{2\nu} \nabla \cdot (\chi^s \int \widehat{\Gamma}_\delta(\cdot, p) w(dp, dt)), \\ U_\varepsilon(r, t, \chi^s) & = & \int K_\varepsilon(r - q) \chi^s(t, dq). \end{array} \right. \tag{4.2.12}$$

Since $\chi(t)$ is measure valued here, what we mean by (4.2.12) is that $\chi^s(t)$ satisfies

$$\left\{\begin{array}{rcl} d < \chi^s(t), f > & = & [\nu < \chi^s(t), \Delta f > + < \chi^s(t), (U_\varepsilon \cdot \nabla f) >] \, dt \\ & & + \sqrt{2\nu} < \chi^s(t), \int \widehat{\Gamma}_\delta(\cdot, p) w(dp, dt) \cdot \nabla f >, \\ U_\varepsilon(r, t, \chi^s) & = & \int K_\varepsilon(r-q)\chi^s(t, dq), \end{array}\right. \tag{4.2.13}$$

for all $f \in C_b^2(\mathbb{R}^2, \mathbb{R})$.

For any real number λ, set $\lambda^+ = \max(0, \lambda)$ and $\lambda^- = -\min(0, \lambda)$.

Lemma 4.2.5 *Suppose that* $\sum_{a_i \geq 0} a_i = a^+$ *and* $\sum_{a_i < 0} a_i = -a^-$. *Let* $\chi(t) \in M(a^+, a^-)$ *be associated with (4.1.5), that is*

$$\chi^{\pm}(t) = \sum_{i=1}^{N} (a_i)^{\pm} \delta_{r^i(t)}$$

Then $\chi(t) \in M$ *for all* $t \geq 0$ *and the empirical signed measure* $\chi^s(t)$ *defined by*

$$\chi^s(t) = \chi^+(t) - \chi^-(t) = \sum_{i=1}^{N} a_i \delta_{r^i(t)},$$

satisfies (4.2.13). That is χ *is a weak solution of (4.2.12).*

Proof. Let $f \in C_b^2(\mathbb{R}^2, \mathbb{R})$. We use assumption C1 and the standard multi-dimensional version of Itô's formula, e.g. Karatzas [48, p. 153], in order to calculate $< \chi^s(t), f >= \sum_{j=1}^{N} a_j f(r^j(t))$.

Set $U_\varepsilon^N(r, t) = U_\varepsilon(r, t, \chi^s)$. Then equation (4.1.5) can be written as $r^i(t) = r^i(0) + \int_0^t U_\varepsilon(r^i(s), s, \chi^s) ds + \sqrt{2\nu} M^i(t)$, $1 \leq i \leq N$, where the $\mathbb{R}^2$-valued martingale M^i has components $M_{ij} = \int_0^t \int \tilde{\Gamma}_\delta(r^i(s), p) w_j(dp, ds),\ j = 1, 2.$ Using representation (4.1.8) for the quadratic variation of M^i, one obtains,

$$\begin{aligned} d < \chi^s(t), f > &= \sum_{i=1}^N \sum_{j=1}^2 a_i \frac{\partial}{\partial x_j} f(r^i(t)) U_{\varepsilon,j}(r^i(t), t) dt \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^2 a_i \frac{\partial^2}{\partial^2 x_j} f(r^i(t)) (\sqrt{2})^2 \nu dt \\ &\quad + \sqrt{2\nu} \sum_{i=1}^N \sum_{j=1}^2 a_i \frac{\partial}{\partial x_j} f(r^i(t)) \int \widetilde{\Gamma}_\delta(r^i(t), p) w_j(dp, dt) \\ &= [\nu < \chi^s(t), \Delta f > + < \chi^s(t), (U_\varepsilon \cdot \nabla f) >] \, dt \\ &\quad + \sqrt{2\nu} < \chi^s(t), \int \widehat{\Gamma}_\delta(\cdot, p) w(dp, dt) \cdot \nabla f >, \end{aligned}$$

which is exactly the weak form (4.2.13). □

Let $\chi(0) = (\chi^+(0), \chi^-(0)), \eta(0) = (\eta^+(0), \eta^-(0)) \in M_f = M_f(a^+, a^-)$. It follows that there exist solutions $x(t)$ and $y(t)$ of the SODE's (4.1.5) with initial conditions $x(0)$ and $y(0)$ so that the respective empirical measures are $\chi^s(0) = \chi^+(0) - \chi^-(0)$ and $\eta^s(0) = \eta^+(0) - \eta^-(0)$.

Denote by $\chi(t)$ and $\eta(t)$ the bivariate measures associated with $x(t)$ and $y(t)$ respectively. As usual denote by $\chi^s(t) = \chi^+(t) - \chi^-(t)$ and $\eta^s(t) = \eta^+(t) - \eta^-(t)$ the associated empirical signed measures.

The following lemma will allow us to extend the solution of discrete initial conditions to arbitrary initial conditions in $M = M(a^+, a^-)$.

Lemma 4.2.6 *For an any $T > 0$, there exist $c' = c'(T), c'' = c''(T) > 0$,* *independent of* $\chi(0), \eta(0) \in M_f$*, such that*

$$E \sup_{0 \le t \le T} \gamma_2^2(\chi(t), \eta(t)) \le c' \gamma_2^2(\chi(0), \eta(0)). \qquad (4.2.14)$$

Moreover

$$E \sup_{0 \le t \le T} \sup_{\|f\|_L \le 1} < \chi^s(t) - \eta^s(t), f >^2 \le c'' \gamma_2^2(\chi(0), \eta(0)). \qquad (4.2.15)$$

Proof. We consider the following two $\mathbb{R}^2$-valued Itô equations with deterministic initial conditions $y, z \in \mathbb{R}^2$.

$$\begin{cases} dr(t) = \int K_\varepsilon(r(t) - p)\chi^s(t, dp)dt + \sqrt{2\nu} \int \hat{\Gamma}_\delta(r(t), p) w(dp, dt), \\ r(0) = y. \\ dq(t) = \int K_\varepsilon(q(t) - p)\eta^s(t, dp)dt + \sqrt{2\nu} \int \hat{\Gamma}_\delta(q(t), p) w(dp, dt), \\ q(0) = z. \end{cases}$$

When $y = x^i(0)$ (the ith two-dimensional component of $x(0)$, then $r(t, y)$ is the position of the ith vortex starting at x^i, that is $r(t, y) = x^i(t)$, using uniqueness property in Lemma 4.2.1. Similarly, $q(t) = y^i(t)$, if $z = y^i(0)$.

Let $Q^\pm(0)$ be joint representations of $(\chi^\pm(0), \eta^\pm(0))$. The following expressions define joint representations $Q^\pm(t)$ of $(\chi^\pm(t), \eta^\pm(t))$ for every $t \ge 0$:

$$\int \int f(y, z) Q^\pm(t, dy, dz) = \int \int f(r(t, y), q(t, z)) Q^\pm(0, dy, dz),$$

$f \in C_b(\mathbb{R}^4, \mathbb{R})$.

To see that $Q^+(t)$ is indeed a representation for $(\chi^+(t), \eta^+(t))$, remark that

$$\int \int f(y) Q^+(t, dy, dz) \quad = \quad \int \int f(r(t, y)) Q^+(0, dy, dz)$$

$$
\begin{aligned}
&= \int f(r(t,y))\chi^+(0,dy) \\
&= \sum_{i;a_i\geq 0} a_i f(r(t,x^i(0))) \\
&= \sum_{i;a_i\geq 0} a_i f(x^i(t)) \\
&= \int f(y)\chi^+(t,dy).
\end{aligned}
$$

Similarly,

$$\int f(y)Q^-(t,dy,dz) = \int f(y)\chi^-(t,dy)$$

and

$$\int f(z)Q^\pm(t,dy,dz) = \int f(z)\eta^\pm(t,dz).$$

It follows that

$$
\begin{aligned}
\gamma_2^2(\chi(t),\eta(t)) \;\leq\; & \int \rho^2(r(t,y),q(t,z))Q^+(0,dy,dz) \\
& + \int \rho^2(r(t,y),q(t,z))Q^-(0,dy,dz).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|r(t)-r(0)-q(t)+q(0)\| &\geq \rho(r(t)-q(t),r(0)-q(0)) \\
&\geq \rho(r(t),q(t))-\rho(r(0),q(0)).
\end{aligned}
$$

Next we calculate $\|r(t)-r(0)-q(t)+q(0)\|^2$.

Proceeding as in Lemma 4.2.1, we have

$$E\sup_{0\leq t\leq T}\left\|\int_0^t\int \widehat{\Gamma}_\delta(r(s),p)w(dp,ds)-\int_0^t\int \widehat{\Gamma}_\delta(q(s),p)w(dp,ds)\right\|^2$$

$$\leq 2c \int_0^T E\rho^2(r(s), q(s))ds,$$

and

$$\left\| \int_0^t \int K_\varepsilon(r(s) - p)\chi^s(s, dp)ds - \int_0^t \int K_\varepsilon(q(s) - p)\eta^s(s, dp)ds \right\|^2$$
$$\leq 2 \left\| \int_0^t \int (K_\varepsilon(r(s) - p) - K_\varepsilon(q(s) - p))\chi^s(s, dp)ds \right\|^2$$
$$+2 \left\| \int_0^t \int K_\varepsilon(q(s) - p)(\chi^s(s, dp) - \eta^s(s, dp))ds \right\|^2.$$

Since $\|K_\varepsilon\|_L \leq C_\varepsilon$, one has that

$$\left\| \int_0^t \int (K_\varepsilon(r(s) - p) - K_\varepsilon(q(s) - p))\chi^s(s, dp)ds \right\|^2$$
$$\leq 4T(aC_\varepsilon)^2 \int_0^t \rho^2(r(s), q(s))ds.$$

In addition, it follows from (4.2.11) that

$$\left\| \int_0^t \int K_\varepsilon(q(s) - p)(\chi^s(s, dp) - \eta^s(s, dp))ds \right\|^2$$
$$\leq 2T(aC_\varepsilon)^2 \int_0^t \gamma_2^2(\chi(s), \eta(s))ds.$$

Hence

$$\begin{aligned} E \sup_{0\leq t\leq T} \rho^2(r(t), q(t)) &\leq E \sup_{0\leq t\leq T} \|r(t) - r(0) - q(t) + q(0)\|^2 + 2\rho^2(y, z) \\ &\leq 2\rho^2(y, z) \\ &\quad +d \int_0^T E \sup_{0\leq s\leq T} \rho^2(r(s), q(s))ds \\ &\quad\quad +d \int_0^T \gamma_2^2(\chi(s), \eta(s))ds, \end{aligned}$$

where $d = 8T\,(c + (2aC_\varepsilon)^2)$.

By Gronwall inequality, we have

$$\begin{aligned} E \sup_{0\le t\le T} \rho^2(r(t), q(t)) &\le 2e^{dT}\rho^2(y,z) \\ &\qquad + e^{dT}\int_0^T \gamma_2^2(\chi(s), \eta(s))ds. \end{aligned}$$

Integrating the last inequality with respect to $Q^+ + Q^-$, one obtains

$$\begin{aligned} E \sup_{0\le s\le t} \gamma_2^2(\chi(t), \eta(t)) &\le E \sup_{0\le t\le T} \rho^2(r(t), q(t))\{Q^+ + Q^-\}(dy, dz) \\ &\le 2e^{dT}\int \rho^2(y,z)\{Q^+ + Q^-\}(dy, dz) \\ &\qquad + ae^{dT}\int_0^T \gamma_2^2(\chi(s), \eta(s))ds. \end{aligned}$$

Taking the infimum over all $Q^\pm \in C(\chi^\pm(0), \eta^\pm(0))$, one gets

$$\begin{aligned} E \sup_{0\le s\le t} \gamma_2^2(\chi(t), \eta(t)) &\le 2e^{dT}\gamma_2^2(\chi(0), \eta(0)) \\ &\qquad + ae^{dT}\int_0^T \gamma_2^2(\chi(s), \eta(s))ds. \end{aligned}$$

Using Gronwall inequality again yields (4.2.14). Finally, (4.2.15) is obtained by combining (4.2.11) and (4.2.14) $\square$

Notice that this lemma yields the unicity of solution to equation (4.2.13) when the initial position is given.

The proof of the lemma also holds in the initial conditions are random. Therefore we have the following.

Corollary 4.2.7 *For an any $T > 0$, and any random initial conditions $\chi(0), \eta(0) \in M_f$ a.s.,*

$$E \sup_{0 \le t \le T} \gamma_2^2(\chi(t), \eta(t)) \le c' E \gamma_2^2(\chi(0), \eta(0)). \tag{4.2.16}$$

Moreover

$$E \sup_{0 \le t \le T} \sup_{\|f\|_L \le 1} < \chi^s(t) - \eta^s(t), f >^2 \le c'' E \gamma_2^2(\chi(0), \eta(0)). \tag{4.2.17}$$

The next result, which is new in the literature, shows that if the positive and negative initial parts $\chi^{\pm}(0)$ have discrete disjoint supports, then they remain apart forever and $\chi^{\pm}(t)$ forms the Jordan decomposition of $\chi^s(t)$. Note that the case where the initial parts do not have discrete support, remains an important open problem, especially in view of the fact that many in the physics litterature assume it to be true.

Theorem 4.2.8 *Suppose that the initial conditions satisfy*

$$E \sum_{i,j=1\ i \ne j}^{N} |\ln \|r_i(0) - r_j(0)\|| < \infty.$$

Then

$$E \sum_{i,j=1,\ i \ne j}^{N} \sup_{t \le T} |\ln \|r_i(t) - r_j(t)\|| < \infty,$$

for every $T > 0$.

In particular, the positive and negative parts in the Jordan decomposition of the empirical measure process $\chi_N(t)$ have none-explosive continuous trajectories.

Proof.

Rewrite model (4.2.9) as

$$\begin{cases} x^i(t) = \int_0^t F(x^i(s), \chi(s))ds + \sqrt{2\nu} \int \hat{\Gamma}_\delta(x^i, p)w(dp, dt), \\ x^i(0) = x_0^i, \ i = 1, 2, \ldots, N, \end{cases}$$

where $F(x^i(s), \chi(s)) = \int K_\varepsilon(x^i - y)\chi(s, dy)$, and $\chi(s, dy) = \sum_{i=1}^N a_i \delta_{x^i(t)}(dy)$.

We have

$$\begin{aligned} \|F(x^i(s), \chi(s)) - F(x^j(s), \chi(s))\| &= \int (K_\varepsilon(x^i - y) - K_\varepsilon(x^j - y))\chi(s, dy) \\ &\leq c_\varepsilon \|x^i - x^j\| \int \|\chi(s, dy)\| \\ &= a c_\varepsilon \|x^i - x^j\|. \end{aligned}$$

It suffices to look at distance between x^1 and x^2. To this end, let $r_t = \|x^1(t) - x^2(t)\|$ and assume r_0 is some fixed positive number. For any choice of $0 < b < r_0 < B < \infty$, define the stopping times $\rho_b = \inf\{t \geq 0; r_t \leq b\}$ and $\sigma_B = \inf\{t \geq 0; r_t \geq B\}$. Itô's formula yields

$$\begin{aligned} \ln r(t \wedge \rho_b \wedge \sigma_B) &= \ln r_0 + N_{t \wedge \rho_b \wedge \sigma_B} \\ &\quad + \int_0^{t \wedge \rho_b \wedge \sigma_B} \frac{1}{r_s^2}(x^1(s) - x^2(s)) \cdot \\ &\quad [F(x^1(s), \chi(s)) - F(x^2(s), \chi(s))]ds, \end{aligned}$$

where the martingale N defined by

$$N_t = \int_0^t \frac{1}{r_s^2}(x^1(s) - x^2(s)) \cdot \int [\widehat{\Gamma}_\delta(x^1(s) - y) - \widehat{\Gamma}_\delta(x^2(s) - y)]w(dy, ds)$$

is actually a square-integrable martingale, with quadratic variation

$$< N_t >= 2\int_0^t \frac{1}{r_s^2}(1 - e^{-r_s^2/8\delta})ds \leq \frac{t}{4\delta},$$

because of $1-e^{-x} \leq x$. Therefore

$$\sup_{t\leq T} \| \int_0^t \frac{1}{r_s^2}(x^1(s)-x^2(s)) \cdot [F(x^1(s))-F(x^2(s))]ds\| \leq c^*T,$$

and we can conclude that

$$\sup_{t\leq T} \| \ln r(t\wedge \rho_b \wedge \sigma_B)\| \leq \| \ln r_0\| + \sup_{t\leq T} \|N_{t\wedge\rho_b\wedge\sigma_B}\| + c^*T.$$

Next, the Burkholder-Davis-Gundy inequality (see Metivier and Pellaumail [68]) yields

$$E\sup_{t\leq T} \| \ln r(t\wedge \rho_b \wedge \sigma_B)\| \leq \| \ln r_0\| + 8\sqrt{\frac{T}{\delta}} + c^*T.$$

Since the trajectory of r is everywhere continuous, by monotone convergence the left-hand side in the last inequality converges to

$$E\sup_{t\leq T} \| \ln r(t\wedge \rho_0 \wedge \sigma_\infty)\|,$$

as b decreases to 0 and B increases to ∞. It remains true if we put $b=0$ and $B=\infty$ or both; hence $P(\rho_0 \wedge \sigma_\infty = \infty) = 1$. The case where r_0 is random follows by conditioning on the value of r_0. This completes the proof. $\square$

Recall that $M = M(a^+, a^-)$ and $M_f = M_f(a^+, a^-)$. Further let
$\mathcal{M}$ be the space of M-valued random variables with metric γ defined by

$$\gamma(\chi,\eta) = \left[E\gamma_2^2(\chi,\eta)\right]^{1/2},$$

$\mathcal{M}_f$ be the subset of M_f-valued random variables,

$\mathcal{M}[0,T]$ be the space of continuous $C([0,T];M)$-valued random variables with metric $\gamma_{[0,T]}$ defined by

$$\gamma_{[0,T]}(\chi,\eta) = \left[E \sup_{0\le t\le T} \gamma_2^2(\chi(t),\eta(t))\right]^{1/2}.$$

Theorem 4.2.9 *$\gamma(\chi,\eta)$ is a distance on $\mathcal{M}$ and $(\mathcal{M},\gamma)$ is a metric space.*

Proof.

We prove the triangle inequality here. Let $\chi,\eta,\psi \in \mathcal{M}$ we have:

$$\begin{aligned}
&(\gamma(\chi,\psi)+\gamma(\eta,\psi))^2\\
&= ([E\gamma_2^2(\chi,\psi)]^{1/2} + [E\gamma_2^2(\eta,\psi)]^{1/2})^2\\
&= E\gamma_2^2(\chi,\psi) + E\gamma_2^2(\eta,\psi) + 2\,[E\gamma_2^2(\chi,\psi)]^{1/2}\,[E\gamma_2^2(\eta,\psi)]^{1/2}\\
&\ge E\gamma_2^2(\chi,\psi) + E\gamma_2^2(\eta,\psi) + 2E(\gamma_2(\chi,\psi)\gamma_2(\eta,\psi))\\
&= E(\gamma_2(\chi,\psi)+\gamma_2(\eta,\psi))^2\\
&\ge E(\gamma_2^2(\chi,\eta))\\
&= \gamma^2(\chi,\eta)
\end{aligned}$$

So we have $\gamma(\chi,\psi)+\gamma(\eta,\psi) \ge \gamma(\chi,\eta)$

Note since (M,γ_2) is complete, so $(\mathcal{M},\gamma)$ is complete too. □

Similarly we can prove the following.

Theorem 4.2.10 *$\gamma_{[0,T]}(\chi,\eta)$ is a distance on $\mathcal{M}[0,T]$ and $(\mathcal{M}[0,T],\gamma_{[0,T]})$ is a complete metric space.*

Note that $\mathcal{M}_f$ is dense in $\mathcal{M}$.

Theorem 4.2.11 *The map $\chi(0) \mapsto \chi(\cdot)$ from $\mathcal{M}_f$ into $\mathcal{M}[0,T]$ extends uniquely to a map $\chi(0) \mapsto \chi(t)$ from $\mathcal{M}$ into $\mathcal{M}[0,T]$. Moreover, for any $\chi(0), Y_0 \in \mathcal{M}$,*

$$\gamma_{[0,T]}(\chi, \eta) \leq c' \gamma(\chi(0), \eta(0))$$

and $\chi^s = \chi^+ - \chi^-$ satisfies (4.2.13), that is χ is a weak solution of the stochastic Navier-Stokes equations with initial condition $\chi(0) \in \mathcal{M}$.

Proof.

To this end, let $\chi(0) \in \mathcal{M}$ be given. Since $\mathcal{M}_f$ is dense in $\mathcal{M}$, there exists a sequence $\chi_n(0) \in \mathcal{M}_f$ so that $\gamma(\chi(0), \chi_n(0)) \to 0$, as $n \to \infty$. Using Corollary 4.2.7, it follows that χ_n is a Cauchy sequence in $\mathcal{M}[0,T]$. Thus there exists $\chi \in \mathcal{M}[0,T]$ so that $\gamma_{[0,T]}(\chi, \chi_n) \to 0$. Since the limit does not depend on the sequence, the mapping is well-defined.

It also follows that for any $\chi(0), \eta(0) \in \mathcal{M}$, the corresponding paths $\chi, \eta \in \mathcal{M}_{[0,T]}(\chi, \eta)$ satisfy

$$\gamma_{[0,T]}(\chi, \eta) \leq c' \gamma(\chi(0), \eta(0)).$$

Next we will show this extension gives a weak solution of the stochastic Navier-Stokes equation (4.2.12). Using 4.2.17 and the last inequality, one can conclude that

$$E \sup_{0 \leq t \leq T} \sup_{\|f\|_L \leq 1} < \chi^s(t) - \eta^s(t), f >^2 \leq c'' \gamma(\chi(0), \eta(0)).$$

In particular,

$$\lim_{n \to \infty} E \sup_{0 \leq t \leq T} \sup_{\|f\|_L \leq 1} < \chi^s(t) - \chi_n^s(t), f >^2 = 0.$$

The next step is to show that for any $f \in C_b^2(\mathbb{R}^2, R)$ and $\chi(0) \in M_0$, $< \chi^s(t), f >$ satisfies (4.2.13).

Note that by the choice of f, f and ∇f are bounded, so the right-hand side of the stochastic Navier-Stokes equation is defined for $< \chi(t), f >$. Moreover, since $\|f\|_L < \infty$ and $\|\nabla f\|_L < \infty$, it follows that

$$\lim_{n\to\infty} E \sup_{0\le t\le T} < \chi^s(t) - \chi_n^s(t), f >^2 = 0.$$

Similarly, $E \sup\limits_{0\le t\le T} \sup\limits_{p} |U_\varepsilon(p, t, \chi^s(t)) - U_\varepsilon(p, t, \chi_n^s(t))|^2 \to 0$ and one can prove that all righthand side terms of (4.2.13) tends to zero, as n tends to infinity. This completes the proof. $\square$

4.3 Macroscopic limit

In this section we prove convergence results for our particle systems as $N \to \infty$. First we restate our problem as follows:

Assume the number of particles is N, their initial positions $x_{i,N}$ and their vorticities $a_{i,N}, i = 1, 2, \ldots N$.

Suppose that $\omega_0(x)$ has compact support inside the interval $[-M, M]^2$ and $\omega_0(x) \in L^2(\mathbb{R}^2)$. For fixed N, we use the regularly spaced points $\{x_{i,N}, i = 1, 2, ..., N\}$ on $[-M, M]^2$ to build a partition or this square into $(2M)^2$ small square grids of individual surface area $\triangle_N = (2M)^2/N$.

Lemma 4.3.1 *If $\omega_0(x) \in L^2(\mathbb{R}^2)$ is continuous and has support in $[-M, M]^2$, we have:*

$$\lim_{N\to\infty} \sum_{i=1}^{N} (\omega_0(x_{i,N}))^\alpha f(x_{i,N}) \triangle_N = \int (\omega_0(x))^\alpha f(x) dx$$

for any $f \in C_b^2(\mathbb{R}^2, R)$, and $\alpha = 1, 2$.

Moreover

$$\lim_{N\to\infty} \sum_{i=1}^{N} (\omega_0(x_{i,N})^\pm)^\alpha f(x_{i,N}) \triangle_N = \int (\omega_0(x)^\pm)^\alpha f(x) dx$$

for any $f \in C_b^2(\mathbb{R}^2, R)$, and $\alpha = 1, 2$.

From now on, let $a_{i,N} = \omega_0(x_{i,N})\triangle_N$. Under the same assumption as above, we have

Lemma 4.3.2

$$\sum_{N=1}^{\infty} \left(\sum_{i=1}^{N} a_{i,N}^2 \right)^2 < \infty$$

Proof.

First,

$$\sum_{i=1}^{N} a_{i,N}^2 = \triangle_N \sum_{i=1}^{N} (\omega_0(x_{i,N}))^2 \triangle_N \,.$$

Since $c = \int (\omega_0(x))^2 dx < \infty$, Lemma (4.3.1) implies that when N is large enough,

$$\sum_{i=1}^{N} a_{i,N}^2 = \triangle_N \sum_{i=1}^{N} (\omega_0(x_{i,N}))^2 \triangle_N \le 2(c+1) \triangle_N \,.$$

Therefore $\left(\sum_{i=1}^N a_{i,N}^2\right)^2 = O(N^{-2})$, proving the result. $\square$

The following lemma, which is a direct consequence of the famous Borel-Cantelli lemma, is used in our strong convergence proof.

Lemma 4.3.3 *Let $\{U_n\}_{n=1}^\infty$ be a sequence of random variables, and let U be a random variable. If for any $\varepsilon > 0$ we have*

$$\sum_{n=1}^\infty P\{|U_n - U| > \varepsilon\} < \infty,$$

then $P\left(\lim_{n\to\infty} U_n = U\right) = 1$.

Next, consider the following particle model:

$$\begin{cases} dx_{i,N}(t) &= \sum_{j=1}^N a_{j,N} K_\varepsilon(x_{i,N} - x_{j,N})dt + \sqrt{2\nu}db_i(t), \\ x_{i,N}(0) &= x_{i,N}, \ i = 1, 2, \dots N, \end{cases} \tag{4.3.18}$$

where the $b_i(t)$, $i = 1, 2, \dots N$, are independent two-dimensional Brownian motions.

Set $a^\pm = \displaystyle\int \omega_0(x)^\pm dx$ and set

$$\lambda_N^\pm = \sum_{i=1}^n a_{i,N}^\pm.$$

Since $\lambda_N^\pm \to a^\pm$, by normalizing the $a_{i,N}$'s if necessary, there is no loss of generality if we assume that $\lambda_N^\pm = a^\pm$.

As before we may define the empirical measure-valued processes as

$$\omega_N^\pm(t) = \sum_{i=1}^N a_{i,N}^\pm \delta_{x_{i,N}(t)}.$$

It follows that $\omega_N(t) = (\omega_N^+(t), \omega_N^-(t)) \in M(a^+, a^-)$. Moreover, from Lemma 4.3.1, $\omega_N(0)$ converge to a bivariate measure with density (ω_0^+, ω_0^-).

As usual, set $\omega_N^*(t) = \omega_N^+(t) - \omega_N^-(t)$. Then using Itô's formula, one can conclude that for any $f \in C_b^2(\mathbb{R}^2, R)$,

$$\left\{\begin{array}{rcl} < \omega_N^*(t), f > & = & < \omega_N^*(0), f > +\nu \int_0^t < \omega_N^*(s), \Delta f > ds \\ & & \quad + \int_0^t < \omega_N^*(s), u^N(s) \cdot \nabla f > ds \\ & & \quad\quad + \sqrt{2\nu} \sum_{i=1}^N \int_0^t a_{i,N} \nabla f(x_{i,N}(s)) \cdot db_i(s) \\ u^N(x,t) & = & \int K_\varepsilon(x-q)\omega_N^*(dq, s) \\ \omega_N^*(0) & = & \sum_{i=1}^N a_{i,N}\delta_{x_{i,N}} \end{array}\right. \tag{4.3.19}$$

Theorem 4.3.4 *Under the assumption above, when $N \to \infty$, the sequence of laws on $\mathcal{M}[0,T]$ of the random elements ω_N^* converges to the deterministic measure in $\mathcal{M}[0,T]$ that is the solution of (3.2.14).*

Proof. Let $\Phi_N(t) = \sum_{i=1}^N \int_0^t a_{i,N} \nabla f(x_{i,N}(s)) db_i(s)$. Obviously Φ_N is square-integrable martingale and its quadratic variation is given by

$$< \Phi_N >_t = \sum_{i=1}^N a_{i,N}^2 \int_0^t \|\nabla f(x_{i,N}(s))\|^2 ds \leq t \sup_{x \in \mathbb{R}^2} \|\nabla f(x)\|^2 \sum_{i=1}^N a_{i,N}^2.$$

Therefore, using by Lemma 4.3.2 and setting $C_f = \sup_{x \in \mathbb{R}^2} \|\nabla f(x)\|^2$, one obtains

$$\sum_{N=1}^\infty < \Phi_N >_t^2 \leq tC_f \sum_{N=1}^\infty \left(\sum_{i=1}^N a_{i,N}^2 \right)^2 < \infty.$$

Now

$$\sum_{N=1}^\infty P\left\{ \sup_{t \leq T} |\Phi_N(t)| > \varepsilon \right\} \leq \sum_{N=1}^\infty \frac{E(\Phi_N^4(T))}{\varepsilon^4} \leq \frac{C_4}{\varepsilon^4} \sum_{N=1}^\infty E(< \Phi_N >_T^2) < \infty,$$

by using the Burkholder-Davis-Gundy inequality (see theorem 2.0.2). It follows from Lemma 4.3.3 that $\lim_{n\to\infty}\sup_{0\le t\le T}|\Phi_N(t)| = 0$ with probability one.

Using the same techniques as in Lemma 4.2.6 and Theorem 4.2.11, it follows that the laws of ω_N^* converge to ω in $\mathcal{M}[0,T]$, with ω solution to

$$\left\{\begin{array}{rcl} <\omega(t), f> &=& <\omega(0), f> + \nu\int_0^t <\omega(s), \Delta f> ds \\ && \quad + \int_0^t <\omega(s), u(s)\cdot\nabla f> ds \\ u(x,t) &=& \int K_\varepsilon(x-q)\omega(dq,s) \\ \omega^{\pm}(0)(dx) &=& \omega_0^{\pm}(x) \end{array}\right. \tag{4.3.20}$$

Since $\omega \in \mathcal{M}[0,T]$ we have $\omega^{\pm}(t)(\mathbb{R}^2) = a^{\pm}, 0 \le t \le T$. The mass is conserved.

By the uniqueness of solution of the 2D Navier-Stokes equation in vortex form, the proof is now complete.

□

We conjecture that a similar statement for the Brownian sheet case holds but the proof eludes us for the moment.

Chapter 5

New results on Navier-Stokes equations in 3D

We discuss the stochastic structure of the Navier-Stokes equation (3.1.5) in vorticity form in $\mathbb{R}^3$. While we cannot yet prove that it can be approximated by means of a finite-dimensional stochastic process, we do prove it for the regularized version.

5.1 The 3-D case with independent Brownian motion as noise

Let $\omega^\varepsilon(t)$ be a solution of the 3D regularized Navier-Stokes equation in vortex form (3.1.9). Consider the stochastic process satisfying the stochastic differential equation

$$\begin{cases} d\phi_\alpha^\varepsilon(t) = u_\alpha^\varepsilon(\phi^\varepsilon(t),t)dt + \sqrt{2\nu}db_\alpha(t), & \text{almost surely,} \\ \phi_\alpha^\varepsilon(0) = x_\alpha,\ \alpha = 1,2,3, \end{cases} \tag{5.1.1}$$

where $\phi^\varepsilon(t) = (\phi_1^\varepsilon(t), \phi_2^\varepsilon(t), \phi_3^\varepsilon(t)) \in \mathbb{R}^3$, and where b_1, b_2 and b_3 are independent standard Brownian motions. Recall in passing that $u_\alpha^\varepsilon = k_\alpha^\varepsilon * \omega_\alpha^\varepsilon$.

For any $t \in [0,T]$ the matrix-valued process $(\nabla\phi^\varepsilon)_{\alpha\beta}(x,t) = \frac{\partial\phi_\alpha^\varepsilon(x,t)}{\partial x_\beta}$ is defined to be the derivative of $\phi_\alpha^\varepsilon(x,t)$ with respect to the space variable x. It satisfies almost surely the differential equation in flow form (see [57])

$$\begin{cases} \frac{d}{dt}(\nabla\phi^\varepsilon)_{\alpha\beta} = \sum_\gamma (\nabla u_t^\varepsilon(\phi^\varepsilon(t),t)_{\alpha\gamma}(\nabla\phi^\varepsilon)_{\gamma\beta}, \\ \nabla{\phi^\varepsilon}_{\alpha\beta}(t=0) = \delta_{\alpha\beta}. \end{cases} \tag{5.1.2}$$

Using (3.5.18) and (5.1.2), it is easy to get the following estimates:

$$\|\nabla u_t^\varepsilon\|_\infty \le c\,\|\omega_t^\varepsilon\|_1 < ca_\varepsilon,$$

and

$$\|\nabla\phi_t^\varepsilon\|_\infty \le e^{ca_\varepsilon T}. \tag{5.1.3}$$

The following useful representation result was first stated in Esposito and Pulvirenti [31, equation (2.12)].

Lemma 5.1.1 *If $\omega^\varepsilon(t)$ is the solution of the regular Navier-Stokes equation in vortex form (3.1.9) and $\phi^\varepsilon(t,x)$ satisfies (5.1.1), then for any $\varepsilon > 0$ and any vector-valued bounded C_2^3 (twice continuously and boundedly differentiable in every image component) function f from $\mathbb{R}^3$ into itself, we have*

$$\int \omega^\varepsilon(t,x)\cdot f(x)dx = \int E(f(\phi_t^\varepsilon(x))\cdot\nabla\phi_t^\varepsilon(x))\cdot\omega_0(x)dx\,. \tag{5.1.4}$$

Proof. Define a measure $\overline{\omega}^\varepsilon(t)$ on $\mathbb{R}^3$ as the unique one for which the following holds : for any $f \in C_2^3$ on $\mathbb{R}^3$, we have

$$\overline{\omega}^\varepsilon(t)(f) = \int E(f(\phi_t^\varepsilon(x)) \cdot \nabla\phi_t^\varepsilon(x)) \cdot \omega_0(x)dx. \tag{5.1.5}$$

Using Itô's formula, we obtain

$$\begin{aligned}
\frac{d}{dt}\overline{\omega}^\varepsilon(t)(f) &= \int E[u_t^\varepsilon(\phi_t^\varepsilon(x), t) \cdot (\nabla f)(\phi_t^\varepsilon(x)) \cdot \nabla\phi_t^\varepsilon(x)]\omega_0(x)dx \\
&\quad + \int E[f(\phi_t^\varepsilon(x)) \cdot \nabla u_t^\varepsilon(\phi_t^\varepsilon(x)) \cdot \nabla\phi_t^\varepsilon(x)]\omega_0(x)dx \\
&\quad + \nu \int E[(\Delta f)(\phi_t^\varepsilon(x)) \cdot \nabla\phi_t^\varepsilon(x)] \cdot \omega_0(x)dx
\end{aligned}$$

or symbolically

$$\frac{d}{dt}\overline{\omega}^\varepsilon(t)(f) = \overline{\omega}^\varepsilon(t)[(u_t^\varepsilon \cdot \nabla)f + \nu\Delta f + \nabla u_t^\varepsilon \cdot f].$$

But since equation (3.5.17) has a unique weak solution $\omega^\varepsilon(t)$ in the distributional sense, we have equality (as tempered distributions) of $\overline{\omega}^\varepsilon(t) = \omega^\varepsilon(t)$. The proof is completed. □

In the special case where we put $\nu = 0$ equation (5.1.4) reduces to the well-known Lagrangian representation for the Euler flow.

The stochastic particle system to approximate $\omega_t^\varepsilon(x)$ is given by equation (3.5.19). Recall the weighted empirical process defined in (3.5.2) by

$$\omega_t^N(dx) = \sum_{j=1}^N \omega_j(t) \cdot \delta_{r_j(t)(dx)}. \tag{5.1.6}$$

Since (3.5.19) has quadratic tems in ω, we cannot say at present whether the solution of (3.5.19) is unique or even if it exists at all. So we consider the following modification of the vortex model (3.5.19), a truncated form :

$$\begin{cases} \tilde{R}_i(t) = R_i(0) + \int_0^t \sum_{j=0}^N K^\varepsilon(\tilde{R}_i(s) - \tilde{R}_j(s))Y_j(s)\omega_j(0)ds + \sqrt{2\nu}b_i, \\ \frac{dY_i(t)}{dt} = [\sum_{j=0}^N \nabla K^\varepsilon(\tilde{R}_i(t) - \tilde{R}_j(t))Y_j(t)\omega_j(0)] \cdot \chi_M(Y_i(t)), \\ Y_i(0) = I, \quad i = 1, 2, \ldots, N, \end{cases} \tag{5.1.7}$$

where the only modification is the introduction of a censoring operator, i.e., χ_M is a bounded smooth (infinitely differentiable) matrix-valued mapping on the space of 3×3 matrices, such that:

$$(\chi_M(A))_{\alpha,\beta} := \begin{cases} A_{\alpha,\beta} & \|A\| \leq \frac{M}{2} \\ \frac{MA_{\alpha,\beta}}{\|A\|} & \|A\| > M \end{cases} \tag{5.1.8}$$

The stochastic particle system (5.1.7) has a unique solution for any positive t, since the resulting system has Lipschitz coefficients. Moreover, if we set $M = \infty$, $\omega_j(t) = Y_j(t)\omega_j(0)$, then $\{\tilde{R}_j(t), \omega_j(t)\}$ is the solution of (3.5.19), in a formal sense only of course.

Our goal in this section will be to show that the censored system (5.1.7) converges (in a sense made precise below) when both M and N are large but ε remains fixed, to the unique solution of the smooth Navier-Stokes equation (3.1.9).

The ultimate goal pursued was to show that the uncensored system (3.5.19) approximated the classical Navier-Stokes equation (3.1.5) as well when $\varepsilon = \varepsilon(M, N)$ is allowed to go to 0 at some speed but this has not been achieved yet.

Let Ω be the set of the continuous trajectories $[0,T] \mapsto \mathbb{R}^3$. If we think of $\omega_0(x)$ as a vector-valued sign measure on $\mathbb{R}^3$, let $\omega_0^+(x)$ and $\omega_0^-(x)$ be Jordan decomposition of $\omega_0(x)$. Suppose $a_\alpha^+ = \|\omega_{0\alpha}^+(x)\|$, $a_\alpha^- = \|\omega_{0\alpha}^-(x)\|$. Note that in our case $a_\alpha^+, a_\alpha^- \in \mathbb{R}^3, \alpha = 1,2,3$.

Let $M(\Omega)$ the set of finite vector-valued signed measures on Ω such that $\mu_\alpha^\pm(\mathbb{R}^3) = a_\alpha^\pm, \alpha = 1,2,3$, where $\mu_\alpha^\pm$ are respectively the positive and negative part of the Jordan decomposition of $\mu \in M(\Omega)$.

Given $\mu \in M(\Omega)$ and $p \in \Omega$, let

$$\begin{cases} x_t^\mu(p) = p(t) + \int_0^t ds \int K^\varepsilon(x_s^\mu(p) - x_s^\mu(q)) \cdot X_s^\mu(q) \cdot \mu(dq), \\ X_t^\mu(p) = I + \int_0^t ds \int \nabla K^\varepsilon(x_s^\mu(p) - x_s^\mu(q)) \cdot X_s^\mu(q) \cdot \mu(dq) \cdot \chi_M(X_s^\mu(p)). \end{cases} \tag{5.1.9}$$

The system of equations (5.1.9) is well-defined (as can be seen through an iteration scheme or a contraction principle) and therefore implicitly defines a map $p \mapsto (x_t^\mu(p), X_t^\mu(p))$. It provides a description of the evolution of the vorticity when the initial measure μ does not have discrete support, as it did in the finite system (5.1.7). Further, the structure of (5.1.9) remains unchanged whether the particle system it describes comprises a finite or an infinite number of particles, a useful observation first made by Kotelenez [53].

Denoting by $M(\mathbb{R}^3)$ the set of the $\mathbb{R}^3$-valued signed measures on $\mathbb{R}^3$, Esposito and Pulvirenti define the Tanaka-like stochastic linear operator T as a mapping from $M(\Omega)$ into the set of $M(\mathbb{R}^3)$-valued trajectories as follows: for any continuous and bounded $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$,

$$(T_\mu)_t(f) = \int f(x_t^\mu(p)) \cdot X_t^\mu(p) \cdot \mu(dp). \tag{5.1.10}$$

Given the standard Wiener measure $P_x(db)$ on Ω starting at x, denote by $\widetilde{P}_x(db(t)), x \in \mathbb{R}^3$ the (apropriately rescaled) Wiener measure starting at x, that is :

$$\widetilde{P}_x(db) = P_x((\sqrt{2\nu})^{-1}db) \tag{5.1.11}$$

and set

$$\lambda(db) = \int dx\, \omega_0(x)\widetilde{P}_x(db)\,. \tag{5.1.12}$$

The purpose of the introduction of operator T lies in the fact that it yields an explicit form for the solution to the censored system (5.1.7) as a function of the starting law, according to the following proposition.

Proposition 5.1.2 *For any smoothing value $\varepsilon > 0$ and any time $T > 0$ before which a solution to equation (5.1.7) is known to exist, there exists a constant $M = M(\varepsilon, T, \omega^\varepsilon) < \infty$ which depends on the actual solution chosen, such that for all $t \leq T$,*

$$(T\lambda)_t = \omega^\varepsilon(t) \tag{5.1.13}$$

Proof. From the definition of $\phi^\varepsilon(t,x)$, $\nabla\phi^\varepsilon(t,x)$, equation (5.1.2) and their relationship with $\omega^\varepsilon(t)$ through (5.1.4), when M is chosen to be larger than $2\sup_{0\leq t\leq T} \|\nabla\phi^\varepsilon(t,x)\|$, these two functions solve the following system of equations:

$$\begin{cases} \phi^\varepsilon(t,x) = x + \int_0^t \int K^\varepsilon(\phi^\varepsilon(s,x)-y)\omega^\varepsilon(s,dy) + \sqrt{2\nu}db(t), \\ \nabla\phi^\varepsilon(t,x) = I + \int_0^t ds \int \nabla K^\varepsilon(\phi^\varepsilon(s,x)-y)\cdot\omega^\varepsilon(s,dy)\cdot\chi_M(\nabla\phi^\varepsilon(s,x)), \\ \int \omega^\varepsilon(t)\cdot f(x)dx = \int dxE(f(\phi^\varepsilon_t(x))\cdot\nabla\phi^\varepsilon_t(x))\cdot\omega_0(x). \end{cases} \tag{5.1.14}$$

Furthermore

$$\int (T\lambda)_t(dx)\cdot f(x) = \int f(x^\lambda_t(p))\cdot X^\lambda_t(p)\cdot\lambda(dp) = \int dxE(f(x^\lambda_t)\cdot X^\lambda_t)\cdot\omega_0(x).$$

and, denote $p(t) = x+\sqrt{2\nu}b(t)$, by definition of $x^\mu_t(p)$ and $X^\mu_t(p)$, one obtains

$$\begin{aligned} x^\lambda_t(p,x) &= p(t) + \int_0^t ds \int K^\varepsilon(x^\lambda_s(p)-y)\cdot(T\lambda)_s(dy) \\ X^\lambda_t(p,x) &= I + \int_0^t ds \int \nabla K^\varepsilon(x^\lambda_s(p)-y)\cdot(T\lambda)_s(dy)\cdot\chi_M(X^\lambda_s(p)). \end{aligned}$$

Therefore $x^\lambda_t(p), X^\lambda_t(p)$ and $(T\lambda)_t$ coincide with $\phi^\varepsilon(t)$, $\nabla\phi^\varepsilon(t)$ and $\omega^\varepsilon(t)$, by the uniqueness of the equations (5.1.14), the proof of the proposition is complete. □

Proposition 5.1.3 *Let $b_j \in \Omega, b_j(0) = 0, j = 1,...,N$ and $\omega_j \in \mathbb{R}^3, j = 1,...,N$, and set*

$$\lambda^N(db) = \sum_{j=1}^N \omega_j \delta_{x_j+\sqrt{2\nu}b_j}(db). \tag{5.1.15}$$

Then

$$(T\lambda^N)_t(dx) = \sum_{j=1}^N \widetilde{\omega}_j(t)\delta_{\widetilde{x}_j(t)}(dx). \tag{5.1.16}$$

where $\widetilde{x}_j(t)$ is a solution of (5.1.7) with $\widetilde{x}_j(0) = x_j, \widetilde{\omega}_j(0) = \omega_j$ and $\widetilde{\omega}_j(t) = Y_j\cdot\omega_j$.

Proof.

$$\int (T\lambda^N)_t(dx)\cdot f(x) = \sum_{j=1}^{N} f(x_t^{\lambda^N}(p_j))\cdot X_t^{\lambda^N}(p_j)\cdot \omega_j.$$

where $p_j = x_j + \sqrt{2\nu} b_j$, and b_j is starting at the origin. Then

$$\begin{aligned}
x_t^{\lambda^N}(p_i) &= p_j(t) + \int_0^t ds \sum_{j=1}^{N} K^\varepsilon(x_s^{\lambda^N}(p_i) - x_s^{\lambda^N}(p_j))\cdot X_s^{\lambda^N}(p_j)\cdot \omega_j, \\
X_t^{\lambda^N}(p_i) &= I + \int_0^t ds \sum_{j=1}^{N} \nabla K^\varepsilon(X_s^{\lambda^N}(p_i) - X_s^{\lambda^N}(p_j))\cdot X_s^{\lambda^N}(p_j) \\
&\quad \cdot\omega_j \cdot \chi_M(X_s^{\lambda^N}(p_j)).
\end{aligned}$$

and therefore are solution of (5.1.7). Uniqueness of the solution (5.1.7) then proves the proposition. $\square$

We are now in a position to prove the continuity properties of the map T and the macroscopic limit theorem.

Let $\mu, \nu \in M(\Omega)$ and $\mu_\alpha^\pm, \nu_\alpha^\pm$ denote the decomposition of μ_α, ν_α respectively into Jordan positive and negative parts.

For each $\alpha = 1, 2, 3$, let $P_\alpha^\pm(db, dp)$ be joint representation of $\mu_\alpha^\pm$ and $\nu_\alpha^\pm$.

By the definition of joint representation we have

$$\begin{aligned}
\int\int P_\alpha^\pm(db, dp)\phi(b) &= \int \mu_\alpha^\pm(db)\phi(b), \\
\int\int P_\alpha^\pm(db, dp)\phi(p) &= \int \nu_\alpha^\pm(dp)\phi(p).
\end{aligned}$$

Let P denote the measure defined by

$$P(db, dp) = \sum_{\alpha=1}^{3}(P_\alpha^+(db, dp) + P_\alpha^-(db, dp)).$$

Further let $\mathcal{H}(\mu\ \nu)$ be the set of positive measure P on $\Omega\times\Omega$, constructed as above.

We introduce the space $\Omega_\infty = \prod_{i=1}^{\infty} \Omega_i$ where $\Omega_i = \Omega$. On each Ω_i let P_i denote the Wiener measure starting at the origin. We denote by $P_\infty = \prod_{i=1}^{\infty} P_i$ the product measure on Ω_∞. We will now prove the following lemma.

Lemma 5.1.4 *Assume that* $\{x_j^N\}_{j=1}^N$ *and* $\left\{\omega_j^N\right\}_{j=1}^N$ *are such that for each continuous and bounded function* $f : \mathbb{R}^3 \to \mathbb{R}^3$*, the following condition holds true:*

$$\lim_{N\to\infty} \sum_{j=1}^{N} \omega_j^N \cdot f(x_j^N) = \int \omega_0(x) \cdot f(x) dx.$$

Let λ *and* λ^N *be defined by equation (5.1.12)(5.1.15) and set*

$$d(b,p) = \sup_{0\le t\le T} |b(t) - p(t)|.$$

Then

$$\liminf_{N\to\infty} \inf_{P\in\mathcal{H}(\lambda,\lambda^N)} \int d(b,p)P(db,dp) = 0$$

with P_∞ *probability* 1*. Equivalently,* λ^N *converges weakly to* λ*.*

Proof. Let $\{\omega_\alpha^{\pm}\}_{\alpha=1}^3$ be the Jordan decomposition of $\{\omega_{0\alpha}\}_{\alpha=1}^3$ respectively. Further let $\{\omega_{j,\alpha}^{N,\pm}\}_{j=1,\alpha=1}^{N,3}$ satisfy

$$\sum_{j=1}^{N} \omega_{j,\alpha}^{N,\pm} = \| \omega_\alpha^{\pm}\|_1, \omega_{j,\alpha}^{N,+} - \omega_{j,\alpha}^{N,-} = \omega_{j,\alpha}^{N}.$$

Finally define the following measures on $M(\Omega)$:

$$\lambda_\alpha^{\pm}(db) = \int \omega_\alpha^{\pm}(x)\widetilde{P}_{x,\alpha}(db)$$

$$\lambda_\alpha^{N,\pm}(db) = \sum_{j=1}^{N} \omega_{j,\alpha}^{N,\pm} \delta_{x_j^N + \sqrt{2}b_j}(db)$$

where $\widetilde{P}_{x,\alpha}, (\alpha = 1, 2, 3)$ are components of $\widetilde{P}_x$(see 5.1.11).

From the law of large number we may see that $\lambda_\alpha^{N,\pm}(db)$ converges weakly to $\lambda_\alpha^{\pm}(db)$ for P_∞-almost all $\{b_j\}_{j=1}^N$. So for each ε, there are joint measures $P_\alpha^{N,\pm} \in \mathcal{H}(\lambda_\alpha^{N,\pm}, \lambda_\alpha^{\pm})$ such that:

$$\int P_\alpha^{N,\pm}(db, dp) d(b, p) < \varepsilon$$

So we get the result.

We now prove a continuity property for map T. Given $\mu, \nu \in M(\Omega)$, define

$$\zeta_t(b, p) = \|x_t^\mu(b) - x_t^\nu(p)\| + \|X_t^\mu(b) - X_t^\nu(p)\|,$$

and, for $P \in \mathcal{H}(\mu\ \nu)$, define

$$y(t) = \int \sup_{0 \le s \le t} \zeta_s(b, p)\, P(db, dp).$$

Lemma 5.1.5 *There is $\phi(M, \varepsilon)$ diverging as $\varepsilon \to 0$ such that for $0 \le t \le T$,*

$$y(t) \le \phi(M, \varepsilon) \int d(p, b) P(db, dp).$$

Proof. We first note $\nabla K^\varepsilon(x)$ is bounded from (3.1.8). By Gronwall lemma we have

$$\|X_t^\mu\| \le \exp\left\{\frac{CMt}{\varepsilon^4}|\mu|(1)\right\},\ t \ge 0,$$

where for any continuous bounded real-valued function ϕ ,

$$|\mu|(\phi) = \sum_{\alpha} (\mu_\alpha^+(\phi) + \mu_\alpha^-(\phi)).$$

We have

$$\begin{aligned} x_t^\mu(b) - x_t^\nu(p) &= b(t) - p(t) + \int_0^t ds \{ \int K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \cdot X_s^\mu(e) \cdot \mu(de) \\ &\quad - \int K^\varepsilon(x_s^\mu(b) - x_s^\nu(e)) \cdot X_s^\nu(e) \cdot \nu(de) \\ &\quad + \int [K^\varepsilon(x_s^\mu(b) - x_s^\nu(e)) - K^\varepsilon(x_s^\nu(p) - x_s^\nu(e))] \\ &\quad \cdot X_s^\nu(e) \cdot \nu(de) \}. \end{aligned} \tag{5.1.17}$$

$$\begin{aligned} X_t^\mu(b) - X_t^\nu(p) &= \int_0^t ds \int \nabla K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \\ &\quad \cdot X_s^\mu(e) \cdot \mu(de) \cdot \{\chi_M(X_s^\mu(b)) - \chi_M(X_s^\nu(p))\} \\ &\quad + \int_0^t ds \{ \int \nabla K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \cdot X_s^\mu(e) \cdot \mu(de) \\ &\quad - \int \nabla K^\varepsilon(x_s^\mu(b) - x_s^\nu(e)) \cdot X_s^\nu(e) \cdot \nu(de) \\ &\quad + \int [\nabla K^\varepsilon(x_s^\mu(b) - x_s^\nu(e)) - \nabla K^\varepsilon(x_s^\nu(p) - x_s^\nu(e))] \\ &\quad \cdot X_s^\nu(e) \cdot \nu(de) \} \cdot \chi_M(X_s^\nu(p). \end{aligned} \tag{5.1.18}$$

The difference of the first two integrals in the r.h.s (5.1.17) can be thought as a sum of integrals w.r.t.$\{P_\alpha^\pm\}_{\alpha=1}^3$, joint representations of $\mu_\alpha^\pm$ and $\nu_\alpha^\pm$.

Therefore

$$\begin{aligned}
\|x_t^\mu(b) - x_t^\nu(p)\| \quad &\leq \quad \|b(t) - p(t)\| \\
&+ \int_0^t ds \| \sum_\alpha \int (K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \cdot X_s^\mu(e) \\
&-K^\varepsilon(x_s^\mu(b) - x_s^\nu(e^{'})) \cdot X_s^\nu(e^{'}))_\alpha (P_\alpha^+(de, de^{'}) \\
&\quad -P_\alpha^-(de, de^{'}))\| \\
&\quad + \int_0^t ds \int \|K^\varepsilon(x_s^\mu(b) - x_s^\nu(e)) \\
&\qquad -K^\varepsilon(x_s^\nu(p) - x_s^\nu(e))\| \| |X_s^\nu(e)\| \|\nu\|(de) \\
&\leq \| |b(t) - p(t)\| + \int_0^t ds \int \|K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \cdot X_s^\mu(e) \\
&-K^\varepsilon(x_s^\mu(b) - x_s^\nu(e^{'})) \cdot X_s^\nu(e^{'})\| P(de, de^{'}) \\
&\quad + \exp\left\{\frac{CMt}{\varepsilon^4}|\nu|(1)\right\} c_\varepsilon \| |\nu\| \int_0^t \| |x_s^\mu(b) - x_s^\nu(p)\| |ds,
\end{aligned}$$

where A_α is αth row of matrix A. Analogously

$$\begin{aligned}
\|X_t^\mu(b) - X_t^\nu(p)\| \quad &\leq \quad \int_0^t ds \sup_\alpha \int \|\partial_\alpha K^\varepsilon(x_s^\mu(b) - x_s^\mu(e))\| \\
&\quad \|X_s^\mu(e)\| \|\chi_M(X_s^\mu(b)) - \chi_M(X_s^\nu(p))\| \|\mu\|(de) \\
&\quad +M \int_0^t ds \| \sum_\alpha \int (\nabla K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \\
&\qquad \cdot X_s^\mu(e) - \nabla K^\varepsilon(x_s^\mu(b) - x_s^\nu(e^{'})) \cdot X_s^\nu(e^{'}))_\alpha \\
&\qquad \times (P_\alpha^+(de, de^{'}) - P_\alpha^-(de, de^{'}))\| \\
&\quad +M \int_0^t ds \sup_\alpha \int \|\partial_\alpha K^\varepsilon(x_s^\mu(b) - x_s^\nu(e)) \\
&\qquad -\partial_\alpha K^\varepsilon(x_s^\nu(p) - x_s^\nu(e))\| \|X_s^\nu(e)\| \|\nu\|(de) \\
&\leq \exp\left\{\frac{CMt}{\varepsilon^4}|\nu|(1)\right\} c_\varepsilon \|\mu\| M \int_0^t (\|X_s^\mu(b)) - (X_s^\nu(p))\| \\
&\quad +\|x_s^\mu(b) - x_s^\nu(p)\|)ds \\
&\quad +M \int_0^t ds \sup_\alpha \int \|\partial_\alpha K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \cdot X_s^\mu(e)
\end{aligned}$$

$$-\partial_\alpha K^\varepsilon(x_s^\mu(b) - x_s^\nu(e^{'})) \cdot X_s^\nu(e^{'})\|P(de, de^{'}),$$

and

$$\|K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \cdot X_s^\mu(e) - K^\varepsilon(x_s^\mu(b) - x_s^\nu(e^{'})) \cdot X_s^\nu(e^{'})\|$$

$$\begin{aligned}
\leq\ & \|K^\varepsilon(x_s^\mu(b) - x_s^\mu(e))\|\|X_s^\mu(e) - X_s^\nu(e^{'})\| + \|K^\varepsilon(x_s^\mu(b) - x_s^\mu(e)) \\
& -K^\varepsilon(x_s^\mu(b) - x_s^\nu(e^{'}))\|\|X_s^\nu(e^{'})\| \\
\leq\ & c_\varepsilon(\|X_s^\mu(e) - X_s^\nu(e^{'})\| + \exp\left\{\frac{CMt}{\varepsilon^4}|\nu|(1)\right\}\|x_s^\mu(e) - x_s^\nu(e^{'})\|).
\end{aligned}$$

A similar bound holds for the integrand of the second term of (5.1.18). So we have

$$\zeta_t(b,p) \leq \|b(t) - p(t)\| + B\left\{\int_0^t \zeta_s(b,p)ds + \int_0^t ds \int \zeta_s(e,e')P(de,de')\right\},$$

for some constant B depending on $\varepsilon, M, T, c_\varepsilon, \|\nu\|, \|\mu\|$.

Hence for $\tau > t$, we have

$$\zeta_t(b,p) \leq \|b(t) - p(t)\| + B\left\{\int_0^t \zeta_s(b,p)ds + \int_0^\tau y(s)ds\right\}.$$

By Gronwall lemma:

$$\zeta_t(b,p) \leq e^{BT}\{d(b,p) + B\int_0^\tau y(s)ds\}. \tag{5.1.19}$$

Taking the sup for $t \leq \tau$ and integrating with respect to $P(db, dp)$, one obtains

$$y(\tau) \leq e^{BT}\{\int d(b,p)P(db,dp) + B\widetilde{a}\int_0^\tau y(s)ds\},$$

where $\widetilde{a}$ depend on $a_\alpha^\pm$.

Therefore another application of Gronwall lemma proves the result with $\phi(M,\varepsilon) = \exp\{BTe^{BT}\}$. □

Proposition 5.1.6 *With P_∞ probability 1, when N is large enough, the system of stochastic ordinary equations (3.5.19) describing the particles movement have a unique solution.*

Proof. Because of (5.1.3) we can choose $M = M_\varepsilon$ such that

$$\|\nabla \phi_t^\varepsilon\|_\infty \leq \frac{M_\varepsilon}{4},\ t \leq T.$$

By (5.1.19) at Lemma 5.1.5 we have

$$\begin{aligned}\|\nabla \phi_t^\varepsilon(b) - X_t^{\lambda^N}(b)\| &\leq \zeta_t(b,b) \leq C\int_0^\tau y(s)ds \\ &\leq \phi(M,\varepsilon) \inf_{P\in\mathcal{H}(\lambda,\lambda^N)} \int P(db,dp)d(p,b) \\ &\to 0 \text{ as } N \to \infty.\end{aligned}$$

Hence $\|X_t^{\lambda^N}(b)\| \leq \frac{M}{2}$ for N large enough and therefore $\chi_M(X_t^{\lambda^N})(b) = X_t^{\lambda^N}(b)$. So $(x_t^{\lambda^N}(b_j), X_t^{\lambda^N}(b_j)\omega_j)$ coincides with $(x_j(t), \omega_j(t))$ for $a.s.$ $\{b_j\}_{j=1}^N$. This concludes the proof. □

Theorem 5.1.7 *For every $\varepsilon > 0$ fixed, there holds $\omega_t^N(dx) \to \omega_t^\varepsilon(dx)$ almost surely as $N \to \infty$.*

Proof. For any $f \in C_b^2(R^3, R^3)$, from (5.1.12) and (5.1.15) we have:

$$\int f(x)\omega_t^N(dx) = \int f(x)(T\lambda^N)(t)dx = \int f(x_t^{\lambda^N}(p)) \cdot X_t^{\lambda^N}(p)\lambda^N(dp)$$

and:

$$\int f(x)\omega_t^\varepsilon(dx) = \int f(x)(T\lambda)(t)dx = \int f(x_t^\lambda(p)) \cdot X_t^\lambda(p)\lambda(dp)$$

Let $P(de, dq)$ be a joint representation of $\lambda^N(dp)$ and $\lambda(dq)$. Then

$$\left|\int f(x)\omega_t^N(dx) - \int f(x)\omega_t^\varepsilon(dx)\right|$$

$$= \left|\int f(x_t^{\lambda^N}(p)) \cdot X_t^{\lambda^N}(p)\lambda^N(dp) - \int f(x_t^\lambda(p)) \cdot X_t^\lambda(p)\lambda(dp)\right|$$

$$\leq \int \left\|f(x_t^{\lambda^N}(p)) \cdot X_t^{\lambda^N}(p) - f(x_t^\lambda(q)) \cdot X_t^\lambda(q)\right\| P(de, dq)$$

$$\leq \int \left\|f(x_t^{\lambda^N}(p))\right\| \left\|X_t^{\lambda^N}(p) - X_t^\lambda(q)\right\| P(dp, dq)$$

$$+ \int \left\|f(x_t^{\lambda^N}(p)) - f(x_t^\lambda(q))\right\| \left\|X_t^\lambda(q)\right\| P(de, dq)$$

$$\leq \|f\| \int \left\|X_t^{\lambda^N}(p) - X_t^\lambda(q)\right\| P(dp, dq)$$

$$+ \|\nabla \cdot f\| \int \left\|x_t^{\lambda^N}(p) - x_t^\lambda(q)\right\| \left\|X_t^\lambda(q)\right\| P(de, dq).$$

Actually, $X_t^\lambda(q)$ is just $\nabla\phi_t^\varepsilon$ from (5.1.3) and it is bounded. Because of lemmas (5.1.4) and (5.1.5) we may say that as $N \to \infty$, the right hand side of the formula above will go to zero and this finishes our proof.

Next we use the result above to prove that, if instead of independent Brownian motion, we use common Brownian sheet as noise, the solution of the corresponding SPDEs still exists and is unique.

5.2 Particle System with Brownian Sheet as noise term

For simplicity we just treat the diagonal matrix form for the correlation function and the form is specified.

Define the correlation function $\tilde{\Gamma}_\delta : \mathbb{R}^6 \mapsto R_+$ in the following way:

$$\tilde{\Gamma}_\delta(r,p) = \left[\frac{1}{(2\pi)^{\frac{3}{2}}\delta^{\frac{3}{2}}}\exp\left(-\frac{|r-p|^2}{2\delta}\right)\right]^{\frac{1}{2}}, \quad r,p \in \mathbb{R}^3, \tag{5.2.20}$$

If we fix r, $\tilde{\Gamma}^2_\delta(r,p)$ is just joint density function of three $\mathbb{R}$-valued independent normal random variables. We have:

$$\int \tilde{\Gamma}^2_\delta(r,p)dp = 1. \tag{5.2.21}$$

Set $\widehat{\Gamma}_\delta(r,p) = \tilde{\Gamma}_\delta(r,p) \cdot I$, where I is unit matrix. Let w_l, $l = 1,2,3$ be independent Brownian sheets, and set $w(r,t) = (w_1(r,t), w_2(r,t), w_3(r,t))^\top$.

We define the following stochastic particles system for the 3-D vortex form of NSE

$$\begin{cases} R_i(t) &= R_i(0) + \int_0^t \sum_{j=0}^N K^\varepsilon(R_i(s)-R_j(s))\omega_j(s)ds \\ & \qquad +\sqrt{2\nu}\int_0^t\int \widehat{\Gamma}_\delta(R_i(s)-p)w(dp,ds), \\ \frac{d\omega_i(t)}{dt} &= \omega_i(t)\cdot\sum_{j=0}^N \nabla K^\varepsilon(R_i(t)-R_j(t))\omega_j(t), \\ \omega_i(0) &= \omega_i \quad i=1,2,\ldots,N. \end{cases} \tag{5.2.22}$$

We also define the following empirical signed vector measure-valued process

$$\omega^N_{\delta,\varepsilon}(t)(dx) = \sum_{j=1}^N \omega_j(t)\cdot\delta_{R_j(t)}(dx). \tag{5.2.23}$$

For the 3D vortex form of NSE, we have to calculate $\omega_i(t)$ but we cannot guarantee there is non-explosion. As in the independent Brownian case at last section, we define the following modified particle system:

$$\begin{cases} \tilde{R}_i(t) &= R_i(0) + \int_0^t \sum_{j=0}^N K^\varepsilon(\tilde{R}_i(s) - \tilde{R}_j(s))Y_j(s)\omega_j ds \\ & \qquad +\sqrt{2\nu}\int_0^t \int \widehat{\Gamma}_\delta(R_i(s) - p)w(dp, ds), \\ \frac{dY_i(t)}{dt} &= \left[\sum_{j=0}^N \nabla K^\varepsilon(\tilde{R}_i(t) - \tilde{R}_j(t))Y_j(t)\omega_j\right] \cdot \chi_M(Y_i(t)), \\ Y_i(0) &= I \;\; i = 1, 2, \ldots, N, \end{cases} \tag{5.2.24}$$

where χ_M is the censoring map (5.1.8). Just like in the previous section, the stochastic particle system (5.2.24) has a unique non explosive solution for all positive times t.

Moreover, if we set $M = \infty$ and $\omega_j(t) = Y_j(t)\omega_j$, then $\{\tilde{R}_j(t), \omega_j(t)\}$ is the solution of (5.2.22).

It is easy to check the boundedness of $Y_i(t)$ over compact time sets.

Proposition 5.2.1 *Given fixed $T > 0$ and M, $Y_i(t)$ are bounded for $t \leq T$,$i = 1, 2, \ldots N$.*

Proof. Since

$$Y_i(t) = I + \int_0^t [\sum_{j=0}^N \nabla K^\varepsilon(\tilde{R}_i(s) - \tilde{R}_j(s))Y_j(s)\omega_j] \cdot \chi_M(Y_i(s))ds.$$

As a result of boundness of ∇K^ε:

$$\|Y_i(t)\| \leq Mc\varepsilon^4 \int_0^\top \sum_{j=1}^N \|Y_j(s)\|ds + 1,$$

so by the Gronwall's inequality, one finally obtains

$$\|Y_i(t)\| \leq C' \exp(M't) \;\; t \geq 0.$$

C' and M' depend on T, M, ε.

Theorem 5.2.2 *With probability* 1*, when* N *is large enough and* δ *is fixed, the system of stochastic ordinary equations (5.2.22) describing the particles movement have a unique solution.*

Proof. From the proof of Proposition 5.1.6 we may chose M_0, such that when N is big enough, $Y_i(t), i = 1, 2, ...N$, the solution of (5.1.7) is smaller than M_0. So the solution of (3.5.19) in which the stochastic terms are independent Brownian motions is exist and unique.

Now we fix M_0, we use the classic iteration method to get solution of (5.2.24). Let $R_i^n(t), i = 1, 2, ...N$ be the nth iteration value in the iteration process, and set

$$\beta_i^n(t) := \int\int_0^t \hat{\Gamma}_\delta(R_i^n(s) - p)w(dp, ds) \quad i = 1, 2, ...N.$$

From the property of stochastic integral with respect to Brownian sheet, $\beta_i^n(t)$ $i = 1, 2, ...N$ are Brownian motions but not necessary independent.

For clarification we recall that $\Omega_\infty = \prod_{i=1}^{\infty} \Omega_i$ where $\Omega_i = \Omega$. On each Ω_i we define P_i, the Wiener measure starting at the origin. We denote $P_\infty = \prod_{i=1}^{\infty} p_i$ the product measure on Ω_∞, and $\{b_i(t)\}_{i=1}^{\infty}$ are independent Brownian motions on Ω_∞ under P_∞.

We have a fact that the support of $\{\beta_i^n(t)\}_{i=1}^N$ is same as support of $\{b_i(t)\}_{i=1}^N$. Because of that and from the proof of Proposition 5.1.6 we have: when N is big enough, $R_i^{n+1}(t)$ and $Y_i^{n+1}(t), i = 1, 2, ...N$ that satisfy the following $SDEs$:

$$\begin{cases} R_i^{n+1}(t) = R_i(0) + \int_0^t \sum_{j=0}^N K^\varepsilon(R_i^{n+1}(s) - R_j^{n+1}(s))Y_j^{n+1}(s)\omega_j ds + \sqrt{2\nu}\beta_i^n(t) \\ \frac{dY_i^{n+1}(t)}{dt} = [\sum_{j=0}^N \nabla K^\varepsilon(R_i^{n+1}(t) - R_j^{n+1}(t))Y_j^{n+1}(t)\omega_j] \bullet \chi_{M_0}(Y_i^{n+1}(t)) \\ Y_i(0) = I \quad i = 1, 2, \ldots, N. \end{cases}$$

are with P_∞ probability 1 smaller than M_0. So we may say that $R_i^{n+1}(t)$ and $Y_i^{n+1}(t)$ are the solution of the following $SPDEs$:

$$\begin{cases} R_i^{n+1}(t) = R_i(0) + \int_0^t \sum_{j=0}^N K^\varepsilon(R_i^{n+1}(s) - R_j^{n+1}(s))Y_j^{n+1}(s)\omega_j ds \\ +\sqrt{2\nu} \int \int_0^t \hat{\Gamma}_\delta(R_i^n(s) - p)w(dp, ds) \\ \frac{dY_i^{n+1}(t)}{dt} = [\sum_{j=0}^N \nabla K^\varepsilon(R_i^{n+1}(t) - R_j^{n+1}(t))Y_j^{n+1}(t)\omega_j] \bullet Y_i^{n+1}(t) \\ Y_i(0) = I \quad i = 1, 2, \ldots, N. \end{cases}$$

And remember the fact that M_0 does not depend on N and $Y_i^{n+1}(t)$ is bounded uniformly when $n > N$.

Because of the boundedness of $\nabla K^\varepsilon(x), K^\varepsilon(x)$ and $\hat{\Gamma}_\delta(r)$ next we may use the classic iteration method to prove $R_i^{n+1}(t), Y_i^{n+1}(t)$ are convergence respectively as $n \to \infty$. The limits are the solution of (5.2.22).

5.2.1 Macroscopic limit

We fix ε and N. Let $\{R_{\varepsilon,\delta,N}(t)\}$ be the $\mathbb{R}^{3N}$-valued solution of (5.2.22), which start in the same initial position $\{R_{\varepsilon,\delta,N}(0)\}$, for any $\delta > 0$.

Lemma 5.2.3 *For every fixed choice of ε and N, $\{R_{\varepsilon,\delta,N}(t)\}_{\delta>0}$ is relatively compact on $C([0,T^*], \mathbb{R}^{3N})$.*

Proof. Since K^ε and $Y_i(t)$ are bounded and

$$E\|\int_0^t\int\widehat{\Gamma}_\delta(R^i_{\varepsilon,\delta,N}(s)-p)w(dp,ds)\|^2=3t, \tag{5.2.25}$$

we have

$$P\{\|R^i_{\varepsilon,\delta,N}(t)\|>M'\}\le\frac{E\|R^i_{\varepsilon,\delta,N}(t)\|^2}{M'^2}\le\frac{c(\varepsilon,N,t)}{M'^2}.$$

For the modulus of continuity

$$E[\|R^i_{\varepsilon,\delta,N}(t)\|^2|F_s]\le 3N^2(c\varepsilon^{-4})^2(t-s)^2+3(t-s)$$

so it follows from Ethier and Kurtz [32, Theorem 3.8.6], that $\{R_{\varepsilon,\delta,N}\}_{\delta>0}$ is relatively compact on $C([0,T^*],\mathbb{R}^{3N})$. □

Lemma 5.2.4 *Define the continuous square integrable martingales*

$$M^i_{\varepsilon,\delta,N}=\int_0^t\int\widehat{\Gamma}_\delta(R^i_{\varepsilon,\delta,N}(s)-p)w(dp,ds).$$

Then $M_{\varepsilon,\delta,N}=(M^1_{\varepsilon,\delta,N},M^2_{\varepsilon,\delta,N},\ldots,M^i_{\varepsilon,\delta,N})$ converges in law to β_N on $C([0,T^*],\mathbb{R}^{3N})$, as $\delta\to0$, where β_N is a standard independent $\mathbb{R}^{3N}$-valued Brownian motion.

Proof. By Ethier and Kurtz [32, Theorem 7.14], we only need to show that the mutual quadratic variation of $M_{\varepsilon,\delta,N}$: $< M^{i,l}_{\varepsilon,\delta,N}, M^{j,k}_{\varepsilon,\delta,N} >$ tends to $t\delta_{i,j}\delta_{l,k}$ in probability, for any $t\ge0$, $i,j=1,2,\ldots,N$ and $l,k=1,2,3$.

When $i=j,k=l$, the result follows from (5.2.25).

From Lemma 5.2.3, for any $\eta>0$, there is compact set $K_\eta\in C([0,T^*],\mathbb{R}^{3N})$, such that $P\{R_{\varepsilon,\delta,N}\in K_\eta\}\ge1-\eta$. For $i,j\in\{1,2,\ldots,N\}$ and $l,k=1,2,3$, write the mapping

$$G^{ijlk}=< M^{i,l}_{\varepsilon,\delta,N}, M^{j,k}_{\varepsilon,\delta,N} >: [0,1]\times C([0,T^*],\mathbb{R}^{3N})\mapsto C([0,T^*],R)$$

in the following way:

$$G^{ijlk}(\delta, q_N)(t) = \begin{cases} \int_0^t g_\delta(q^i(s) - q^j(s))ds & \delta > 0 \\ t\delta_{ij}\delta_{lk} & \delta = 0 \end{cases}$$

where $q_N(t) \in \mathbb{R}^{3N}$, $t \le T^*$.

The G^{ijlk}'s are continuous from $[0,1] \times C([0,T^*], \mathbb{R}^{3N})$ into $C([0,T^*], R)$. Since $K_\eta \in C([0,T^*], \mathbb{R}^{3N})$ is compact, the restriction of G^{ijlk} on $[0,1] \times K_\eta$ is uniformly continuous. In particular, for any $\rho > 0$, and $i \ne j, l \ne k$ there is an $\varepsilon_{ijlk} > 0$, such that for all $\delta \le \varepsilon_{ijlk}$,

$$\sup_{q_N(\cdot) \in K_\eta} \sup_{0 \le t \le T^*} C^{ijlk}(\delta, q_N(\cdot))(t) \le \rho\,.$$

Hence for $\delta \le \varepsilon_{ijlk}$ and $i \ne j, l \ne k$, $P\{\omega :< M^{i,l}_{\varepsilon,\delta,N}, M^{j,k}_{\varepsilon,\delta,N} > (t) > \rho\} \le \eta$. This completes the proof. □

Next we consider the particle system described in the last section. As we proved in Proposition 5.1.5, when the number N of particles is large enough, the solution of following $SDEs$ is exist and unique almost surely:

$$\begin{cases} r_i(t) &= r_i(0) + \int_0^t \sum_{j=0}^N K^\varepsilon(r_i(s) - r_j(s))X_j(s)\omega_j ds \\ & \qquad + \sqrt{2\nu}\beta_i(t) \\ \frac{dX_i(t)}{dt} &= \left[\sum_{j=0}^N \nabla K^\varepsilon(r_i(t) - r_j(t))X_j(t)\omega_j\right] \cdot X_i(t) \\ X_i(0) &= I.\ \ i = 1, 2, \ldots, N, \end{cases} \tag{5.2.26}$$

For $q_N(t) \in C([0,T^*], \mathbb{R}^{3N})$, define

$$F(q_N)(t) = (F^1(q_N)(t), F^2(q_N)(t), \ldots, F^N(q_N)(t))$$

where

$$\begin{cases} F^i(q_N)(t) &= \sum_{j=0}^N K^\varepsilon(q^i(t) - q^j(t))X_j(t)\omega_j \\ \frac{dX_i(t)}{dt} &= \left[\sum_{j=0}^N \nabla K^\varepsilon(q^i(t) - q^j(t))X_j(t)\omega_j\right] \cdot (X_i(t)), \\ X_i(0) &= I, \quad i = 1, 2, \ldots, N. \end{cases}$$

Define a continuous map $\Psi : C([0, T^*], \mathbb{R}^{3N}) \mapsto C([0, T^*], \mathbb{R}^{3N})$ by

$$\Psi(q_N)(t) = \int_o^t F(\Psi(q_N)(s)ds + q_N(t).$$

For $q_N(t, \omega) = M_{\delta,N}(t, \omega) + R_N(0)$, Ψ is solution of (5.2.22). For $q_N(t, \omega) = \beta_N(t, \omega) + r_N(0)$, Ψ is solution of (5.2.26).

So by continuous mapping theory we just proved the following result.

Theorem 5.2.5 *Let ε be fixed and N big enough, If $R_N(0) = r_N(0)$, then $\tilde{R}_{\delta,N}$ converges in law to $\tilde{r}_N = \{\tilde{r}_i(t)\}_{i=0}^N$ on $C([0, T^*], \mathbb{R}^{3N})$, as $\delta \to 0$.*

Conclusion

In this thesis we have studied collections of solutions to stochastic Navier-Stokes equations under the following dichotomic lenses : the simple 2D vorticity picture against the complex 3D one, classical solutions and their smoothed counterparts, approximating particle systems and their macroscopic scaling limits.

We reviewed the most important steps taken in the last thirty years in the development of rigourous results, notably the introduction by Marchioro and Pulvirenti [64] in 1982 of a particle system driven by independent Brownian motions which yields the full 2D Navier-Stokes dynamics. We saw how Beale and Majda ([4], [5], [6], [7]) generalized (in a highly non trivial way) the particle system in question and were able to analyzed the smoothed Navier-Stokes vorticity equations in three dimensions. We also had to come to terms with the fact that the major papers in the eighties on the 3D dynamics, notably those of Esposito and Pulvirenti [31] and the series of papers by Kotelenez ([53], [54], [55]), while replete with interesting new ideas, unfortunately contain major flaws and several incorrect statements.

Our main contributions are : the clarification of what is true and what is dubious in these important papers; the proof of the absence of collisions amongst the particles of the finite systems that yield smoothed 3D deterministic fluid flows in the macroscopic limit; the proof of said macroscopic limit.

What is left to be done is of course to give a rigourous treatment of the so-called theorems of Esposito and Pulvirenti [31] stated in section 3.5, as well as their counterparts when the driving noises are changed from independent Brownian motions to ones moving in a random medium. For the present, we had to be content with a full analysis of the smoothed versions only.

Appendix

This appendix contains the proofs of the more technical results of this thesis. By organizing the material of this thesis in such a fashion, our hope is that the reader will get a better overview of the subject at hand without getting bogged down in small details that could hamper his understanding of the connexions between the various results and contributions.

PROOF OF THEOREM (3.1.3).

Proof. First we calculate ω_1.

Since $\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$, we apply operator $\frac{\partial}{\partial x_2}$ to both sides of the third equation in (3.1.1) and $\frac{\partial}{\partial x_3}$ to both sides of the second equation. Then we substract one from the other. For example, we give the calculation for each term next:

$$-\frac{\partial}{\partial x_3}(\frac{\partial u_2}{\partial t}) + \frac{\partial}{\partial x_2}(\frac{\partial u_3}{\partial t}) = \frac{\partial}{\partial t}(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) = \frac{\partial}{\partial t}\omega_1$$

$$\frac{\partial}{\partial x_3}[(u \cdot \nabla)u_2] = (u \cdot \nabla)\frac{\partial u_2}{\partial x_3} + \frac{\partial}{\partial x_3}[(u \cdot \nabla)]u_2 = (u \cdot \nabla)\frac{\partial u_2}{\partial x_3} + \frac{\partial u}{\partial x_3} \cdot \nabla u_2$$

$$\frac{\partial}{\partial x_2}[(u \cdot \nabla)u_3] = (u \cdot \nabla)\frac{\partial u_3}{\partial x_2} + \frac{\partial}{\partial x_2}[(u \cdot \nabla)]u_3 = (u \cdot \nabla)\frac{\partial u_3}{\partial x_2} + \frac{\partial u}{\partial x_2} \cdot \nabla u_3$$

And:

$\frac{\partial u}{\partial x_3} \cdot \nabla u_2 = \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3}$

$\frac{\partial u}{\partial x_2} \cdot \nabla u_3 = \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}$

So:

$\frac{\partial u}{\partial x_2} \cdot \nabla u_3 - \frac{\partial u}{\partial x_3} \cdot \nabla u_2$

$= \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3} - \frac{\partial u_3}{\partial x_3}\frac{\partial u_2}{\partial x_3}$

$= \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) + \frac{\partial u_3}{\partial x_3}(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3})$

$= \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} + \omega_1(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3})$

Since $div\ u = 0$, we have:

$\frac{\partial u}{\partial x_2} \cdot \nabla u_3 - \frac{\partial u}{\partial x_3} \cdot \nabla u_2$

$= \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} - \omega_1\frac{\partial u_1}{\partial x_1}$

$= \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} - \omega_1\frac{\partial u_1}{\partial x_1}$

$= \frac{\partial u_1}{\partial x_2}(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}) + \frac{\partial u_1}{\partial x_3}(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}) - \omega_1\frac{\partial u_1}{\partial x_1}$

$= -\omega_2\frac{\partial u_1}{\partial x_2} - \omega_3\frac{\partial u_1}{\partial x_3} - \omega_1\frac{\partial u_1}{\partial x_1}$

$= -(\omega \cdot \nabla)u_1$

or:

$\frac{\partial u}{\partial x_2} \cdot \nabla u_3 - \frac{\partial u}{\partial x_3} \cdot \nabla u_2$

$= \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} - \omega_1\frac{\partial u_1}{\partial x_1}$

$= \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial u_3}{\partial x_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_1} - \omega_1\frac{\partial u_1}{\partial x_1}$

$= \frac{\partial u_3}{\partial x_1}(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}) + \frac{\partial u_2}{\partial x_1}(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}) - \omega_1\frac{\partial u_1}{\partial x_1}$

$= -\omega_3\frac{\partial u_3}{\partial x_1} - \omega_2\frac{\partial u_2}{\partial x_1} - \omega_1\frac{\partial u_1}{\partial x_1}$

$= -(\omega\frac{\partial u}{\partial x_1})$

And:

$(u \cdot \nabla)\frac{\partial u_3}{\partial x_2} - (u \cdot \nabla)\frac{\partial u_2}{\partial x_3} = (u \cdot \nabla)(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) = (u \cdot \nabla)\omega_1$

$\frac{\partial}{\partial x_2}(\triangle u_3) - \frac{\partial}{\partial x_3}(\triangle u_2) = \triangle(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) = \triangle\omega_1$

where we use $curl\ (\nabla p) = 0$. Finally we get the equation for ω_1

$$\frac{\partial \omega_1}{\partial t} + (u \cdot \nabla)\omega_1 - \nu \triangle \omega_1 = (\omega \cdot \nabla)u_1 \ \ or \ \ \omega \cdot \frac{\partial u}{\partial x_1} \tag{5.2.27}$$

Following the process above while making the appropriate substitutions yields the equations that ω_2 and ω_3 satisfy

$$\frac{\partial \omega_2}{\partial t} + (u \cdot \nabla)\omega_2 - \nu \triangle \omega_2 = (\omega \cdot \nabla)u_2 \ \ or \ \ \omega \cdot \frac{\partial u}{\partial x_2}$$
$$\frac{\partial \omega_3}{\partial t} + (u \cdot \nabla)\omega_3 - \nu \triangle \omega_3 = (\omega \cdot \nabla)u_3 \ \ or \ \ \omega \cdot \frac{\partial u}{\partial x_3}$$

and together all three equations can be written in the vector form given in the statement of the theorem. □

PROOF OF THEOREM (3.1.4).

Proof. We give the proof in the case of a smooth function but the definition of the weak derivative allows the argument to go through without change. The only additional verification in that case is the validity of the manipulations in the proof of the previous theorem when all derivatives are interpreted in the weak sense. Let $\phi := (\phi_1, \phi_2, \phi_3) \in C_b^2(R^3, R^3)$ have finite support. Multiply by ϕ_1 and integrate over R^3 on both side of (5.2.27) to get

$$\frac{\partial}{\partial t}\langle \omega_1, \phi_1\rangle + \langle (u\cdot\nabla)\omega_1, \phi_1\rangle - \nu < \triangle\omega_1, \phi_1 >= \langle (\omega\cdot\nabla)u_1, \phi_1\rangle$$

and

$$\langle (u\cdot\nabla)\omega_1, \phi_1\rangle = \left\langle u_1 \frac{\partial}{\partial x_1}\omega_1, \phi_1\right\rangle + \left\langle u_2 \frac{\partial}{\partial x_2}\omega_1, \phi_1\right\rangle + \left\langle u_3 \frac{\partial}{\partial x_1}\omega_1, \phi_1\right\rangle$$

$$= -(\left\langle \omega_1, \frac{\partial(u_1\phi_1)}{\partial x_1}\right\rangle + \left\langle \omega_1, \frac{\partial(u_2\phi_1)}{\partial x_2}\right\rangle + \left\langle \omega_1, \frac{\partial(u_3\phi_1)}{\partial x_3}\right\rangle)$$

$$= -(\left\langle \omega_1, u_1\frac{\partial\phi_1}{\partial x_1} + \phi_1\frac{\partial u_1}{\partial x_1}\right\rangle + \left\langle \omega_1, u_2\frac{\partial\phi_1}{\partial x_2} + \phi_1\frac{\partial u_2}{\partial x_2}\right\rangle + \left\langle \omega_1, u_3\frac{\partial\phi_1}{\partial x_3} + \phi_1\frac{\partial u_3}{\partial x_3}\right\rangle)$$

$$= -\langle \omega_1, u\cdot\nabla\phi_1 + \phi_1(divu)\rangle$$

$$= -\langle \omega_1, u\cdot\nabla\phi_1\rangle$$

$$\langle \triangle\omega_1, \phi_1\rangle = \langle \omega_1, \triangle\phi_1\rangle$$

$$\langle (\omega\cdot\nabla)u_1, \phi_1\rangle = \left\langle \omega_1\frac{\partial u_1}{\partial x_1}, \phi_1\right\rangle + \left\langle \omega_2\frac{\partial u_1}{\partial x_2}, \phi_1\right\rangle + \left\langle \omega_3\frac{\partial u_1}{\partial x_3}, \phi_1\right\rangle$$

$$= \left\langle u_1, \frac{\partial(\omega_1\phi_1)}{\partial x_1}\right\rangle + \left\langle u_1, \frac{\partial(\omega_2\phi_1)}{\partial x_2}\right\rangle + \left\langle u_1, \frac{\partial(\omega_3\phi_1)}{\partial x_3}\right\rangle$$

$$= \left\langle u_1, \frac{\partial\omega_1}{\partial x_1}\phi_1 + \omega_1\frac{\partial\phi_1}{\partial x_1}\right\rangle + \left\langle u_1, \frac{\partial\omega_2}{\partial x_2}\phi_1 + \omega_2\frac{\partial\phi_1}{\partial x_x}\right\rangle + \left\langle u_1, \frac{\partial\omega_3}{\partial x_3}\phi_1 + \omega_3\frac{\partial\phi_1}{\partial x_3}\right\rangle$$

$$= \langle u_1, \phi_1(div\omega) + \omega\cdot\nabla\phi_1\rangle$$

$$= \langle u_1, \omega\cdot\nabla\phi_1\rangle$$

because of $div\ \omega = 0$.

So we have:

$$\frac{\partial}{\partial t}\langle \omega_1, \phi_1 \rangle - \langle \omega_1, u \cdot \nabla \phi_1 \rangle - \nu \langle \omega_1, \triangle \phi_1 \rangle = \langle u_1, \omega \cdot \nabla \phi_1 \rangle \tag{5.2.28}$$

and we also have similar equation for ω_2 and ω_3

$$\frac{\partial}{\partial t}\langle \omega_2, \phi_2 \rangle - \langle \omega_2, u \cdot \nabla \phi_2 \rangle - \nu \langle \omega_2, \triangle \phi_2 \rangle = \langle u_2, \omega \cdot \nabla \phi_2 \rangle \tag{5.2.29}$$

$$\frac{\partial}{\partial t}\langle \omega_3, \phi_3 \rangle - \langle \omega_3, u \cdot \nabla \phi_3 \rangle - \nu \langle \omega_3, \triangle \phi_3 \rangle = \langle u_3, \omega \cdot \nabla \phi_3 \rangle \tag{5.2.30}$$

Add (5.2.28), (5.2.29) and (5.2.30) to get the result. □

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