Optimal Hedging of Defaults in CDOs

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Thanks

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Résumé

Un CDO (Collateralized Debt Obligation) est la titrisation d'actifs financiers de nature diverse dans le but de créer un nouvel instrument structuré de dettes. Les CDOs ont été maintes fois blâmées pour les lourdes pertes dont ont souffert plusieurs institutions financières. Un grand nombre d'échange dans le marché de CDO a conduit à des pertes massives et ceci est dû principalement à la qualité inférieure des actifs sous-jacents. Cependant, même dans le cas de CDOs à collatéraux de qualité, plusieurs échanges ont mal tourné car la plupart des traders de CDO n'ont pas réussi à comprendre les risques associés à leurs positions d'échanges. C'est dans ce contexte que cette recherche tente d'éclaircir le comportement des tranches de CDO et les moyens optimaux de couvrir le risque des défauts.

Tout d'abord, nous allons créer un cadre pour simuler les défauts. Ce cadre est basé sur les données disponibles du marché tel que le taux d'aléa, les corrélations des actifs ainsi que les volatilités. Les résultats seront utilisés dans de nombreux problèmes d'optimisation afin de trouver la stratégie optimale de couverture qui minimise une certaine mesure de risque. Les résultats de ces stratégies optimales seront analysés et comparés entre eux afin que le lecteur ait une meilleure compréhension de la dynamique des tranches de CDO ainsi que le coût et les bénéfices des stratégies de couverture.

Nous allons ensuite exécuter la même procédure d'optimisation pour différents groupes de facteurs qui affectent les défauts de paiement ou la richesse d'un trader de CDO afin d'examiner la sensibilité des positions dans les CDO et les stratégies reliées à ces facteurs clés. Ces tests de sensibilité offriront un meilleur aperçu sur les risques présents dans un échange typique de CDO ainsi que les implications de changements de facteurs de marché sur ces échanges.

Mots-clefs: *Collateralized Debt Obligation* (CDO), CDO Synthétique, Couverture, *Hedged Monte-Carlo*, Corrélation des Défauts, Copule, *Structural Variance Gamma*, Couverture Optimale, Risque, Gestion de Risque

Abstract

A Collateralized Debt Obligation (CDO) is the securitization of a pool of assets in order to create a new structured debt instrument. These complex products are at the heart of the ongoing financial crisis and have been widely blamed for the heavy losses that many of the biggest financial institutions have incurred. This has led to a dramatic decrease in the issuance of CDOs and many financial institutions are avoiding new positions in this market. A great number of CDO trades have resulted in heavy losses due to the inferior quality of their underlying assets, most notably subprime mortgages. Yet, even in the case of CDOs with quality collateral, many trades have gone wrong because most of the CDO traders have failed to fully appreciate the quite unique risks of their positions especially when confronted with correlated defaults. In this context, this work tries to shed some light on the behavior of CDO tranches and the optimal ways to hedge the defaults.

First, we shall create a framework for simulating defaults based on the available market data such as hazard rates, asset correlations, and volatilities. These simulated results will be used in a number of optimization problems in order to find the optimal hedging strategy for minimizing a given measure of risk. The results of these optimal strategies will be analyzed and compared with each other to provide the reader with a better understanding of the dynamics of CDO tranches and the costs and benefits of related hedging strategies.

Second, we will run the same optimization processes for different sets of factors that affect the defaults or the wealth of a CDO trader in order to examine the sensitivity of CDO positions and their related hedging strategies to these key factors. These sensitivity tests offer more insight about the risks present in a typical CDO trade and the implications of changes in market environment on these trades.

Keywords: Collateralized Debt Obligation (CDO), Synthetic CDO, Hedging, Hedged Monte-Carlo, Default Correlation, Copula, Structural Variance Gamma, Optimal Hedging, Risk, Risk Management

1. Introduction

In the past few years the credit market has witnessed the introduction of a vast array of new credit derivatives and an increase in the use of both new and more traditional derivatives by market participators. The size and sophistication of this market has grown enormously with notional growing from $1 to $20 trillion dollars from 2000 to 2006 and derivatives varying from single name (CDS, CLN) to full blown portfolio based ones (CDO, CBO, FtD, Synthetic loss tranches).

These derivatives were initially used for risk management purposes by bank loan managers but over the years they have also become extremely popular within the ranks of insurance companies, hedge funds, asset managers, etc. The main function of these derivatives is to isolate the credit risk from the financial instruments and thus allowing the managers to hedge the credit risk. However, the extensive use of these derivatives by speculators and arbitragers has outpaced industry's understanding of these instruments and the associated risks. Unfortunately, the lack of a comprehensive and deep understanding of certain credit derivatives, notably the CDOs, is not a phenomenon limited to individual traders and small financial institutions. Recent events such as the Abacus CDO case and the ongoing financial crisis indicate that even among sizeable and sophisticated investment institutions and banks there are many who enter the credit derivatives' market without a clear view of the associated risks. The scale and gravity of such risky practices further highlight the need to model the risks properly and understand the behavior of these instruments in the face of market volatility and correlated default arrivals.

One of the most popular and widely used instruments in this market is the Collateralized Debt Obligation (CDO). We are particularly interested in CDOs and in ways hedging the default risk in these instruments because of their mportant role in the ongoing financial crisis and the controversy surrounding them. The CDO market witnessed an exponential growth in the past decade right until the arrival of the financial crisis where the market witnessed an abrupt collapse in issuance of CDOs (see figure 1.1). Many experts and agents in the market have blamed the extensive and reckless use of these instruments as one of the most important contributing factors to the downturn of the credit markets and the ensuing financial crisis. In this context, it would be most appropriate to take a new look at the problem of correctly hedging the default risk in CDOs and to try to improve our understanding of these complicated instruments.

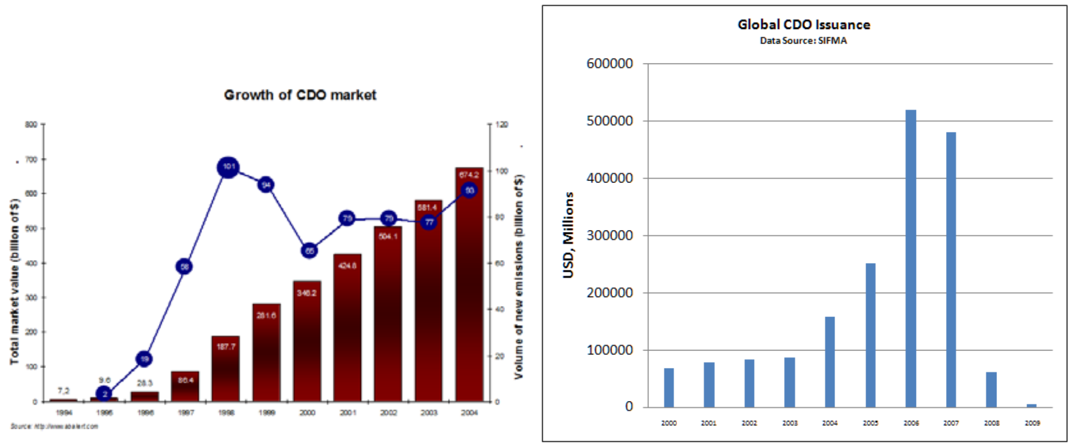
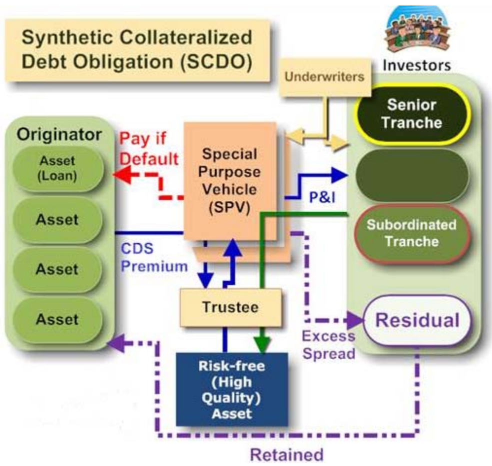


Figure 1.1 CDO market

A collateralized debt obligation (CDO) is a structured financial product which pools together assets from different sources as collateral or source of cash flows in order to issues new debt obligations with different risks and returns. This pooling of assets and creating liabilities based on them is done either for balance sheet purposes or arbitrage purposes. In balance sheet CDOs one is trying to remove certain assets from the balance sheet without losing the benefits of those assets. In arbitrage CDOs the creators believe that repackaging the assets into a CDO and tranching them will add value to the assets. Another major characteristic of CDOs is whether they are “Funded” or “Synthetic”. Synthetic CDOs, as opposed to funded CDOs, are not backed by any physical assets, but rather by synthetically created ones like CDSs or CDS indices.

Most of the CDOs follow a *waterfall* structure which means that the more senior tranches get paid first and then the subordinate tranches are paid. This means that the lower the tranche, the more the chance of default. This waterfall mechanism is illustrated in figure 1.2a where the more senior tranches are the safest and the lowest ones are the riskier ones. Figure 1.2b shows the structure of a synthetic CDO and how a Special Purpose Vehicle (SVP) is created to manage the CDO and its payments.



(a) **(b)**

Figure 1.2. Waterfall and Synthetic CDOs (sources: www.vimeo.com and www.bionicturtle.com)

Repackaging the assets and implementing the waterfall provisions results in a novel and complicated product which is difficult to price and the associated risks are hard to asses. This situation becomes even more complicated when the assets that form the collateral are from different sources and have different risks. The problem we attempt to tackle is to how to optimally hedge the default risk in the synthetic CDO contracts so that the agent (a CDO trader) who sells or purchases insurance for a CDO tranche could be able to make a fair bet on the price and minimize a given error measure in his hedging strategy which occurs due to jumps in credit spreads or defaults. In other words, our problem is one of replicating/hedging in the face of defaults or jumps in the spread and assessing the costs and irreducible errors of such strategy.

One challenge that makes hedging of the CDOs more difficult and interesting is the jump to default which follows a discrete distribution (i.e. Poisson distribution) as opposed to continuous distributions which are much easier to replicate. These sudden jumps may occur due to a change in the basic market variables which, especially in the case of more exotic pricing models, make the whole hedging process non-trivial and all these problems are compounded when the instrument references multiple issuers.

While this hedging problem has been addressed to some extent in the past by some researchers, their approach has been one of quite simplistic constraints and has been generally silent on the challenges presented by defaults, jumps, or diffusion. Most of the researchers to this date have used a *risk-neutral* approach consisting of normal copulas. One attraction of the aforementioned method is the fact that results are not sensitive to the choice of risk measure. In contrast to that practice, we do not assume that perfect replication is feasible and instead set our goal on explicitly illustrating a specific static hedging scheme and the irreducible residuals risks associated with it.

Our approach to this problem will be one of creating a zero P&L strategy while at the same time minimizing the hedging error measure. It should be noted that the choice of the hedging error measure has profound effects on our final results. While it would be interesting to try to find out the best and most appropriate choice of risk measure, this choice is still subject to the preference of the trader and what he deems to be the correct risk measure for his specific needs and therefore it is beyond the scope of this work.

The two risk measures used in this wok are the standard deviation and the expected shortfall (ES) of final wealth. Standard deviation is chosen because it is the most widely used measure of risk by practitioners and ES is used because it is a coherent measure of risk and, as we will show, the behavior of a CDO tranche resembles a digital (or a barrier) option and thus, using an incoherent measure of risk (like VaR) would give us misleading results.

The two models used in this work to generate times to default are Reduced Form Poisson-Normal-Copula (PNC) and Structural Variance Gamma (SVG). In the PNC model we use a flat hazard rate based on Moody's estimates of corporate bonds' default rates. The parameters used in the SVG model are chosen in a way that the average number of defaults and the variance of pool loss are identical to those in the PNC model. We create a CDS index which resembles the North American investment grade CDS index by markit™ (CDX.NA.IG) referencing 125 corporate bonds. In our work these bonds are homogenous but using this methodology we can generate reliable results for any other pool of bonds as long as the right asset correlation and hazard rate are used.

The hedging technique used in this work is based on the methods developed by many scholars who have studied the problem of hedging in discrete time and hedging discrete distributions, most notably Bouchaud and Potters and Bouchaud and Pochart. However, after implementation and extensive testing of the aforementioned method, we have concluded that this method suffers from systematic errors and a simpler one-period method gives us better results. This method involves a simultaneous optimization process in which the average wealth is kept at zero while the expected shortfall or the variance of the portfolio is minimized.

Using these tools, we create a portfolio consisting of a synthetic CDO tranche and the underlying bonds of the reference CDS index (i.e. CDX.NA.IG) and try to optimize our hedging strategy in a way that the average final wealth of the protection seller would be equal to zero while his measure of risk in minimized. The results that we have obtained in this work provide us with an insight to the costs and errors of static hedging of a synthetic CDO contract as well as the resulting carry of such hedging strategy.

Our results show that since the equity tranche is sensitive to the very first defaults and absorbs the full impact of such losses, hedging this tranche involves a large hedge notional. In contrast, the mezzanine and senior tranches require a much smaller hedge notional. Also, we show that since these CDO tranche trades follow a barrier type distribution, using variance as the measure of risk would generate results which diverge significantly from the results obtained using ES. This is especially visible in the case of equity tranche where there is a big no-default carry and variance penalizes these positive gains as well as the losses which occur due to defaults. We will see that the behavior of the more senior tranches resemble that of traditional bonds and thus, the two different hedge notional calculated under variance and ES start to converge in the more senior tranches.

2. Literature Review

2.1 Hedging

Hedging techniques have evolved greatly in the past few decades and now they include many different methods each developed for a specific purpose. These techniques serve as tools for market participants who want to protect themselves against a plethora of factors such as direction of the market, volatility, or default. In this section we review some of these methods and their usefulness.

2.1.1 Black and Scholes

Fischer Black and Myron Scholes published their now famous model in a 1973 paper called "*The Pricing of Options and Corporate Liabilities*" in which they develop the first closed-form solution for pricing European style options. In complete, frictionless capital markets with no transaction costs and where the underlying securities follow geometric Brownian motions, the Black-Scholes formula provides an elegant and tractable solution for pricing derivative securities. Apart from a pricing formula, the Black-Scholes model also provides for the straightforward calculation of the derivatives of the option price with regard to different factors like the underlying price and time to maturity. These derivatives (or Greeks) are the first and most important information that a market agent needs in order to hedge his position. In other words, one needs to know how the price of an option fluctuates with the underlying factors so that he can hedge the risk of a change in those factors.

Unfortunately, actual financial markets are far more complex and empirical testing of the Black-Scholes model has highlighted its' many shortcomings. It is well documented (Fama, 1965, Mandelbrot, 1963, Schwert, 1989) that the observed properties of financial time series are not consistent with the underlying assumptions of the Black-Scholes framework. Time-varying volatility, the presence of higher-order moments and serial correlation are now well established characteristics of asset returns. Moreover, liquidity constraints, market frictions, transaction costs and discrete-time hedging lead to sub-optimal replication of the option's payoff function (Dufie and Huang, 1985, Huang, 1985). Furthermore, Boyle and Emanuel (1980), Gilster (1990), Mello and Neuhaus (1998) and Buraschi and Jackwerth (2001) demonstrate that unrealistic assumptions about continuous-time hedging can lead to large hedging errors and residual hedging risk.

2.1.2 Monte-Carlo methods

To this date Monte-Carlo simulations have been extensively used for pricing options and derivatives where the underlying asset follows non-Gaussian distributions or in more complex situations including where multiple factors dictate the price of the option, where the price is path-dependant, or where there are jumps and diffusions in the underlying factors. While even the simple Monte-Carlo simulations are theoretically capable of providing a reasonable approximation of the option price, the required computation (i.e. number of simulated trajectories) and the resulting variance are critical factors when one needs to price complex instruments.

The work of Longstaff and Schwartz (2001) has tackled the problem of pricing an exotic option through Monte Carlo simulation. The key to their approach is the use of least squares to estimate the conditional expected payoff to the option holder from continuation. Using simulation for pricing exotic options has several advantages. For example, simulation can be easily applied to options whose value depend on multiple factors or to value derivatives with both path-dependant and American-exercise features. Simulation also allows state variables to follow general stochastic processes such as jump-diffusion, as in Merton (1976), non Markovian processes, as in Heath, Jarrow and Morton (1992), and even general semi-martingales, as in Harrison and Pliska (1981).

However, this ‘risk-neutral Monte-Carlo’ (RNMC) method assumes that risk can be completely eliminated while in reality, except for very special cases, the risk in option trading cannot be eliminated. This has incited other researchers to search for similar methods that also take account of the residual risk and its implications for a risk-averse trader.

2.1.2.1 Minimum variance hedging

When using MC methods we can, and should, choose a risk measure in order to deal with the residual risk. The notion of risk reduction becomes even more critical when we acknowledge that perfect replication is not possible and thus "Risk-neutral Monte-Carlo (RNMC)" is not relevant. The most widely used risk measure by the researchers, as well as traders, is the variance of wealth balance. Schweizer (1993) is a major contributor to the fundamentals of variance-optimal hedging in discrete time. In his work he uses an underlying asset with stochastic price behavior and a contingent claim on this asset and he hedges this contingent claim by holding a position in the underlying asset. He tries to minimize a quadratic error measure for this trading strategy and shows that in the framework of a bounded mean-variance tradeoff such optimal trading strategy does exist and can be implemented. This work plays an immensely important role in what other researchers have done for replicating CDOs and other exotic instruments.

However, it should be noted that Schweizer (1993) uses a linear gain function for the hedging strategy and assumes frictionless trading in his work. These assumptions are far from realistic. Yet, showing the existence and structure of an optimal hedging strategy is the contribution of his work and this has had a crucial role in later works about replication of hedge funds and CDOs, especially in the work of V. Kapoor (2006) which will be our main focus in this work.

2.1.2.2 Hedged Monte-Carlo

Potters and Bouchaud (2001) have built on the work of Longstaf and Schwartz by introducing a hedged Monte-Carlo (HMC) approach which intends to minimize a given measure of risk (i.e. the variance of the wealth balance). It is interesting to see that when the objective probabilities are log-normal and continuous time is taken, this method gives us exactly the same results as Black-Scholes model.

This method has several advantages over RNMC including:

- First and most important is considerable variance reduction. The standard deviation of HMC is typically five to ten times smaller than with RNMC, meaning that we can reduce the number of Monte-Carlo trajectories by up to a hundred times and still get the same precision.
- "HMC provides numerical estimates of the price of the derivative and ALSO of the optimal hedge and the residual risk."
- This method does not rely on the notion of risk-neutral measures and thus can be used for any model of true dynamics of the underlying.
- "HMC allows us to use purely historical data to price the derivatives and does not require modeling of the underlying asset fluctuations.

The general nature of this approach allows it to be easily modified and extended to other risk measure such as expected shortfall, as in Pochart and Bouchaud (2003), and thus satisfying the needs of different trading strategies.

Despite all the aforementioned advantages of HMC, we have concluded that this method suffers from systematic errors which are compounded as the frequency of rehedging is increased. In this method the option price and the hedge are approximated with a number of basis functions which inherently contain some errors. These errors are compounded as in the next instance of rehedging one uses the same approximated numbers to again approximate the new price and hedge. While these errors could be justifiable for pricing an overcomplicated path-dependant option, in the case of our CDO we do not need to estimate the price and the hedge before the maturity of our model and thus, we use a simpler and yet more efficient method. For details of HMC method and its implementation see appendix B.

2.3 Risk

In this section we will discuss our choice of measure of risk and the implications of this choice on our results. After deciding what measure of risk is appropriate for the purpose of this work, we will also discuss the risk of hedging a digital option. This particular risk is relevant to our work because, as we will show, CDO tranches behave as barrier options and thus, the quite unique risk of hedging such options should be considered.

2.3.1 Measures of Risk

The structure of CDO is a barrier type derivative which results in jumps in the P&L distribution. In addition, if we assume a fixed or a number of discrete rates of recovery for the underlying bonds, the losses due to defaults will also have a discrete nature. As a result, perfect replication is not feasible in this kind of CDO trade and the commonly used mechanics of replication/hedging, which are based on continuous-diffusion-process, will be misleading. The notion that we cannot introduce a perfect replication of a CDO tranche using the underlying bonds means that there is some risk which can only be reduced and not completely eliminated. This is in contrast to the models that assume a continuous distribution for the underlying asset, like the Black-Scholes model, where a perfect replication strategy does exist and can be implemented.

In this context, our objective becomes one of minimizing the risk as much as we can. To this end we need to choose an appropriate measure of risk. The most commonly used measure of risk among the practitioners (and even academics) is variance and we shall use it as one of our measures of risk in this work.

However, the fact that variance penalizes both profits and losses does not sound very appealing. In addition, variance fails to strongly penalize extreme losses. To address these concerns, one should turn to other measures of risk. VaR is another popular measure of risk which is also a requirement of Basel II accord.

Artzner et al. (1998) argues that a measure of risk can only be coherent if it satisfies the following criteria: Consider a set V of real-valued random variables. A function $\rho: V \rightarrow R$ is called a coherent risk measure if it is

(i) Monotonous: $X \in V,\ X \geq Y \Rightarrow \rho(X) \leq \rho(Y)$,

(ii) Sub-Additive: $X, Y, X + Y \in V \Rightarrow \rho(X + Y) \leq \rho(X) + \rho(Y)$,

(iii) Positively homogeneous: $X \in V, h > 0, hX \in V \Rightarrow \rho(hX) = h\,\rho(X)$, *and*

(iv) Translation invariant: $X \in V, a \in R \Rightarrow \rho(X + a) = \rho(X) - a$.

In their work, they show that VaR is not coherent because it is not sub-additive.

Acerbi et al (2001) introduced Expected Shortfall as a coherent measure of risk which does not suffer from the shortcomings of VaR. Furthermore, Crouchy et al. (2000) conduct a comparative analysis of measures of risk for credit risk models and show that for defaults, which follow a Poisson process, ES is the more appropriate measure of risk.

It should be noted that when the profit-loss distribution is normal, VaR does not suffer from the problems pointed out by Artzner *et al.* (1997). First, under the normality assumption, VaR does not have the problem of tail risk. When the profit-loss distribution is normal, expected shortfall and VaR are scalar multiples of each other, because they are scalar multiples of the standard deviation. Therefore, VaR provides the same information about the tail loss as does expected shortfall.

2.4 Default Correlation

Default correlation is one of the key characteristics in modeling the risk for a basket of corporate bonds. The complicated nature of these correlations is indeed what makes the modeling of CDOs a nontrivial problem. The main approach to tackling this problem in the past few years has been through use of copulas. In this section we shall review the developments in this field and also the use of structural Variance Gamma as an alternative way of modeling correlated defaults.

2.4.1 Copulas

Application of copulas in modeling default correlation first appeared in the work of Li (1999). Before his work, default correlation was defined based on discrete events which dichotomize according to survival or non-survival at critical time such as one year.

This discrete event approach, as used by Lucas (1995), is what we call the discrete default correlation. However, this approach has several disadvantages. First, default is a time dependant event and thus, default correlation is also time dependant. This means that as we increase the time horizon in which we observe defaults, the correlation increase too. Second, by studying only one single period of 1 year we waste important information. An example of this missed information would be the empirical results which suggest that default tendency of corporate bonds is linked to their age since issue. Third, in most cases we need the joint distribution of survival times for the next couple of years in order to estimate the value of a credit derivative. Fourth, default rates can be calculated as simple proportions only when there is no censoring of data in the one year period.

Li (1999) introduces a random variable called "time-until-default" to denote the survival time of each defaultable entity of financial instrument. Then, he defines the default correlation of two entities as the correlation between their survival times. The marginal distribution of conditional survival times can be calculated using a credit curve. This credit curve is usually derived from the risky bond spread curve or asset swap spreads observed currently from the market.

In this method, Li defines the time-until-default, T, for security A and $F(t)$ as the distribution function of T. Then a survival function is defined as $S(t) = 1-F(t)$. This survival function gives the probability that a security will attain age t. He also introduces notations of conditional probability of default. That is, the probability that the security A will default within the next t years conditional on its survival for x years.

Another equivalent function, which is most frequently used by the statisticians, is the hazard rate function which gives the instantaneous default probability for a security that has attained age x. This hazard function can be derived as the first derivative of the survival function which has a conditional probability density interpretation: "it gives the value of conditional probability density function of T at exact age x, given survival to that time."

Typically it is assumed that the hazard rate is a constant, h, which results in a density function that shows that survival time follows an exponential distribution with parameter h.

Using hazard rate function has a number of advantages. First, it provides information about immediate default risk of each entity. Second, it makes it easier to compare a group of individuals. Third, models based on hazard rate function can be useful in complicated situation such as the cases where there is censoring. Fourth, the similarities between the hazard rate and the short rate mean that we can borrow the techniques which are already developed for modeling the short rate.

Following this approach, one can define the default correlation of two different entities based on their survival times. Li (1999) calls this definition of default correlation as the survival time correlation. This correlation is much more general concept compared to the discrete default correlation based on one year period. If one has the joint distribution of the survival times of two different assets, he can also calculate the discrete default correlation.

Li (1999) uses a copula approach to link the univariate marginals to their full multivariate distribution. One of the most widely used copulas in finance is the Gaussian copula which is used by Li (1999) because it is basically the same approach that RiskMetrics uses. Li uses multivariate normal copulas with correlation factors that can be interpreted as the asset correlation between two the used in CreditMetrics.

Using these tools one can simulate the default times following these steps:

1. Calculate asset correlation based on historical data.
2. Simulate $Y_1, Y_2, \ldots, Y_n$ from a multivariate normal distribution using the asset correlation matrix.
3. Transform the equity returns to survival times.

While this approach, consisting of Gaussian copulas, is the industry standard, many researchers have tried to introduce more sophisticated models that produce more realistic models of default correlation. Ddempster et al. (2007) use a minimum entropy copula approach in order to find the best copula fit for the market data. Their approach has the advantage of providing justification for the choice of copula, providing good fits to data, and performing well out-of-sample. However their method still assumes that the dependence structure remains static over time. Also, this method is a computationally intensive procedure.

There are many other extensions to Gaussian copula model such as local correlations, stochastic correlation, and Levy processes. In addition, some dynamic models have been developed to address the realities of the market, such as stochastic intensity models and dynamic loss models. For the purpose of this work, we shall follow the more standard approach of Li (1999) to link marginal distribution of survival times to their multivariate distribution.

2.4.2 Variance Gamma Process

The risk neutral approach to valuing derivatives was first introduced by Black-Scholes (1973) and it remains the standard paradigm in finance. However, this model in known have some biases such as volatility smile and skewness premia. The presence of a volatility smile suggests that the risk neutral density has a kurtosis above that of a normal distribution. In addition, the existence of skewness premia further suggests that the left tail of the return distribution is fatter than the right tail.

Madan et al. (1998) propose the use of a variance gamma (*VG*) process to model the returns of equities to address the abovementioned concerns. This model is a three parameter generalization of Brownian motion as a model for the dynamics of the logarithm of the stock price. The *VG* process is obtained by evaluating Brownian motion at a random time change given by a gamma process. The *VG* process has no continuous martingale component and, in contrast, it is a pure jump process.

"The *VG* process $X(t;\sigma,\upsilon,\theta)$ is defined in terms of the Brownian motion with drift $b(t;\theta,\ \sigma)$ and the gamma process with unit mean rate, $\gamma(t;1,\ \sigma)$ as:"

$$X(t;\sigma,\upsilon,\theta) = b(\gamma(t;1,\ \sigma);\theta,\upsilon)$$

"The *VG* process has three parameters: (i) σ the volatility of the Brownian motion, (ii) υ the variance rate of the gamma time change and (iii) θ the drift in Brownian motion with drift." The *VG* process has the advantage of providing us with two more dimensions of control on the distribution. The control over the skew is obtained via θ and υ controls the kurtosis. Madan et al. show in their work that this additional control on the distribution allows us to correct the biases of Black-Scholes model.

We shall use this process as an alternative way of modeling the underlying asset fluctuations in our CDO trade. By combining the VG process with the structural default model of Merton, we will have a robust model for simulating time-to-default for every issuer in our reference pool without suffering from the biases of Black-Scholes model.

2.5 Optimal Static Hedging of Defaults in CDOs

V. Kapoor (2006) tackles the problem of static hedging of a CDO tranche position with a portfolio of bonds that constitute the CDO reference pool. The goal is to find the hedge ratio and tranche price that result in a fair bet on the average and minimize the hedging error measure.

According to Kapoor, CDO trading offers some attractive features:

- Chase opportunity sloshing across the capital structure
- Evading credit spread delta radar and making carry
- Tradeoff-hedge systematic and idiosyncratic spread convexity

But this trade also has its own problems, namely: 1) The standard CDO model does not directly address the cost of hedging & 2) When attempting to replicate a CDO tranche by CDS we encounter some difficulties:

- Jumps in spread and jump to default
- Uncertain recovery
- Random realized spread-spread correlation

Kapoor uses the two different models to simulate time-until-default for the pool bonds: (i) Reduced form Poisson-Normal-Copula (PNC), and (ii) Structural Varian Gamma (SVG). In his work, the parameters of SVG are chosen in a way that the resulting average times to default and the variance of pool loss are identical to those of PNC.

For calculating the fair price and the appropriate hedge, he uses the work of Bouchad and Potters (2001) and Bouchaud and Pochart (2003). This provides us with the needed tools to calculate the price and the hedge at the same time. However, his results, for the mezzanine and senior tranches, show jumps in the value of ES for different hedge notional (figure 2.1). There is no justification for these jumps since in his model the hedging error has a continuous nature for all tranches.

Nevertheless, his framework provides us with a satisfactory ensemble of tools for pricing CDO tranches and calculating the hedge notional. That is why we will follow his approach to hedging CDOs while trying to correct the presumed errors.

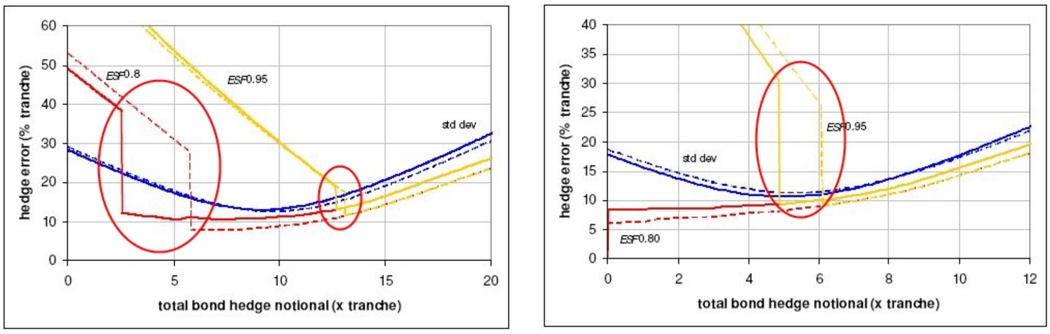


Figure 2.1. Results obtained by V. Kappor

3. Model and Methodology

Our model is based on the work of Kapoor (2006) where a CDO trader sells protection on a given tranche of a synthetic CDO and holds a basket of underlying bonds of the reference CDS index to hedge his position. The price (upfront/spread) and the hedge are chosen in a way that average final wealth of this trade is zero and a given risk measure (variance/ES) is minimized.

To this end, we will have to design two main components in our model: **(1)** Default model and **(2)** Hedging Technique. The default model is of immense importance if we are to obtain realistic results. One of the huge challenges that one faces in modeling the defaults is the lack of adequate historical data. Defaults, by their nature, are relatively rare and finding a sizeable sample of defaults for similar enterprises is almost impossible and comparing defaulting firms in different sectors brings up the idiosyncratic characteristics of the sectors into calculation which further complicate the matters. The two default models that have been employed by Kapoor (2006) are the Reduced Form Poisson-Normal-Copula (PNC) and the Structural Variance Gamma (SVG).

After having modeled the defaults using both aforementioned methods, we shall proceed to a simulation where we create different trajectories based on these default models and then we try to find an optimal strategy of hedging which minimizes the risk measure for the trader. Later it will be explained how this simulation can provide us not only with the fair price, but also with the optimal hedge en route to maturity. This method is in sharp contrast with the *risk-neutral* approach where it is assumed that perfect replication is possible and risk aversion does not change our hedging strategy.

To simplify the notations in our CDO hedging/pricing problem, we use a framework of continuous premium payments and constant interest rates. This method will lead us through a static hedging process as opposed to dynamic hedging. This is a reasonable simplification because: “1) there has not been much work done on direct hedge performance of CDO tranches and to tackle the dynamic hedging problem we should first understand the static hedging; 2) In the face of jumps of uncertain timing and magnitude, dynamic hedging is not even theoretically possible (perfect replication) so understanding residual hedging errors is essential as they will be used in dynamic hedging; 3) a dynamic analysis requires a coupled model of spreads and defaults which is beyond the scope of this work."

In this work we will study three tranches of a CDO: equity, mezzanine, and senior. The trading book that the CDO trader holds and the related cash flows are shown in figure 3.1 where solid lines represent the premium cash flow stream and dotted lines represent contingent cash flows. The trade shown in figure 3.1 depicts a spread delta-neutral sell equity protection trade where the CDO trader sells protection on an equity tranche referencing a credit index, and purchases protection on the credit index to hedge away the spot spread delta. In this method the pool expected loss refers to the sum of the contingent legs of the CDS in the reference index.

The CDO trader receives the premium cash flows from the equity tranche (consisting of a possible upfront and a stream of premiums) and pays any contingent claims due to default. He also shorts a portfolio of bonds, which the CDS index is referencing, and thus, he pays coupons of those bonds and receives money when any of those bonds defaults.

We use a similar trading book for other tranches (mezzanine and senior), with the difference that for equity tranche the premiums consist of an initially unknown upfront (to be calculated) and a known fixed spread while mezzanine and senior tranches only pay the spread which we should calculate.

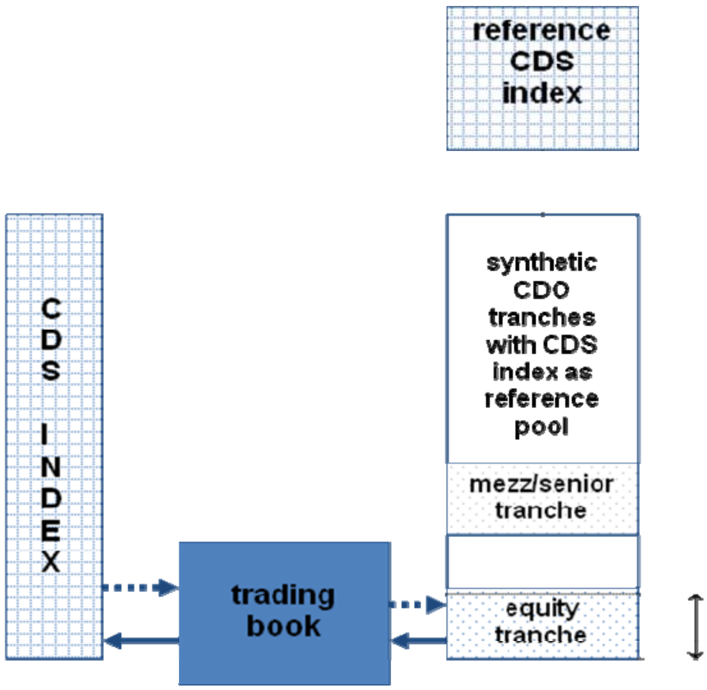


Figure 3.1. CDO trading book

3.1 Change in Wealth of a CDO Trader

In this section we will detail the structure of our CDO, the tranche specifications, and the portfolio of reference pool bonds. $N = \sum_{i=1}^{\infty} n_i$

3.1.1 Reference Pool

We will consider a pool of n bonds of notional n_i ($1 \leq i \leq n$). "The total reference pool notional $N = \sum_{i=1}^{n} n_i$. These reference bonds can default, and in the event of default recover a fraction R_i of notional. The pool loss process, $pL(t)$, is a superposition of delta-functions centered at τ_i, the time to default of reference bond i, and with each default contributing a loss of $(1-R_i)n_i$."

$$pL(t) = \sum_{i=1}^{n} n_i (1 - R_i) \delta(\tau_i)$$

The commutative pool loss (*cpL*) and recovered amount (*cpR*) are superposition of Heaviside functions corresponding to a time integral of the loss and recovery process:

$$cpL(t) = \sum_{i=1}^{n} n_i (1 - R_i) I_i(t);\ cpR(t) = \sum_{i=1}^{n} n_i R_i I_i(t);\ I_i = \begin{Bmatrix} 0\ if\ \ \tau_i > t \\ \\ 1\ if\ \ \tau_i \le t \end{Bmatrix}$$

3.1.2 Tranche

The tranche specifications for our synthetic CDO are the lower strike (k_1), upper strike (k_2), upfront payment fraction (u) and running spread (s). The tranche notional as a function of time, $tn = k_1^*(t) - k_2^*(t)$, where:

$$k_1^*(t) = \min[\max[k_1, cpL(t)], k_2]; \qquad k_2^*(t) = \max[\min[k_2, N - cpR(t)], k_1];$$

By defining our tranche in this way, we are creating the most common model of CDO structure where the tranche amortizes from the bottom to the top. “The standardized synthetic CDOs based on credit indexes follow the aforementioned amortization rule.”

“The present value of the cash-flows for the tranche investor (i.e. the tranche default protection seller) consists of received upfront payments and tranche spread on outstanding tranche notional and outgoing default contingent payments.” The present value of these received premiums is

$$u(k_2 - k_1) + s\int_0^T tn(\tau) e^{-r\tau} d\tau$$

and the present value of contingent payments by the CDO trader is

$$\int_0^T \frac{d(k^*(\tau))}{d\tau} e^{-r\tau} d\tau$$

Thus, the final wealth of a tranche investor is

$$\Delta W_{tranche} = u(k_2 - k_1) + s\int_0^T tn(\tau)\exp[-r\tau]d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau} e^{-r\tau} d\tau$$

This is the first part of our trading book and we will try to hedge it with a portfolio of reference pool bonds which make up the second part of our trading book.

3.1.3 Portfolio of Reference Pool Bonds

"We will attempt to hedge the default risk of a CDO tranche with a position in the reference bonds of notional h_i, market price equal to $f_i \times h_i$, and coupon c_i. The change in wealth of this bond portfolio is given by":

$$\Delta W_{bond} = \sum_{i=1}^{n} \Delta W_i$$

$$\Delta W_i = \begin{cases} no\ default\ over\ t \in [0,T]:\ h_i\left\{- f_i + \exp[-rT] + c_i \int_o^T \exp[-r\tau]d\tau \right\} \\ default\ at\ \tau_i \in [0,T]:\ h_i\left\{- f_i + R_i \exp[-r\tau_i] + c_i \int_o^{\tau_i} \exp[-r\tau]d\tau \right\} \end{cases}$$

The position in the underlying bonds has a total net present value of

$$H\left[- f_i + PV(Bond\,Portfolio)\right]$$

where H is the hedge notional. "The hedging could be done by a portfolio of CDS on the CDO reference issuers, as is customary with much synthetic CDO trading activity.

Hedging the default risks with a CDS is theoretically identical to hedging with a par bond under certain conditions."

By putting the components of our CDO trade together, the total change in wealth of the CDO trader will be

$$\Delta W = u(k_2 - k_1) + \left[s\int_0^T tn(\tau)e^{-r\tau}d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau \right] - H\left[-f_i + PV(Bond\,Portfolio)\right]$$

Equation (3.1)

where $u(k_1\text{-}k_2)$ is the upfront, s is the tranche spread, and H is the hedge notional.

3.2 Optimal Static Hedging

Now that the CDO structure and the resulting portfolio have been defined, we set our goal to find bond hedge notional (*H*) and tranche pricing (*u* and *s*) such that the change in wealth of the hedged portfolio is zero on the average and a certain hedging error measure Θ is as small as possible:

$$\Delta W = \Delta W_{bond} + \Delta W_{tranche}$$

$$\overline{\Delta W} = 0$$

Minimize [Θ]

3.2.1 Monte-Carlo simulation

Our Monte Carlo method is based on the work of Pochart et Bouchaud (2003) but with some major modifications. As mentioned earlier, the HMC method used by Bouchaud et Potters (2001) and in the consequent work of Pochart et Bouchaud (2003) suffers from systematic errors. So, we use a simpler, one-period method which satisfies our needs and is more accurate. In this method we find the price (*u* or *s*) and the hedge (H) in a way that the average wealth of the CDO trader on all trajectories is equal to zero while a given measure of risk is minimized.

In doing so, we work backwards in time using the known factors at maturity time T (which are tranche payoffs, present value of bond portfolio, and contingent payments) to calculate the fair price and hedge at time 0.

We have two main categories of tranches in our CDO: **(i)** Equity tranche which has a fixed spread and an unknown upfront, and **(ii)** Mezzanine and senior tranches which do not have any upfront but only an unknown spread. Thus, for each category we have a different minimization problem. In each of these two cases we have two unknown factors to calculate (u and H for equity tranche, and s and H for other tranches) and also two equation (average $\Delta W=0$, and Θ minimized).

In order to minimize the standard deviation, we use a simple OLS regression while for ES we use a numeric method. One such numeric method is the *fmincon* command in MATLAB which is useful for solving constrained nonlinear minimization problems.

3.2.2 Minimizing Standard Deviation

The first of the two measures of risk (Θ) which we want to minimize in this trade is standard deviation:

$$\Theta = \sigma_{\Delta W} \equiv \left(E[(\Delta W - \overline{\Delta W})^2]\right)^{1/2}$$

We will run a regression to minimize variance of wealth.

(i) For the equity tranche we need to estimate the upfront u and the hedge notional H. So, since the average wealth is equal to zero, we rearrange the wealth equation (3.1) in the following way:

$$-\left[s\int_0^T tn(\tau)e^{-r\tau}d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau \right] = u(k_2 - k_1) - H\left[-f_i + PV(Bond\ Portfolio)\right]$$

and estimate the following regression:

$$-\left[s\int_0^T tn(\tau)e^{-r\tau}d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau \right] = \beta_0 + \beta_1\left[-f_i + PV(Bond\,Portfolio)\right] + \varepsilon$$

where β_0 estimates u and β_1 estimates H.

(ii) For the mezzanine and senior tranches we need to estimate spread s and the hedge notional H. Again we rearrange the wealth equation (3.1) in the following way

$$\int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau = s\int_0^T tn(\tau)e^{-r\tau}d\tau - H\left[-f_i + PV(Bond\,Portfolio)\right]$$

and estimate the following regression:

$$\int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau = \beta_0\left(\int_0^T tn(\tau)e^{-r\tau}d\tau\right) + \beta_1\left[-f_i + PV(Bond\,Portfolio)\right] + \varepsilon$$

where β_0 estimates s and β_1 estimates H.

This simple and efficient regression will provide us with the best price and hedge that minimize the variance of our CDO trade.

3.2.3 Minimizing Expected Shortfall

The second measures of risk (Θ) which we want to minimize in this trade is expexted shortfall (ES):

$$\Theta = ES_\alpha \equiv -E[\Delta W \mid \Delta W \leq -VaR_\alpha]$$

$$\Pr\{\Delta W < -VaR_\alpha\} = 1 - \alpha$$

We use the *fmincon* command in MATLAB to minimize this measure of risk while keeping the average wealth equal to zero. To this end, we need to express equation (3.1) in form of a number of matrices where the price and hedge are separated from the rest of equation in a single matrix.

(i) For the equity tranche we define the following matrices for N trajectories:

$$A = \begin{bmatrix} 1 & \left[-f_i + PV(Bond\ Portfolio)_1\right] \\ 1 & \left[-f_i + PV(Bond\ Portfolio)_2\right] \\ & \vdots \\ 1 & \left[-f_i + PV(Bond\ Portfolio)_N\right] \end{bmatrix}_{N\times 2}$$

$$Y = \begin{bmatrix} u \\ H \end{bmatrix}_{2\times 1}$$

$$B = \begin{bmatrix} \left(s\int_0^T tn(\tau)e^{-r\tau}d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau\right)_1 \\ \left(s\int_0^T tn(\tau)e^{-r\tau}d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau\right)_2 \\ \vdots \\ \left(s\int_0^T tn(\tau)e^{-r\tau}d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau\right)_N \end{bmatrix}_{N\times 1}$$

$$C = \left[1 \quad E\left[-f_i + PV(Bond\ Portfolio)\right]\right]_{1\times 2}$$

$$D = -E\left[s\int_0^T tn(\tau)e^{-r\tau}d\tau - \int_0^T \frac{d(k_1^*(\tau))}{d\tau}e^{-r\tau}d\tau\right]$$

Using these matrices we can calculate the vector of N final wealth as:

$$\Delta W = A \times Y + B$$

and

$$ES_\alpha(\Delta W) = ES_\alpha(A \times Y + B)$$

while the zero average wealth condition is satisfied by

$$C \times Y = D$$

Now, with the help of these matrices, we can use the *fmincon* command in MATLAB to solve our minimization problem on matrix *Y*.

(ii) For the mezzanine and senior tranches we define the following matrices for N trajectories:

$$A = \begin{bmatrix} \left(\int_0^T tn(\tau)e^{-r\tau}d\tau\right)_1 & \left[-f_i + PV(Bond\ Portfolio)_1\right] \\ \left(\int_0^T tn(\tau)e^{-r\tau}d\tau\right)_2 & \left[-f_i + PV(Bond\ Portfolio)_2\right] \\ & \vdots \\ \left(\int_0^T tn(\tau)e^{-r\tau}d\tau\right)_N & \left[-f_i + PV(Bond\ Portfolio)_N\right] \end{bmatrix}_{N\times 2}$$

$$Y = \begin{bmatrix} s \\ H \end{bmatrix}_{2\times 1}$$

$$B = \begin{bmatrix} \left(-\int_0^T \frac{d(k_1^*(\tau))}{d\tau} e^{-r\tau}d\tau\right)_1 \\ \left(-\int_0^T \frac{d(k_1^*(\tau))}{d\tau} e^{-r\tau}d\tau\right)_2 \\ \vdots \\ \left(-\int_0^T \frac{d(k_1^*(\tau))}{d\tau} e^{-r\tau}d\tau\right)_N \end{bmatrix}_{N\times 1}$$

$$C = \left[E\left[\int_0^T tn(\tau) e^{-r\tau} d\tau \right] \quad E\left[-f_i + PV(Bond\ Portfolio) \right] \right]_{1\times 2}$$

$$D = E\left[\int_0^T \frac{d(k_1^*(\tau))}{d\tau} e^{-r\tau} d\tau \right]$$

Using these matrices we can calculate the vector of N final wealth as:

$$\Delta W = A \times Y + B$$

and

$$ES_\alpha(\Delta W) = ES_\alpha(A \times Y + B)$$

while the zero average wealth condition is satisfied by

$$C \times Y = D$$

Again, the *fmincon* command in MATLAB is used to solve our minimization problem on Y.

3.3 Default Models

The two default models used in this work are “Poisson-Normal-Copula” and “Structural Variance Gamma Process”. The final results will include calculated prices and hedges using both these methods.

3.3.1 Poisson-Normal Copula

One approach to modeling the defaults is a reduced form model of Poisson arrival of defaults with Normal Copula based dependence. This reduced form Poisson-Normal-Copula (PNC) method sets up a mark-to-model dynamic and takes hold in the accounting of synthetic CDO trading P&L. This approach uses the historic defaults and correlations between different tranches and assets. In this work we will use a flat default hazard rate and a uniform asset correlation. Time to default is simulated using a latent variable single factor approach with both market and idiosyncratic drivers of randomness taking place via standard Normal independent random variates.

The PNC parameters (hazard rate and asset correlation) used in this work are those used by Vivek Kapoor (2006) and Li (1999) which are calculated using Moody's estimates for corporate bond defaults.

In this method, Li defines the time-until-default, *T*, and *F(t)* the distribution function of *T*, as

$$F(t) = \Pr(T \leq t), \qquad t \geq 0$$

and the survival time function is defined as

$$S(t) = 1 - F(t) = \Pr(T > t), \qquad t \geq 0$$

This survival function gives the probability that a security will attain age *t*.

The probability density function is defined as follows

$$f(t) = F'(t) = -S'(t) = \lim_{\Delta \to 0^+} \frac{\Pr[t \leq T \leq t + \Delta]}{\Delta}$$

Li (1999) also introduces two more notations

$$_t q_x = \Pr[T - x \leq t \mid T. > x], \qquad t \geq 0$$
$$_t p_x = 1 - _t q_x = \Pr[T - x > t \mid T. > x], \qquad t \geq 0$$

where $_t q_x$ can be interpreted as the conditional probability that the security will default within the next *t* years conditional on its survival for *x* years. In the special case of *x=0*, we have

$$_t p_0 = S(t) \qquad x \geq 0$$

We can use the distribution function *F(t)* or the survival function *S(t)* to specify the distribution of random variable time-until-default. Another equivalent function, which is most frequently used by the statisticians, is the hazard rate function which gives the instantaneous default probability for a security that has attained age *x*.

$$\Pr[x < T \le x + \Delta x \mid T > x] = \frac{F(x+\Delta x) - F(x)}{1 - F(x)} \approx \frac{f(x)\Delta x}{1 - F(x)}$$

This leads us to the following hazard rate function

$$h(x) = \frac{f(x)\Delta x}{1 - F(x)} = -\frac{S'(x)}{S(x)}$$

Which has a conditional probability density interpretation: "it gives the value of conditional probability density function of T at exact age x, given survival to that time."

Using the aforementioned derivation of hazard rate, the survival function can be expressed in terms of the hazard rate function,

$$S(t) = e^{-\int_0^t h(s)ds}$$

Consequently, ${}_tq_x$ and ${}_tp_x$ can be expressed in terms of the hazard rate function as follows

$$_t\,p_x = e^{-\int_0^t h(s+x)ds}$$

$$_t\,q_x = 1 - e^{-\int_0^t h(s+x)ds}$$

In addition,

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t h(s+x)ds}$$

and

$$f(t) = S(t).h(t)$$

which is the density function for *T*.

Typically it is assumed that the hazard rate is a constant, *h*, which gives the density function

$$f(t) = he^{-ht}$$

This shows that survival time follows an exponential distribution with parameter h. Thus, the survival probability over any time interval with the length t is

$$ {}_t p = e^{-ht} $$

Now, we can define the default correlation of two entities *A* and *B* based on their survival times T_A and T_B as follows

$$ \rho_{AB} = \frac{Cov(T_A, T_B)}{\sqrt{Var(T_A)Var(T_B)}} = \frac{E(T_A T_B) - E(T_A)E(T_B)}{\sqrt{Var(T_A)Var(T_B)}} $$

This is the survival time correlation as defined by Li (1999) and he uses a Gaussian copula to link the univariate marginals to their full multivariate distribution. A bivariate normal copula is defined as

$$ C(u,v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v), \rho), \quad -1 \le \rho \le 1 $$

where Φ_2 is the bivariate normal distribution function with correlation coefficient ρ and Φ^{-1} is the inverse of a univariate normal distribution function. In this method, Li uses a ρ which can be interpreted as the asset correlation between two credits used in CreditMetrics.

Using these tools and in order to generate default times using PNC, we simulate a pool of 125 bonds which follow a multivariate normal distribution with asset correlation equal to 25%. Then we use the probability integral transform to create variables with uniform distributions. In the end, we calculate time-to-default for each issue using

$$ T_i = F_i^{-1}(\Phi(Y_i)) $$

where

$$ F_i^{-1}(U) = -\ln(U)/h $$

3.3.2 Structural Variance Gamma

Merton (1976) has introduced the structural default model which is the second method that we use. This model states that whenever the value of a given firm goes below a certain level the firm, and the bonds it has issued, default. This approach has the potential to integrate spread evolutions and default modeling. Furthermore, we can integrate credit and equity modeling in this approach and gain a quite comprehensive accounting of explicative factors.

The proposed structural model here is one with Variance Gamma (VG) firm value drivers. Several researchers have worked on the structural Variance Gamma applications in CDOs and have developed a risk-neutral description of marginal defaults and fitting VG dependence parameters to observed tranche prices. Cariboni & Schoutens (2004), Luciano & Schoutens (2005), and Moosbrucker (2006) are some examples of using VG processes to this end. This approach is an attractive alternative to the base correlation approach of fitting prices and marking to market non-standardized tranches.

In order to make comparison easier, we have fitted the parameters of our SVG approach in a way that they replicate two key characteristics of defaults under the PNC model: (i) the average number of defaults during the tenor of our CDO, and (ii) the variance of cumulative pool loss.

The details of SVG process and its implementation can be found in appendix C.

4. Results

The CDO that we use in this study is loosely based on the CDX.NA.IG index tranches by markit™. This CDS index references 125 North American investment grade bonds and the first 3 breakpoints for its CDO tranches are 3%, 7%, and 10%. Our main default model is Poisson-Normal Copula and the parameters for the Variance Gamma Process are chosen to fit the first two moments of PNC model. First, we find the optimal hedge for the same set of PNC parameters used by Li (1999) and analyze the results and risks. Then, we proceed to perform a number of sensitivity tests by changing some of these parameters.

Model Parameters

Tenor	$T = 5$ yrs
Interest rate	$r = 5\%$/yr

<u>Pool Information</u>	
Number of issuers	$n = 125$
Reference notional	$n_i = \$0.8\text{m}\ \forall i$
Total pool notional	$N = \$100\text{m}$
Bond coupon	$c_i = 5.78\%/\text{yr}\ \forall i$
Bond unit price	$f_i = 1\ \forall i$
Recovery rate	$R_i = 0.3$

<u>Reduced form Poisson-Normal Copula (PNC)</u>	
Hazard rate	$\lambda_i = 0.65\%/\text{yr}\ \forall i$
Asset correlation	25%

<u>Structural Variance Gamma (SVG)</u>	
GBM drift	$\mu_i = 0$ (1/yr) $\forall i$
GBM volatility	$\sigma_i = 0.20$ (1/yr$^{1/2}$) $\forall i$
Gamma volatility	$\upsilon = 2$ yr
Default threshold	$\varpi = 0.3618$
GBM dependence parameter	$\beta = 0.454$
Gamma dependence parameter	$\kappa = 1$

<u>Tranche Information</u>

Name	k_1	k_2	upfront	Fixed running (bps/yr)
Equity	0%	3%	yes	500
Mezzanine	3%	7%	no	-
Senior	7%	10%	no	-

The aforementioned PNC and SVG set of parameters results in similar default distributions which are shown in figure 4.1. In both models around 30% of the trajectories are no-default cases with the SVG model having a slightly higher no-default percentage. Despite the differences in skewness and kurtosis, both models have the same average number of defaults and variance.

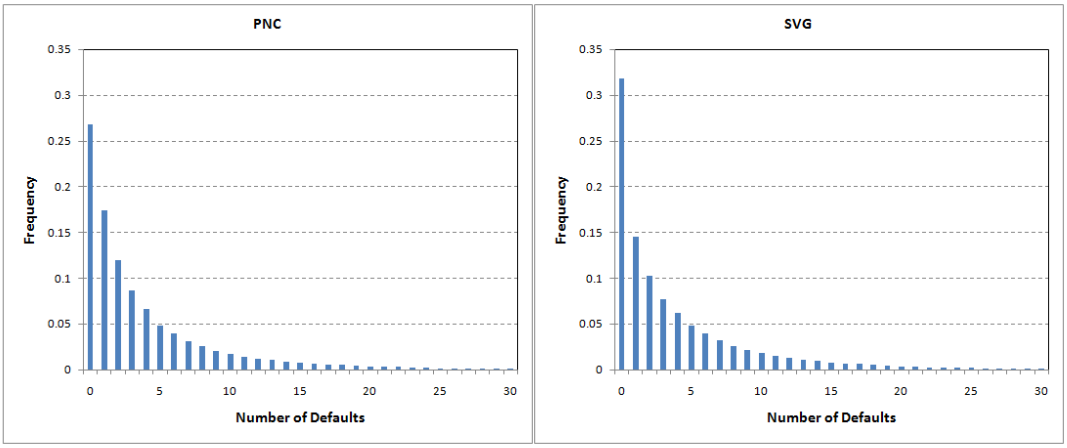


Figure 4.1. Number of defaults (portfolio of 125 bonds, 100,000 trajectories)

4.1 Selling Equity Tranche Protection

All of the distributions presented in this analysis have a zero average change in wealth. Figure 4.2 shows two risk measures (ES_{80} and STD) for different combinations of hedge notional and prices as well as a cross section (the red flat surface) which represents the combinations that result in a zero average wealth.

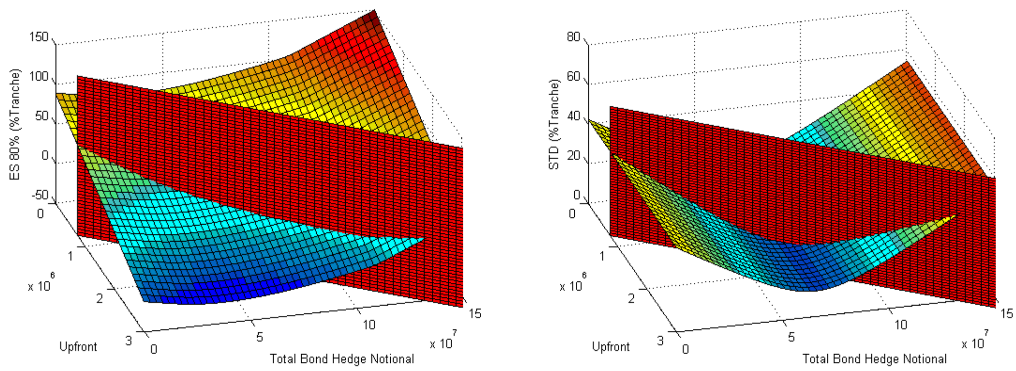


Figure 4.2. Selling equity (0%-3%) tranche protection: Risk measures for diffrernet price and hedge combinations (PNC model)

Figure 4.3 shows the sell equity tranche protection P&L distributions for four different hedge notional ranging from zero to extremely high hedge notional (50x). The ES_{80} optimal and STD optimal hedges are within this range. The spikes seen in the P&L probability distributions are associated with the no-default carry. As we can see in Figure 3, the zero hedging strategy has a very high positive no-default carry (+45%) but after only 3 defaults in the portfolio, the P&L becomes negative and the extreme losses reach the very negative ends of the distribution (-80%). In this zero hedging strategy, the break-even upfront only compensates for the difference between received spreads and contingent payments due to default so that the average change in wealth is kept at zero.

Increasing the hedge notional will introduce the cost of hedging to our portfolio. This cost is the average total net present value of the underlying unit bonds times the hedge notional ($H \times \Delta W_{Bonds}$). Figure 4.4b shows that as the hedge notional increases, one would need a higher upfront to break-even this cost. Also, it can be seen in Figure 4.3 that as the hedge notional increases from zero to optimal hedge levels and beyond, this cost pushes the no-default spikes to the left. This inverse relationship between hedge notional and no-default carry is depicted in Figure 4.4c, where the no default carry starts from more than 15%/yr for zero hedge, reaching 0% at about 25x hedge notional, and keeps on decreasing for higher hedge notional.

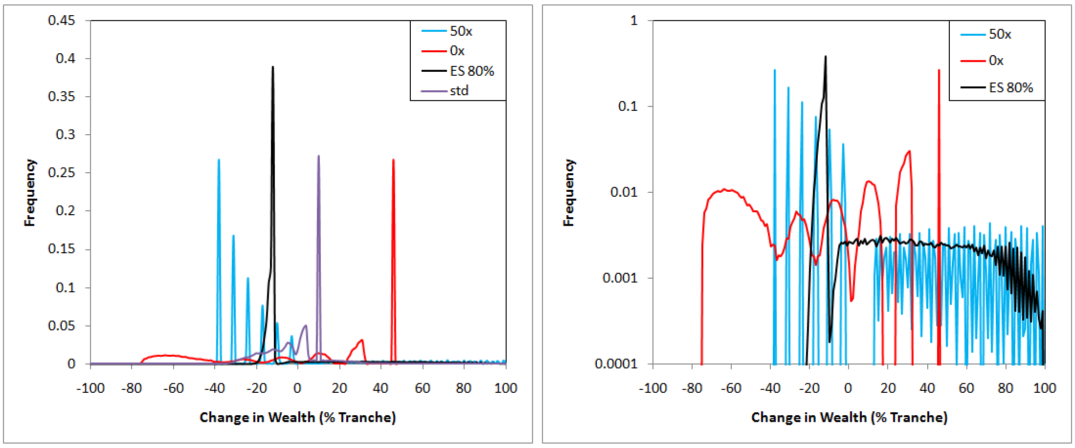
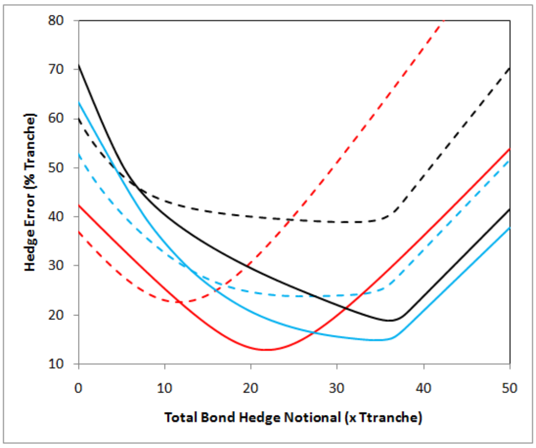
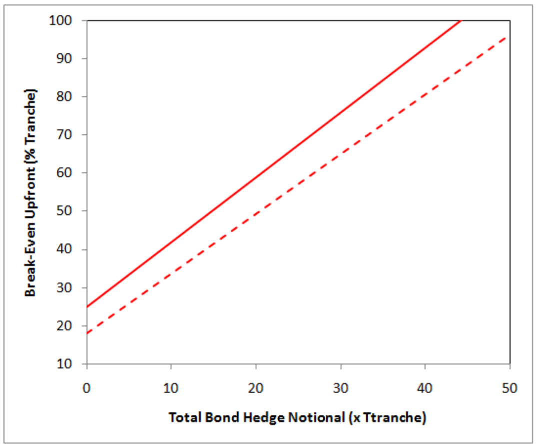


Figure 4.3. Selling equity (0%-3%) tranche protection: P&L distribution (PNC model, bin size = 1% tranche notional)

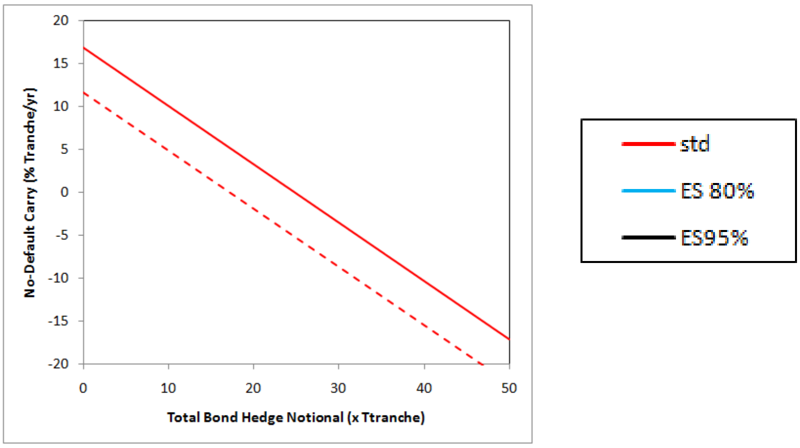
In fact, by including a hedging position on the underlying bonds in our portfolio, the effects of defaults on the P&L distribution are partially reversed and defaults become profitable. The magnitude of this reversal is directly controlled by the hedge notional. If we chose a hedge notional equal to the total pool notional ($100M, or 33.3x tranche notional), these effects are completely reversed. However, as mentioned before, any increase in the hedge notional will also increase the cost of hedging and will push the no-default carry to the left. This can be seen in Figure 4.3 where for the zero hedge strategy all of the P&L distribution is at the left side of no-default carry while for the 50x hedge the situation is completely reversed.



(a)



(b)



(c)

Figure 4.4. Selling equity (0%-3%) tranche protection (solid lines: PNC, dashed lines: SVG)

Figure 4.4a depicts hedging errors as a function of hedge notional and as we can see, STD minimization results in a lower hedge notional (22x) as compared to ES minimization (35x) because variance penalizes both gains and losses while expected shortfall only penalizes the worst α percent losses. Between 22x and 35x hedges, the variance is increasing while expected shortfall (for both 80% and 95% confidence levels) is still decreasing. This is the grey zone where our choice of risk measure makes a difference.

The tail losses in both ES optimal and STD optimal hedges are significantly less than the zero hedge and 50x hedge strategies. The STD optimal hedge has a positive no-default carry while ES_{80} optimal hedge results in a negative no-default carry. Table 4.1 shows the calculated upfront, hedge notional and hedging errors (STD, ES_{80}, and ES_{95}) for each hedging strategy.

Model	Strategy	std	ES_{80}	ES_{95}	upfront	Hedge (×tranche)
PNC	Min std	12.9%	19.3%	26.4%	62.0%	21.8
	Min ES_{80}	27.0%	14.8%	16.7%	83.7%	34.6
	Min ES_{95}	29.6%	15.1%	16.1%	86.5%	36.2
	50x	53.9%	37.7%	37.7%	110.0%	50.0
	0x	42.4%	63.4%	70.9%	24.9%	0.0
VG	Min std	22.7%	30.5%	38.1%	36.4%	11.7
	Min ES_{80}	43.2%	23.7%	29.3%	59.3%	26.3
	Min ES_{95}	62.1%	24.9%	26.2%	72.3%	34.7
	50x	98.9%	51.5%	54.9%	96.3%	50.0
	0x	37.0%	52.8%	60.0%	18.2%	0.0

Table 4.1. Selling equity (0%-3%) tranche protection: Upfront, hedge, and error measures for different strategies (all numbers in %tranche unless specified)

The ES_{80} optimal strategy concentrates all the trajectories with 0 to 6 defaults in the -20% to 0% P&L range and the trajectories with more than 6 defaults will result in a positive carry. For the STD optimal hedge the situation is similar except that the no-default carry is positive and with 1 or 2 defaults the P&L becomes negative. This kind of "n^{th} to default" analysis may appear attractive for those interested in the CDO trade and the relevant risks. However, this kind of analysis only works when defaults arrive one by one rather than in batches of simultaneous defaults.

More precisely, when defaults arrive one by one, after a certain number of defaults the tranche has defaulted completely, all of the contingent payments are paid, and our hedging position is liquidated and thus, susequent defaults do not affect our P&L. In contrast, if the number of defaults that arrive simultaneously is bigger than what is needed to default the whole tranche, they may result in a positive carry since we make money out of those extra defaults. For example, in the equity tranche, the ES_{80} optimal hedge strategy needs more than 6 defaults to show a positive carry but if these defaults arrive one by one, we will liquidate the position after the tranche has defaulted (6^{th} default) and no more money can be made out of the defaults that arrive after this point. In the same tranche if we have 10 simultaneous defaults, 6 of them will cause the whole tranche to default and the other 4 will only increase portfolio's profit since there are no more contingent payments to be made. If we refuse to liquidate our hedging position after all the contingent payments are made, the optimal price will be usually lower than the liquidation strategy but at higher hedging errors. All the results presented here follow the liquidation strategy but a comparison of results with the no-liquidation strategy can be found in appendix D.

In short, with this set of parameters, in the extreme case of zero hedging strategy, the break-even upfront is low, the no-default carry is high and both risk measures are high. On the other extreme, when we use an extremely high hedge notional (50x), the break-even upfront is high, no-default carry is low, and both risk measures are again high. The optimal hedge notional, measured either by ES or STD, falls somewhere between these two extremes and is determined by the choice of risk measure.

4.2 Selling Mezzanine Tranche Protection

In studying the mezzanine tranche we can see that the risk measures, plotted as functions of price and hedge notional, show less convexity as compared to the equity tranche (Figure 4.5). In this tranche, unlike the equity tranche, both ES and STD optimal hedges result in negative no-default carries (Figure 4.6). In the mezzanine tranche the first few defaults, which make up the majority of cases, are absorbed by the subordinate tranche (equity) and thus, the no-default spike in the zero hedging strategy is much bigger. This relative resilience against defaults means that minimizing ES at low confidence levels may involve little or no hedging at all. This behavior becomes even more apparent as we move on to more senior tranches. In this tranche the price and hedge for ES optimal and STD optimal strategies are very close (table 4.2) while, similar to the equity tranche, the STD optimal strategy still results in a slightly lower price and hedge than the ES optimal strategy (Figure 4.7a).

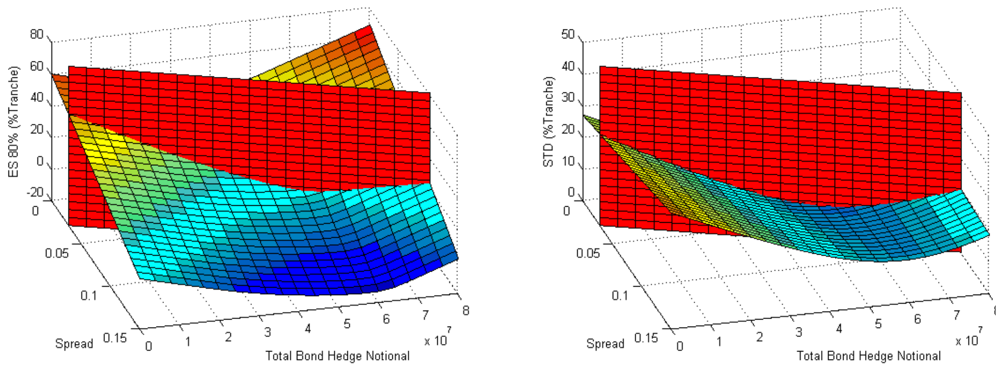


Figure 4.5. Selling mezzanine (3%-7%) tranche protection: Risk measures for diffrernet price and hedge combinations (PNC model)

Figure 4.7a also shows that if a zero hedging strategy is adopted, the difference between ES_{80} and ES_{95} risk measures is huge, while in the equity tranche this difference was relatively small. Again, this is due to the fact that some of the defaults are absorbed by the subordinate tranche and thus, the percentage of no-default cases in the mezzanine tranche is much higher than in the equity tranche; the no-default trajectories make up about 27% of all trajectories in the equity tranche while this number for the mezzanine tranche is about 80%.

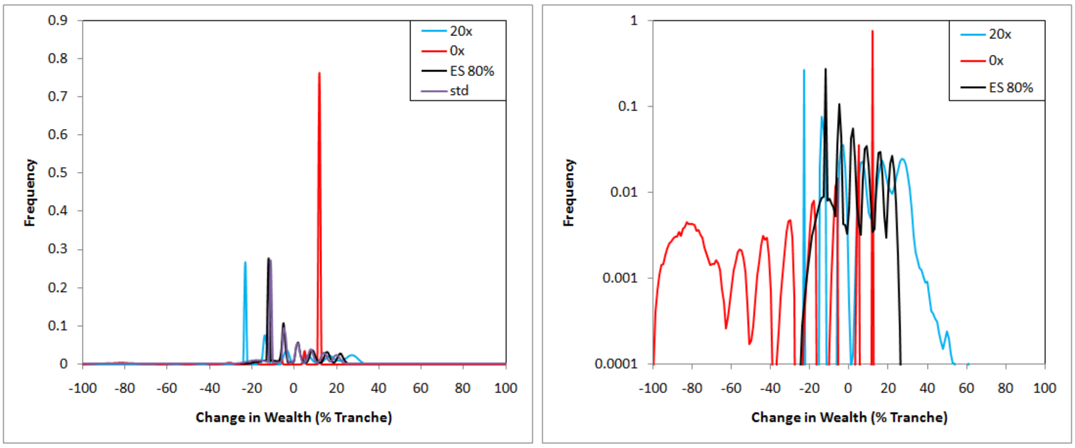
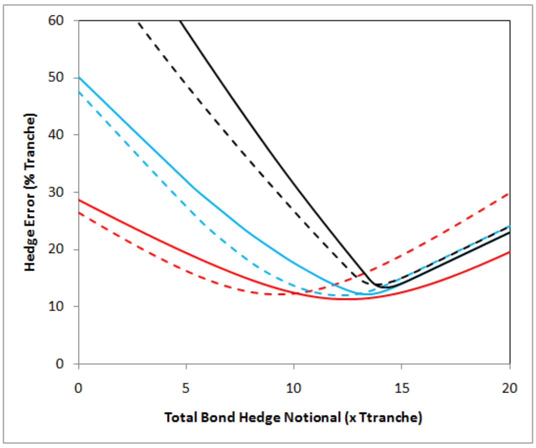


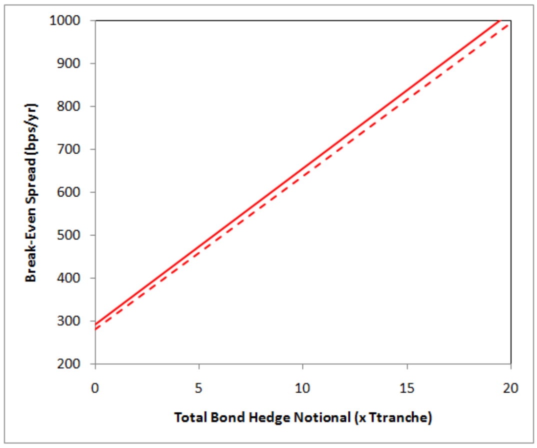
Figure 4.6. Selling mezzanine (3%-7%) tranche protection: P&L distribution (PNC model, bin size = 1% tranche notional)

Model	Strategy	STD	ES_{80}	ES_{95}	spread	Hedge (×tranche)
PNC	Min STD	11.4%	13.6%	20.3%	7.25%	12.4
	Min ES_{80}	11.6%	12.2%	15.3%	7.78%	13.4
	Min ES_{95}	12.0%	12.8%	13.4%	8.09%	14.2
	20x	19.6%	23.0%	23.0%	10.19%	20.0
	0x	28.7%	50.1%	87.2%	2.91%	0.0
VG	Min STD	12.2%	15.3%	30.3%	6.01%	9.2
	Min ES_{80}	14.3%	11.9%	18.0%	7.15%	12.2
	Min ES_{95}	16.8%	13.0%	13.9%	7.72%	13.8
	20x	29.9%	24.1%	24.1%	9.94%	20.0
	0x	26.5%	47.7%	75.5%	2.81%	0.0

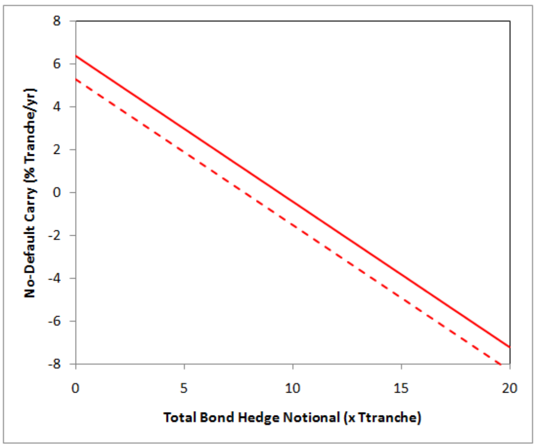
Table 4.2. Selling mezzanine (3%-7%) tranche protection: Spread, hedge, and error measures for different strategies (all numbers in %tranche unless specified)

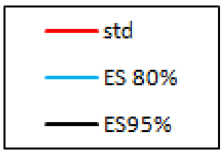
Similar to the equity tranche, the higher the hedge notional, the higher the break-even price of the tranche (in this case the spread) (Figure 4.7b) and the lower the no-default carry (Figure 4.7c). Also like the equity tranche, increasing the price (spread) has a linear shifting effect on the position of the P&L distribution while changing the hedge deforms the distribution; as the hedge increases, the left tail of the zero hedge distribution first approaches to the no-default carry and then it expands to the right side of the no-default carry.





(a) **(b)**





(c)

Figure 4.7. Selling mezzanine (3%-7%) tranche protection (solid lines: PNC, dashed lines: SVG)

In the zero hedging strategy, a minimum of 6 defaults are required to obtain a negative carry. Again, the optimal strategies reverse the zero hedge tail losses and concentrate them in the center of the distribution and in the positive region. Both ES_{80}

optimal and STD optimal strategies have a negative carry for 0 and 1 default while 2 defaults or more result in a positive carry.

4.3 Selling Senior Tranche Protection

The behavior of the senior tranche is very similar to the mezzanine tranche in the sense that it is quite resilient to defaults and the risk measures show less convexity with regard to price and hedge (Figure 4.8). Again, both ES optimal and STD optimal hedging strategies result in negative no-default carries (Figure 4.9).

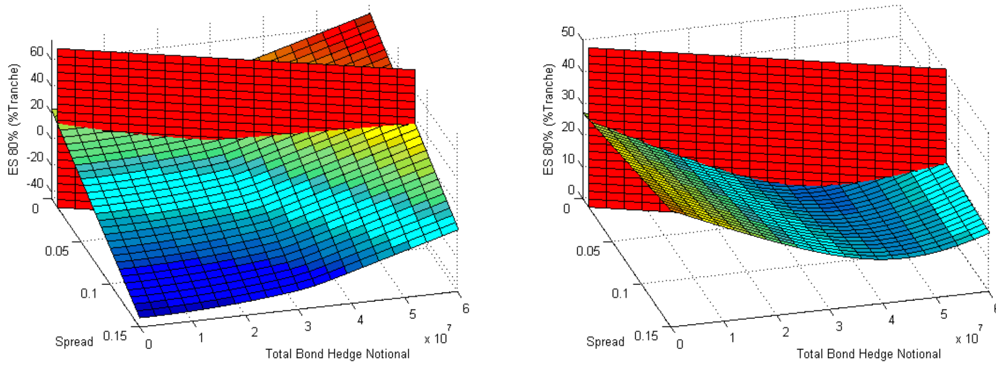


Figure 4.8. Selling senior (7%-10%) tranche protection: Risk measures for diffrernet price and hedge combinations (PNC model)

The senior tranche's resilience against defaults is even greater than the mezzanine tranche; about 93% of the trajectories result in a default-free senior tranche. It takes at least 13 defaults in the reference pool for the senior tranche to show a negative carry.

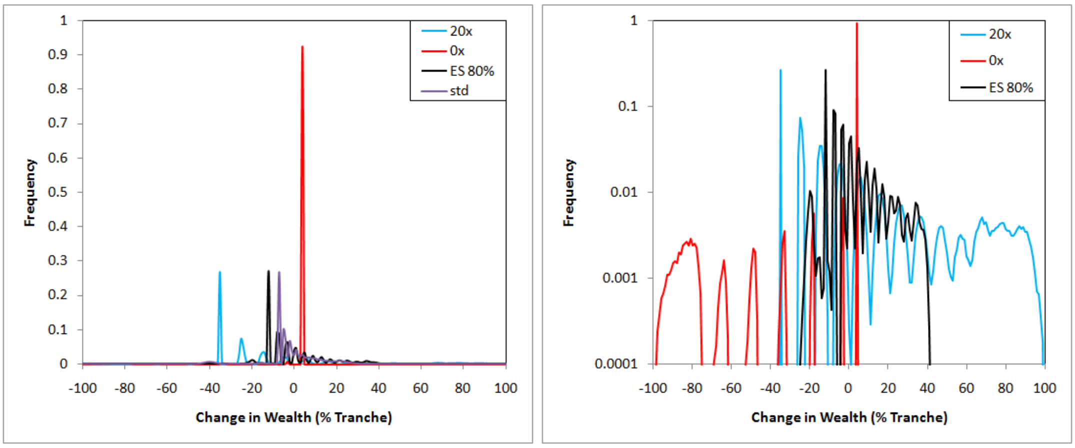
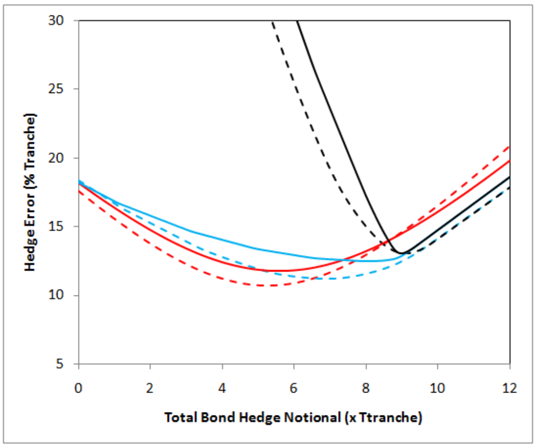


Figure 4.9. Selling senior (7%-10%) tranche protection: P&L distribution (PNC model, bin size = 1% tranche notional)

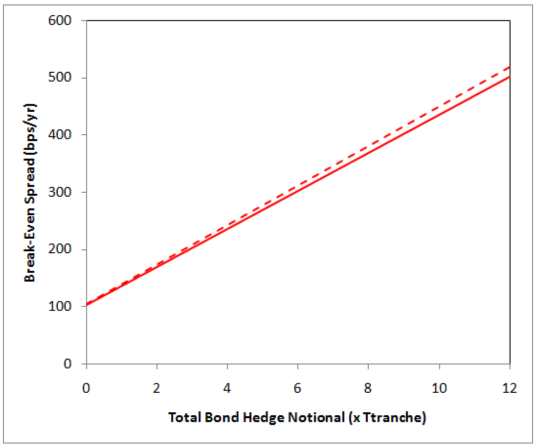
Model	Strategy	std	ES_{80}	ES_{95}	spread	Hedge (×tranche)
PNC	Min std	11.7%	13.7%	34.1%	2.74%	5.6
	Min ES_{80}	13.3%	12.5%	16.9%	3.71%	8.1
	Min ES_{95}	14.5%	12.9%	13.0%	4.02%	9.0
	20x	37.6%	34.1%	34.1%	7.68%	20.0
	0x	18.2%	18.2%	76.3%	1.03%	0.0
VG	Min std	10.7%	11.9%	31.0%	2.80%	5.3
	Min ES_{80}	11.6%	11.2%	19.6%	3.44%	6.9
	Min ES_{95}	14.8%	12.6%	13.1%	4.18%	9.1
	20x	40.4%	32.9%	32.9%	7.95%	20.0
	0x	17.6%	18.4%	73.4%	1.04%	0.0

Table 4.3. Selling senior (7%-10%) tranche protection: Spread, hedge, and error measures for different strategies (all numbers in %tranche unless specified)

In Figure 4.10a we can see a continuation of the trend that we witnessed in the mezzanine tranche with regard to big differences between ES_{80} and ES_{95} for the zero hedging strategy. Obviously, since the senior tranche experiences even fewer defaults than the mezzanine tranche, this difference in expected shortfalls is also greater. We can see that in the senior tranche, hedging the tail losses at high levels of confidence is much more efficient than in the equity tranche. That is, in the equity tranche by increasing the hedge notional from zero to \$100M (33x tranche) the ES_{80} decrease from 70% to 16%, whereas in the senior tranche we need only a \$24M hedge notional (8x tranche) to decrease ES_{80} from 76% to 17%. This increase in hedging efficiency is a continuous trend as we move from subordinate tranches to the more senior ones.



(a)



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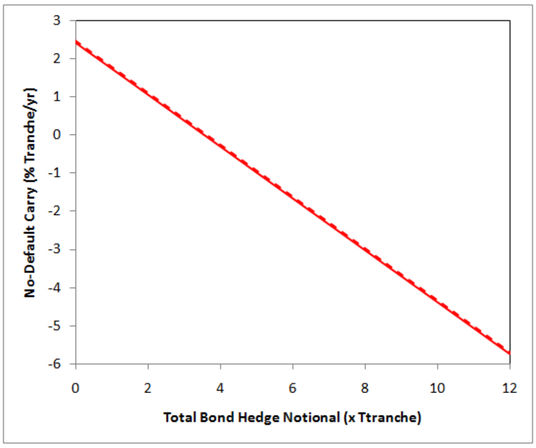
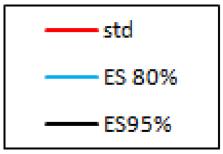




(c)

Figure 4.10. Selling senior (7%-10%) tranche protection (solid lines: PNC, dashed lines: SVG)

Increasing the hedge notional in the senior tranche has the same effects as for the equity and mezzanine tranches; as we increase the hedge notional the break-even spread increases and the no-default carry decreases (Figure 4.10b ad 4.10c).

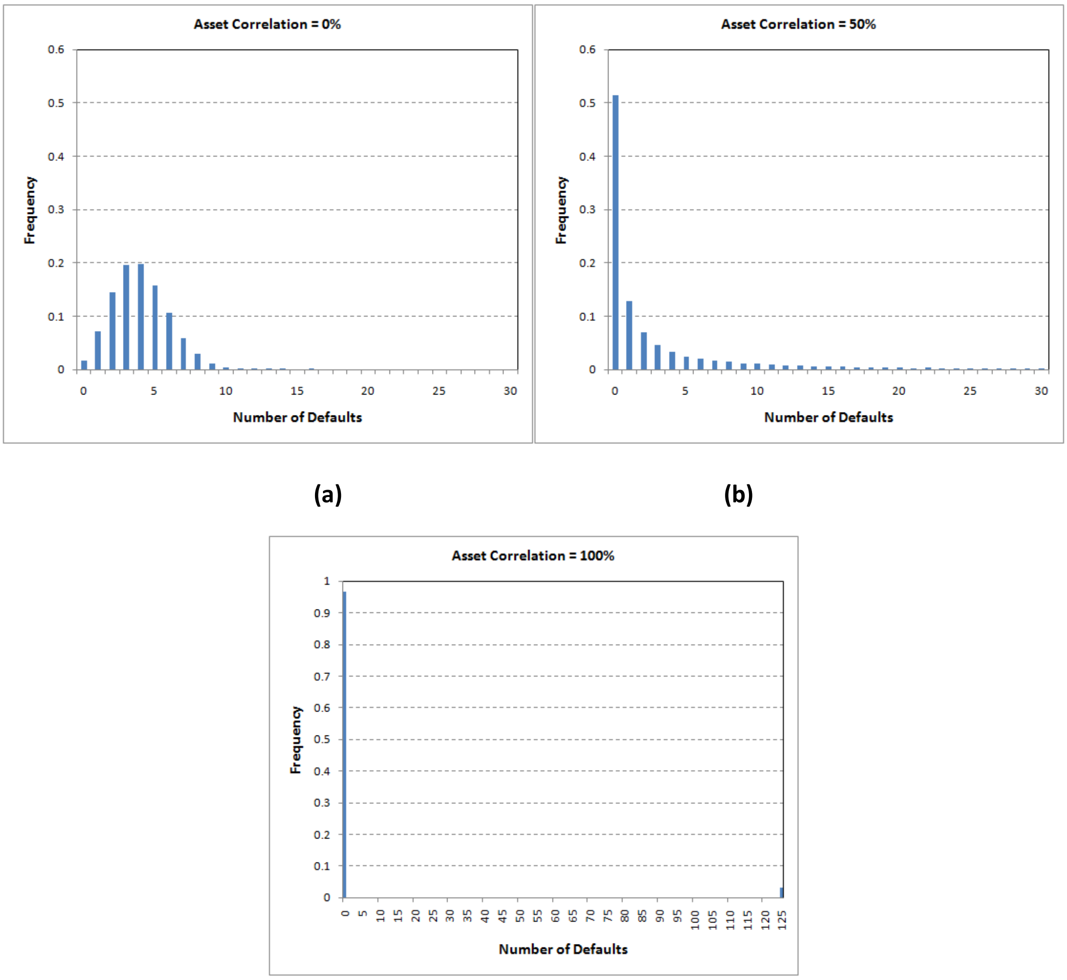
4.4 Sensitivity Tests

Until now we have calculated the optimal price and hedge for each tranche given the parameters presented at the beginning of this section. We have also studied how increasing the hedge notional would affect different aspects of our trade including the no-default carry, tail losses, and STD. Now we are interested in examining how sensitive these results are with regard to changes in these key parameters.

4.4.1. Sensitivity to Hazard Rate and Asset Correlation

As we discussed earlier, asset correlations and the hazard rates control the distribution of defaults in the PNC model. Hazard rate dictates the average number of defaults while asset correlation only affects their standard deviation. Figure 4.11 shows how different asset correlations change the shape of defaults distribution.

When the asset correlation is zero we expect the default arrivals to be independent from one another and indeed this is the case as we can see in figure 4.11a where the defaults have an almost normal distribution and in more than 99% of the trajectories there are less than 10 defaults. As we increase the asset correlation to 50% (figure 4.11b), we see a very different picture: variance, skewness, and kurtosis increase dramatically and we start to see some cases with big numbers of defaults. At an asset correlation equal to 100% (figure 4.11c), the default distribution becomes a binary case where either there is no default or all of the bonds default. In short, increasing the asset correlation reduces the number of defaults around the center of distribution and pushes the defaults towards the tails of distribution (zero defaults or 125 defaults).

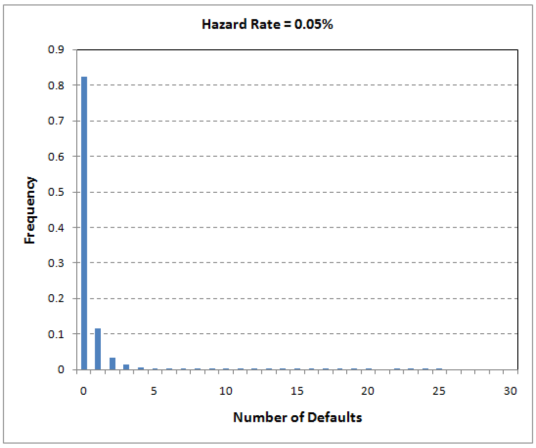


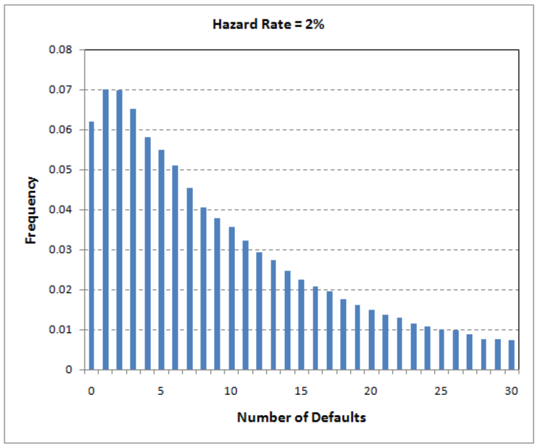
(a) **(b)**

(c)

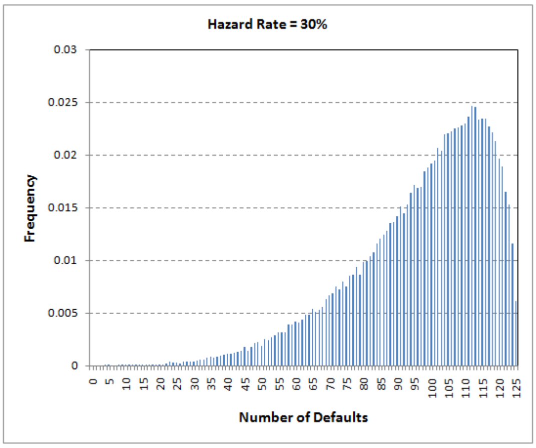
Figure 4.11. Distribution of number of defaults for different asset correlations (hazard rate = 0.65%/yr)

At low hazard rates most of the trajectories are default free or have few defaults (figure 4.12a). This concentration of defaults at the very left side of the distribution translates into low standard deviations. As we increase the hazard rate to 2%/yr, the no-default cases diminish and we witness more cases with big numbers of defaults (figure 4.12b). At this rate the distribution is becoming more and more flat since there is a decrease in concentration of defaults at the left side. This means that in addition to an increase in number of defaults, increasing the hazard rate would also result in higher standard deviations. However, this is not a monotonic trend as at very high hazard rates we expect to see a reduction in standard deviation since the situation has reversed with regard to concentration of defaults; this time the defaults are concentrated at the very right end of the distribution (figure 4.12c).





(a) **(b)**



(c)

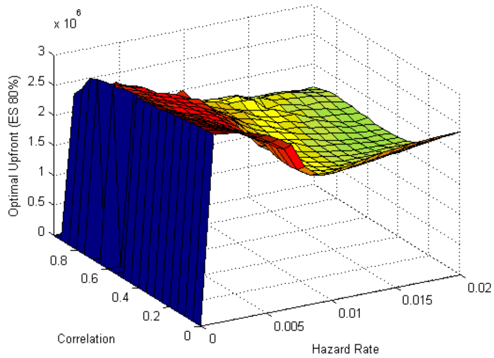
Figure 4.12. Distribution of number of defaults for different hazard rates (asset correlation = 25%)

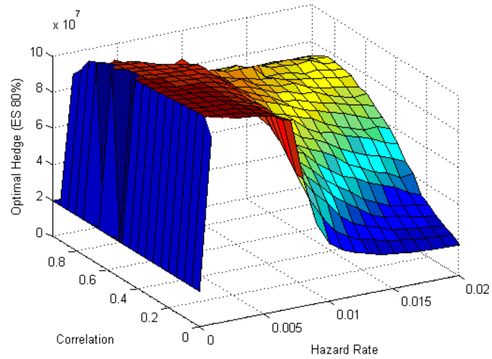
With this insight about the effects of asset correlation and hazard rate on the defaults distribution, we can now analyze how each tranche would react to changes in these parameters.

Selling Equity Protection

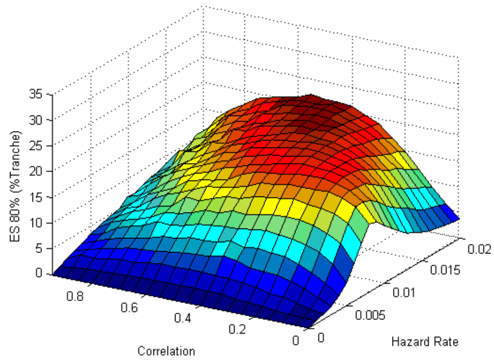
Figure 4.13a shows that, if expected shortfall is taken as our measure of risk, at every asset correlation the price is almost surely a decreasing function of the hazard rate. To understand the reason behind this we should remember the "n^{th} to default" analysis that was presented earlier and stated that if a big number of defaults arrive simultaneously, our hedging position becomes very profitable as it pays much more than the outgoing contingent payments.

At higher hazard rates the average number of defaults increases and consequently our hedging position catches more defaults and becomes more profitable, hence a lower upfront would be needed. This increase in efficiency of our hedging position can be seen in figure 4.13b where the optimal hedge notional is a decreasing function of hazard rate at every asset correlation. The decrease in price and hedge notional is much more abrupt at low asset correlations. Remember that low correlations result in a bell curve shaped default distribution and as we increase the hazard rate the center of this bell curve moves to the right. This means that at high hazard rates and low correlation the average number of defaults is high and their standard deviation is low, resulting in a very profitable hedging position that in most cases catches high numbers of simultaneous defaults. In contrast, at high hazard rates and high correlations the two tails of the distribution are fatter and consequently we have more trajectories with extremely low or extremely high numbers of defaults while there are fewer trajectories with moderately high number of defaults. This means that at higher correlations one would need a greater hedge notional in order to make enough profit out of the many trajectories with lower number of defaults.





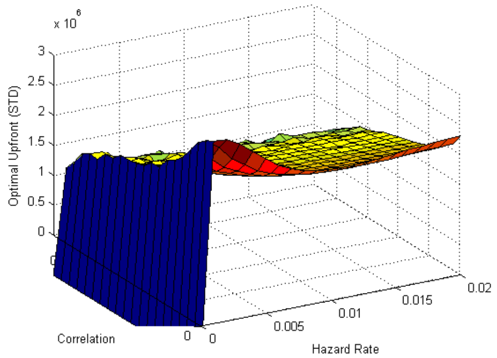
(a) **(b)**



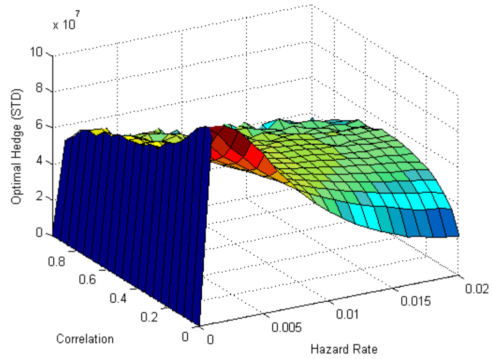
(c)

Figure 4.13. Selling equity (0%-3%) tranche protection: ES80 optimal strategy for different asset correlations and hazard rates (recovery rate = 30%)

The same analysis is useful in interpreting the minimum expected shortfall achieved at each of these points (figure 4.13c). At low correlations the minimum ES increases with the hazard rate until the hedging position starts to catch more defaults than what is needed to compensate for the contingent payments. This is where the optimal ES reaches its maximum and then it becomes a decreasing function of hazard rate. As the asset correlation increases, this maximum is achieved at higher and higher hazard rates since trajectories with extremely low numbers of defaults become more frequent. As the asset correlation approaches 100%, our defaults distribution shows a large number of no-default trajectories and a very small number of trajectories with extremely high default rates. This is where our hedging position makes huge profits in case of defaults and thus the ES diminishes.



(a)



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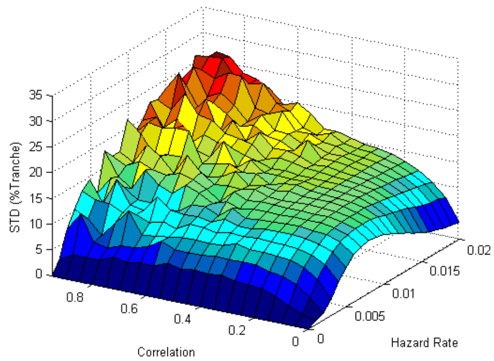


(c)

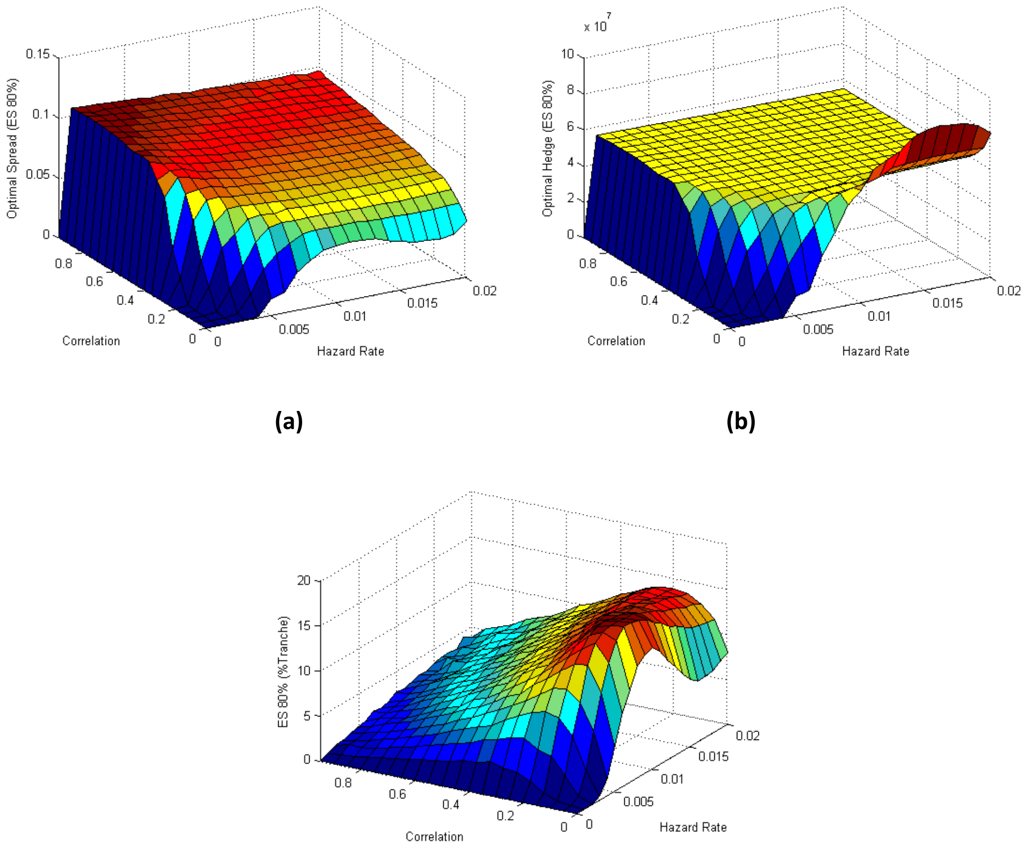
Figure 4.14. Selling equity (0%-3%) tranche protection: STD optimal strategy for different asset correlations and hazard rates (recovery rate = 30%)

If we choose standard deviation as our risk measure, the upfront decreases faster as the default correlation increase as compared to the ES optimal strategy (figure 4.14a). At every hazard rate higher correlations result in higher standard deviations which cannot be hedged away (figure 4.14c). Again, the optimal hedge decrease faster at lower correlations because of the increase in hedging efficiency. Figure 4.14c also shows that at low correlations the standard deviation initially increases with the hazard rate and after reaching its maximum it diminishes again while at very high asset correlations the STD is only increasing. This happens due to the increase in defaults' variance at higher correlations which we discussed at the beginning of this section.

Selling Mezzanine Protection

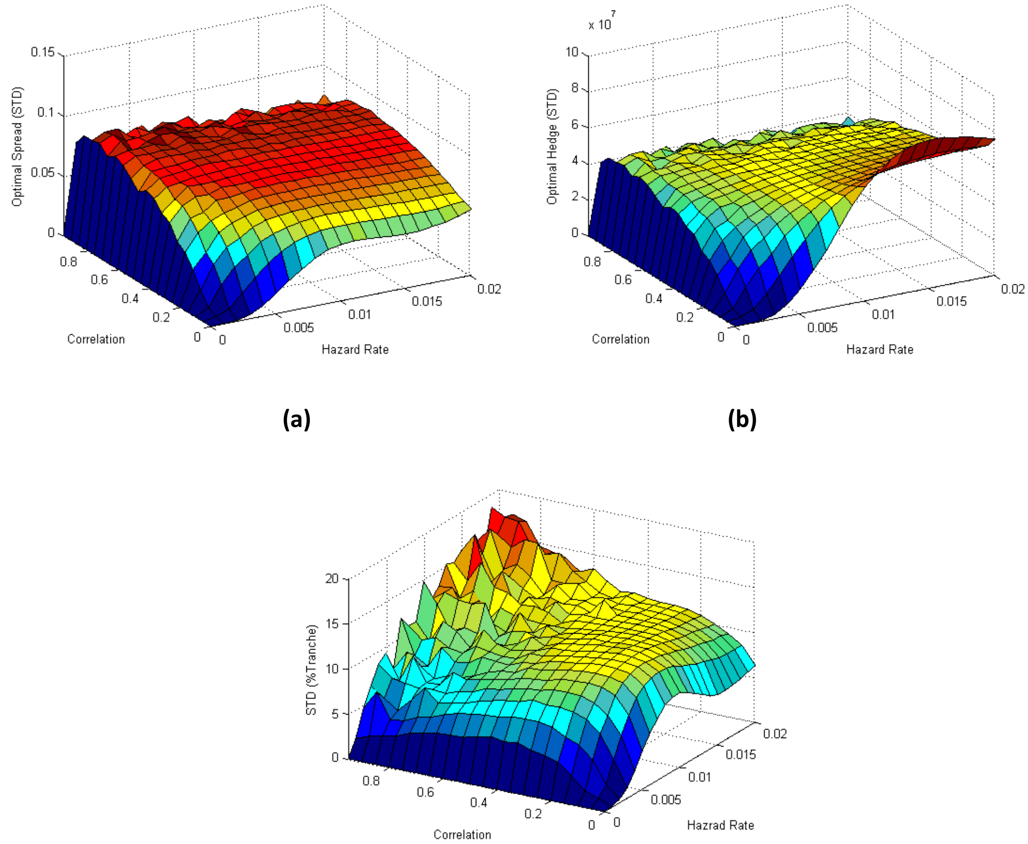
The mezzanine tranche is quite different from the equity tranche since it is immune to the first few defaults in the reference pool. This means that at low correlations and low hazard rates this tranche does not experience any defaults and consequently, very low spreads and hedge notional are needed (figures 4.15a, 4.15b). Also, the expected shortfall is very low in this region (figure 4.15c). In the mezzanine tranche the price (spread) is a strictly increasing function of correlation because as the correlation increases, there is higher probability that the defaults arrive in big batches and surpass the defense barrier of equity tranche.

Figure 4.15b shows how at low correlations the optimal hedge notional increases with hazard rate while at higher correlations this hedge notional is constant after an initial jump. This is because at low correlations the defaults have a bell curve shaped distribution which moves to the right as the hazard rate increases. As the main bulk of the distribution reaches the point where the average number of defaults is enough to hit the mezzanine tranche, we would need a higher hedge notional to compensate for the outgoing contingent payments. But at higher asset correlations the trajectories which contain enough defaults to hit the mezzanine tranche are already spread to the right and left sides of the distribution and hence, even at lower hazard rates we have many trajectories where the defaults do reach the mezzanine tranche. So, as the correlation increases, our hedging position becomes very efficient in countering the effect of defaults and the optimal expected shortfall decreases (figure 4.15c). In short, low variance in number of defaults (low correlation) results in high outgoing contingent payments and moderate hedging profits which in return result in high expected shortfalls. In contrast, high variance in number of defaults (high correlation) results in the same average amount of outgoing contingent payments while the hedging profit is also very high because of the simultaneous arrival of many defaults an thus, the expected shortfall is low.



(a) (b) (c)

Figure 4.15. Selling mezzanine (3%-7%) tranche protection: ES80 optimal strategy for different asset correlations and hazard rates (recovery rate = 30%)



(a) **(b)**

(c)

Figure 4.16. Selling mezzanine (3%-7%) tranche protection: STD optimal strategy for different asset correlations and hazard rates (recovery rate = 30%)

In the STD optimal strategy, the same logic applies to price and hedge notional (figures 4.16a, 4.16b) while the hedging errors show a different picture. Increasing the correlation initially increases the standard deviation since the now fatter right tail starts to hit the mezzanine tranche and cause outgoing contingent payments. But as the correlation increases further, the right tail becomes even fatter and most of the trajectories fall in the range where they cause both outgoing contingent payments and incoming hedging profits. At extremely high correlations there are many no-default trajectories and a few trajectories with extremely high numbers of defaults and, as a result, we will have a binary distribution where for the no-default trajectories we pay a high hedging cost and no contingent payments while in the mass default cases we receive a huge profit out of our hedging position.

Selling Senior Protection

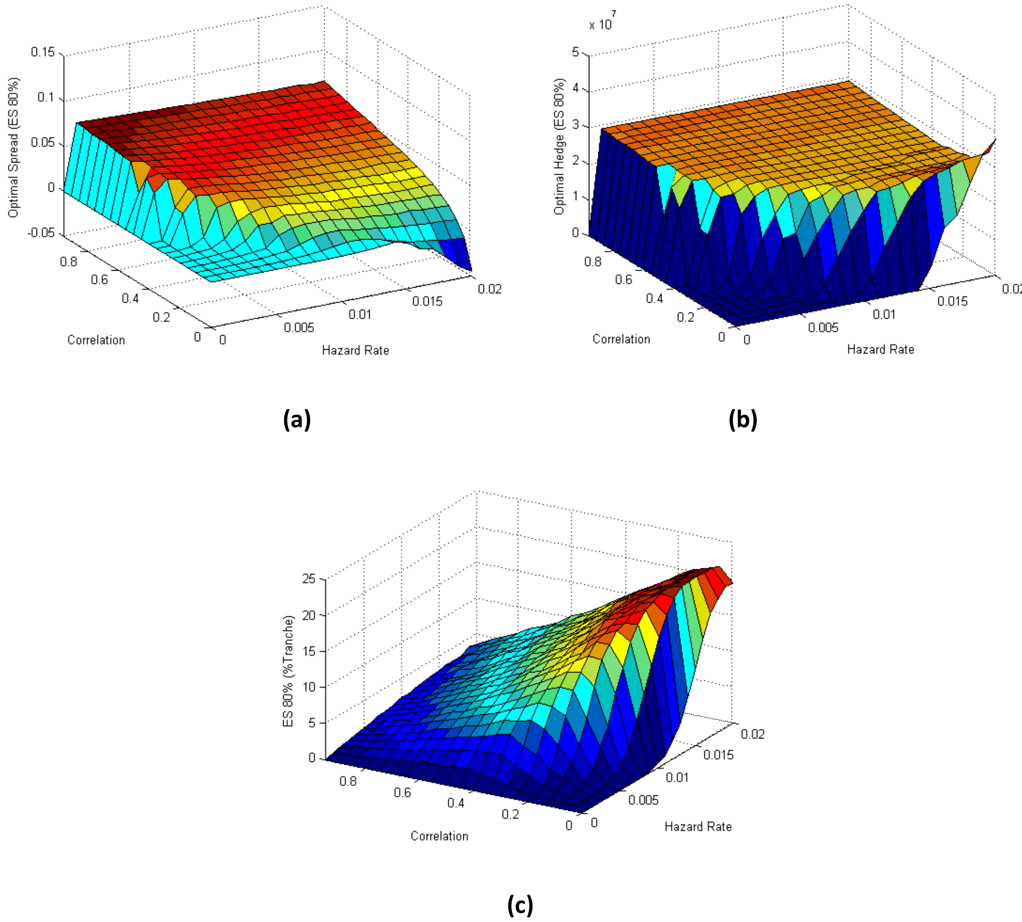
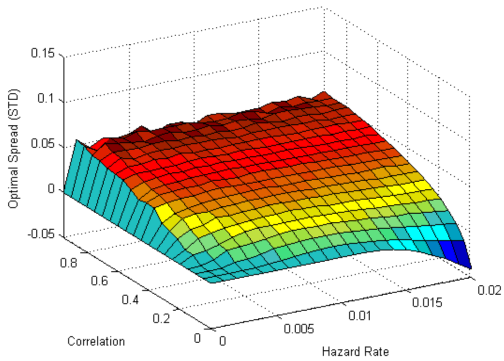
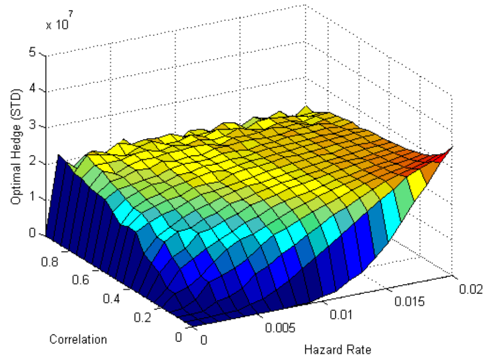


Figure 4.17. Selling senior (7%-10%) tranche protection: ES80 optimal strategy for different asset correlations and hazard rates (recovery rate = 30%)

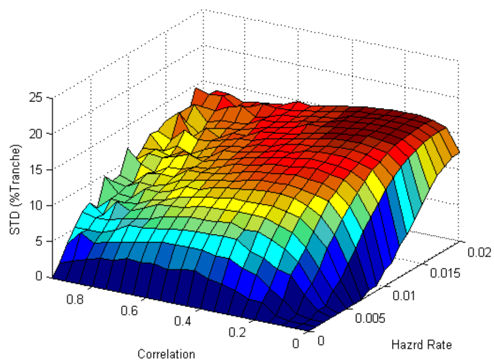
Senior tranche basically follows the same rules that govern the mezzanine tranche with the difference that we start to see a rise in the price and hedge notional at higher hazard rates and asset correlations (figures 4.17a, 4.17b and 4.18a, 4.18b), which is again due to relative resilience of the senior tranche against the first default arrivals. Also, the optimal expected shortfall behaves very similar to the mezzanine tranche (figure 4.17c) as the same dynamics apply. We can also see that standard deviation shows the same characteristics of the mezzanine tranche except that at low hazard rates increasing the correlation would not result in a rise in variance (figure 4.18c). This is of course because the defaults start to hit the senior tranche at higher hazard rates.



(a)



(b)



(c)

Figure 18. Selling senior (7%-10%) tranche protection: STD optimal strategy for different asset correlations and hazard rates (recovery rate = 30%)

Figures 4.17a and 4.18 a show that at low correlations and high hazard rates the price may become negative, that is, we should pay a spread (rather than receiving it) to make it a fair bet. The explanation for this is that at high hazard rates and low correlations the average number of defaults is high but they are concentrated around their mean which is still behind the number of defaults needed to hit the senior tranche. As a result, the hedging position is making a profit out of defaults without us paying contingent payments. However, as correlation increases, the right tail becomes fatter and some of the defaults hit the senior tranche, resulting in an increase in price and decrease in hedge notional.

4.4.2. Sensitivity to Recovery Rate

Another key parameter that impacts our results is the recovery rate. Figure 4.19 shows the effects of different recovery rates on the results obtained for the equity tranche. As the recovery rate increases, the cost of hedging increases since we are paying the same coupons on the underlying bonds while the hedging profit is decreasing. This initially results in higher prices but as the recovery rate approaches 100%, the losses due to defaults approach zero and the tranche spread exceeds our losses to the point that at a 100% recovery rate one would find it profitable to pay an upfront (rather than receiving it) in return for receiving the tranche spreads.

Similarly, at very high recovery rates the optimal hedge notional decreases as our contingent payments are decreasing. Note that both ES and STD are decreasing functions of recovery rate.

In the mezzanine and senior tranches (figures 4.20 and 4.21), we are not receiving a fixed spread. So, increasing the recovery rate only decreases the spread, hedge notional, and hedging errors. However, these reductions start to appear at lower recovery rates compared to the equity tranche. This is due to absorption of the first few defaults by the equity tranche and the profit that our hedging position makes before the defaults hit these senior tranches. Alternatively stated, at higher recovery rates it takes higher numbers of defaults in the reference pool to affect the more senior tranches and consequently the probability of defaults in the more senior tranches becomes even greater. As a result, one would need a lower price and hedge notional to compensate for the contingent payments. The more senior the tranche, the lower the recovery rate needed to be in the safe zone.

The hedging errors for mezzanine and senior tranches are quite different from the equity tranche in the sense that their rate of decline in the face of increasing recovery rates is less than in the equity tranche, which is again due to the absorption of first defaults by the equity tranche.

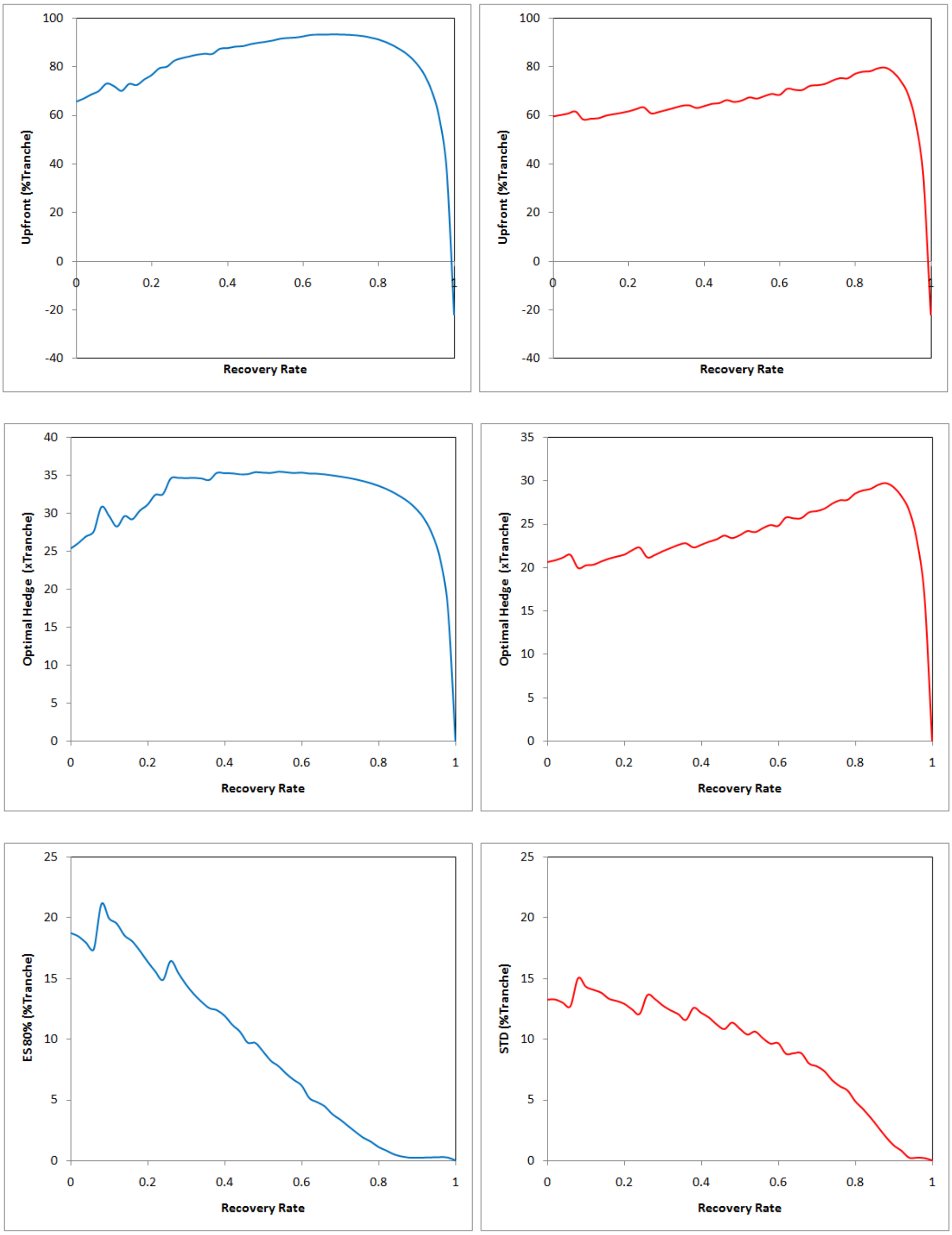


Figure 4.19. . Selling equity (0%-3%) tranche protection at different recovery rates (left column: ES80 optimal strategy, right column: STD optimal strategy, PNC model, asset correlation = 25%, hazard rate = 0.65%)

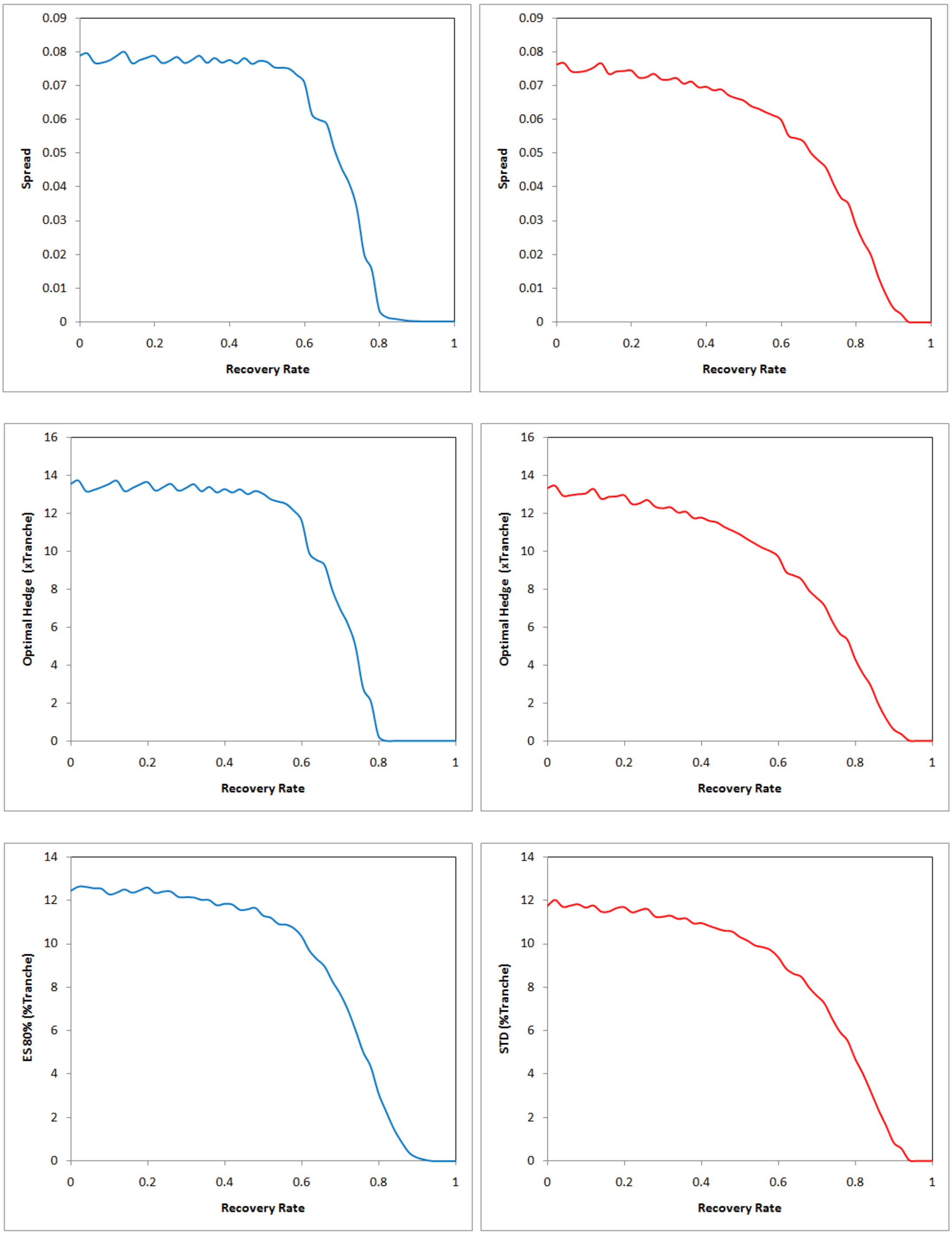


Figure 4.20. Selling mezzanine (3%-7%) tranche protection at different recovery rates (left column: ES80 optimal strategy, right column: STD optimal strategy, PNC model, asset correlation = 25%, hazard rate = 0.65%)

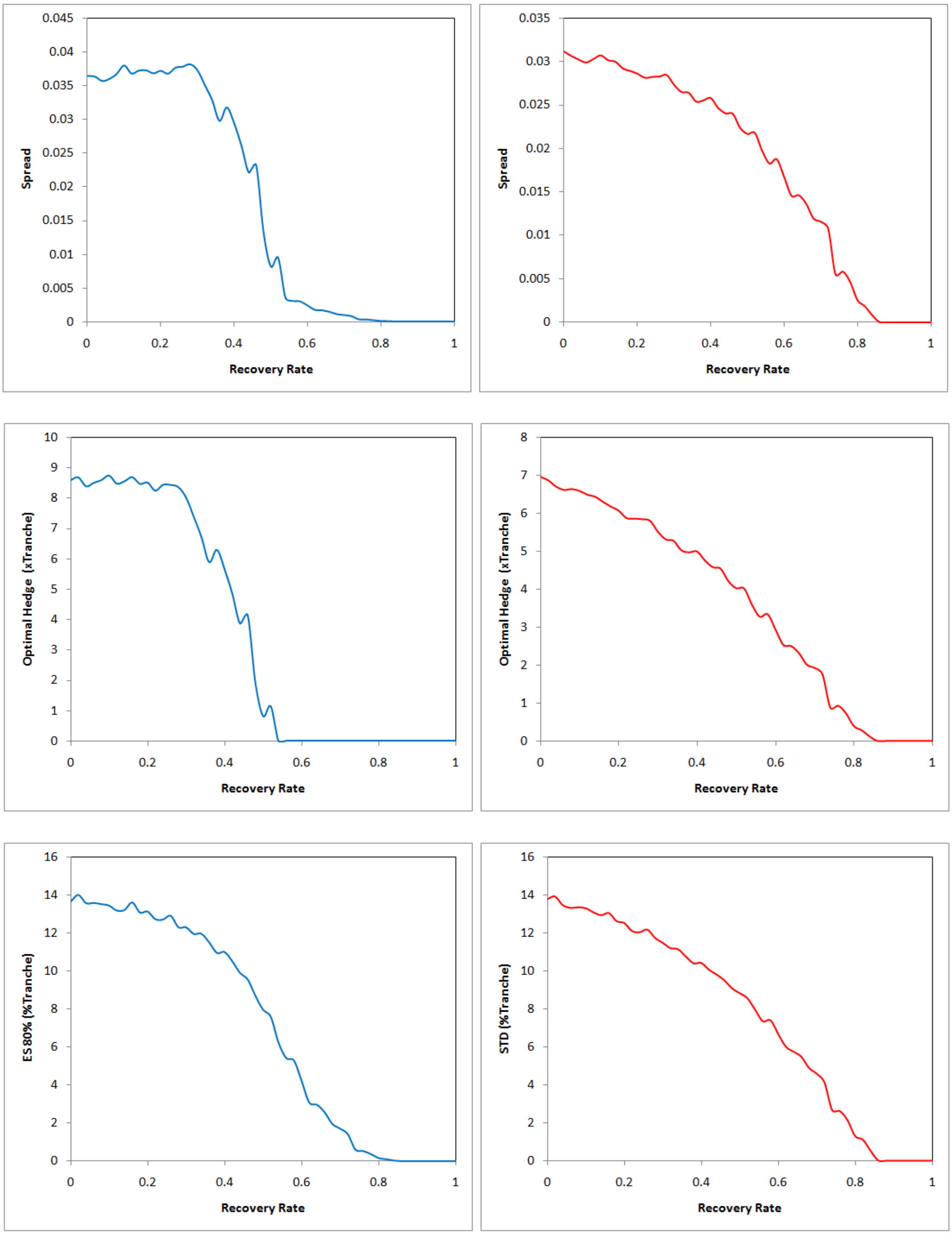


Figure 4.21. Selling senior (7%-10%) tranche protection at different recovery rates (left column: ES80 optimal strategy, right column: STD optimal strategy, PNC model, asset correlation = 25%, hazard rate = 0.65%)

5. Conclusion

We created a practical framework for analyzing some of the most important aspects of a typical CDO trade such as P&L distribution, optimal hedging strategies given a measure of risk, and sensitivity of these CDO trades and their pertaining hedging strategies to changes in market factors. This simulation based approach helps CDO traders to analyze their target CDOs and their potential positions in different tranches of those CDOs.

The default models presented in this work can be adjusted to any CDO given the right asset pool information such as hazard rates, asset correlations, or volatilities. This choice of default model has a profound effect on the pricing and related hedging strategies for a CDO trader. We showed that the return distributions of CDO trades are far from normal and resemble barrier type options. Therefore, our choice of measure of risk becomes another decisive factor in determining the optimal hedging strategy since only coherent measures of risk are capable of correctly capturing the risks of a CDO trade. We demonstrate that using the variance as measure of risk will result in neglecting the huge tail losses that are very common in CDO trades while using expected shortfall as our measure of risk results in high cost of hedging.

The sensitivity tests conducted in this work provide us with further insight regarding the resilience of our trade positions in the face of changes in market factors as well as the profitability of such trades. These scenario tests offer valuable information to the CDO traders regarding the severity of their losses in case there is a change in the base factors and can help them better manage their risk taking practices.

The fact that all these results are obtained through simulation means that this framework can be easily modified and augmented to meet the assumptions and needs of almost any CDO trader and to assess alternative scenarios that one deems to be relevant.

Appendix A - Measures of Risk

Value at Risk (VaR)

VaR is generally defined as "possible maximum loss over a given holding period within a fixed confidence level." That is, mathematically, VaR at the *100(1 – α)* percent confidence level is defined as the lower *100α* percentile of the profit-loss distribution. Artzner *et al.* (1999) define VaR at the *100(1 – α)* percent confidence level ($VaR_\alpha(X)$) as

$$VaR_\alpha(X) = -\inf\{x | P[X \leq x] > \alpha\}$$

When the returns follow a normal distribution:

$$Var_\alpha(X) = q \times \sigma$$

Where q is the upper *100α* percentile of the normal distribution.

Where X is the profit-loss of a given portfolio, $\inf\{x | A\}$ is the lower limit of x given event A, and $\inf\{x | P[X \leq x] > \alpha\}$ indicates the lower 100α percentile of profit-loss distribution. This definition can be applied to discrete profit-loss distributions as well as to continuous ones. Since the loss is defined to be negative (profit positive), –1 is multiplied to obtain a positive VaR number when one incurs a loss within a given confidence interval.

Using this definition, VaR can be negative when no loss is incurred within the confidence interval because the 100α percentile is positive in this case.

Expected Shortfall (ES)

Artzner et al. (1997) have proposed the use of expected shortfall (also called "conditional VaR," "mean excess loss," "beyond VaR," or "tail VaR") to alleviate the problems inherent in VaR. Expected shortfall is the conditional expectation of loss given that the loss is beyond the VaR level (Figure A.1). The expected shortfall is defined as follows:

Suppose X is a random variable denoting the profit-loss of a given portfolio and $VaR_\alpha(X)$ is the VaR at the $100(1-\alpha)$ percent confidence level. $ES_\alpha(X)$ is defined by the following equation:

$$ES_\alpha(X) = E[-X \mid -X \geq VaR_\alpha(X)]$$

And when the returns follow a normal distribution:

$$ES_\alpha(X) = \frac{e^{-\frac{-q^2}{2}}}{\alpha\sqrt{2\pi}}\sigma_x$$

When the profit-loss distribution is normal, expected shortfall and VaR are scalar multiples of each other, because they are scalar multiples of the standard deviation. Therefore, VaR provides the same information about the tail loss as does expected shortfall.

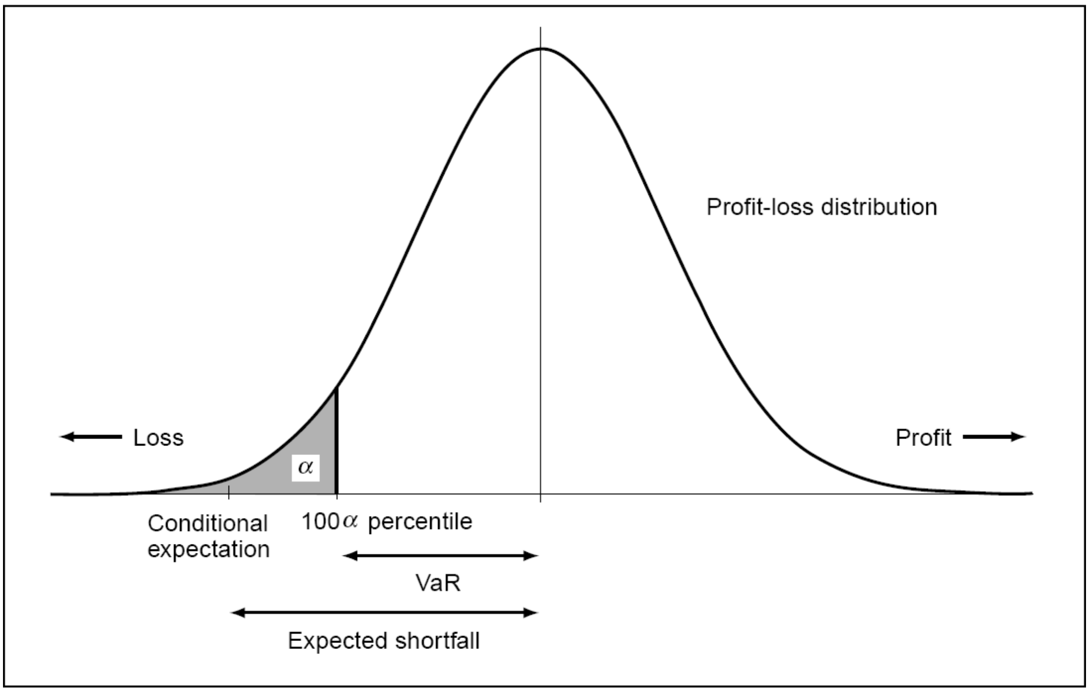


Figure A.1

Coherent Measures of Risk

Artzner *et al.* (1998) argues that a measure of risk can only be coherent if it satisfies the following criteria: Consider a set V of real-valued random variables. A function $\rho : V \rightarrow R$ is called a coherent risk measure if it is

(i) Monotonous: $X \in V,\ X \geq Y \Rightarrow \rho(X) \leq \rho(Y)$,

(ii) Sub-Additive: $X, Y, X + Y \in V \Rightarrow \rho(X + Y) \leq \rho(X) + \rho(Y)$,

(iii) Positively homogeneous: $X \in V, h > 0, hX \in V \Rightarrow \rho(hX) = h\,\rho(X)$, *and*

(iv) Translation invariant: $X \in V, a \in R \Rightarrow \rho(X + a) = \rho(X) - a$.

They show in their work that VaR is not coherent because it is not sub-additive while Expected Shortfall is a coherent measure of risk.

Non-Normality and Problems of VaR

When the profit-loss distribution is normal, VaR does not have the problems pointed out by Artzner *et al.* (1997). First, with the normality assumption, VaR does not have the problem of tail risk. When the profit-loss distribution is normal, expected shortfall and VaR are scalar multiples of each other, because they are scalar multiples of the standard deviation. Therefore, VaR provides the same information about the tail loss as does expected shortfall.

Second, sub-additivity of VaR can be shown as follows. Suppose that there are two portfolios whose profit-loss obeys multivariate normal distribution. With the normality assumption, as we mentioned earlier, VaR is a scalar multiple of the standard deviation, which satisfies sub-additivity. Thus, VaR also satisfies sub-additivity.

Therefore, with the normality assumption, expected shortfall has no advantage over VaR, since VaR satisfies sub-additivity and provides the same information about the tail loss as does expected shortfall. In fact, in this case the ES becomes only a more conservative choice compared to VaR.

However, when the profit-loss distribution is not normal we can easily show that VaR is not coherent because it does not satisfy the sub-additivity criteria. We use the same examples of Artzner et al. to show this shortcoming of VaR.

Example 1: Short position on digital options

Consider the following two digital options on a stock, with the same exercise date *T*. The first option denoted by *A* (initial premium *u*) pays 1,000 if the value of the stock at time *T* is more than a given *U*, and nothing otherwise. The second option denoted by *B* (initial premium *l*) pays 1,000 if the value of the stock at time *T* is less than *L* (with $L < U$), and nothing otherwise. Since the payoffs of those options are not linear, it is clear that the profit-loss distributions are not normal even though the price of the underlying assets obeys normal distribution.

Stock price	Probability (percent)	Option A	Option B	Option A+B
S	*0.8*	u	$-1000 + l$	$-1000 + l$
$L \leq S \leq U$	*98.4*	u	l	$u + l$
$U < S$	*0.8*	$-1000 - u$	l	$-1000 + u + l$
VaR		$-u$	$-l$	$-1000 - u - l$

Table A.1 Payoff and VaR of Digital Options

Suppose *L* and *U* are chosen such that $\Pr(S < L) = \Pr(S > U) = 0.008$, where *ST* is the stock price at time *T*. Consider two traders, trader A and trader B, writing one unit of option *A* and option *B*, respectively. VaR at the 99% confidence level of trader A is –*u*, because the probability that *ST* is more than *U* is 0.8%, which is beyond the confidence level. Similarly, VaR at the 99% confidence level of trader B is $-l$. This is a clear example of the tail risk. VaR disregards the loss of options A and B, because the probability of the loss is less than one minus the confidence level.

Now consider the combined position on options A and B to show that VaR is not sub-additive. VaR at the 99% confidence level of this combined position (option A plus option B) is $1{,}000 - u - l$, because the probability that ST is more than U or less than L is 0.016, which is more than one minus the confidence level (0.01). Therefore, since the sum of VaR of individual positions (option A and B) is $-u - l$, it is clear that VaR is not sub-additive (Table A.1).

Example 2: Concentrated credit portfolio

Suppose that there are 100 corporate bonds, all with the same maturity of one year. Also suppose that all bonds have a coupon rate of 2%, a yield-to-maturity of 2%, a default probability of 1%, and a recovery rate of zero. Furthermore, it is assumed that the occurrences of defaults are mutually independent.

First, we consider investing US$1 million into 100 corporate bonds, each with an equal amount of US$10,000. The default of only one bond does not lead to a loss since the net profit is: 99*200 – 10,000 = 9,800. However, if two bond default, the net profit is: 98*200 – 20,000 = -400. The probability of 2 bonds or more defaulting is 26.4% (1 – the probability that all bonds do not default – the probability that only one bond defaults = $1-0.99^{100}-100\times0.99^{99}\times0.01$). Thus, for this diversified investment, VaR at the 95% confidence level is positive since the probability of loss is more than 5%.

Second, we consider investing US$1 million into only one of those corporate bonds. For this concentrated investment, we are 95% sure that this investment will earn US$20,000, because the default probability is 1%. Therefore, VaR at the 95% confidence level is –US$20,000. This exemplifies the tail risk of VaR, since VaR disregards the potential loss of default. Furthermore, VaR is not sub-additive, because the VaR of the diversified portfolio is larger than the VaR of the concentrated portfolio.

Our CDO trade has a non-normal distribution which closely resembles a digital option and these two examples show why we need to be very careful in choosing our measure of risk and make sure that a coherent one is being applied. For all these reasons, we decide to use expected shortfall as our second measure of risk in this work.

Appendix B - Hedged Monte Carlo

Notations

"We suppose that the price of the option only depends on the current price (x_k) of the asset and call it $C_k(x_k)$ at time t_k The interest rate is assumed to be constant and equal to r. Averaging (denoted by angled brackets $\langle\ldots\rangle$) will in the following always refer to the *objective* (real world) probability measure under which we observe the distribution of the asset returns and *not* any abstract risk neutral measure."

Principles

The implementation of this HMC method requires working backward in time. It means that we start by the final pay-off function, which is known at time T, and calculate the option price and the appropriate hedge for the previous period of time *(T-1)* using the price at time T.

"We denote by $\phi_k(x_k)$ the fraction of the underlying asset in the portfolio at time k, when the asset price is x_k. Between time t_k and t_{k+1} the self-financing condition leads to a local wealth balance given by":

$$\Delta W = e^{rh}C_k(x_x) - C_{k+1}(x_{x+1}) + \phi_k(x_x)(x_{x+1} - e^{rh}x_x)$$

We should use a local risk function in order to measure the quality of our replication. In our work we will implement this method with two hedging error measures:

1) Standard Deviation

$$\Theta = \sigma_{\Delta W} \equiv \left(E[(\Delta W - \overline{\Delta W})^2]\right)^{1/2}$$

The method involving minimization of standard deviation can be implemented by a simple regression. We rearrange our equation as:

$$C_{k+1} = e^{rh}C_k + \varphi(x_{k+1} - e^{rh}x_k)$$

Where C_{k+1} is known and we search for C_k and φ which minimize the standard deviation. This leads us to the following regression:

$$e^{-rh}C_{k+1} = \beta_0 + \beta_1(e^{-rh}x_{k+1} - x_k) + \varepsilon$$

Where β_0 estimates C_0 and β_1 estimates the hedge. This method is very efficient and returns high quality results when tested with European vanilla options.

The matrices used in this minimization are:

$$Y = \begin{bmatrix} C_0 \\ H \end{bmatrix}_{2\times1} \quad , A = \begin{bmatrix} e^{\rho} & x_1^1 - e^{\rho}x_0^1 \\ e^{\rho} & x_1^2 - e^{\rho}x_0^2 \\ & \vdots \\ e^{\rho} & x_1^n - e^{\rho}x_0^n \end{bmatrix}_{N\times2} \quad , Aeq = \begin{bmatrix} N\times e^{\rho} & \sum(x_1 - e^{\rho}x_0 \end{bmatrix}_{1\times2}$$

$$beq = \sum C_1 \quad , \; Payoff = \begin{bmatrix} C_1^1 \\ C_1^2 \\ \vdots \\ C_1^N \end{bmatrix}_{N\times1}$$

2) Expected Shortfall (as used by Pochart and Bouchaud [2003])

Expected shortfall is a coherent measure of risk which does not suffer from the inherent problems of VaR especially when it comes to derivative that have non-Gaussian underliers. The expected shortfall is defined as:

$$\Theta = ESF_{\alpha} \equiv -E[\Delta W \mid \Delta W \le -VaR_{\alpha}]$$

$$\Pr\{\Delta W < -VaR_{\alpha}\} = 1 - \alpha$$

$$\Re_k = \sum_{l=i}^{N} \left[\Delta_0 - \Delta W_K^l\right]^+$$

which penalizes losses exceeding the Δ_0 threshold.

If we decide to use the method of Pochart and Bouchaud (2003), Following Longstaff and Schwartz (2001) and Potters et al. (2001), the functions $C_k(x_k)$ and $\varphi_k(x_k)$ are decomposed with the help of p basis functions:

$$\phi_k(x) = \sum_{a=1}^{p} \varphi_k^a F_k^a(x)$$

$$C_k(x) = \sum_{a=1}^{p} \gamma_k^a C_k^a(x)$$

Each of the F basis function is defined as linear function between two separation points B_1 and B_2 such that at B_1 its value is zero and at B_2 it reaches 1. The authors propose that the separation points be chosen in a way that the same number of trajectories fall in each interval. That is, if we have N simulation trajectories and p basis functions, there shall be $(p+2)$ intervals and $N/(p+2)$ trajectories fall in each of these intervals.

The C functions share the same separation points with the F functions and are the integrals of function F. More precisely, for i and j going from 1 to p, the ith and jth columns of the matrices F and C are calculated using the following algorithm:

$$F_i = \left[\frac{x_k - B_1}{B_2 - B_1}\right] \mathrm{I}_{x_k > B_1 \cap x_k < B_2} + \mathrm{I}_{x_k > B_2}$$

$$C_j = \left[\frac{x_k - B_1}{2}\right] \mathrm{I}_{x_k > B_1 \cap x_k < B_2} + \left[\frac{B_2 - B_1}{2} + x_k - B_2\right] \mathrm{I}_{x_k > B_2}$$

Where B_i is the separation point corresponding to the basis function and x_k is the price vector. Thus we have two matrices of dimension $N{\times}p$ which we can multiply by the column vectors φ and γ when doing the optimization.

The optimization at each of the rebalancing points is done in two steps. The first step is finding the φ^a which minimizes the risk function. We can approximate the function $C_k(x_k)$by $C_{k+1}(x_k)$ which is known from the previous iteration. For example, first we approximate $C_k(x_k)$ by the cash flow function of the option at its maturity and then go one step backward in time and use the calculated coefficients from the previous period.

After having calculated the φ^as, the only remaining task would be minimizing the local wealth to find the γ^as. The problem:

$$\min_{\gamma} \sum_{l=1}^{N} \left[\sum_{a=1}^{p} \gamma_k^a C_k^a(x_k^l) - e^{-rh}(C_k(x_{k+1}^l) - \phi_k(x_k^l)(x_{k+1}^l - e^{rh} x_k^l)) \right]^2$$

After this step, we will have to do the optimizations of the last point of rebalancing, which will give us the price and the strategy at time 0. This is done just like before except that this time the functions $\phi_k(x)$ and $C_k(x)$ are replaced by constants (there is only one price at time 0).

We should clarify some points about this methodology. While the vectors φ^a and γ^a are initialized at zero for the first point of readjustment, for the next rebalancing points we take the optimized values of the previous iteration as the starting point. In Pochart and Bouchaud (2001) the authors have not mentioned whether they have imposed any constraints. Nevertheless, we have chosen to constrain the φ^as and γ^as as non-negative to prevent negative gamma for the option. Also, it seems reasonable to impose another constraint requiring that the sum of φ^as be equal to 1 so that the optimal hedge function is a weighted average of the basis functions. However to this moment this formulation has not worked in our framework because this constraint creates errors in the first step which propagate to the rest of the readjustment process and produce absurd results for the hedge function.

Another important consideration would be the use of gradients. The authors suggest using the following gradient for the first optimization:

$$\frac{\partial \Re_k}{\partial \varphi_k^a} = -\sum_{l=1}^{N} \left[x_{k+1}^l - e^{rh} x_k^l \right] F_k^a(x_k^l) \mathrm{I}_{\Delta W_k^l < \Delta_0}$$

Even with optimized programming (i.e. doing most of the calculation out of the loops), using the gradient increases the optimization time significantly. Yet, we have observed that using gradients makes the optimization more robust. Nevertheless, after having constrained the φ^as to be non-negative we observe that the optimization without gradients results in an acceptable level of robustness. However, it should be mentioned that in many cases using the gradients results in numeric problems in Matlab and we do not get any results (we just get error messages). In the cases where there are no numerical problems, the optimization using gradients takes 6 to 7 times more time than in the other case. For now we are just using the explained method (without gradients) and constrain φ^as to non-negative values.

Despite all the aforementioned efforts, we have observed that this method suffers frm the problem of compounding errors. That is, at the first instance of hedging (T-1) we are approximating $C_k(x_k)$ by $C_{k+1}(x_k)$ which obviously has some errors and then these errors are carried to the next instance of rehedging and again a similar approximation is made and so on. This leads to the compounding of errors and we have observed that as we increase the number of rehedging instances the accuracy of our estimates drop dramatically. The only reason why one would opt for more than one hedging instance is to create a mesh of time and prices which is useful for path dependant derivatives. Our CDO trade problem is studied in a static frame work and has a quite simple path dependency structure which can be addressed through bond defaults in the simulation process. For this reason we have tried to simplify this method for use in our final work. The two resulting versions are explained as follows:

Version 1: In this method we simulate <u>one</u> set of underlier prices and we solve the following minimization problem:

$$\min \sum \max \left[\left(\Delta_0 - (e^{\rho} C_0 - C_1 + H(x_1 - e^{\rho} x_0))\right), 0\right]$$

Subject to:

$$\overline{\Delta W} = 0$$

This can be done using the *fmincon* function in MatLab:

```
[Y,fval,exitflag] = fmincon(@(Y) sum(max(delta0-(A*Y-Payoff),0)), Y, [], [], Aeq, beq,
zeros(2,1), [], [], [])
```

The matrices used in this minimization are:

$$Y = \begin{bmatrix} C_0 \\ H \end{bmatrix}_{2\times 1} \quad , A = \begin{bmatrix} e^{\rho} & x_1^1 - e^{\rho} x_0^1 \\ e^{\rho} & x_1^2 - e^{\rho} x_0^2 \\ & \vdots \\ e^{\rho} & x_1^n - e^{\rho} x_0^n \end{bmatrix}_{N\times 2} \quad , Aeq = \begin{bmatrix} N \times e^{\rho} & \sum (x_1 - e^{\rho} x_0 \end{bmatrix}_{1\times 2}$$

$$beq = \sum C_1 \quad , \; Payoff = \begin{bmatrix} C_1^1 \\ C_1^2 \\ \vdots \\ C_1^N \end{bmatrix}_{N\times 1}$$

where C_0 and x_0 are respectively the option price and the underlier price at $t=0$ and C_1 and x_1 are respectively the option price and the underlier price at $t=T$ (maturity). x_t^n represents the underlier price at time t and the n^{th} simulation trajectory. The same notation is used for C_1.

With these matrices the *fmincon* function minimizes the expected shortfall while the *Aeq* and *beq* matrices enforce the condition requiring the zero average change of wealth. This method results in accurate estimates of both the option price and the hedge. Furthermore, the efficiency of this method is higher than a naïve Monte-Carlo, that is, the standard deviation of this method is half the standard deviation of a naïve Monte-Carlo.

Version 2: In this method we simulate two sets of underlier prices and we calculate the optimal hedge and the option price in two consecutive steps as opposed the previous method where we enforced both conditions (ES and average wealth) in a simultaneous manner. In fact, this method is quite identical to that of Pochart and

Bouchaud (2003) in the sense that it is done in two steps (first calculating the optimal hedge and then the option price).

In the first step we use the (known) payoffs for both sets of data at t=T to calculate the hedge:

$$\min_{H} \sum \max \left[(\Delta_0 - (C_1 - C_2 + H(x_2 - x_1))), 0 \right]$$

where x_1 and C_1 denote the trajectories for the first set of data and x_2 and C_2 the second set. This minimization gives us the hedge (H).

In the second step we use the calculated hedge and one of the two sets of data to calculate the option price and enforce the zero average wealth condition. We use a regression to achieve this. The zero average wealth condition is rearranged as:

$$e^{-\rho} C_1 - H(e^{-\rho} x_1 - x_0) = C_0$$

and the following regression is done:

$$e^{-\rho} C_1 - H(e^{-\rho} x_1 - x_0) = \beta_0 + \varepsilon$$

where β_0 is the estimate of C_0. This method is almost two times faster than version 1 and shows almost exactly the same standard deviation and accuracy.

Our results show that Version 1 performs much better for our CDO simulation. However, one can also use Version 2 for assets with Gaussian distributions since it gives good results for such assets.

All of the results presented in this dissertation are produced using Version 1.

Appendix C - Structural Variance Gamma

The Variance-Gamma structural (SVG) approach that we use in this work is developed by Madan et al (1998). The Variance-Gamma (VG) process is a three parameter generalization of Brownian motion as a model for the dynamics of the logarithm of the stock price. To obtain this process, a Brownian motion (with constant drift and volatility) is evaluated at a random time change given by a gamma process. Each unit of time is given by an independent random variable that has a gamma density with unit mean and positive variance. The resulting stochastic process provide us with a robust three parameter model that in addition to the volatility of the Brownian motion control for (i) kurtosis and (ii) skewness. It can be shown that lognormal density and the Black-Scholes formula are parametric special cases of this process.

The VG process has three parameters: (i) σ the volatility of the Brownian motion, (ii) υ the variance rate of the gamma time change and (iii) θ the drift in the Brownian motion with drift. "The process therefore provides two dimensions of control over and above that of the volatility." Control over skew is attained via θ while υ controls kurtosis.

In the SVG approach defaults are based on evolution of firm's value return which follows a geometric Brownian motion (with drift and volatility parameters μ_i and σ_i) evaluated at stochastic time clocks governed by increments of gamma process:

$$\frac{\Delta f_i}{f_i} = \mu_i g_i(\Delta t;1,\upsilon) + \sigma_i W_i(g_i(\Delta t;1,\upsilon))$$

Our approach for correlating gamma stochastic clock processes and the Brownian motion underlying the firm-value evolution is a single factor one:

$$W_i = \beta W_m + \sqrt{1-\beta^2 Z_i}$$

where W_m and Z_i are independent standard Weiner processes. W_m represents the Weiner process of market at any given instance if time and Zi is the idiosyncratic Weiner process for each issuer. The increments of the gamma processes for different issuers follow

$$g_i(\Delta t;1,\upsilon) = g_{market}(\Delta t;\kappa,\upsilon\kappa) + u_i(\Delta t;1-\kappa,\upsilon(1-\kappa))$$

The process g_{market} and u_i are increments of independent gamma processes. g_{market} and gives the tome clocks at which the market evolution is evaluated while u_i is the time clock for each different issuer.

In this work a default-barrier model is used where the first time that the firm's value goes bellow a given barrier (ω as a fraction of initial firm-value) the firm (and the issued bonds) default. In order to facilitate the comparison of features of the VG structural model and the Poisson-Normal Copula reduced form model, we calibrate the VG model to produce the same first two moments of the PNC model. That is, **(1)** T period default probability for issuers; **(2)** T period portfolio loss standard deviation. The PNC model used in the main body of this work has a hazard rate of 0.65%/yr which results in a 5 year default probability of 0.0319. This marginal 5 year default probability can be attained using the following set of VG parameters:

$$\mu_i = 0\ (1/yr);\ \sigma_i = 0.20\ (1/yr^{1/2});\ \upsilon = 2\ yr;\ \omega = 0.3618$$

With recovery set to 30% and asset correlation set to 25% for the PNC model, a pool of 125 initially homogenous bonds has a 5 year pool loss standard deviation of 3.32% of the initial pool notional. To achieve this portfolio-loss standard deviation, we should calibrate β and κ. We obtain this standard deviation using:

$$\kappa = 1;\ \beta = 0.454$$

Note that with $\kappa=1$, the idiosyncratic component of the gamma process for different issuers (u_i) becomes zero. That is, all issuers are evaluated at the same time increments which are the market clocks (g_{market}).

Appendix D – Liquidated vs. Non-Liquidated Hedging Positions

Liquidated Position						
Model	**Strategy**	**std**	**ES_{80}**	**ES_{95}**	**upfront**	**Hedge (×tranche)**
PNC	**Min std**	12.9%	19.3%	26.4%	62.0%	21.8
	Min ES_{80}	27.0%	14.8%	16.7%	83.7%	34.6
	Min ES_{95}	29.6%	15.1%	16.1%	86.5%	36.2
	50x	53.9%	37.7%	37.7%	110.0%	50.0
	0x	42.4%	63.4%	70.9%	24.9%	0.0
VG	**Min std**	22.7%	30.5%	38.1%	36.4%	11.7
	Min ES_{80}	43.2%	23.7%	29.3%	59.3%	26.3
	Min ES_{95}	62.1%	24.9%	26.2%	72.3%	34.7
	50x	98.9%	51.5%	54.9%	96.3%	50.0
	0x	37.0%	52.8%	60.0%	18.2%	0.0
Non-Liquidated Position						
Model	**Strategy**	**Std**	**ES_{80}**	**ES_{95}**	**upfront**	**Hedge (×tranche)**
PNC	**Min std**	26.0%	36.4%	44.2%	40.4%	11.4
	Min ES_{80}	62.8%	26.2%	30.1%	66.4%	30.6
	Min ES_{95}	76.2%	27.3%	28.7%	73.0%	35.5
	50x	117.4%	55.2%	55.2%	92.6%	50.0
	0x	42.5%	63.2%	70.6%	24.9%	0.0
VG	**Min std**	24.1%	34.5%	43.1%	33.0%	10.1
	Min ES_{80}	57.7%	25.9%	30.0%	61.3%	28.9
	Min ES_{95}	71.7%	27.0%	28.5%	69.5%	34.3
	50x	113.7%	54.7%	54.7%	93.1%	50.0
	0x	37.0%	53.1%	60.5%	17.8%	0.0

Table A.2. Sell equity (0%-3%) tranche protection upfront, hedge, and error measures for different strategies (all numbers in %tranche unless specified). Liquidated vs. Non Liquidated position.

Liquidated Position						
Model	**Strategy**	**STD**	**ES_{80}**	**ES_{95}**	**spread**	**Hedge (×tranche)**
PNC	**Min STD**	11.4%	13.6%	20.3%	7.25%	12.4
	Min ES_{80}	11.6%	12.2%	15.3%	7.78%	13.4
	Min ES_{95}	12.0%	12.8%	13.4%	8.09%	14.2
	20x	19.6%	23.0%	23.0%	10.19%	20.0
	0x	28.7%	50.1%	87.2%	2.91%	0.0
VG	**Min STD**	12.2%	15.3%	30.3%	6.01%	9.2
	Min ES_{80}	14.3%	11.9%	18.0%	7.15%	12.2
	Min ES_{95}	16.8%	13.0%	13.9%	7.72%	13.8
	20x	29.9%	24.1%	24.1%	9.94%	20.0
	0x	26.5%	47.7%	75.5%	2.81%	0.0
Non-Liquidated Position						
Model	**Strategy**	**STD**	**ES_{80}**	**ES_{95}**	**spread**	**Hedge (×tranche)**
PNC	**Min STD**	12.8%	16.5%	33.7%	5.79%	9.3
	Min ES_{80}	14.9%	13.6%	21.5%	6.81%	12.0
	Min ES_{95}	18.5%	15.1%	15.7%	7.49%	14.1
	20x	32.3%	26.4%	26.4%	9.40%	20.0
	0x	28.6%	50.1%	86.8%	2.91%	0.0
VG	**Min STD**	12.5%	16.0%	33.4%	5.75%	8.7
	Min ES_{80}	15.3%	12.6%	20.5%	7.00%	11.9
	Min ES_{95}	19.4%	14.1%	14.9%	7.78%	14.2
	20x	32.9%	24.6%	24.6%	9.83%	20.0
	0x	26.5%	47.6%	75.7%	2.80%	0.0

Table A.2. Sell mezzanine (3%-7%) tranche protection upfront, hedge, and error measures for different strategies (all numbers in %tranche unless specified). Liquidated vs. Non Liquidated position.

Model	Strategy	std	ES_{80}	ES_{95}	spread	Hedge (×tranche)
Liquidated Position						
Model	**Strategy**	**std**	ES_{80}	ES_{95}	**spread**	**Hedge (×tranche)**
PNC	**Min std**	11.7%	13.7%	34.1%	2.74%	5.6
	Min ES_{80}	13.3%	12.5%	16.9%	3.71%	8.1
	Min ES_{95}	14.5%	12.9%	13.0%	4.02%	9.0
	20x	37.6%	34.1%	34.1%	7.68%	20.0
	0x	18.2%	18.2%	76.3%	1.03%	0.0
VG	**Min std**	10.7%	11.9%	31.0%	2.80%	5.3
	Min ES_{80}	11.6%	11.2%	19.6%	3.44%	6.9
	Min ES_{95}	14.8%	12.6%	13.1%	4.18%	9.1
	20x	40.4%	32.9%	32.9%	7.95%	20.0
	0x	17.6%	18.4%	73.4%	1.04%	0.0
Non-Liquidated Position						
Model	**Strategy**	**std**	ES_{80}	ES_{95}	**spread**	**Hedge (×tranche)**
PNC	**Min std**	10.6%	12.0%	28.2%	2.46%	5.0
	Min ES_{80}	11.3%	11.3%	20.2%	2.97%	6.3
	Min ES_{95}	14.6%	13.0%	13.7%	3.64%	8.5
	20x	44.2%	36.0%	36.0%	7.23%	20.0
	0x	17.9%	17.7%	75.2%	1.00%	0.0
VG	**Min std**	10.5%	11.5%	29.9%	2.72%	5.1
	Min ES_{80}	11.2%	10.9%	20.6%	3.26%	6.5
	Min ES_{95}	14.9%	12.7%	13.5%	4.12%	9.0
	20x	42.0%	33.0%	33.0%	7.91%	20.0
	0x	17.5%	18.0%	73.4%	1.02%	0.0

Table A.3. Sell senior (7%-10%) tranche protection upfront, hedge, and error measures for different strategies (all numbers in %tranche unless specified). Liquidated vs. Non Liquidated position.

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