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Stratégies de réplication et applications en structuration de portefeuille

par

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RÉSUMÉ

Cette thèse s'articule autour de quatre essais rédigés en format article. Ces articles sont présentés en anglais ayant été soumis à publication. Les quatres articles ont été rédigés en collaboration avec mes directeurs de recherche, Nicolas Papageorgiou et Bruno Rémillard.

Le premier chapitre constitue l'article clé de cette thèse *Replicating the Properties of Hedge Fund Returns*. Cet article propose une extension des méthodes de couverture d'options à la réplication distributionnelle, appliquée spécifiquement aux fonds de couverture. Une nouvelle mesure de performance est proposée afin d'évaluer la valeur ajoutée d'un fonds de couverture à un portefeuille initial en fonction de sa densité marginale des rendements, et de sa structure de dépendance avec le portefeuille considéré. Une méthodologie de réplication est alors dérivée, permettant la construction d'un portefeuille de densité bivariée de rendements ayant des propriétés identiques à celles du fonds de couverture dans son contexte d'intégration au portefeuille de référence.
Le deuxième chapitre vise à illustrer les innovations présentées dans l'article précédent en comparant certains résultats de tarification et de réplication à une méthodologie plus classique dérivée des hypothèses du modèle de Black-Scholes (1973), adaptée dans le même contexte par Kat et Palaro (2005) dans le cadre de leurs travaux de recherche. Cet article, intitulé *Optimal Hedging Strategies with an Application to Hedge Fund Replication*, est une courte documentation technique ayant pour objectif une démonstration de l'efficacité de la méthodologie de couverture.
Le troisième chapitre est une application des techniques proposées à l'assurance de por-

tefeuille. L'article intitulé *The Payoff Distribution Model : A Portfolio Insurance Approach* met l'emphase sur la gestion dynamique d'un protocole d'assurance et compare différentes stratégies classiques de gestion des pertes avec une approche par contrôle de densité. Ceci représente une extension du modèle de réplication présenté dans les deux premiers articles, et propose d'intégrer une option d'assurance en déformant la distribution univariée des rendements du portefeuille initial non couvert.

Le quatrième chapitre propose d'intégrer une meilleure modélisation du processus du sous-jacent à un algorithme de réplication d'options européennes cohérent. L'article intitulé *Option Pricing and Hedging for Regime-Switching Models* se concentre sur la modélisation du processus de rendements inhérent à la stratégie de réplication. Une modélisation par processus à changements de régimes est proposée avec la dérivation d'une méthodologie de couverture en temps discret appropriée. Des résultats en-échantillon et hors-échantillon illustreront l'efficacité du modèle en terme de réactivité aux conditions de marché.

Classification JEL : G10, G13, G20, G28, C15, C16, C22

Mots clés : fonds de couverture, réplication de distributions, stratégie de réplication, assurance de portefeuille, portefeuille synthétique, processus à changements de régimes, chaînes de Markov cachées, couverture en temps discret

ABSTRACT

This thesis focuses on four essays. These articles have been submitted for publication. The four articles are written in collaboration with my research directors, Nicolas Papageorgiou and Bruno Rémillard.

The first chapter is the key section of this thesis *Replicating the Properties of Hedge Fund Returns*. In this paper, we implement a multivariate extension of Dybvig (1988) Payoff Distribution Model that can be used to replicate not only the marginal distribution of most hedge fund returns but also their dependence with other asset classes. In addition to proposing ways to overcome the hedging and compatibility inconsistencies in Kat and Palaro (2005), we extend the results of Schweizer (1995) and adapt American options pricing techniques to evaluate the model and also derive an optimal dynamic trading (hedging) strategy. The proposed methodology can be used as a benchmark for evaluating fund performance, as well as to replicate hedge funds or generate synthetic funds.

The second section aims to illustrate the innovations proposed in the previous article comparing some results of pricing and replication with a more conventional methodology derived from the model assumptions of Black-Scholes (1973). This article, *Optimal Hedging Strategies with an Application to Hedge Fund Replication*, is a short technical documentation that demonstrates the effectiveness of the proposed methodology.

The third chapter is an application of the proposed techniques to portfolio insurance. We propose an innovative approach for dynamic portfolio insurance that overcomes many of the limitations of the earlier techniques. In this paper, *The Payoff Distribution Model : A Portfolio Insurance Approach*, we transform the Payoff Distribution Model, originally introduced by Dybvig (1988) as a performance measure, to a fund management tool. This approach allows us to generate funds with pre-specified distributional properties. Specifically, we generate funds that are characterized by a Left Truncated Gaussian distribution and then demonstrate out-of-sample that this approach to managing market exposure results in more reliable portfolio protection at a lower cost than more popular techniques such as the CPPI.

Chapter four proposes to integrate a better modeling of the underlying process in the hedging algorithm. In this paper, *Option Pricing and Hedging for Regime-Switching Models*, we implement optimal (mean-variance) dynamic hedging in discrete time for a class of regime-switching models. This methodology for pricing and hedging options is robust and flexible and overcomes the main drawbacks of the Black-Scholes-Merton model. We compare our discrete time methodology to a continuous time model approximation using regime-switching geometric Brownian motion, for which it has recently been shown that the optimal hedging and associated pricing can be deduced from a risk neutral distribution. We provide both in-sample and out-of-sample results to support our approach.

JEL Classification : G10, G13, G20, G28, C15, C16, C22

KeyWords : Hedge Funds, Distributional Replication, Hedging Strategy, Portfolio Insurance, Synthetic Funds, Switching Regimes, Hidden Markov Chains, Discrete Time Hedging

CONTRIBUTION DES AUTEURS

Les quatre articles ont été écrits en co-rédaction avec mes directeurs de thèse, Nicolas Papageorgiou et Bruno Rémillard. Les auteurs ont apporté une contribution égale aux travaux de recherche et de rédaction.

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Chapitre 1

Introduction

L'objectif principal de cette contribution de recherche est de répondre à un enjeu essentiel en finance de marché soit la structuration du profil de risque du portefeuille d'un investisseur. L'idée sous-jacente est de faire correspondre les besoins (rendements) et capacité (risque) d'un investisseur privé ou institutionnel au profil de rendements de son portefeuille de marché dont la composition en actifs est une fonction de ses préférences. De nombreux intervenants construisent leur modèle d'affaire autour de la définition du processus de déformation des intrants (rendements des actifs) en extrants (rendements du portefeuille profilé). On citera de façon non exhaustive les banquiers personnels et conseillers en épargne retraite avec un mandat de structuration d'un portefeuille de décaissement optimal en fonction du portefeuille de l'épargnant, les banquiers d'affaire avec un mandat de contrôle de la volatilité des revenus de placement et les courtiers avec un mandat de gestion de pertes d'un portefeuille institutionnel. Ces concepts prennent naissance dans la théorie des options et dans le mécanisme de transfert de risque. Afin de caractériser notre approche, une simplification de la problématique s'impose. Nous considérons le cas d'un investisseur ayant accès à un sous-jacent risqué S et dont la fonction de préférence est déterminée selon des spécifications reliées à l'évolution de S. Ceci définit une hypothèse de travail selon laquelle la valeur du portefeuille de l'investisseur sera déterminée par une fonction possiblement non-linéaire en S. Cet enjeu est classiquement adressé par la définition d'un " payoff " optionnel écrit sur le sous-jacent S (option d'achat, option de vente, option exotique, combinaison d'options). L'investisseur détermine les spécificités de son profil de rendement à terme selon ses préférences, soit des prix d'exercice, maturité, et toutes autres caractéristiques particulières à la définition de l'option. L'option est alors caractérisée comme un instrument de transfert de risque d'une partie de la distribution des rendements de S sur l'horizon fixé. Le mécanisme de transfert de risque peut alors être confié au marché, en transigeant l'option avec une contrepartie, ou géré dynamiquement par l'investisseur, en transigeant activement S selon un protocole déterminé qui déterminera le " payoff " à maturité. L'équivalence théorique de ces deux approches est cruciale. La déformation du profil de risque de S se fait selon un protocole Π d'investissement en S à un coût C_0. Le coût C_0 du transfert de risque est donc intimement lié au protocole de structuration du " payoff ". On appellera C_0 le coût de l'option et Π la stratégie dynamique de réplication de l'option. Notre approche est donc de proposer une méthodologie conforme à ce cadre de travail, en contribuant successivement aux différentes problématiques du sujet. La littérature a vastement couvert la théorie des options. Depuis Black-Scholes (1973) qui ont défini la théorie des options selon leurs hypothèses de marché, de nombreux auteurs ont proposé des extensions et améliorations dans le but de palier aux inconsistences du modèle initial. Le modèle Black-Scholes a permis le développement de solutions analytiques simples pour la tarification de produits dérivés standard. Cependant les hypothèses restrictives de volatilité constante, processus gaussien en temps continu et de rendements indépendants, ont rapidement été identifiées comme non adaptées à la réalité des marchés (Fama (1965), Mandelbrot (1963), Schwert (1989)). De plus, Boyle et Emanuel (1980), Gilster (1990), Mello et Neuhaus (1998) et Buraschi et Jackwerth (2001) ont caractérisé le biais d'erreur de réplication introduit par l'hypothèse de couverture en temps continu. De nombreux modèles en temps discret basés sur l'optimisation de différentes fonctions objectifs ont été proposés. On citera Owen (2002), Potters, Bouchaud et Sestovic (2001) et Pochart et Bouchaud (2004). Notre approche est d'identifier la stratégie de réplication en temps discret optimale auto financée telle que définie par Cox et Ross (1976) et Harrison et Kreps (1979). A partir des travaux de Follmer et Schweizer (1990) et Schweizer (1992, 1995), nous dériverons une stratégie dynamique optimale de couverture en temps discret basée sur la minimisation de l'erreur quadratique de réplication.

L'idée est alors de travailler dans un premier temps sur la définition d'une nouvelle fonction de " payoff ", dont le déterminant ne sera, non pas le niveau de S mais le rendement à terme de S. A partir des travaux de Dybvig (1988), nous caractériserons la fonction objectif de densité de rendements périodiques de l'investisseur. Ce " payoff " de densité sera appliqué dans un contexte univarié, en tant que protocole d'assurance de portefeuille, et dans un contexte bivarié, dans un cadre de réplication de densité de rendements mensuels de fonds de couverture apparié à un portefeuille de référence. Une seconde contribution sera apportée en définissant une nouvelle méthodologie de réplication de " payoff ". Nous viendrons ici répondre à la problématique de couverture en temps discret par un processus d'investissement en S d'une fonction objectif définie sur S. Ceci sera illustré tant dans un cadre de réplication d'options d'achat et de vente classiques que dans un cadre de réplication de densité de rendements. Une des caractéristiques de cette méthodologie est sa définition dans un environnement non gaussien sous probabilité physique. Nous venons alors répondre aux lacunes du modèle Black-Scholes restreint à un environnement de réplication en temps continu sous un processus de rendements indépendants et identiquement distribuées de loi gaussienne. Afin d'illustrer cette caractéristique du modèle, nous définirons la stratégie optimale de réplication sous un processus de rendements du sous-jacent suivant dans un premier temps une mixture de lois gaussiennes et dans un second temps un processus à changements de régimes. Cette thèse rédigée au format article sera composée de trois essais et d'une note technique. Dans le but de soumettre ces articles aux revues scientifiques spécialisées, ces articles seront rédigés en anglais et une bibliographie spécifique à chaque article sera proposée.

La première partie sera intitulée *Replicating the Properties of Hedge Fund Returns* et a fait l'office d'une publication dans la revue " Journal of Alternative Investments " édition " Fall 2008 ". Cet article illustre la méthodologie de réplication de " payoff " de densité bivariée sous processus de rendements suivant une mixture de lois gaussiennes. L'idée est ici de permettre à l'investisseur de reproduire la densité marginale de rendements de fonds de couverture ainsi que la structure de dépendance entre son portefeuille de référence et le fonds de couverture considéré. Par cette approche l'investisseur pourra tarifer la distribution de rendement d'un fonds de couverture conditionnellement aux caractéristiques spécifiques de son portefeuille. Une règle de décision sera établie afin d'évaluer l'opportunité de répliquer cette distribution par une stratégie d'investissement en temps discret dans les portefeuilles d'actifs liquides appropriés. En adressant l'évaluation et la réplication de fonds de couverture, cette contribution trouve sa place dans la littérature de l'investissement alternatif. De cet article, une note technique *Optimal Hedging Strategies with an Application to Hedge Fund Replication* est publiée dans l'édition de Janvier-Février 2008 de " Wilmott Magazine ". Cette note a pour but d'illustrer l'avantage de la méthodologie proposée en comparaison avec une approche Black-Scholes, spécifiquement dans la minimisation de l'erreur de couverture et de sa variance, dans un contexte où la structure de dépendance discutée est définie par une fonction de Copules.

Le deuxième article est intitulé *The Payoff Distribution Model : An Application to Dynamic Portfolio Insurance*. L'idée est ici d'appliquer le modèle décrit précédemment dans un contexte d'assurance de portefeuille. La stratégie de réplication est appliquée à un portefeuille composé d'une combinaison d'actif risqué S et d'actif sans risque dont l'objectif est la protection d'un rendement minimum périodique tout en contrôlant la volatilité des rendements résultants. La fonction de " payoff " sera définie comme une densité de rendements de loi gaussienne tronquée. Le niveau de troncature déterminera le seuil de garantie. La méthodologie sera comparée aux approches plus classiques de gestion d'assurance, soit un modèle de " stop-loss ", un modèle de " CPPI " et une réplication d'option de vente écrite sur S sous un environnement Black-Scholes. Ces approches avaient été précédemment étudiées par Brennan et Schwartz (1979), Rubinstein et Leland (1981), Black et Jones (1987) et Black et Perold (1992).

Le troisième article est intitulé *Option Pricing and Dynamic Hedging for Regime-Switching Geometric Random Walks Models*. La contribution est de proposer un algorithme de réplication d'option d'achat et de vente consistant avec un processus de rendements à changements de régimes. Chaque état est caractérisé par une loi gaussienne spécifique. Des études en échantillons et hors échantillons seront proposées, ainsi qu'une analyse de robustesse de la méthodologie. L'approche sera comparée à une approche Black-Scholes classique ainsi qu'à une modélisation par une mixture de lois gaussiennes. Les processus à changements de régime popularisés par Hamilton (1990) et Kim, Piger et Startz (2008) permettent une caractérisation intuitive des états perturbateurs des rendements du portefeuille, en associant une fonction de passage d'un état à un autre, assurant alors la conditionnalité des états.

Chapitre 2

Replicating the Properties of Hedge Fund Returns

2.1 Introduction

The impressive growth of the hedge fund industry has naturally led to an increased scrutiny of the fund managers and of their investment strategies. Given the often exorbitant management and performance fees charged by hedge fund managers, it is not surprising that investors are starting to question what they are actually getting for their money. Shrewd investors and institutional fund of funds are becoming increasingly careful about paying alpha fees for beta returns. The challenge that investors and researchers are therefore confronted with is how to reliably separate the funds that are generating alpha returns from the ones that are simply repackaging beta.

The approach that has generally been favored by academics and practitioners in order to extract information about hedge fund returns is the factor model approach. The underlying idea is to try and separate the returns that are due to systematic exposure to risk factors (beta returns) from those that are due to managerial skill (alpha returns). Once the relevant risk factors have been identified, one can evaluate whether the funds exhibit abnormal returns based on the intercept of a linear regression of the fund returns against the factor returns. A further advantage of this methodology is that if the linear model is well-specified, one can attempt to replicate the returns of the hedge fund by investing in the appropriate portfolio of factors. A recent paper by Hasanhodzic and Lo (2007) provides some evidence that linear replication can be successful for certain strategies whilst offering certain advantages to hedge fund investing. These include more transparency, increased liquidity and fewer capacity constraints. However the authors warn that the heterogeneous risk profile of hedge funds and the non-linear risk exposures greatly reduce the ability of these models to consistently replicate hedge fund returns. Over the last few months, several banks including Goldman Sachs, JP Morgan and Merril Lynch have launched linear replication funds.

Certain generic hedge fund characteristics help explain some of the difficulty in identifying a well specified linear model. The use of financial derivatives, the use of dynamic leverage, the use of dynamic trading strategies and the asymmetric performance fee structures are some of the most obvious sources of non-linearities in hedge fund returns. Several recent papers, such as Mitchell and Pulvino (2001), Fung and Hsieh (2001), Agarwal and Naik (2004), Chen and Liang (2006), Kazemi and Schneeweis (2003) have dealt with the inclusion of risk premia and conditional betas that attempt to account for these non-linearities. The inclusion of the above option-based factors significantly improves the explanatory power of factor models, however, most of these factors are not tradable and therefore cannot be used to construct a replicating portfolio.

In order to circumvent the issue of identifying tradable risk factors, an interesting alternative approach was proposed by Amin and Kat (2003) and more recently extended by Kat and Palaro (2005). Based on earlier work by Dybvig (1988), the authors evaluate hedge fund performance not by identifying the return generating betas, but rather by attempting to replicate the distribution of the hedge fund returns. The underlying idea is based on the hypothesis that much of the trading activity undertaken by hedge funds is not creating value, just altering the timing of the returns available from traditional assets. In effect, many hedge funds are simply distorting readily available asset distributions. So the real challenge is whether or not we can find a more efficient method to distort these distributions than by investing in hedge funds. Armed with their new efficiency measure, Kat and Palaro (2005) show that hedge fund returns are by no means exceptional and that for the majority of funds an alternative dynamic strategy would have provided investors with superior returns. This methodology not only provides a model free benchmark for evaluating hedge funds, it can also be used to create synthetic funds with predetermined distributional properties.

The efficiency measure as presented by Kat and Palaro (2005) is however subject to several shortcomings and inconsistencies. The most significant of these relates to the way that the daily trading strategies are derived from the distribution of monthly returns. The properties of the estimated monthly distributions and copula functions proposed by the authors are not infinitely divisible and therefore the true properties of the daily returns are not known. As a result, the replicating strategy will not be precise. A further weakness pertains to the fact that although the hedge fund returns and traded assets are clearly non-normal, the efficiency measure is calculated within the confines of the Black-Scholes-Merton world, hence ignoring the higher moments of the distributions.

In this paper, we will implement a multivariate extension of Dybvig (1988) Payoff Distribution Model that can be used to replicate not only the marginal distribution of most hedge fund returns but also their dependence with other asset classes. In addition to proposing ways to overcome the hedging and compatibility inconsistencies in previous papers, we extend the results of Schweizer (1995) and adapt American options pricing techniques to evaluate the model and also derive an optimal dynamic trading (hedging) strategy. The proposed methodology can be used as a benchmark for evaluating fund performance, as well as to replicate hedge funds or generate synthetic funds.

The rest of the paper will be structured as follows. Section 2 will explain the intuition behind the multivariate extension of Dybvig's Payoff Distribution model. Section 3 presents the technical details relating to the modeling and estimation of the distributions. Section 4 presents the payoff function. Section 5 presents the replication issues and presents the optimal dynamic trading strategy. Section 6 presents some numerical results. Section 7 concludes.

2.2 The Multivariate Payoff Distribution Model

In Kat and Palaro (2005), the authors show that given two risky assets $S^{(1)}$ and $S^{(2)}$, it is possible to "reproduce" the statistical properties of the joint return distribution of asset $S^{(1)}$ and a third asset $S^{(3)}$. Let's assume asset $S^{(1)}$ is the investor portfolio, asset $S^{(2)}$ is a tradable security and asset $S^{(3)}$ is a hedge fund, this result implies that we can generate the distribution of the hedge fund and its dependence with the investor portfolio, by only investing in the tradable security $S^{(2)}$ and the investor portfolio $S^{(1)}$. Note that we do not replicate the month by month returns of the hedge fund, but instead we replicate its distributional properties (i.e. expectation, volatility, skewness and kurtosis) as well as dependence measures with respect to the returns of the investor portfolio (i.e. Pearson, Spearman correlations...).

Essentially, there exist a payoff function that will allow us to transform the joint distribution of assets $S^{(1)}$ and $S^{(2)}$ into the bivariate distributions of $S^{(1)}$ and $S^{(3)}$. This payoff function is easily shown to be calculable using the marginal distribution functions F_1, F_2 and F_3 of $S_T^{(1)}$, $S_T^{(2)}$, $S_T^{(3)}$, and the copulas $\mathcal{C}_{1,2}$ and $\mathcal{C}_{1,3}$ associated respectively with the joints returns $\left(R^{(1)}_{0,T}, R^{(2)}_{0,T}\right)$ and $\left(R^{(1)}_{0,T}, R^{(3)}_{0,T}\right)$. The exact expression for the payoff function is given in section 2.4.

The challenge that we are confronted with is how to best evaluate this function, and this is by no means a trivial problem. The problem can however be broken down into three separate components. The first part relates to the proper modeling of the distributions and copula functions. The second part consists in calculating the payoff function. The third part consists in selecting an approach that will allow us to generate a dynamic trading strategy that provides us with the best possible approximation of the payoff function.

2.3 Modeling the returns

In order to provide a robust solution in this framework, we propose the following steps. First, we will model the joint daily distribution of $S^{(1)}$ and $S^{(2)}$ using bivariate Gaussian mixtures. Since we will be trading these assets on a daily basis, it is imperative that the distribution of the monthly returns for both the investor portfolio and the reserve asset are consistent with the distribution of the daily returns. We need to be sure that by dynamically trading the assets based on the joint daily distributions we will be able to generate the desired monthly properties. We will therefore estimate the parameters of the bivariate Gaussian mixtures of R_t, (investor portfolio and reserve asset) using the historical daily returns of $S^{(1)}$ and $S^{(2)}$. We can then solve for the law of the monthly returns that is compatible with the law of daily returns. Furthermore, the daily dependence which is modeled with the bivariate mixtures will allow us to obtain the desired monthly dependence. This would not have been possible if we used univariate laws to model the marginal distributions and a copula to model the dependence structure. Although copula provide us with much flexibility in terms of modeling the dependence, there is however no proof to this day that the statistical properties of copula functions are divisible. Finally, we need to estimate the monthly distribution of the hedge fund returns as well as the dependence between the hedge fund and the investor portfolio. There are no particular restrictions regarding the choice of the distribution of $S^{(3)}$ and the copula $\mathcal{C}_{1,3}$. We have developed statistical tests that allow us to select the most appropriate marginal distribution and copula function. We now consider each of these steps in detail.

2.3.1 Mixtures of Gaussian distributions

The choice of Gaussian mixtures to model the bivariate distribution of investor portfolio and the reserve asset is due to both the flexibility of the mixtures in capturing high levels of skewness as well as the fact that the bivariate distribution is infinitely divisible. In this section, we will first provide a brief description of bivariate Gaussian mixtures and discuss the goodness-of-fit test that we developed in order to estimate the mixtures and select the optimal number of regimes.

Definition of mixtures of Gaussian bivariate vectors

A bivariate random vector X is a Gaussian mixture with m regimes and parameters $(\pi_k)_{k=1}^m$, $(\mu_k)_{k=1}^m$ and $(A_k)_{k=1}^m$, if its density is given by

$$f(x) = \sum_{k=1}^{m} \pi_k \phi_2(x; \mu_k, A_k)$$

where $\phi_2(x; \mu, A) = \frac{e^{-\frac{1}{2}(x-\mu)^\top A^{-1}(x-\mu)}}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}$ is the density of a bivariate Gaussian vector with mean vector $\mu = (\mu_1, \mu_2)^\top$ and covariance matrix $A = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. Its distribution function is

$$F(x_1, x_2) = \sum_{k=1}^{m} \pi_k \Phi_2\left(\frac{x_1 - \mu_{k1}}{\sigma_{k1}}, \frac{x_2 - \mu_{k2}}{\sigma_{k2}}; \rho_k\right),$$

where $\Phi_2(\cdot,\cdot;\rho)$ is the bivariate standard Gaussian distribution function with correlation ρ. Some of the important properties of mixtures of bivariate Gaussian variables are discussed in Appendix 2.7.

Estimation and goodness-of-fit

In order to choose the optimal number of regimes, we need to first estimate the parameters of the model, and then provide a goodness-of-fit test to evaluate whether a greater number of regimes is required. The estimation method is based on the EM algorithm of (Dempster et al., 1977).

A new goodness-of-fit test is proposed to assess the suitability as well as to select the number of mixture regimes m. The proposed test, based on the work in Genest et al. (2009), uses the Rosenblatt's transform [1].

For the selection of the number m of regimes, the following two steps procedure is suggested :

(a) Find the first m_0 for which the P-value of the test is larger than 5%.

(b) Estimate parameters for m_0+1 regimes and apply the likelihood ratio test to check if the null hypothesis $H_0 : m = m_0$ vs $H_1 : m = m_0+1$. If H_0 is rejected at the 5% level, repeat steps (a) and (b) starting at $m = m_0+1$. However, if the parameters under H_1 yield a degenerate density (e.g., $|\rho_k| = 1$), stop and set $m = m_0$.

2.3.2 Choice/estimation of the marginal distribution F_3

There are no restrictions on the choice of F_3, which is the distribution of the hedge fund that we seek to replicate (or the desired distribution in the case of a synthetic fund). Unlike the reserve asset and investor portfolio that require divisible laws, we are

1. The derivation of the goodness-of-fit test for Gaussian mixtures is available on request from the authors.

only interested in monthly return distribution and hence can introduce any distribution. In the case of the replication of an existing hedge fund, goodness-of-fit is important and therefore we test using a Durbin type test [2].

2.3.3 Choice/estimation of the copula $\mathcal{C}_{1,3}$

Again, there are no restrictions on the choice of copula function $\mathcal{C}_{1,3}$, between the monthly returns of the hedge fund and the investor portfolio. Suppose that we have historical monthly returns $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ belong to a copula family C_θ. To estimate θ, one often uses the so-called IFM method. However, we do not recommend it as the parameters of the copula function rely on the estimated marginal distributions. Any mis-specification of the marginal distributions will bias the choice of copula. For reasons of robustness, it is therefore preferable to use normalized ranks, i.e. if R_{i1} represents the rank of Y_i among $Y_1, \ldots, Y_n$ and if R_{i2} represents the rank of Z_i among $Z_1, \ldots, Z_n$, with $R_{ij} = 1$ for the smallest observations, then set

$$U_i = \frac{R_{i1}}{n+1}, \quad V_i = \frac{R_{i2}}{n+1}, \quad i = 1, \ldots, n.$$

To estimate θ one could try to maximize the pseudo-log-likelihood

$$\sum_{i=1} \log c_\theta(U_i, V_i),$$

as suggested in Genest et al. (1995). For example, if the copula is the Gaussian copula with correlation ρ, the pseudo-likelihood estimator for ρ yields the famous van der Waerden coefficient defined to be the correlation between the pairs $\{\Phi^{-1}(U_i), \Phi^{-1}(V_i); i = 1, \ldots, n\}$. For other families that can be indexed by Kentall's tau, e.g., Clayton, Frank and Gumbel families, one could estimate the parameter by inversion of the sample Kendall's tau. See, e.g., Genest et al. (2006).

2. The derivation of the goodness-of-fit test for the choice of copula is available on request from the authors.

Finally, to test for goodness-of-fit, one can use Cramér-von Mises type statistics for the empirical copula or for the Rosenblatt's transform. The latter could be the best choice given that $\frac{\partial}{\partial u}\mathcal{C}_{1,3}(u,v)$ needed to be calculated for the evaluation of the payoff function. These tests are described in Genest et al. (2009) and in view of their results, we recommend to use the test statistic $S_n^{(B)}$.

2.4 The payoff function

Having estimated the necessary distribution and copula functions, one must now calculate the payoff's return function g. As deduced by Kat and Palaro (2005), its formula is given by

$$g(x,y) = Q\left\{x, P\left(R_{0,T}^{(2)} \leq y | R_{0,T}^{(1)} = x\right)\right\},$$

where $Q(x,\alpha)$ is the order α quantile of the conditional law of $R_{0,T}^{(3)}$ given $R_{0,T}^{(1)} = x$, i.e., for any $\alpha \in (0,1)$, $q(x,\alpha)$ satisfies

$$P\left\{R_{0,T}^{(3)} \leq Q(x,\alpha) | R_{0,T}^{(1)} = x\right\} = \alpha.$$

Using properties of copulas, e.g. Nelsen (1999), the conditional distributions can be expressed in terms of the margins and the associated copulas.

$$P\left(R_{0,T}^{(2)} \leq y | R_{0,T}^{(1)} = x\right) = \frac{\partial}{\partial u}\mathcal{C}_{1,2}(u,v)\bigg|_{u=F_1(x), v=F_2(y)}.$$

Note that $\frac{\partial}{\partial u}\mathcal{C}_{1,2}(u,v) = P\left\{F_2\left(R_{0,T}^{(2)}\right) \leq v | F_1\left(R_{0,T}^{(1)}\right) = u\right\}$.

In our methodology, since the monthly returns $\left(R_{0,T}^{(1)}, R_{0,T}^{(2)}\right)$ are modeled by a Gaussian mixtures with parameters $(\pi_k)_{k=1}^m$, $(\mu_k)_{k=1}^m$ and $(A_k)_{k=1}^m$, the conditional distributions can be expressed as follows

$$P\left(R_{0,T}^{(2)} \leq y | R_{0,T}^{(1)} = x\right) = \sum_{k=1}^{m} \tilde{\pi}_k(x)\phi\{y; \tilde{\mu}_k(x), \tilde{\sigma}^2\}$$

where $\tilde{\pi}_k(x)$, $\tilde{\mu}_k(x)$ and $\tilde{\sigma}^2$ are given by formulas (2.7) and (2.8) in Appendix 2.7

2.5 Dynamic replication

Having solved for the payoff function, we need to find an optimal dynamic trading strategy that will replicate the payoff function. We do so by selecting the portfolio (V_0, φ) such as to minimize the expected square hedging error

$$E\left[\beta_T^2 \left\{V_T(V_0,\varphi) - C_T\right\}^2\right],$$

where β_T is the discount factor and $C_T = 100 \exp\left\{g\left(R_{0,T}^{(1)}, R_{0,T}^{(2)}\right)\right\}$ is the payoff at maturity.

In order to achieve this, we develop extensions of the results of Schweizer (1995). Note that there is no "risk-neutral" evaluation involved in our approach and that all calculations are carried out under the objective probability measure.

If the dynamic replication is successful, i.e., $V_T = C_T$, then return of the investment can be decomposed as

$$\log(V_T/V_0) = \log(100/V_0) + g\left(R_{0,T}^{(1)}, R_{0,T}^{(2)}\right).$$

Therefore, as proposed in Kat and Palaro (2005), one can view $\alpha = \log(100/V_0)$ as a measure of performance. For, if $\alpha = 0$, we generate exactly the target distribution, while if $\alpha > 0$, we outperform the target distribution ; if $\alpha < 0$, then the fund outperforms the replication strategy. However, whatever the value of α, statistics based on centered moments are not affected ; only the value of the expectation depends on α.

2.5.1 Optimal hedging

Suppose that $(\Omega, P, \mathcal{F})$ is a probability space with filtration $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$, under which the stochastic processes are defined. For the moment, assume that the price process S_t is d-dimensional, i.e. $S_t = \left(S_t^{(1)}, \ldots, S_t^{(d)}\right)$. In the next section, one will come back with the case $d = 2$.

Before defining what is meant by a dynamic replicating strategy, let β_t denote the discount factor, i.e. β_t is the value at period 0 to be invested in the non risky asset so that it has a value of 1$ at period t. By definition, $\beta_0 = 1$. It is assumed that the process β is predictable, i.e. β_t is $\mathcal{F}_{t-1}$-measurable for all $t = 1, \ldots, T$.

A dynamic replicating strategy can be described by a (deterministic) initial value V_0 and a sequence of random weight vectors $\varphi = (\varphi_t)_{t=0}^T$, where for any $j = 1, \ldots, d$, $\varphi_t^{(j)}$ denotes the number of parts of assets $S^{(j)}$ invested during period $(t-1, t]$. Because φ_t may depend only on the values values $S_0, \ldots, S_{t-1}$, the stochastic process φ_t is assumed to be predictable. Initially, $\varphi_0 = \varphi_1$, and the portfolio initial value is V_0. It follows that the amount initially invested in the non risky asset is

$$V_0 - \sum_{j=1}^{d} \varphi_1^{(j)} S_0^{(j)} = V_0 - \varphi_1^\top S_0.$$

Since the hedging strategy must be self-financing, it follows that for all $t = 1, \ldots, T$,

$$\beta_t V_t(V_0, \varphi) - \beta_{t-1} V_{t-1}(V_0, \varphi) = \varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1}). \tag{2.1}$$

Using the self-financing condition (2.1), it follows that

$$\beta_T V_T = \beta_T V_T(V_0, \varphi) = V_0 + \sum_{t=1}^{T} \varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1}). \tag{2.2}$$

The replication strategy problem for a given payoff C is thus equivalent to finding the strategy (V_0, φ) so that the hedging error

$$G_T(V_0, \varphi) = \beta_T V_T(V_0, \varphi) - \beta_T C \tag{2.3}$$

is as small as possible. In this paper, we choose the expected square hedging error as a measure of quality of replication. It is therefore natural to suppose that the prices $S_t^{(j)}$ have finite second moments. We further assume that the hedging strategy φ satisfies a similar property, namely that for any $t = 1, \ldots, T$, $\varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1})$ have finite second moments. Note that these two technical conditions were also made by Schweizer (1995).

For simplicity, set

$$\Delta_t = S_t - E(S_t|\mathcal{F}_{t-1}), \qquad t = 1, \ldots, T.$$

Under the above moment conditions, the conditional covariance matrix Σ_t of Δ_t exists and is given by

$$\Sigma_t = E\left\{\Delta_t \Delta_t^\top | \mathcal{F}_{t-1}\right\},\ 1 \le t \le T.$$

In Schweizer (1995), the author treats the case $d = 1$ and assumes a restrictive boundedness condition. Here, in contrast, we treat the general d-dimensional case and we only suppose that Σ_t is invertible for all $t = 1, \ldots, T$. This was implicitly part of the boundedness condition of Schweizer (1995).

If Σ_t is not invertible for some t, there would exists a $\varphi_t \in \mathcal{F}_{t-1}$ such that $\varphi_t^\top S_t = \varphi_t^\top E(S_t|\mathcal{F}_{t-1})$, that is, $\varphi_t^\top S_t$ is predictable. Our assumption can be interpreted as saying that the genuine dimension of the assets is d. One may now state the main result whose proof is given in Appendix 2.7.

Theorem 1 *Suppose that Σ_t is invertible for all $t = 1, \ldots, T$.*

Then the risk $E\{G^2(V_0, \varphi)\}$ is minimized by choosing recursively $\varphi_T, \ldots, \varphi_1$ satisfying

$$\varphi_t = (\Sigma_t)^{-1} E\left(\{S_t - E(S_t|\mathcal{F}_{t-1})\} C_t | \mathcal{F}_{t-1}\right),\ t = T, \ldots, 1, \tag{2.4}$$

where $C_T, \ldots, C_0$ are defined recursively by setting $C_T = C$ and

$$\beta_{t-1} C_{t-1} = \beta_t E(C_t|\mathcal{F}_{t-1}) - \varphi_t{}^\top E(\beta_t S_t - \beta_{t-1} S_{t-1}|\mathcal{F}_{t-1}), \tag{2.5}$$

for $t = T, \ldots, 1$.

Moreover the optimal value of V_0 *is* C_0*, and*

$$E(G^2) = \sum_{t=1}^{T} E\left(\beta_t^2 G_t^2\right),$$

where $G_t = \varphi_t{}^\top \{S_t - E(S_t|\mathcal{F}_{t-1})\} - \{C_t - E(C_t|\mathcal{F}_{t-1})\}$*,* $1 \le t \le T$*.*

Having found the optimal hedging strategy, according to the mean square error criterion, one might ask what the link is between the price given by C_0, as in Theorem 1, and the price suggested by the martingale measure method. The answer is given by the following result proven in Appendix 2.7.

Corollary 1 *For any* $t = 1\ldots, T$*, set*

$$U_t = 1 - \Delta_t^\top (\Sigma_t)^{-1} E\left(S_t - \beta_{t-1}S_{t-1}/\beta_t|\mathcal{F}_{t-1}\right). \tag{2.6}$$

Further set $M_0 = 1$ *and* $M_t = U_t M_{t-1}$*,* $1 \le k \le n$*. Then* $(M_t, \mathcal{F}_t)_{t=0}^T$ *is a (not necessarily positive) martingale and*

$$\beta_{t-1}C_{t-1} = E(\beta_t C_t U_t|\mathcal{F}_{t-1}).$$

In particular $\beta C_t M_t$ *is a martingale and* $C_0 = E(\beta_T C_T M_T|\mathcal{F}_0)$*.*

Moreover $E(\beta_t S_t U_t|\mathcal{F}_{t-1}) = \beta_{t-1}S_{t-1}$*, so* $\beta_t S_t M_t$ *is a martingale.*[3]

The Markovian case

If the price process S is Markovian, i.e., the law of S_t given $\mathcal{F}_{t-1}$ is $\nu_t(S_{t-1}, dx)$, and if the terminal payoff $C_T = C$ only depends on the terminal prices, that is $C = f_T(S_T)$, then the Markov property, together with Theorem 1, yield that $C_t = f_t(S_t)$ and $\varphi_t =$

3. When the market is complete, there is a unique martingale measure Q and every claim is attainable, so the risk associated with the optimal strategy is zero. Therefore M_t, as defined in Corollary 1 is positive, and as a by-product of our method, we have an explicit representation of the density of Q with respect to P.

$\psi_t(S_{t-1})$, where

$$\begin{aligned}
L_{1t}(s) &= E(S_t|S_{t-1}=s) = \int x\nu_t(s,dx), \\
L_{2t}(s) &= E(S_t S_t^\top|S_{t-1}=s) = \int xx^\top \nu_t(s,dx), \\
A_t(s) &= L_{2t}(s) - L_{1t}(s)L_{1t}(s)^\top, \\
\psi_t(s) &= A_t(s)^{-1}E\left[\{S_t - L_{1t}(s)\}f_t(S_t)|S_{t-1}=s\right] \\
&= A_t(s)^{-1}\int (x - L_{1t}(s))f_t(x)\nu_t(s,dx), \\
U_t(s,x) &= 1 - (L_{1t}(s) - \beta_{t-1}s/\beta_t)^\top A_t(s)^{-1}(x - L_{1t}(s)), \\
f_{t-1}(s) &= \frac{\beta_t}{\beta_{t-1}}E\{U_t(s,S_t)f_t(S_t)|S_{t-1}=s\} \\
&= \frac{\beta_t}{\beta_{t-1}}\int U_t(s,x)f_t(x)\nu_t(s,dx).
\end{aligned}$$

Note that $E(S_t|\mathcal{F}_{t-1}) = L_{1t}(S_{t-1})$ and $\Sigma_t = A_t(S_{t-1})$. Explicit calculations can be done when the returns are assumed to be a finite Markov chain. In most models, one can write $S_t = \omega_t(S_{t-1}, \xi_t)$ where ξ_t is independent of $\mathcal{F}_{t-1}$ and has law P_t. When μ_t has an infinite support, there are ways to approximate ψ_t and f_t.

The importance of Theorem 1 to the replication problem of hedge funds is obvious, particularly under the Markovian setting. All that is needed is a way to calculate or approximate the value of f_0 and of the deterministic functions $\psi_t(s)$, $f_t(s)$, $t = 1, \ldots$. In particular $V_0 = f_0$ and $\varphi_t = \psi_t(s)$ gives the optimal hedging strategy when $S_{t-1} = s$.

In the Markovian case, one can use the methodology developed by Del Moral et al. (2006) to calculate both the φ_t's and the C_t's. The algorithm for implementing the dynamic trading strategy is based on Monte Carlo simulations and linear interpolation and is detailed in Appendix 2.7.

2.5.2 A comparison between optimal hedging and hedging under Black-Scholes setting

To compare the two methods, simply take $T = 1$ and $r = 0$ and $d = 1$. In this case, the solution for optimal hedging yields $\varphi^\star = \text{Cov}\{\Delta S_1, C(S_1)\}/\text{Var}(\Delta S_1)$, where $\Delta S_1 = S_1 - S_0$, and $V_0^\star = E\{C(S_1)\} - \varphi^\star E(\Delta S_1)$.

For the Black-Scholes setting, we have

$$V_0^{BS} = E\left\{C\left(S_0 e^{\sigma Z - \sigma^2/2}\right)\right\} \quad \text{and} \quad \varphi^{BS} = E\left\{e^{\sigma Z - \sigma^2/2} C'\left(S_0 e^{\sigma Z - \sigma^2/2}\right)\right\},$$

with $\sigma^2 = \text{Var}\{\log(S_1/S_0)\}$, where $Z \sim N(0,1)$, provided C is differentiable. See, e.g., Broadie and Glasserman (1996). In general, $\varphi^\star \neq \varphi^{BS}$ and $V_0^\star \neq V_0^{BS}$, so

$$E\left[\{V_1(V_0^\star, \varphi^\star) - C(S_1)\}^2\right] < E\left[\left\{V_1(V_0^{BS}, \varphi^{BS}) - C(S_1)\right\}^2\right].$$

For an analysis of the (discrete) hedging error in a Black-Scholes setting, see, e.g., Wilmott (2006). To illustrate the difference in an hedge funds context, we performed a numerical experiment in which we tried (10 000 times) to reproduce a synthetic fund with centered Gaussian distribution with annual volatility 12% and correlation 30% with the portfolio. The distribution of the daily returns of the (portfolio, reserve) pair are modeled by a mixture of 4 regimes for the daily returns distribution with parameters given in Exhibit 2.I. With this choice of parameters, it turns out that the associated monthly returns are best modeled by a bivariate Gaussian with parameters given in Exhibit 2.II.

As said previously, we simulated 10 000 values of $g\left(R_{0,T}^{(1)}, R_{0,T}^{(2)}\right)$, $\log(V_T^\star/100)$ (under optimal hedging) and $\log(V_T^{BS}/100)$ (under delta hedging). Some sample characteristics of these three variables are given in Exhibit 2.III, together with the corresponding true values, while for each dynamic trading method, the estimated mean hedging error and square root mean square error are given in Exhibit 2.IV.

TABLE 2.I – Parameters for the Gaussian mixture with 4 regimes used for modeling daily returns

π_k	μ_{k1}	μ_{k2}	σ_{1k}	σ_{2k}	ρ_k
0.0956	0.0016	0.0008	0.0039	0.0016	0.9754
0.4673	0.0000	0.0002	0.0069	0.0032	0.7981
0.0763	-0.0003	-0.0005	0.0115	0.0054	0.6964
0.3607	0.0006	0.0005	0.0037	0.0027	0.4613

TABLE 2.II – Estimation of the parameters of the Gaussian model compatible with the daily returns

μ_1	μ_2	σ_1	σ_2	ρ
0.007892797	0.0068086	0.029334999	0.014646356	0.700295314

By construction, optimal hedging always produces an hedging error with zero mean. However, this is not the case in general for delta hedging. Note how far the delta hedging method is off the goal of a zero mean of the replicating portfolio, while the optimal hedging error is much smaller.

As our proposed method is optimal for minimizing the square hedging error, it is not surprising that it dominates delta hedging. However, since the theoretical setting is very close to the Black-Scholes setting, all monthly returns being Gaussian, it is worth noting that the square root Mean Square Error of the optimal hedging is 150% less than the one of the delta hedging.

Finally, the distribution of the respective hedging errors is illustrated in Exhibit 2.5. From that graph, it appears that the values of the replication portfolio with the methodology proposed in Kat and Palaro (2005) are almost always smaller than the target values.

Table 2.III – Replication results based on 10 000 trajectories for $g\left(R^{(1)}_{0,T}, R^{(2)}_{0,T}\right) = \log(C_T/100)$ and $\log(V_T/100)$ under optimal hedging and delta hedging.

Parameter	True value	g	Optimal hedging	Delta hedging
Mean	0	3.957E-07	3.574E-07	-0.000422735
Std. dev.	0.034641016	0.034957842	0.034961135	0.034985553
Skewness	0	-0.058910418	-0.064053039	-0.063978046
Kurtosis	0	0.029916203	0.032479236	0.032374552
ρ	0.3	0.30283895	0.30279462	0.30288552

Table 2.IV – Replication results based on 10 000 trajectories for the payoff $\tilde{g}$ and $\log(V_T/100)$ under optimal hedging and delta hedging.

Parameter	Optimal hedging	Delta hedging	$\|OH/DH\|$
Mean hedging error	0.000004009	-0.042061101	10491.66889
Square root MSE	0.017861376	0.045665732	2.556674977

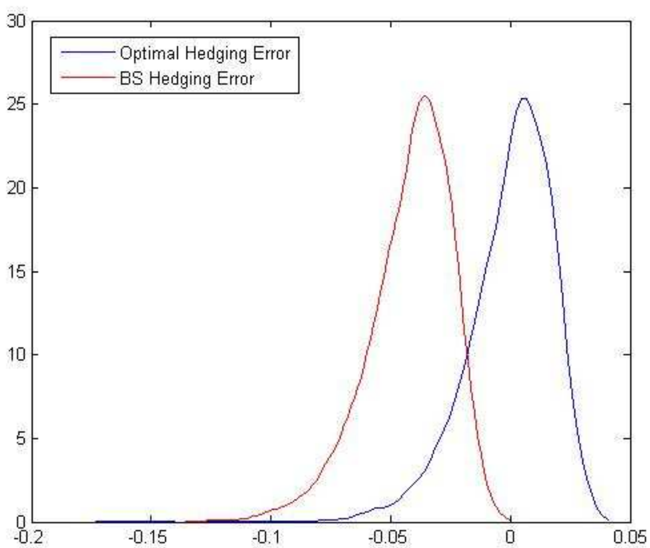


Figure 2.5 – Kernel density estimation of hedging errors for optimal hedging and delta hedging.

2.6 Replication of hedge fund indices

In this section we will provide some empirical evidence regarding the ability of the model to replicate hedge fund returns. For the sake of parsimony, we will present results for the (in-sample) replication of the EDHEC indices and HFR indices. We will look at the models ability to replicate the statistical properties of the monthly returns of the different indices over the ten year period from 01/30/1997 to 12/29/2006 (120 months), as well as for 2 subperiods ranging respectively from 01/30/1997 to 12/29/2001 (59 months) and from 12/30/2001 - 12/29/2006 (61 months).

2.6.1 Portfolio and Reserve assets

The first step is to select the assets that will make up the investor portfolio $S^{(1)}$, and the reserve asset $S^{(2)}$. Because these two portfolios are dynamically traded on a daily basis, we seek very liquid instruments with low transaction costs. We therefore restrict the components of these two assets to be either Futures contracts or Exchange Traded Funds (ETF).

All futures data comes from CRB Trader database. The cash rate is the BBA Libor 1 month rate. Log-returns on futures are calculated from the reinvestment of a rolling strategy in the front contract. The front contract is the nearest to maturity, on the March/June/September/December schedule and is rolled on the first business day of the maturity month at previous close prices. Each future contract is fully collateralized, so that, the total return is the sum the rolling strategy's return and the cash rate. The ETF data is obtained from Bloomberg.

The investor portfolio, which is meant to be a proxy for a typical institutional portfolio, will be an equal-weighted portfolio of S&P500 futures contracts and 30 year US Treasury Bond futures contracts. In order to illustrate the sensitivity of the methodology to the choice of reserve asset, we will perform the study using two very different reserve assets. The first asset (Reserve 1) is made up of 50% PowerShares Dynamic Small Cap Value Portfolio, 25% iShares Lehman 20 Year Treasury Bond Fund and 25% Citigroup Treasury 10 Year Bond Fund. The second asset (Reserve 2) is an equally weighted portfolio Two Year Treasury Notes, Ten Year Treasury Notes, S&P500, and Goldman Sachs Commodity Index future contracts.

Exhibit 2.VI presents some of the statistical properties of our investor portfolio and the two reserve assets for the entire ten year period and the two sub-periods. We report the mean, standard deviation, skewness, robust skewness [4], kurtosis, robust kurtosis [5]

TABLE 2.VI – Summary statistics for the portfolio and the reserve assets over the three time periods.

Asset	Statistics	Period 1 (97–06)	Period 2 (97–01)	Period 3 (02–06)
Portfolio	Mean	0.0035	0.0047	0.0024
	S.Dev	0.0244	0.0289	0.0192
	Skew	-0.2150	-0.2697	-0.2482
	R. Sk	-0.0813	-0.2665	-0.1097
	Kurt	3.2109	2.6942	3.6637
	R. Kurt	3.2467	2.7483	3.6386
Reserve 1	Mean	0.0094	0.0095	0.0093
	S.Dev	0.0225	0.0260	0.0187
	Skew	0.3006	0.5346	-0.3480
	R. Sk	0.0362	0.0552	0.0159
	Kurt	5.0025	5.0399	3.2161
	R. Kurt	3.2419	4.0561	2.9244
	Corr. with Port.	0.6749	0.7054	0.6206
Reserve 2	Mean	0.0031	0.0016	0.0047
	S.Dev	0.0195	0.0219	0.0168
	Skew	0.0338	0.3193	-0.3886
	R. Sk	-0.0891	-0.0161	-0.2345
	Kurt	3.4509	3.3083	3.7213
	R. Kurt	3.3207	3.3894	3.4959
	Corr. with Port.	0.6040	0.7231	0.3989

4. Defined by $\{E(X) - Q(1/2)\} \big/ E\{|X - Q(1/2)|\}$, where Q_α is the α-quantile.

5. Defined by $0.09 + \{Q(.975) - Q(.025)\} \big/ \{Q(.75) - Q(.25)\}$.

As explained in Section 2.3.1, we have chosen to model the daily returns of the pairs (portfolio, reserve) by bivariate Gaussian mixtures with m regimes, denoted by BGM(m).

In Exhibit 2.VII, the distributions of the daily and monthly returns for the (portfolio, reserve) pairs are given, over the three time periods. These results were obtained by using the estimation and goodness-of-it procedures described in Section 2.3.1.

TABLE 2.VII – Distribution of the daily and monthly returns for the two pairs (portfolio, reserve), over the three time periods.

Returns	Period 1 (97–06)		Period 2 (97–01)		Period 3 (02–06)	
	Reserve 1	Reserve 2	Reserve 1	Reserve 2	Reserve 1	Reserve 2
Daily	BGM(5)	BGM(5)	BGM(5)	BGM(5)	BGM(3)	BGM(4)
Monthly	BGM(2)	BGM(2)	BGM(4)	BGM(2)	BGM(2)	BGM(3)

It may seems odd at first that the model for the joint monthly returns is a (bivariate) Gaussian mixture with fewer regimes than for the daily returns. However, as explained in Remark 2.7.1, it is quite normal. In fact, in view of the central limit theorem, the number of regimes would possibly be 1 if we were to consider returns over a two months period.

2.6.2 Hedge fund indices

For the sake of comparison, we chose to replicate the 13 EDHEC indices and the 22 HFRI indices. According to the procedures described in Sections 2.3.2 and 2.3.3, the marginal distribution F_3 and the copula $C_{1,3}$ were estimated for each hedge fund index.

For the marginal distributions, we considered (univariate) Gaussian mixtures with m regimes, denoted GM(m) and Johnson distribution. For the copula families, we selected the Gaussian, Student, Clayton, Frank and Gumbel. In each case, we estimated

Kendall's tau, which measures the dependence between the hedge fund returns and the portfolio returns. Except for the Student copula, which is dependent on two parameters, the other families only depend on one parameter.

The best fitting models are displayed in Exhibits 2.VIII–2.X.

2.6.3 Performance of the replication

There are two important issues that need to be addressed when analyzing the models ability to replicate hedge fund returns. The first issue concerns the models ability to effectively replicate hedge fund indices. The second issue pertains to the choice of the reserve asset and it's impact on the models performance.

To study the effectiveness of the replication strategies, there are two main factors to consider : the initial investment V_0 that is required to replicate each index as well as the actual quality of the replication. In order to obtain the payoff distribution of the hedge fund indices, we follow the approach used by Kat and Palaro (2005)- we calculate the monthly returns assuming an investment of 100 at the beginning of each month. Therefore, if the value V_0 of the replicating strategy is below 100, this would lead us to conclude that the replicating strategy offers a cheaper alternative to the hedge fund index, and therefore is the better investment choice. This analysis can however be misleading if we do not also examine the precision of the replication strategy. Before dismissing the hedge fund indices as poor-performers, we need to properly evaluate whether the properties of the replication strategies and hedge fund indices are truly the same. A proper examination of both the cost and the precision of the replication strategy is fundamental before any strong conclusion can be drawn about the model's ability to replicate hedge fund indices.

Then arises the question of the reserve asset. Does the reserve asset impact the performance of the model, and if so does it affect only V_0 or also the ability of the model to replicate the statistical properties of the hedge fund indices ? In other words, does the choice of reserve asset impact the performance measure and/or for the quality of the replication ?

Exhibits 2.XI–2.XIII present the values of V_0 for the HFRI and EDHEC hedge fund indices. It is quite clear that even without correcting for the well documented biases in hedge fund indices, the replicating strategies still out-perform a large number of the hedge fund indices over the entire period as well as over the two sub-periods. In order to show that the replication strategies are effectively reproducing the statistical properties of the hedge fund indices, Exhibits 2.15–2.19 present the target mean, volatility, Kendall's tau, skewness and kurtosis of the indices as well as those for the replication strategies. It is quite clear that independently of the period that is considered, the volatility and Kendall's tau are reproduced with great precision. It is important to note that the only moment that is sensitive to the choice of reserve asset is the return of the replication strategy - the other moments as well as the dependence coefficient appear to be insensitive to the choice of reserve asset. Our results clearly indicate that the reserve asset plays a role in the measure of performance, V_0, but it has almost no effect on the quality of the replication.

In order to further examine the model's ability to replicate the statistical properties of the hedge fund indices, Exhibit 2.XIV presents the results obtained by regressing the statistical properties of the replication portfolios against the estimated parameters of both EDHEC and HFRI indices for the three samples periods. If the replications were perfect, the slope would then be 1 and the intercept would be 0. As one can see, the fit is very impressive for both reserve assets. The volatility and dependence measures (Kendal's tau and Spearman's Rho) are perfectly replicated, and the regression coefficients for the higher moments, although not perfect, support the model's ability to replicate the statistical properties of hedge fund returns.

The final stage of the analysis consists of breaking down the costs and other potential sources of error associated with the dynamic replicating strategy. We quantify three potential costs/errors associated with our methodology. The first is the transaction costs related to the dynamic trading ; the second is the rounding error that results from not being able to trade fractions of futures contracts ; the third, and most significant, is the profit/loss that is due to the hedging error of the discrete hedging strategy.

The transaction costs are assumed to be 1 basis point for the sale/purchase of all futures contracts. Obviously, the amount of trading required to replicate the different indices can vary substantially. In Exhibit 2.XX we present the average monthly transaction costs (in terms of basis points) incurred for each replicating portfolio over the whole sample period. Note that the average monthly transaction costs for the replication strategies is approximately 5 basis points.

The rounding error that results from the inability to buy or sell fractions of futures contracts depends very much on the size of the replication portfolio and this error tends to zero as the portfolio increases in size. For a replicating strategy with $100 Million invested, the average monthly rounding error is approximately 1 basis point.

Finally, we calculate replicating errors, that is the average difference between the value of the replicating strategy and the value of the hedge fund index. The results are presented in Exhibits 2.XXI–2.XXIII. Note that the average monthly hedging error on all replications as defined in Equation 2.3 is around 3 basis points.

2.7 Conclusion

In this paper, we implement a multivariate extension of Dybvig (1988) Payoff Distribution Model that can be used to replicate not only the marginal distribution of hedge fund returns but also their dependence with other asset classes. In addition to proposing ways to overcome the hedging and compatibility inconsistencies in Kat and Palaro (2005) we extend the results of Schweizer (1995) and adapt American options pricing techniques to evaluate the model and also derive an optimal dynamic trading (hedging) strategy. In section 2.5.2 we demonstrate the superiority of the hedging algorithm that is used to generate the dynamic replicating strategy. We successfully replicate the statistical properties of the HFRI and EDHEC indices over the period from 1997-2006, as well as for two 60 month sub-periods. Even without correcting for the well-documented biases in hedge fund index returns, the indices can be readily replicated using this methodology. The volatility and the dependence coefficients are replicated with great precision ; the skewness and kurtosis are also captured by the model, however with slightly less accuracy.

Contrary to the conclusions put forth by recent studies at EDHEC (Amenc et al., 2007) and Northwater (Simons and Hussey, 2007), the choice of reserve asset does not impact the model's ability to replicate the statistical properties of the indices. The choice of reserve asset only impacts the initial cost of investing in the replicating portfolio (and hence only impacts the return of the replicating strategy). This is not to say that the return generated by the model is not important, however it is not a measure of the model's success. One must dissociate the technical issues of the replicating methodology (i.e how to best model the returns and solve for the optimal trading strategy) from the choice of the reserve asset. Our contribution is to provide a robust framework for the replication methodology, and address the technical shortcomings of the much publicized research of Kat and Palaro.

As is the case with any investment strategy, the returns depend on the choice of assets. The results in this paper indicate, however, that it is not necessary to select the best performing assets over the sample period in order to replicate and outperform the hedge fund indices. In fact, we show that by using run-of-the-mill exposures in our reserve asset we can nonetheless outperform the majority of hedge fund indices. We purposely selected two reserve assets that have exposures to different yet common market premia over the sample period, and we find that both reserve assets outperform a large percentage of the indices. (reserve 1 being the better of the two). We also find that the EDHEC indices, which are subject to less significant biases, are more easy to replicate that the HFRI indices. It is important to remember that we are comparing an investable trading strategy to non-investable indices- the actual return we would anticipate from investing in a hedge fund index would be considerably lower than the ”non-investable” index returns used in this study. Our results reinforce the notion that on aggregate, hedge funds are on aggregate simply repackaging beta returns.

Bibliography

Agarwal, V. and Naik, N. (2004). Risks and portfolio decisions involving hedge funds. *Review of Financial Studies*, 17 :63–98.

Amenc, N., Géhin, W., Martellini, L., and Meyfredi, J. (2007). The myths and limits of passive hedge fund replication. Technical report, EDHEC Risk and Asset Management Research Center.

Amin, G. and Kat, H. (2003). Hedge fund performanc 1990-2000 : Do the "money machines" really add value. *Journal of Financial and Quantitative Analysis*, 38(2) :251–275.

Broadie, M. and Glasserman, P. (1996). Estimating security price derivatives using simulation. *Management Science*, 42 :260–285.

Chen, Y. and Liang, B. (2006). Do market timing hedge funds time the market ? Technical report, Carroll School of Management, Boston College.

Del Moral, P., Rémillard, B., and Rubenthaler, S. (2006). Monte Carlo approximations of American options. Technical report, GERAD.

Dempster, A. P., Laird, N. M., and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm. *J. Roy. Statist. Soc. Ser. B*, 39 :1–38.

Durbin, J. (1973). Weak convergence of the sample distribution function when parameters are estimated. *Ann. Statist.*, 1(2) :279–290.

Dybvig, P. (1988). Distributional analysis of portfolio choice. *The Journal of Business*, 61(3) :369–393.

Fung, W. and Hsieh, D. (2001). The risk in hedge fund stategies : Theory and evidence from trend followers. *Review of Financial Studies*, 14 :313–341.

Genest, C., Ghoudi, K., and Rivest, L.-P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82 :543–552.

Genest, C., Quessy, J.-F., and Rémillard, B. (2006). Goodness-of-fit procedures for copula models based on the integral probability transformation. *Scand. J. Statist.*, 33 :337–366.

Genest, C. and Rémillard, B. (2005). Validity of the parametric bootsrap for goodness-of-fit testing in semiparametric models. Technical Report G-2005-51, GERAD.

Genest, C., Rémillard, B., and Beaudoin, D. (2009). Omnibus goodness-of-fit tests for copulas : A review and a power study. *Insurance Math. Econom.*, 44 :199–213.

Hasanhodzic, J. and Lo, A. W. (2007). Can hedge-fund returns be replicated ? : The linear case. *Journal of Investment Management*, 5(2) :5–45.

Kat, H. and Palaro, H. (2005). Who needs hedge funds ? A copula-based approach to hedge fund return replication. Technical report, Cass Business School, City University.

Kazemi, H. and Schneeweis, T. (2003). Conditional performance of hedge funds. Technical report, Center for International Securities and Derivatives Markets.

Mitchell, M. and Pulvino, T. (2001). Characteristics of risk and return in risk arbitrage. *Journal of Finance*, LVI :2135–2175.

Nelsen, R. B. (1999). *An introduction to copulas*, volume 139 of *Lecture Notes in Statistics*. Springer-Verlag, New York.

Rosenblatt, M. (1952). Remarks on a multivariate transformation. *Ann. Math. Stat.*, 23 :470–Ű472.

Schweizer, M. (1995). Variance-optimal hedging in discrete time. *Math. Oper. Res.*, 20(1) :1–32.

Simons, N. and Hussey, A. (2007). Northwater capital managementŠs thoughts on hedge fund replication. Technical report, Northwater Capital Management.

Stute, W., Gonzáles Manteiga, W., and Presedo Quindimil, M. (1993). Bootstrap based goodness-of-fit tests. *Metrika*, 40 :243–256.

Wilmott, P. (2006). *Paul Wilmott on Quantitative Finance*, volume 3. John Wiley & Sons.

Appendix A : Replication Tables

Table 2.VIII – Marginal distribution, copula and Kendall's tau for entire period (1997–2006).

Fund	Marginal	Copula	Kendall's tau
EDHEC-Convertible Arbitrage	GM(3)	Frank	0.0927
EDHEC-CTA Global	GM(2)	Gumbel	0.0552
EDHEC-Distressed Securities	GM(2)	Clayton	0.2311
EDHEC-Emerging Markets	Johnson	Frank	0.3394
EDHEC-Equity Market Neutral	GM(2)	Frank	0.2302
EDHEC-Event Driven	GM(3)	Frank	0.3724
EDHEC-Fixed Income Arbitrage	GM(3)	Frank	0.0997
EDHEC-Global Macro	GM(3)	Frank	0.3316
EDHEC-Long/Short Equity	GM(2)	Student	0.4529
EDHEC-Merger Arbitrage	GM(2)	Frank	0.2956
EDHEC-Relative Value	GM(3)	Gaussian	0.3324
EDHEC-Short Selling	GM(2)	Frank	-0.4636
EDHEC-Funds of Funds	GM(4)	Gaussian	0.3536
HFRI Convertible Arbitrage Index	GM(3)	Frank	0.1048
HFRI Distressed Securities Index	GM(3)	Clayton	0.2160
HFRI Emerging Markets (Total)	Johnson	Student	0.3269
HFRI Equity Hedge Index	GM(2)	Clayton	0.4530
HFRI Equity Market Neutral Index	GM(3)	Frank	0.1345
HFRI Equity Non-Hedge Index	GM(3)	Student	0.4770
HFRI Event-Driven Index	GM(3)	Clayton	0.3700
HFRI Fixed Income (Total)	GM(3)	Frank	0.3168
HFRI Fixed Income : Arbitrage Index	GM(3)	Ind.	0
HFRI Fixed Income : High Yield Index	GM(2)	Student	0.2036
HFRI FOF : Conservative Index	Johnson	Frank	0.3021
HFRI FOF : Diversified Index	GM(3)	Frank	0.2945
HFRI FOF : Market Defensive Index	GM(2)	Frank	0.1020
HFRI FOF : Strategic Index	GM(3)	Frank	0.3555
HFRI FOF Composite Index	GM(3)	Frank	0.3327
HFRI FOF Composite Index (Off.)	GM(3)	Frank	0.3180
HFRI Fund Weighted Composite Index	GM(3)	Clayton	0.4403
HFRI Macro Index	GM(2)	Clayton	0.2364
HFRI Merger Arbitrage Index	GM(3)	Frank	0.2568
HFRI Regulation D Index	GM(3)	Gaussian	0.2210
HFRI Relative Value Arbitrage Index	GM(3)	Gaussian	0.2567
HFRI Short Selling Index	GM(3)	Frank	-0.4520

Table 2.IX – Marginal distribution, copula and Kendall's tau for first sub-period (1997–2001).

Fund	Marginal	Copula	Kendall's tau
EDHEC-Convertible Arbitrage	GM(3)	Gumbel	0.0777
EDHEC-CTA Global	GM(2)	Ind.	0
EDHEC-Distressed Securities	GM(3)	Clayton	0.2309
EDHEC-Emerging Markets	GM(3)	Frank	0.3241
EDHEC-Equity Market Neutral	GM(2)	Gaussian	0.3691
EDHEC-Event Driven	Johnson	Clayton	0.3793
EDHEC-Fixed Income Arbitrage	GM(3)	Frank	0.1268
EDHEC-Global Macro	GM(3)	Frank	0.4198
EDHEC-Long/Short Equity	GM(2)	Frank	0.4868
EDHEC-Merger Arbitrage	GM(4)	Gumbel	0.2951
EDHEC-Relative Value	GM(3)	Clayton	0.3454
EDHEC-Short Selling	GM(2)	Frank	-0.4695
EDHEC-Funds of Funds	GM(2)	Frank	0.3934
HFRI Convertible Arbitrage Index	GM(3)	Frank	0.1011
HFRI Distressed Securities Index	GM(3)	Gaussian	0.1939
HFRI Emerging Markets (Total)	GM(3)	Frank	0.3148
HFRI Equity Hedge Index	GM(2)	Frank	0.4880
HFRI Equity Market Neutral Index	GM(2)	Frank	0.1607
HFRI Equity Non-Hedge Index	Johnson	Frank	0.4962
HFRI Event-Driven Index	GM(3)	Frank	0.3461
HFRI Fixed Income (Total)	GM(3)	Frank	0.3078
HFRI Fixed Income : Arbitrage Index	GM(3)	Ind.	0
HFRI Fixed Income : High Yield Index	GM(3)	Frank	0.2367
HFRI FOF : Conservative Index	GM(3)	Frank	0.3310
HFRI FOF : Diversified Index	Johnson	Frank	0.2915
HFRI FOF : Market Defensive Index	GM(3)	Frank	0.1257
HFRI FOF : Strategic Index	GM(3)	Frank	0.3600
HFRI FOF Composite Index	GM(2)	Frank	0.3427
HFRI FOF Composite Index (Off.)	GM(2)	Frank	0.3276
HFRI Fund Weighted Composite Index	GM(3)	Frank	0.4567
HFRI Macro Index	GM(2)	Clayton	0.2975
HFRI Merger Arbitrage Index	Johnson	Gumbel	0.2285
HFRI Regulation D Index	GM(3)	Gaussian	0.2736
HFRI Relative Value Arbitrage Index	GM(3)	Frank	0.2705
HFRI Short Selling Index	GM(2)	Frank	-0.4402

TABLE 2.X – Marginal distribution, copula and Kendall's tau for second sub-period (2002–2006).

Fund	Marginal	Copula	Kendall's tau
EDHEC-Convertible Arbitrage	GM(3)	Gaussian	0.0885
EDHEC-CTA Global	GM(2)	Frank	0.0743
EDHEC-Distressed Securities	GM(2)	Gaussian	0.2224
EDHEC-Emerging Markets	GM(3)	Frank	0.2710
EDHEC-Equity Market Neutral	Johnson	Frank	0.0896
EDHEC-Event Driven	Johnson	Gaussian	0.3052
EDHEC-Fixed Income Arbitrage	GM(3)	Ind.	0
EDHEC-Global Macro	GM(2)	Gaussian	0.1987
EDHEC-Long/Short Equity	GM(2)	Clayton	0.3377
EDHEC-Merger Arbitrage	GM(3)	Clayton	0.3126
EDHEC-Relative Value	GM(2)	Clayton	0.2973
EDHEC-Short Selling	GM(2)	Frank	-0.4266
EDHEC-Funds of Funds	Johnson	Clayton	0.2470
HFRI Convertible Arbitrage Index	GM(3)	Gumbel	0.0743
HFRI Distressed Securities Index	GM(2)	Clayton	0.2109
HFRI Emerging Markets (Total)	GM(3)	Frank	0.2797
HFRI Equity Hedge Index	GM(3)	Frank	0.2993
HFRI Equity Market Neutral Index	GM(2)	Frank	0.0874
HFRI Equity Non-Hedge Index	GM(2)	Frank	0.3687
HFRI Event-Driven Index	GM(3)	Gaussian	0.3377
HFRI Fixed Income (Total)	GM(3)	Gaussian	0.2303
HFRI Fixed Income : Arbitrage Index	GM(3)	Ind.	0
HFRI Fixed Income : High Yield Index	GM(2)	Gumbel	0.1311
HFRI FOF : Conservative Index	GM(2)	Frank	0.2164
HFRI FOF : Diversified Index	GM(2)	Clayton	0.2437
HFRI FOF : Market Defensive Index	GM(2)	Frank	0.0831
HFRI FOF : Strategic Index	GM(3)	Clayton	0.2885
HFRI FOF Composite Index	GM(2)	Clayton	0.2383
HFRI FOF Composite Index (Off.)	GM(2)	Clayton	0.2164
HFRI Fund Weighted Composite Index	GM(2)	Frank	0.3243
HFRI Macro Index	GM(2)	Gumbel	0.0787
HFRI Merger Arbitrage Index	GM(2)	Clayton	0.2984
HFRI Regulation D Index	Johnson	Clayton	0.1552
HFRI Relative Value Arbitrage Index	GM(2)	Clayton	0.2328
HFRI Short Selling Index	GM(2)	Frank	-0.4319

Table 2.XI – Initial investment V_0 in the replication of EDHEC and HFRI indices for both reserve assets over the entire period (1997–2006).

Fund	V_0	
	Reserve 1	Reserve 2
EDHEC-Convertible Arbitrage	99.88746927	100.3546058
EDHEC-CTA Global	99.22395238	100.2822217
EDHEC-Distressed Securities	100.0433158	100.5343205
EDHEC-Emerging Markets	99.20994993	100.5118262
EDHEC-Equity Market Neutral	100.0923959	100.3305248
EDHEC-Event Driven	99.99904541	100.5027729
EDHEC-Fixed Income Arbitrage	99.68524183	100.0620038
EDHEC-Global Macro	99.83012861	100.4453958
EDHEC-Long/Short Equity	99.91948345	100.5253251
EDHEC-Merger Arbitrage	99.94738788	100.3347095
EDHEC-Relative Value	100.044295	100.3582369
EDHEC-Short Selling	97.91881695	99.96879961
EDHEC-Funds of Funds	99.88679097	100.4167799
Percentage of V_0 *under 100$*	76.92%	7.69%
HFRI Convertible Arbitrage Index	99.9104685	100.321649
HFRI Distressed Securities Index	99.9100765	100.446987
HFRI Emerging Markets (Total)	99.1617091	100.497154
HFRI Equity Hedge Index	99.760536	100.537810
HFRI Equity Market Neutral Index	99.8160615	100.178244
HFRI Equity Non-Hedge Index	99.2694693	100.529065
HFRI Event-Driven Index	99.8678282	100.443743
HFRI Fixed Income (Total)	99.8533463	100.180401
HFRI Fixed Income : Arbitrage Index	99.4744962	99.9612590
HFRI Fixed Income : High Yield Index	99.4606113	100.118320
HFRI FOF : Conservative Index	99.8019766	100.171418
HFRI FOF : Diversified Index	99.5428340	100.224120
HFRI FOF : Market Defensive Index	99.6295097	100.290348
HFRI FOF : Strategic Index	99.3496291	100.310468
HFRI FOF Composite Index	99.6186407	100.240115
HFRI FOF Composite Index (Off.)	99.4353982	100.150926
HFRI Fund Weighted Composite Index	99.7328707	100.309632
HFRI Macro Index	99.6917718	100.369990
HFRI Merger Arbitrage Index	99.8584340	100.285088
HFRI Regulation D Index	99.9386375	100.681884
HFRI Relative Value Arbitrage Index	100.055301	100.346992
HFRI Short Selling Index	97.5229297	99.8979799
Percentage of V_0 *under 100$*	95.45%	9.09%

TABLE 2.XII – Initial investment V_0 in the replication of EDHEC and HFRI indices for both reserve assets for first sub-period (1997–2001).

Fund	V_0	
	Reserve 1	Reserve 2
EDHEC-Convertible Arbitrage	100.2944853	100.8987467
EDHEC-CTA Global	99.46788172	100.9472588
EDHEC-Distressed Securities	99.90059626	100.720508
EDHEC-Emerging Markets	98.69192451	100.9835661
EDHEC-Equity Market Neutral	100.3954179	100.7210641
EDHEC-Event Driven	100.0609365	100.868357
EDHEC-Fixed Income Arbitrage	99.59077798	100.2017969
EDHEC-Global Macro	99.97142407	100.979203
EDHEC-Long/Short Equity	100.1375749	101.127196
EDHEC-Merger Arbitrage	100.2331299	100.7861365
EDHEC-Relative Value	100.2203665	100.6965085
EDHEC-Short Selling	99.03421453	102.1095181
EDHEC-Funds of Funds	99.96160577	100.9516279
Percentage of V_0 under 100$	53.84%	0.00%
HFRI Convertible Arbitrage Index	100.2829484	100.8055676
HFRI Distressed Securities Index	99.72936197	100.6646377
HFRI Emerging Markets (Total)	98.09524276	100.9525596
HFRI Equity Hedge Index	100.056951	101.5042088
HFRI Equity Market Neutral Index	100.038409	100.6734399
HFRI Equity Non-Hedge Index	99.05531596	101.2392224
HFRI Event-Driven Index	99.97242706	100.980233
HFRI Fixed Income (Total)	99.75412401	100.3504572
HFRI Fixed Income : Arbitrage Index	99.3254573	100.0324407
HFRI Fixed Income : High Yield Index	99.31890751	100.1544936
HFRI FOF : Conservative Index	99.86644524	100.4768867
HFRI FOF : Diversified Index	99.52279888	100.9361689
HFRI FOF : Market Defensive Index	99.80508973	100.7630553
HFRI FOF : Strategic Index	99.28992499	100.9862717
HFRI FOF Composite Index	99.60846434	100.7634087
HFRI FOF Composite Index (Off.)	99.36188049	100.7094413
HFRI Fund Weighted Composite Index	99.75155852	100.9238131
HFRI Macro Index	99.76812518	100.8842713
HFRI Merger Arbitrage Index	100.1469401	100.7111258
HFRI Regulation D Index	100.5815208	101.5412257
HFRI Relative Value Arbitrage Index	100.1412334	100.6153432
HFRI Short Selling Index	98.51962283	100.8070637
Percentage of V_0 under 100$	72.72%	0.00%

Table 2.XIII – Initial investment V_0 in the replication of EDHEC and HFRI indices for both reserve assets for second sub-period (2002–2006).

Fund	V_0	
	Reserve 1	Reserve 2
EDHEC-Convertible Arbitrage	99.54307232	99.91754606
EDHEC-CTA Global	99.00591261	99.85286557
EDHEC-Distressed Securities	100.2752537	100.645535
EDHEC-Emerging Markets	100.0757102	100.4515871
EDHEC-Equity Market Neutral	99.85680498	99.99383759
EDHEC-Event Driven	99.87363087	100.3162115
EDHEC-Fixed Income Arbitrage	99.88868645	100.0907229
EDHEC-Global Macro	99.84474995	100.2384539
EDHEC-Long/Short Equity	99.60337666	100.1087833
EDHEC-Merger Arbitrage	99.70200103	99.99705359
EDHEC-Relative Value	99.81967336	100.109444
EDHEC-Short Selling	98.05685558	99.04396197
EDHEC-Funds of Funds	99.74332198	100.0559835
Percentage of V_0 under 100$	84.62%	38.46%
HFRI Convertible Arbitrage Index	99.60821174	99.93483497
HFRI Distressed Securities Index	100.2391759	100.6380069
HFRI Emerging Markets (Total)	100.0572944	100.8595669
HFRI Equity Hedge Index	99.58364075	100.014334
HFRI Equity Market Neutral Index	99.66759956	99.85722405
HFRI Equity Non-Hedge Index	99.37314042	100.2792862
HFRI Event-Driven Index	99.80612072	100.3402519
HFRI Fixed Income (Total)	99.95688427	100.1391919
HFRI Fixed Income : Arbitrage Index	100.0072767	100.1695353
HFRI Fixed Income : High Yield Index	100.0771647	100.3417642
HFRI FOF : Conservative Index	99.82149692	100.0377755
HFRI FOF : Diversified Index	99.7547993	100.0216789
HFRI FOF : Market Defensive Index	99.56207483	99.97381601
HFRI FOF : Strategic Index	99.62610152	99.96801828
HFRI FOF Composite Index	99.73892366	100.0563079
HFRI FOF Composite Index (Off.)	99.68519975	100.0475484
HFRI Fund Weighted Composite Index	99.78329249	100.2000232
HFRI Macro Index	99.73199639	100.3030235
HFRI Merger Arbitrage Index	99.66510204	100.0050475
HFRI Regulation D Index	99.47794411	100.3049513
HFRI Relative Value Arbitrage Index	99.94588108	100.1510614
HFRI Short Selling Index	98.37750341	99.15058551
Percentage of V_0 under 100$	81.81%	22.73%

Table 2.XIV – Regression of EDHEC and HFRI indices returns with the replication returns (for reserve assets 1–2) for the following target parameters : volatility, skewness, robust skewness, kurtosis, robust kurtosis, Kendall's tau and Pearson's rho.

Period : (1997–2006)	Reserve 1			Reserve 2		
Target	Intercept	Slope	R^2(%)	Intercept	Slope	R^2(%)
Volatility	0.000624738	0.997485421	99.38	0.000132117	1.034882095	99.38
Skewness	-1.21833672	1.135438624	63.48	-0.660897414	1.017065756	78.82
Robust Skew	0.005285212	0.591422785	38.74	0.049971694	0.845539485	68.79
Kurtosis	1.427089662	1.320048662	26.05	1.738641971	1.116543294	79.34
Robust Kurt	2.057766094	0.48321167	36.19	1.800169291	0.491547026	34.31
Kendall's Tau	0.040820382	1.009779392	98.80	0.034979443	1.024409652	99.36
Pearson's Rho	0.031939885	1.046644073	95.80	0.030103056	1.064383569	96.32

Period : (1997–2001)	Reserve 1			Reserve 2		
Target	Intercept	Slope	R^2(%)	Intercept	Slope	R^2(%)
Volatility	0.000246245	0.999597258	98.15	0.000303651	1.026011825	98.27
Skewness	-0.58025733	0.917232282	32.10	-0.86585092	1.542889847	65.17
Robust Skew	0.044192227	0.916761936	56.14	-0.00965482	0.729453739	54.08
Kurtosis	5.581125649	0.675401646	15.27	2.081736175	1.733323643	20.62
Robust Kurt	-0.85346256	1.451214328	67.35	-0.58104505	1.267471264	63.20
Kendall's Tau	0.0254162	1.019450292	98.52	0.020171502	1.016297547	99.18
Pearson's Rho	0.056163582	1.022429795	91.53	0.021180004	1.06516375	94.76

Period : (2002–2006)	Reserve 1			Reserve 2		
Target	Intercept	Slope	R^2(%)	Intercept	Slope	R^2(%)
Volatility	-0.00015482	0.987878677	99.84	0.000145601	0.984626992	99.59
Skewness	-0.07264573	1.035201227	83.03	0.045240922	1.104989802	80.44
Robust Skew	0.004977508	0.816341527	53.79	0.068954198	1.033664472	61.45
Kurtosis	1.26397635	0.768109088	35.44	0.536774019	0.93664374	67.14
Robust Kurt	1.471815049	0.540835024	45.90	1.421018259	0.566191474	26.63
Kendall's Tau	-0.00043248	1.069855948	98.96	0.019777659	1.034958883	98.97
Pearson's Rho	0.059853575	1.027479829	91.93	0.119157269	1.054127939	91.77

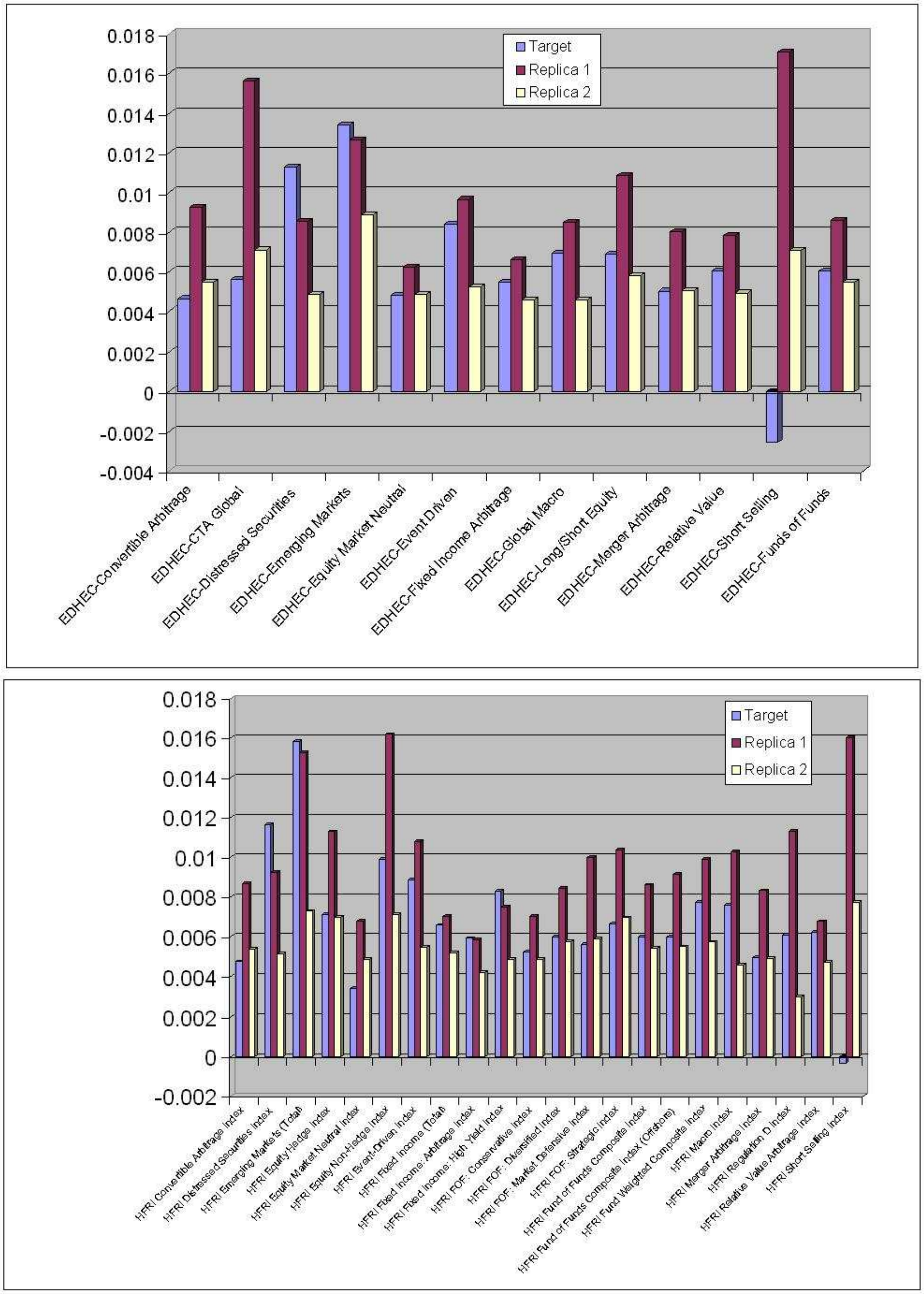
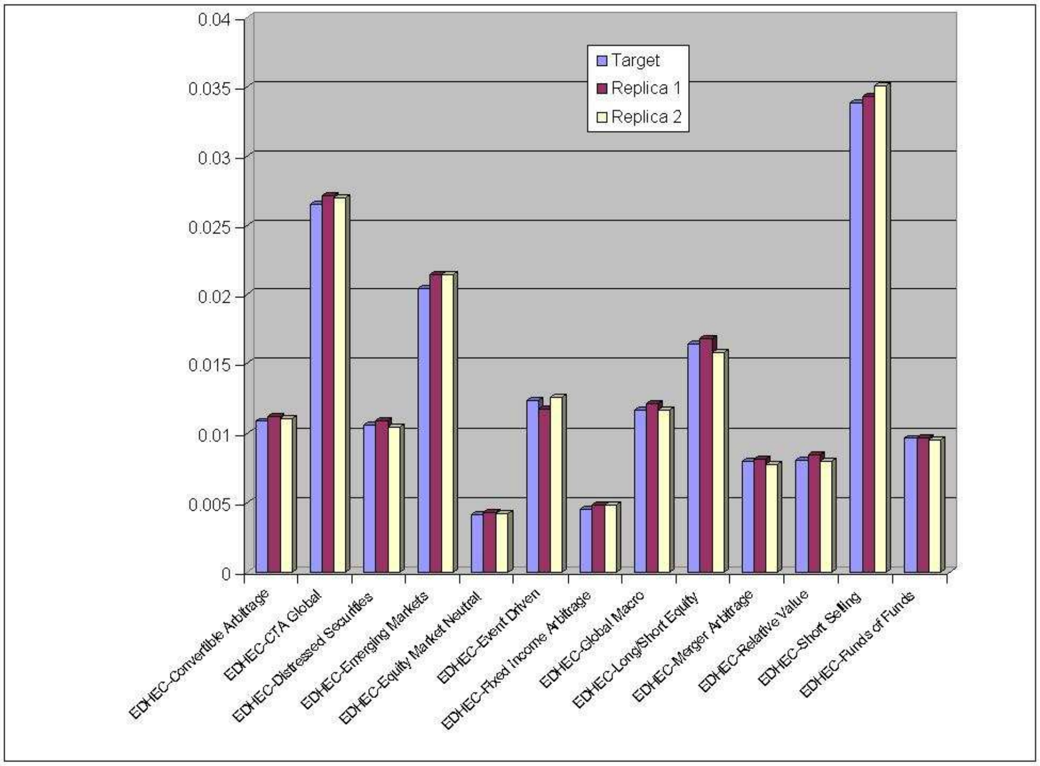


Figure 2.15 – Mean return of replication for both reserve assets vs mean return for EDHEC (top) and HFRI (bottom) indices (2002–2006)



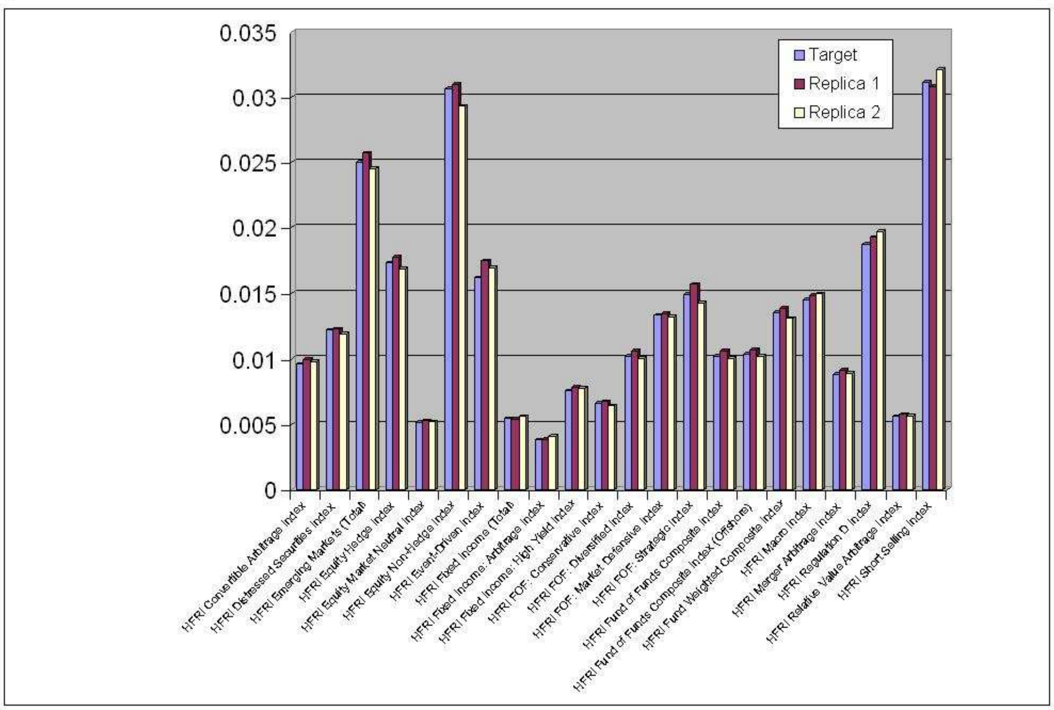


Figure 2.16 – Volatility of the replication with each reserve asset vs target volatility for EDHEC (top) and HFRI (bottom) indices (2002–2006)

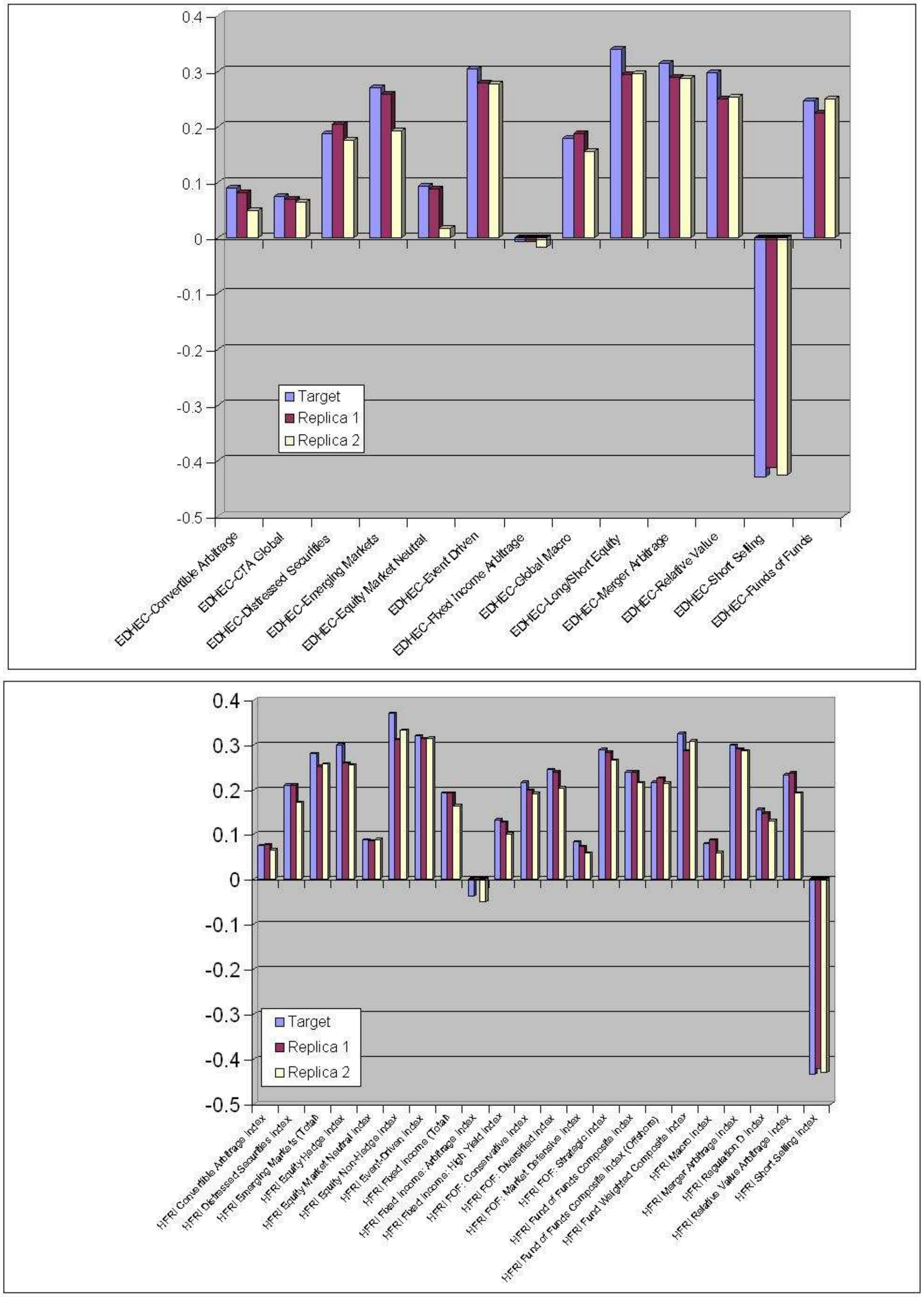
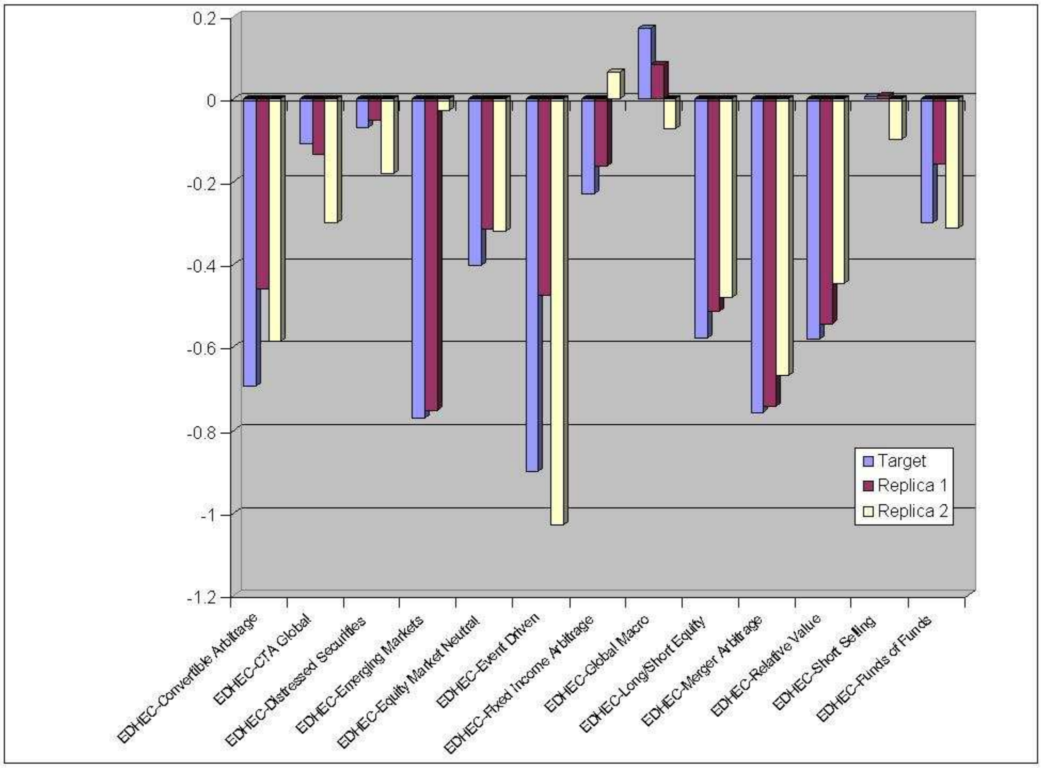


FIGURE 2.17 – Kendall's tau of the replication with each reserve asset vs target Kendall's tau for EDHEC (top) and HFRI (bottom) indices (2002–2006)



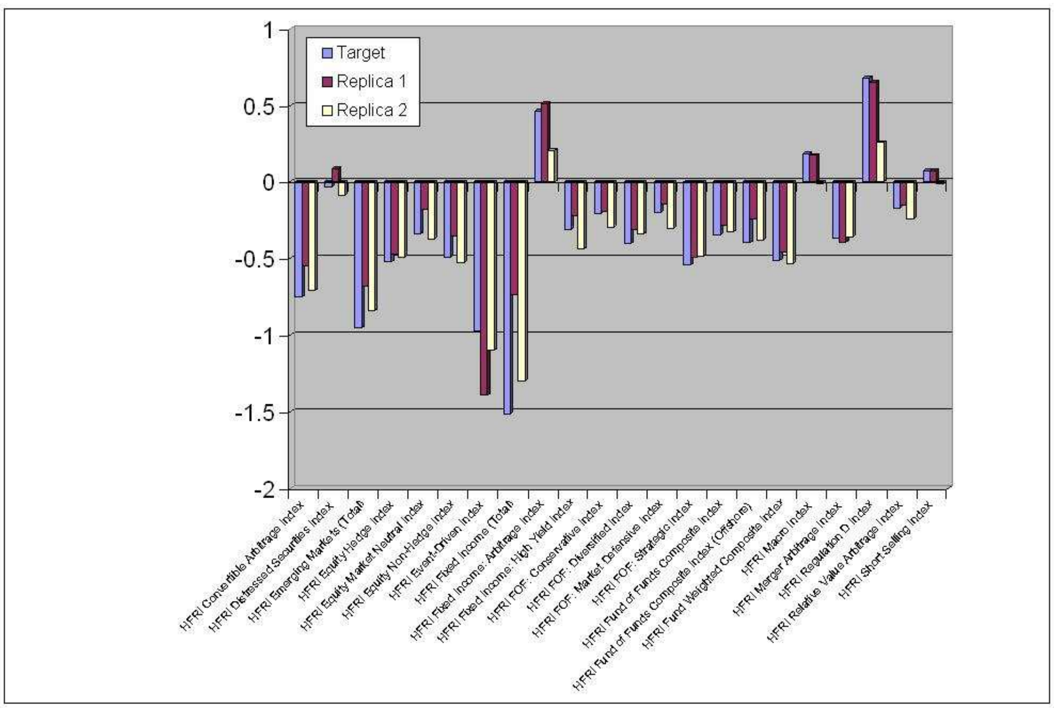


FIGURE 2.18 – Skewness of the replication with each reserve asset vs target skewness for EDHEC (top) and HFRI (bottom) indices (2002–2006)

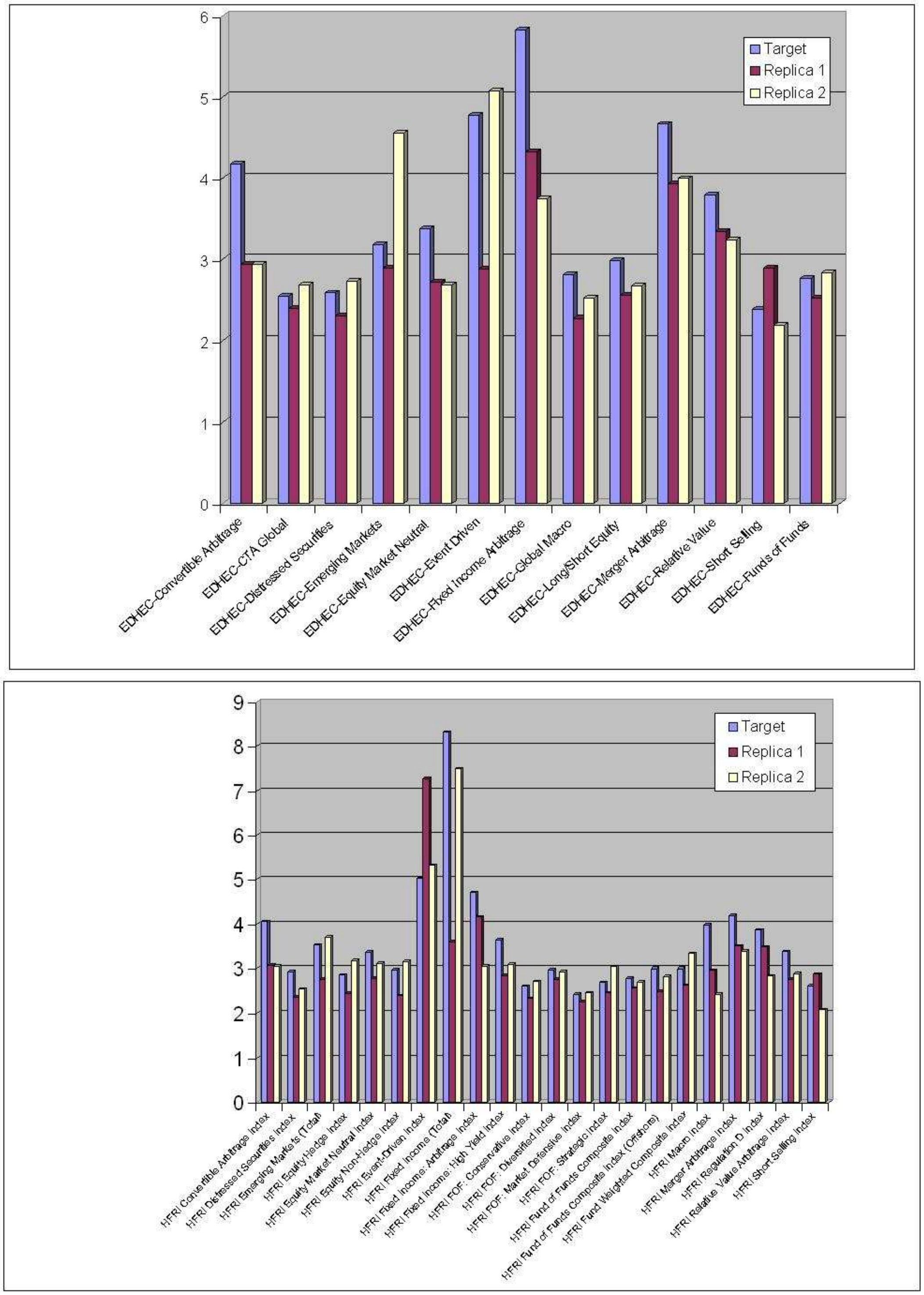


FIGURE 2.19 – Kurtosis of the replication with each reserve asset vs target kurtosis for EDHEC (top) and HFRI (bottom) indices (2002–2006)

Table 2.XX – Transaction costs (basis points) of the EDHEC and HFRI indices for each of two reserve assets over the entire period (1997–2006).

Fund	Transaction costs	
	Reserve 1	Reserve 2
EDHEC-Convertible Arbitrage	-3.5760	-2.6937
EDHEC-CTA Global	-5.1209	-3.6392
EDHEC-Distressed Securities	-3.1461	-2.9916
EDHEC-Emerging Markets	-10.436	-8.5692
EDHEC-Equity Market Neutral	-1.1785	-1.2782
EDHEC-Event Driven	-4.9894	-3.7833
EDHEC-Fixed Income Arbitrage	-5.6955	-3.5177
EDHEC-Global Macro	-3.1539	-3.5487
EDHEC-Long/Short Equity	-3.5405	-3.5815
EDHEC-Merger Arbitrage	-3.5994	-2.7794
EDHEC-Relative Value	-2.1994	-1.8390
EDHEC-Short Selling	-14.472	-12.690
EDHEC-Funds of Funds	-2.5685	-2.7680
Average of the transaction costs over the indices	-4.8982	-4.1292
HFRI Convertible Arbitrage Index	-2.9748	-2.3503
HFRI Distressed Securities Index	-3.7409	-3.1175
HFRI Emerging Markets (Total)	-10.409	-11.231
HFRI Equity Hedge Index	-5.2928	-5.5529
HFRI Equity Market Neutral Index	-1.9814	-1.8804
HFRI Equity Non-Hedge Index	-7.6039	-7.7172
HFRI Event-Driven Index	-3.7228	-3.3989
HFRI Fixed Income (Total)	-2.8376	-2.2500
HFRI Fixed Income : Arbitrage Index	-6.1764	-4.3318
HFRI Fixed Income : High Yield Index	-6.4438	-3.6841
HFRI FOF : Conservative Index	-2.4110	-2.1042
HFRI FOF : Diversified Index	-4.7314	-4.0279
HFRI FOF : Market Defensive Index	-3.6750	-2.8050
HFRI FOF : Strategic Index	-6.2475	-6.1420
HFRI FOF Composite Index	-3.7260	-3.9430
HFRI FOF Composite Index (Off.)	-4.6198	-4.6972
HFRI Fund Weighted Composite Index	-4.2733	-4.2082
HFRI Macro Index	-3.3393	-3.5459
HFRI Merger Arbitrage Index	-3.9681	-2.8237
HFRI Regulation D Index	-3.5011	-3.7099
HFRI Relative Value Arbitrage Index	-2.4469	-1.7284
HFRI Short Selling Index	-19.302	-17.595
Average of the transaction costs over the indices	-5.1557	-4.6747

Table 2.XXI – Hedging errors (basis per points) of the EDHEC and HFRI indices for each of two reserve assets over the entire period (1997–2006).

Fund	Hedging error	
	Reserve 1	Reserve 2
EDHEC-Convertible Arbitrage	-5.022966343	2.689724779
EDHEC-CTA Global	-8.058744042	5.645421806
EDHEC-Distressed Securities	4.124754378	19.47155871
EDHEC-Emerging Markets	-11.21163859	13.259774
EDHEC-Equity Market Neutral	-1.471590683	1.56198415
EDHEC-Event Driven	-3.020763751	6.22406221
EDHEC-Fixed Income Arbitrage	-5.177575949	3.189905767
EDHEC-Global Macro	-4.053867497	4.395207
EDHEC-Long/Short Equity	4.47809413	3.734220311
EDHEC-Merger Arbitrage	-3.442046302	2.242736202
EDHEC-Relative Value	-1.10554998	2.836227619
EDHEC-Short Selling	-24.29013217	18.8506452
EDHEC-Funds of Funds	2.033494462	8.749446216
Average of the hedging errors over the indices	-4.324502488	7.142377997
HFRI Convertible Arbitrage Index	-4.675913708	2.609102503
HFRI Distressed Securities Index	3.722398591	16.58984332
HFRI Emerging Markets (Total)	7.097564556	12.66959323
HFRI Equity Hedge Index	-1.643622346	11.19037495
HFRI Equity Market Neutral Index	-2.258515275	2.472596466
HFRI Equity Non-Hedge Index	7.453328183	4.603254198
HFRI Event-Driven Index	2.862294451	12.29110626
HFRI Fixed Income (Total)	-2.603139406	2.402853856
HFRI Fixed Income : Arbitrage Index	-4.087640896	4.977681096
HFRI Fixed Income : High Yield Index	2.638073684	2.387582196
HFRI FOF : Conservative Index	-2.598299696	2.863947585
HFRI FOF : Diversified Index	-5.7248332	-2.263005293
HFRI FOF : Market Defensive Index	-7.19063663	3.850690389
HFRI FOF : Strategic Index	-8.584197214	7.510126485
HFRI FOF Composite Index	-4.800243375	3.856993799
HFRI FOF Composite Index (Off.)	-7.540482923	5.850515308
HFRI Fund Weighted Composite Index	2.244013529	15.42135204
HFRI Macro Index	-2.491140954	7.274298256
HFRI Merger Arbitrage Index	-3.635490602	2.457954561
HFRI Regulation D Index	-4.354249204	3.999422953
HFRI Relative Value Arbitrage Index	-1.757126584	4.245628798
HFRI Short Selling Index	-30.41350326	21.98728382
Average of the hedging errors over the indices	-3.106425558	6.989782153

TABLE 2.XXII – Hedging errors (basis per points) of the EDHEC and HFRI indices for each of two reserve assets over the first sub-period (1997–2001).

Fund	Hedging error	
	Reserve 1	Reserve 2
EDHEC-Convertible Arbitrage	-5.854414114	7.331726159
EDHEC-CTA Global	-3.261304874	15.48344278
EDHEC-Distressed Securities	-10.63141111	18.21688996
EDHEC-Emerging Markets	-41.58467617	10.40839934
EDHEC-Equity Market Neutral	0.216747837	4.117171088
EDHEC-Event Driven	5.530304616	15.02572238
EDHEC-Fixed Income Arbitrage	-10.48685482	12.72732957
EDHEC-Global Macro	-1.950399253	10.81371999
EDHEC-Long/Short Equity	-5.472302407	8.63029379
EDHEC-Merger Arbitrage	-7.268360093	9.778517204
EDHEC-Relative Value	12.74567524	8.974747668
EDHEC-Short Selling	9.60796941	55.3754198
EDHEC-Funds of Funds	-12.45957574	9.552013774
Average of the hedging errors over the indices	-5.4514308	14.34118411
HFRI Convertible Arbitrage Index	-4.443351952	3.162705469
HFRI Distressed Securities Index	-9.790346341	17.16305189
HFRI Emerging Markets (Total)	-35.23487925	15.15013559
HFRI Equity Hedge Index	-15.50411415	10.73944888
HFRI Equity Market Neutral Index	-3.470329903	4.313327157
HFRI Equity Non-Hedge Index	-23.74524481	12.04989301
HFRI Event-Driven Index	-14.22269812	2.919363113
HFRI Fixed Income (Total)	-8.573296126	5.631050796
HFRI Fixed Income : Arbitrage Index	-4.419486285	7.911636813
HFRI Fixed Income : High Yield Index	-11.13974405	8.52434995
HFRI FOF : Conservative Index	-0.431323396	5.769353502
HFRI FOF : Diversified Index	-35.19300862	10.4684057
HFRI FOF : Market Defensive Index	-10.3549352	11.11226975
HFRI FOF : Strategic Index	-12.24470309	11.52847099
HFRI FOF Composite Index	-7.634859635	9.191341753
HFRI FOF Composite Index (Off.)	-9.434687073	11.45793373
HFRI Fund Weighted Composite Index	-24.92998374	8.097748863
HFRI Macro Index	3.853207823	15.60074834
HFRI Merger Arbitrage Index	-3.887143693	5.073019858
HFRI Regulation D Index	-2.431374814	13.42772207
HFRI Relative Value Arbitrage Index	-12.36688683	-0.033279555
HFRI Short Selling Index	-11.04287744	4.216592611
Average of the hedging errors over the indices	-11.66554849	8.794331377

Table 2.XXIII – Hedging errors (basis per points) of the EDHEC and HFRI indices for each of two reserve assets over the second sub-period (2002–2006).

Fund	Hedging error	
	Reserve 1	Reserve 2
EDHEC-Convertible Arbitrage	-0.216644211	14.71418971
EDHEC-CTA Global	-6.582901453	36.52509979
EDHEC-Distressed Securities	-1.016305328	13.72616412
EDHEC-Emerging Markets	-10.13043018	31.53349596
EDHEC-Equity Market Neutral	-1.061827125	5.534718588
EDHEC-Event Driven	6.84618159	12.12860342
EDHEC-Fixed Income Arbitrage	-0.469146864	9.667485841
EDHEC-Global Macro	-0.423280278	15.97182143
EDHEC-Long/Short Equity	-4.781157282	11.93987295
EDHEC-Merger Arbitrage	2.432702745	11.46724918
EDHEC-Relative Value	1.392743596	6.816342792
EDHEC-Short Selling	8.422391583	44.59416583
EDHEC-Funds of Funds	-0.055444422	12.87295253
Average of the hedging errors over the indices	-0.434085972	17.49939709
HFRI Convertible Arbitrage Index	0.262326757	12.38729712
HFRI Distressed Securities Index	0.409148738	14.75409472
HFRI Emerging Markets (Total)	-9.733473043	18.61313588
HFRI Equity Hedge Index	-7.44974449	20.13641716
HFRI Equity Market Neutral Index	-0.864838511	7.778622358
HFRI Equity Non-Hedge Index	-12.1917044	22.30909531
HFRI Event-Driven Index	2.282364634	16.12105281
HFRI Fixed Income (Total)	0.028468682	4.514146651
HFRI Fixed Income : Arbitrage Index	-0.000830608	8.164211829
HFRI Fixed Income : High Yield Index	0.197970856	10.94636336
HFRI FOF : Conservative Index	-3.168935239	6.353466396
HFRI FOF : Diversified Index	-0.756117028	12.47152293
HFRI FOF : Market Defensive Index	-4.356542614	12.3839458
HFRI FOF : Strategic Index	-5.764923859	19.62428265
HFRI FOF Composite Index	-0.639004576	8.640850198
HFRI FOF Composite Index (Off.)	-0.945047981	9.668544712
HFRI Fund Weighted Composite Index	-4.808300645	13.62398725
HFRI Macro Index	1.031221433	30.93949304
HFRI Merger Arbitrage Index	2.456653636	12.85418557
HFRI Regulation D Index	0.291293742	28.74394205
HFRI Relative Value Arbitrage Index	0.400473588	7.062009818
HFRI Short Selling Index	7.951857962	38.58089671
Average of the hedging errors over the indices	-1.607621953	15.30325292

Appendix B : Some properties of mixtures of bivariate Gaussian variables

One property that is quite important in our setting is the fact that a sum of independent Gaussian mixtures is still a Gaussian mixture. In fact, if $X_1, \ldots, X_n$ are independent and identically Gaussian mixtures with parameter θ, then $X = X_1 + \cdots + X_n$ is also a Gaussian mixture. To describe the associated parameters, let

$$\mathcal{A} = \{\alpha = (\alpha_1, \ldots, \alpha_m); \alpha_j \geq 0 \text{ and } \alpha_1 + \ldots + \alpha_m = n\}.$$

Then $\text{card}(\mathcal{A}) = \binom{n+m-1}{m-1}$ so there are $\binom{n+m-1}{m-1}$ regimes. The parameters of the mixture are $(\pi_\alpha)_{\alpha\in\mathcal{A}}$, $(\mu_\alpha)_{\alpha\in\mathcal{A}}$, $(A_\alpha)_{\alpha\in\mathcal{A}}$, where for each $\alpha \in \mathcal{A}$, π_α is the multinomial probability

$$\pi_\alpha = \pi_{(\alpha_1,\cdots,\alpha_m)} = \frac{n!}{\alpha_1! \cdots \alpha_m!} \prod_{k=1}^{m} \pi_k^{\alpha_k},$$

and the mean vectors μ_α and covariances A_α are respectively given by

$$\mu_\alpha = \sum_{k=1}^{n} \alpha_k \mu_k, \qquad A_\alpha = \sum_{k=1}^{n} \alpha_k A_k.$$

Remark 2.7.1 *If n is moderately large, then m^n is huge and it is computationally impossible to calculate the new parameters. In fact, most probabilities could be very small so in fact, the sum could be a mixture of fewer terms. Therefore, one has to estimate again the joint law of $\left(R_{0,T}^{(1)}, R_{0,T}^{(2)}\right)$ by a Gaussian mixture, using the monthly returns this time. As a result, the marginal distributions F_1 and F_2 are (univariate) Gaussian mixtures and $\mathcal{C}_{1,2}$ is the copula deduced from the bivariate Gaussian mixture.*

Finally, consider the conditional distribution of a bivariate Gaussian mixture $X = (X^{(1)}, X^{(2)})$. Set $\beta_k = \rho_k \frac{\sigma_{k2}}{\sigma_{k1}}$ and $\alpha_k = \mu_{k2} - \beta_k \mu_{k1}$, $k = 1, \ldots, m$. Then it is easy to check that the conditional distribution of $X^{(2)}$ given $X^{(1)} = x_1$ is a Gaussian mixture with parameters $\{\tilde{\pi}_k(x_1)\}_{k=1}^m$, $\{\tilde{\mu}_k(x_1)\}_{k=1}^m$, $\{\tilde{\sigma}_k^2\}_{k=1}^m$, where

$$\tilde{\pi}_k(x_1) = \frac{\pi_k \phi(x_1; \mu_{k1}, \sigma_{k1}^2)}{\sum_{j=1}^m \pi_j \phi(x_1; \mu_{j1}, \sigma_{j1}^2)} \tag{2.7}$$

and

$$\tilde{\mu}_k(x_1) = \alpha_k + \beta_k x_1, \qquad \tilde{\sigma}_k^2 = \sigma_k^2(1 - \rho_k^2). \tag{2.8}$$

Appendix C : Estimation and goodness-of-fit

In this section, we describe the estimation procedure and the goodness-of-fit tests.

C.1 : EM algorithm for bivariate Gaussian mixtures

Let $y_1, \ldots, y_n$ be a random sample from a bivariate Gaussian mixture with parameters $\pi = (\pi_k)_{k=1}^m$, $\mu = (\mu_k)_{k=1}^m$ and $A = (A_k)_{k=1}^m$. Start with an initial estimator $\theta^{(0)}$. Given an estimator $\theta^{(\ell)} = \left(\pi^{(\ell)}, \mu^{(\ell)}, A^{(\ell)}\right)$ of the parameters $\theta = (\pi, \mu, A)$, set

$$\pi_k\left(y_i, \theta^{(\ell)}\right) = \frac{\pi_k^{(\ell)} \phi_2\left(y_i; \mu_k^{(\ell)}, A_k^{(\ell)}\right)}{\sum_{j=1}^m \pi_j^{(\ell)} \phi_2\left(y_i; \mu_j^{(\ell)}, A_j^{(\ell)}\right)}, \quad i = 1, \ldots, n,$$

and define the new estimator $\theta^{(\ell+1)} = \left(\pi^{(\ell+1)}, \mu^{(\ell+1)}, A^{(\ell+1)}\right)$ viz.

$$\pi_k^{(\ell+1)} = \frac{1}{n} \sum_{i=1}^n \pi_k\left(y_i, \theta^{(\ell)}\right),$$

$$\mu_k^{(\ell+1)} = \frac{1}{n} \sum_{i=1}^n y_i \pi_k\left(y_i, \theta^{(\ell)}\right) \Big/ \pi_k^{(\ell+1)},$$

and

$$A_k^{(\ell+1)} = \frac{1}{n} \sum_{i=1}^n \left(y_i - \mu_k^{(\ell+1)}\right)\left(y_i - \mu_k^{(\ell+1)}\right)^\top \pi_k\left(y_i, \theta^{(\ell)}\right) \Big/ \pi_k^{(\ell+1)},$$

for $k = 1, \ldots, m$. As ℓ increases, the numbers $\{\pi_k\left(y_i, \theta^{(\ell)}\right); k = 1, \ldots, i = 1, \ldots, n\}$ stabilize and the estimators converge.

C.2 : Tests of goodness-of-fit

Testing goodness-of-fit is an essential step for modelling data. There are many tests available but to our knowledge, the best ones are based on empirical processes (Genest and Rémillard, 2005, Genest et al., 2009). Here, we only consider two tests based on the so-called Rosenblatt‘s transform. The first one is due to Durbin (1973) but the calculation of P-values is recent (Stute et al., 1993). For the second test designed for testing goodness-of-fit for bivariate data, the validity of the algorithm for calculating P-values follows from Genest and Rémillard (2005).

C.3 : Tests of goodness-of-fit for a univariate parametric distribution

Let $X_1, \ldots, X_n$ be a sample of size n from a (continuous) distribution F on $\mathbb{R}$. Suppose that the hypotheses to be tested are

$$\mathcal{H}_0 : F \in \mathcal{F} = \{F_\theta; \theta \in \Theta\} \qquad \text{vs} \qquad \mathcal{H}_1 : F \notin \mathcal{F}$$

For example, the parametric family $\mathcal{F}$ could be the family of univariate Gaussian mixtures with m regimes.

The proposed test statistic is based on Durbin (1973). Let $\theta_n = T_n(X_1, \ldots, X_n)$ be a regular estimator of θ, in the sense of Genest and Rémillard (2005) and set

$$D_n(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(U_i \leq u), \quad u \in [0,1],$$

where $U_i = F_{\theta_n}(X_i)$, $i = 1, \ldots, n$. To test $\mathcal{H}_0$ against $\mathcal{H}_1$, one may use the Cramér-von Mises type statistic

$$\begin{aligned} S_n &= n \int_0^1 \{D_n(u) - u\}^2 du \\ &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{U_i^2 + U_j^2 - 2\max(U_i, U_j)}{2} + \frac{1}{3} \right\}. \end{aligned}$$

Since the U_i's are "almost uniformly distributed on $[0,1]$" under the null hypothesis, large values of S_n should lead to rejection of the null hypothesis. However, in general the limiting distribution of S_n depend on the unknown parameter θ. To calculate the P-value of S_n, one can use a parametric bootstrap approach as described below.

a) Calculate θ_n and S_n.

b) For some large integer N (say 1000), repeat the following steps for every $k \in \{1, \ldots, N\}$:

(i) Generate a random sample $X_{1,k}, \ldots, X_{n,k}$ from distribution F_{θ_n}.

(ii) Calculate

$$\begin{aligned}
\theta_{n,k} &= T_n\left(X_{1,k}, \ldots, X_{n,k}\right), \\
U_{i,k} &= F_{\theta_{n,k}}(X_{i,k}), \quad i = 1, \ldots, n, \\
S_{n,k} &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{U_{i,k}^2 + U_{j,k}^2 - 2\max(U_{i,k}, U_{j,k})}{2} + \frac{1}{3} \right\}.
\end{aligned}$$

An approximate P-value for the test based on the Cramér–von Mises statistic S_n is then given by

$$\frac{1}{N} \sum_{k=1}^{N} \mathbb{I}(S_{n,k} > S_n).$$

C.4 : Tests of goodness-of-fit for a bivariate parametric distribution

Let $(X_1, Y_1) \ldots, (X_n, Y_n)$ be a sample of size n from a (continuous) distribution F on $\mathbb{R}^2$. Suppose that the hypotheses to be tested are

$$\mathcal{H}_0 : F \in \mathcal{F} = \{F_\theta; \theta \in \Theta\} \qquad \text{vs} \qquad \mathcal{H}_1 : F \notin \mathcal{F}$$

For example, the parametric family $\mathcal{F}$ could be the family of bivariate Gaussian mixtures with m regimes. Denote by G_θ the distribution function of X_i and let H_θ be the conditional distribution function of Y_i given X_i, i.e., $H_\theta(x, y) = P(Y_i \leq y | X_i = x)$.

The proposed test statistic is based on Durbin (1973) and the Rosenblatt's transform (Rosenblatt, 1952).

Suppose that $\theta_n = T_n(X_1, Y_1, \ldots, X_n, Y_n)$ is a regular estimator of θ, in the sense of Genest and Rémillard (2005) and set

$$D_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(U_i \leq u, V_i \leq v), \quad u, v \in [0, 1],$$

where $U_i = G_{\theta_n}(X_i)$, $V_i = H_{\theta_n}(X_i, Y_i)$, $i = 1, \ldots, n$. To test $\mathcal{H}_0$ against $\mathcal{H}_1$, one may use the Cramér-von Mises type statistic

$$\begin{aligned} S_n &= n\int_0^1\int_0^1 \{D_n(u,v)-uv\}^2 du dv \\ &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\Big[\frac{1}{9}-\frac{1}{4}(1-U_i^2)(1-V_i^2)-\frac{1}{4}(1-U_j^2)(1-V_j^2) \\ &\qquad +\{1-\max(U_i,U_j)\}\{1-\max(V_i,V_j)\}\Big]. \end{aligned}$$

Since the pairs (U_i, V_i)'s are "almost uniformly distributed on $[0,1]^2$" under the null hypothesis, large values of S_n should lead to rejection of the null hypothesis. However, in general the limiting distribution of S_n depend on the unknown parameter θ. To calculate the P-value of S_n, one can use a parametric bootstrap approach as described below.

a) Calculate θ_n and S_n.

b) For some large integer N (say 1000), repeat the following steps for every $k \in \{1,\ldots,N\}$:

(i) Generate a random sample $(X_{1,k}, Y_{1,k}),\ldots,(X_{n,k},Y_{n,k})$ from distribution F_{θ_n}.

(ii) Calculate

$$\begin{aligned} \theta^*_{n,k} &= T_n\left(X_{1,k},Y_{1,k},\ldots,X_{n,k},Y_{n,k}\right), \\ U_{i,k} &= G_{\theta_{n,k}}(X_{i,k}), \quad V_{i,k} = H_{\theta_{n,k}}(X_{i,k},Y_{i,k}), \quad i=1,\ldots,n \\ S_{n,k} &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\Big[\frac{1}{9}-\frac{1}{4}(1-U_{i,k}^2)(1-V_{i,k}^2)-\frac{1}{4}(1-U_{j,k}^2)(1-V_{j,k}^2) \\ &\qquad +\{1-\max(U_{i,k},U_{j,k})\}\{1-\max(V_{i,k},V_{j,k})\}\Big]. \end{aligned}$$

An approximate P-value for the test based on the Cramér–von Mises statistic S_n is then given by

$$\frac{1}{N}\sum_{k=1}^N \mathbb{I}(S_{n,k} > S_n).$$

Appendix D : Implementation of the dynamic trading strategy

Before describing the algorithm, it is important to define what is meant by a partition. Here we assume that $S_t = \omega_t(S_{t-1}, \xi_t)$, $\xi_t \sim \mu_t$ being independent of $\mathcal{F}_{t-1}$, $t = 1, \ldots, T$.

Definition 2.7.1 *A partition $\mathcal{P}$ of a compact convex set K, is any finite set $\mathcal{P} = \{S_1, \ldots, S_m\}$ of simplexes with disjoint non empty interiors, so that $K = \bigcup_{j=1}^{m} S_j$. The set of vertices of the partition $\mathcal{P}$ is denoted by $\mathcal{V}(\mathcal{P})$.*

Note that K is then the convex hull generated by $\mathcal{V}(\mathcal{P})$.

The algorithm is based on Monte Carlo simulations, combined with a sequence of approximations on compact sets $K_0, \ldots, K_{T-1}$, determined by partitions $\mathcal{P}_0, \ldots, \mathcal{P}_{T-1}$. The idea behind the algorithm is quite simple : Given approximations $\tilde{f}_t$, of f_t, one first get $\hat{L}_{1t}, \hat{L}_{2t}, \hat{A}_t, \hat{\Delta}_t, \hat{U}_t$ and $\hat{f}_{t-1}$, by estimating these functions at every vertices $x \in \mathcal{V}(\mathcal{P}_{t-1})$, using Monte Carlo simulations, and then, one uses a linear interpolation to extend them at any point $x \in K_{t-1}$. More precisely, one may proceed through the following steps.

D.1 : Algorithm

- Set $\tilde{f}_T = f_T$;
- For each $t = T, \ldots, 1$
  - Generate $\xi_{1,t}, \ldots, \xi_{N_t,t}$ according to μ_t ;

– For every $s \in \mathcal{V}(\mathcal{P}_{t-1})$, calculate

$$\begin{aligned}
\hat{L}_{1t}(s) &= \frac{1}{N_t}\sum_{i=1}^{N_t}\omega_t(s,\xi_{i,t}) \\
\hat{L}_{2t}(s) &= \frac{1}{N_t}\sum_{i=1}^{N_t}\omega_t(s,\xi_{i,t})\omega_t(s,\xi_{i,t})^\top \\
\hat{A}_t(s) &= \hat{L}_{2t}(s) - \hat{L}_{1t}(s)\hat{L}_{1t}(s)^\top \\
\hat{\psi}_t(s) &= \hat{A}_t(s)^{-1}\frac{1}{N_t}\sum_{i=1}^{N_t}\{\omega_t(s,\xi_{i,t}) - \hat{L}_{1t}(s)\}\tilde{f}_t\{\omega_t(s,\xi_{i,t})\} \\
\hat{U}_t(s,x) &= 1 - \{\hat{L}_{1t}(s) - \beta_{t-1}s/\beta_t\}^\top\hat{A}_t(s)^{-1}\{x - \hat{L}_{1t}(s)\} \\
\hat{f}_{t-1}(s) &= \frac{\beta_t}{\beta_{t-1}}\frac{1}{N_t}\sum_{i=1}^{N_t}\hat{U}_t\{s,\omega_t(s,\xi_{i,t})\}\tilde{f}_t\{\omega_t(s,\xi_{i,t})\}.
\end{aligned}$$

– Interpolate linearly $\hat{\Delta}_t$ and $\hat{f}_{t-1}$ over K_{t-1} and extend it to all of $\mathfrak{X}$.

A detailed description of the linear interpolation implementation techniques is given below, but first, the following result adapted from Del Moral et al. (2006), confirms that the algorithm produces good approximations.

Theorem 2 *Suppose that f_T is continuous and that for all $1 \le t \le T$, $\omega_t(\cdot,\xi)$ are continuous for a fixed ξ. Let K_0 be a given compact convex subset of $\mathfrak{X}$. Let $\epsilon > 0$ be given. Then one can find compact convex sets $K_1,\ldots,K_{n-1} \subset \mathfrak{X}$, partitions $\mathcal{P}_0,\ldots\mathcal{P}_{n-1}$ generating respectively $K_0,\ldots,K_{n-1}$, and integers $N_{10},\ldots,N_{n0}$, so that for the simple interpolation method,*

$$\max_{1\le k\le n}\|\psi_t - \tilde{\psi}_t\|_{K_{t-1}} < \epsilon,$$

and

$$\max_{0\le k\le n-1}\|f_t - \tilde{f}_t\|_{K_t} < \epsilon,$$

whenever $N_1 \ge N_{10},\ldots,N_n \ge N_{n0}$.

D.2 : Linear interpolations

Definition 2.7.2 *Given a function h and a partition $\mathcal{P}$ of K, a linear interpolation of h over $\mathcal{P}$ is the (unique) function $\tilde{g}$ defined in the following way :*

If $S \in \mathcal{P}$ is a simplex with vertices $x_1, \ldots, x_{d+1}$, then set

$$\tilde{h}(x) = \sum_{i=1}^{d+1} \lambda_i h(x_i),$$

where the barycenters $\{\lambda_1, \ldots, \lambda_{d+1}\}$ are the unique solution of

$$x = \sum_{i=1}^{d+1} \lambda_i x_i, \quad \sum_{i=1}^{d+1} \lambda_i = 1, \quad \lambda_i \in [0,1], i = 1, \ldots d+1.$$

If $x \notin K$, let x_K be the (unique) closest point to x that belongs to K, and set $\tilde{h}(x) = \tilde{h}(x_K)$. Uniqueness follows from the convexity of K and the strict convexity of the Euclidean norm.

Remark 2.7.2 *Note that since each x_i is extreme in S, the unique solution of*

$$x_i = \sum_{j=1}^{d+1} \lambda_j x_j, \quad \sum_{j=1}^{d+1} \lambda_j = 1, \quad \lambda_j \in [0,1], j = 1, \ldots d+1,$$

is $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \neq i$, yielding $\tilde{g}(x_i) = g(x_i)$ for all $1 \leq i \leq m$. Moreover, $\tilde{g}$ is affine on each simplex, justifying the term "linear interpolation".

Finally, $\tilde{g}$ is continuous and bounded on $\mathfrak{X}$ and

$$\sup_{x \in K} |g(x) - \tilde{g}(x)| \leq \omega(g, K, \mathrm{mesh}(\mathcal{P})),$$

where

$$\mathrm{mesh}(\mathcal{P}) = \max_{\mathrm{S} \in \mathcal{P}} \sup_{\mathrm{x,z} \in \mathrm{S}} \|\mathrm{x} - \mathrm{z}\|$$

and $\omega(g, K, \delta)$ is the modulus of continuity of g over K, i.e.

$$\omega(g, K, \delta) = \sup_{x,z \in K, \|x-z\| \leq \delta} |g(x) - g(z)|.$$

Example 2.7.1 *Suppose* $d = 1$. *Then the linear interpolation* $\tilde{g}$ *of a monotone (respectively convex) function* g *on* $K = [a, b]$ *is monotone (respectively convex). To see that, set* $a_i = a + i(b - a)/m$, $i = 0, \ldots, m$ *and let* $\mathcal{P}$ *be the partition given by* $\mathcal{P} = \{[a_{i-1}, a_i]; i = 1, \ldots, m\}$. *Set* $\Delta_i = \frac{g(a_i) - g(a_{i-1})}{a_i - a_{i-1}}$, $1 \le i \le m$. *Then the linear interpolation of* g *over* K *is given by*

$$\tilde{h}(x) = \begin{cases} h(a), & x \le a, \\ h(a_i) + (x - a_i)\Delta_{i+1}, & x \in [a_i, a_{i+1}], \ i = 0, \ldots, m-1, \\ h(b) & x \ge b. \end{cases}$$

If h *is monotone, the slopes* Δ_i *all have the same sign, so* $\tilde{h}$ *has the same monotonicity. If* h *is convex, the slopes* Δ_i *are non decreasing, so* $\tilde{h}$ *is also convex.*

Example 2.7.2 *Suppose* $d = 2$. *First define interpolation on* $[0, 1]^2$. *Suppose that* h *is known at points* $(0, 0)$, $(0, 1)$, $(1, 0)$ *and* $(1, 1)$. *If one wants to linearly interpolate* h, *as in Definition 2.7.2, a convenient choice for the partition* $\mathcal{P}$ *of* $[0, 1]^2$ *is* $\mathcal{P} = \{S_1, S_2\}$ *where*

$$S_1 = \{(x_1, x_2) \in [0, 1]^2; x_1 \le x_2\} \quad S_1 = \{(x_1, x_2) \in [0, 1]^2; x_1 \ge x_2\}.$$

Any $x \in S_1$ *can be uniquely written as*

$$x = \lambda_1(0, 1) + \lambda_2(1, 1) + \lambda_3(0, 0),$$

with $\lambda_2 = x_1$, $\lambda_1 = x_2 - x_1$, *and* $\lambda_3 = 1 - x_2$, *so one can define*

$$\begin{aligned} \tilde{h}(x) &= \lambda_1 h(0, 1) + \lambda_2 h(1, 1) + \lambda_3 h(0, 0) \\ &= h(0, 0) + x_1\{h(1, 1) - h(0, 1)\} + x_2\{h(0, 1) - h(0, 0)\}. \end{aligned}$$

Similarly, for any $x \in S_2$, *one obtains*

$$\begin{aligned} \tilde{h}(x) &= \lambda_1 h(0, 1) + \lambda_2 h(1, 1) + \lambda_3 h(0, 0) \\ &= h(0, 0) + x_1\{h(1, 0) - h(0, 0)\} + x_2\{h(1, 1) - h(1, 0)\}. \end{aligned}$$

Suppose now that $K = [a_1, b_1] \times [a_2, b_2]$ *is partition into smaller rectangles. On each of these sub-rectangles* $R = [y_1, y_2] \times [z_1, z_2]$, *just use the linear interpolation on* $[0,1]^2$ *by transforming* $x \in R$ *into* $x' = (x'_1, x'_2) \in [0,1]^2$ *through the mapping* $x'_1 = \frac{x_1 - y_1}{y_2 - y_1}, x'_2 = \frac{x_2 - z_1}{z_2 - z_1}$.

Outside K, $\tilde{h}$ *is defined as follows :*

$$\tilde{h}(x) = \begin{cases} \tilde{h}(x_1, a_2) & \textit{if} \quad x \in [a_1, b_1] \times (-\infty, a_2) \\ \tilde{h}(x_1, b_2) & \textit{if} \quad x \in [a_1, b_1] \times (b_2, \infty) \\ \tilde{h}(a_1, x_2) & \textit{if} \quad x \in (-\infty, a_1) \times [a_2, b_2] \\ \tilde{h}(b_1, x_2) & \textit{if} \quad x \in (b_1, \infty) \times [a_2, b_2] \\ \tilde{h}(a_1, a_2) & \textit{if} \quad x \in (-\infty, a_1) \times (-\infty, a_2) \\ \tilde{h}(b_1, a_2) & \textit{if} \quad x \in (b_1, \infty) \times (-\infty, a_2) \\ \tilde{h}(a_1, b_2) & \textit{if} \quad x \in (-\infty, a_1) \times (b_2, \infty) \\ \tilde{h}(b_1, b_2) & \textit{if} \quad x \in (b_1, \infty) \times (b_2, \infty) \end{cases} .$$

Appendix E : Auxiliary results

Throughout this appendix, $L^2 = L^2(\Omega, \mathcal{F}, P)$ is the set of all random variables on $(\Omega, \mathcal{F})$ which are square integrable.

Proposition 1 *Suppose that X is non negative random variable on $(\Omega, \mathcal{F}, P)$ such that $E(X) < \infty$. Suppose $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$ and let $Z = E(X|\mathcal{G}) \geq 0$, P almost surely. Then for any non negative $\mathcal{G}$-measurable random variable ξ, the following equality holds*

$$E(\xi X) = E(\xi Z).$$

Proof In the case of bounded random variable ξ, the result follows from the very definition of the conditional expectation. In particular it is true for $\xi_n = \min(n, \xi) \geq 0$, for any $n \geq 1$. Since $\xi_n \uparrow \xi$, it follows from Beppo-Levy theorem that

$$E(\xi X) = \lim_{n\to\infty} E(\xi_n X) = \lim_{n\to\infty} E(\xi_n Z) = E(\xi Z).$$

Proposition 2 *Suppose that $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}$ are L^2 random variables in $(\Omega, \mathcal{F})$ and suppose that $A = E(\xi\xi^\top|\mathcal{G})$ is invertible, where $\mathcal{G}$ is a sub σ-algebra of $\mathcal{F}$. Then $\varphi \in \mathbb{R}^d$ minimizes $E\{(\varphi^\top\xi - \eta)^2\}$ over all $\varphi \in \mathcal{G}$ such that $\varphi^\top\xi \in L^2$ if and only if $\varphi = A^{-1}b$, where $b = E(\xi\eta|\mathcal{G})$. In particular $\varphi^\top\xi$ is square integrable.*

Proof Set $\varphi = A^{-1}b$. To prove that $\varphi^\top\xi \in L^2$, note that it follows from Proposition 1 that

$$\begin{aligned}
E\left\{(\varphi^\top\xi)^2\right\} &= \sum_{i=1}^d E(\varphi_i^2\xi_i^2) \\
&= \sum_{i=1}^d E\{\varphi_i^2 E(\xi_i^2|\mathcal{G})\} \\
&= \sum_{i=1}^d E(\varphi_i^2 A_{ii}) \\
&= E(b^\top A^{-1} b).
\end{aligned}$$

Since A is symmetric and positive definite, there exist a $d \times d$ matrix $M \in \mathcal{G}$ such that $M^{-1} = M^\top$ and a $d \times d$ diagonal matrix $\Delta \in \mathcal{G}$ such that $A = M\Delta M^\top$. Set $\tilde{\xi} = M^\top \xi$ and $\tilde{b} = M^\top b$. Then $\Delta = E(\tilde{\xi}\tilde{\xi}^\top|\mathcal{G})$, $\tilde{b} = E(\tilde{\xi}\eta|\mathcal{G})$, $E(\tilde{\xi}_i^2|\mathcal{G}) = \Delta_{ii} > 0$ by hypothesis, and

$$
\begin{aligned}
b^\top A^{-1} b &= \tilde{b}^\top \Delta^{-1} \tilde{b} \\
&= \sum_{i=1}^d \frac{E^2(\tilde{\xi}_i \eta|\mathcal{G})}{E(\tilde{\xi}_i^2|\mathcal{G})} \\
&\leq dE(\eta^2|\mathcal{G}) \text{ a.s. },
\end{aligned}
$$

from Cauchy-Schwarz inequality. Hence

$$
E\{(\varphi^\top \xi)^2\} \leq pE(\eta^2) < \infty.
$$

Next, let ψ be any random vector in $\mathcal{G}$ such that $\psi^\top \xi \in L^2$. Then

$$
E\{(\psi^\top \xi - \eta)^2\} = E\left[E\{(\psi^\top \xi - \eta)^2|\mathcal{G}\}\right],
$$

and it is easy to check that

$$
\begin{aligned}
E\{(\psi^\top \xi - \eta)^2|\mathcal{G}\} &= \psi^\top A\psi - 2\psi^\top b + c \\
&= (\psi - \varphi)^\top A(\psi - \varphi) + \varphi^\top A\varphi - 2\varphi^\top b + c \\
&= (\psi - \varphi)^\top A(\psi - \varphi) + E\{(\varphi^\top \xi - \eta)^2|\mathcal{G}\}.
\end{aligned}
$$

Hence the result.

Appendix F : Proof of the main results

In this section, we will prove the two main results, using the propositions proved in Appendix 2.7.

F.1 : Proof of Theorem 1

Recall that the process $\varphi = (\varphi_t)_{t=0}^T$ is predictable. For any $1 \le t \le T$, set $\Delta_t = S_t - E(S_t|\mathcal{F}_{t-1})$ and

$$G_t = {\varphi_t}^\top \Delta_t - \{C_t - E(C_t|\mathcal{F}_{t-1})\}, \tag{2.9}$$

where $C_T = C$ and

$$\beta_{t-1}C_{t-1} = E(\beta_t C_t|\mathcal{F}_{t-1}) - {\varphi_t}^\top E(\beta_t S_t - \beta_{t-1}S_{t-1}|\mathcal{F}_{t-1}). \tag{2.10}$$

It follows from equations (2.9)-(2.10) that

$$\beta_t G_t = \beta_{t-1}C_{t-1} - \beta_t C_t + {\varphi_t}^\top(\beta_t S_t - \beta_{t-1}S_{t-1}), \qquad 1 \le t \le T. \tag{2.11}$$

Note that the $G_t \in \mathcal{F}_t$ and $E(G_t|\mathcal{F}_{t-1}) = 0$, for all $1 \le t \le T$. Moreover, using (2.2)–(2.3) and (2.11), one gets

$$\sum_{t=1}^T \beta_t G_t = C_0 - \beta_T C + \sum_{t=1}^T \varphi_t^\top(\beta_t S_t - \beta_{t-1}S_{t-1}) = G + C_0 - V_0$$

and $E(G) = E(G|\mathcal{F}_0) = C_0 - V_0$, since $E(G_t|\mathcal{F}_{t-1}) = 0$ for all $t = 1, \ldots, T$. It also follows from well known properties of conditional expectations that

$$\begin{aligned} E(G^2) &= E(G^2|\mathcal{F}_0) = (C_0 - V_0)^2 + \sum_{t=1}^T E\left(\beta_t^2 G_t^2|\mathcal{F}_0\right) \\ &= (C_0 - V_0)^2 + \sum_{t=1}^T E\left\{\beta_t^2\, E\left(G_t^2|\mathcal{F}_{t-1}\right)\big|\, \mathcal{F}_0\right\}. \end{aligned} \tag{2.12}$$

Because G_t depends only on $\varphi_t, \ldots, \varphi_T$ through C_t, to minimize $E(G^2)$, it suffices to find φ_T minimizing $E\left(G_T^2|\mathcal{F}_0\right)$, then to find φ_{T-1} minimizing $E\left(G_{T-1}^2|\mathcal{F}_0\right)$ and so on. Doing so, we will find the minimum since each term is non negative. Having found the optimal φ, one obtains that the optimal choice for V_0 is C_0.

First, note that $G_T = \xi_T^\top \varphi_T - \eta_T$, where $\xi_T = \Delta_T = S_T - E(S_T|\mathcal{F}_{T-1})$ and $\eta_T = C - E(C|\mathcal{F}_{T-1}) = C_T - E(C_T|\mathcal{F}_{T-1})$.

Using Proposition 2, one can conclude that

$$\varphi_T = (\Sigma_T)^{-1} E\left(\xi_T \eta_T|\mathcal{F}_{T-1}\right) = (\Sigma_T)^{-1} E\left(\xi_T C_T|\mathcal{F}_{T-1}\right)$$

minimizes $E(G_T^2|\mathcal{F}_0)$. Having found the optimal φ_T, one can define C_{T-1} as in (2.10).

Suppose now that $\varphi_T, \ldots, \varphi_t$ have been defined and define G_{t-1} and C_{t-1} according to (2.9) and (2.10). Then one can use again Proposition (2) to conclude φ_{t-1} given by (2.4) minimizes $E(G_{t-1}^2|\mathcal{F}_0)$.

Therefore the risk $E(G^2|\mathcal{F}_0)$ is minimized by choosing the φ_t's according to (2.4). Finally, using (2.12), the optimal value of V_0 is C_0. This completes the proof.

F.2 : Proof of Corollary

The proof of the representation $C_{t-1} = E(C_t U_t|\mathcal{F}_{t-1})$ follows directly from Theorem 1. In fact, using equations (2.4) and (2.5), one obtains

$$\begin{aligned}
\beta_{t-1} C_{t-1} &= E(\beta_t C_t|\mathcal{F}_{t-1}) - \varphi_t^\top E(\beta_t S_t - \beta_{t-1} S_{t-1}|\mathcal{F}_{t-1}) \\
&= E(\beta_t C_t|\mathcal{F}_{t-1}) \\
&\quad - E\left\{ C_t \Delta_t^\top (\Sigma_t)^{-1} E(\beta_t S_t - \beta_{t-1} S_{t-1}|\mathcal{F}_{t-1}) \middle| \mathcal{F}_{t-1} \right\} \\
&= E(C_t U_t|\mathcal{F}_{t-1}),
\end{aligned}$$

where U_t is defined by (2.6). One can easily see that $E(U_t|\mathcal{F}_{t-1}) = 1$, so $(M_t)_{t=0}^T$ is a martingale.

It only remains to prove that $\beta_t S_t M_t$ is a martingale. All is needed is to prove that $E(\beta_t S_t U_t|\mathcal{F}_{t-1}) = \beta_{t-1} S_{t-1}$. To this end, let $t \in \{1, \ldots, T\}$ be given and set

$\xi_t = E\left(\beta_t S_t - \beta_{t-1} S_{t-1} | \mathcal{F}_{t-1}\right)$. Note that

$$\beta_t S_t U_t = \beta_t S_t - \left\{\Delta_t + E(S_t|\mathcal{F}_{t-1})\right\} \Delta_t^\top \left(\Sigma_t\right)^{-1} \xi_t.$$

Next, since $E(\Delta_t|\mathcal{F}_{t-1}) = 0$, one has

$$\begin{aligned} E(\beta_t S_t U_t|\mathcal{F}_{t-1}) &= E(\beta_t S_t|\mathcal{F}_{t-1}) - E(\Delta_t \Delta_t^\top|\mathcal{F}_{t-1}) \left(\Sigma_t\right)^{-1} \xi_t \\ &\quad - E(S_t|\mathcal{F}_{t-1}) E(\Delta_t^\top|\mathcal{F}_{t-1}) \left(\Sigma_t\right)^{-1} \xi_t \\ &= E(\beta_t S_t|\mathcal{F}_{t-1}) - \Sigma_t \left(\Sigma_t\right)^{-1} \xi_t - 0 \\ &= E(\beta_t S_t|\mathcal{F}_{t-1}) - \xi_t = \beta_{t-1} S_{t-1}. \end{aligned}$$

Hence the result.

Chapitre 3

Optimal Hedging Strategies with an Application to Hedge Fund Replication

3.1 Introduction

Over the last couple of years, considerable attention paid within the hedge fund industry to the development replicating strategies. Many of the large banks have launched beta replication funds that attempt to use a portfolio of liquid assets to replicate the time-series properties of various hedge fund strategies.[1] The tracking portfolio generally consists in exposure to market, credit and liquidity premia. However, the replicating portfolio may consist of assets that are not necessarily employed by managers (e.g., high yield bonds may explain exposure of hedge equity to liquidity risk).

An interesting alternative replication method was proposed by Amin and Kat (2003) and more recently extended by Kat and Palaro (2005). Based on the Payoff Distribution Model put forth by Dybvig (1988), the authors attempt to replicate hedge fund returns not by identifying the return generating betas, but identifying a systematic trading strategy that can be used to generate the distribution of the hedge fund returns. Kat and Palaro (2005) show that for most hedge funds, their statistical properties can be replicated by investing in an alternative dynamic strategy.

The derivation of the bivariate Payoff Distribution Model by Kat and Palaro (2005)

1. ML Factor Index, GS Absolute Return Tracker, Partners Group AB Program, JPM AB Index

represents an interesting contribution to the performance evaluation and asset pricing literature. The implementation proposed by Kat and Palaro is however subject to several shortcomings and inconsistencies. In this paper we will address these problems and propose some techniques for overcoming these issues.

3.2 The Payoff Function

In Kat and Palaro (2005), the authors show that given two risky assets $S^{(1)}$ and $S^{(2)}$, it is possible to "reproduce" the statistical properties of the joint composed returns $R^{(1)}_{0,T} = \log(S^{(1)}_T/S^{(1)}_0)$ and $R^{(3)}_{0,T} = \log(S^{(3)}_T/S^{(3)}_0)$, in the sense that there exists a function g such that the joint distribution of $R^{(1)}_{0,T}$ and $g\left(R^{(1)}_{0,T}, R^{(2)}_{0,T}\right)$ is the same as the joint distribution of $R^{(1)}_{0,T}$ and $R^{(3)}_{0,T}$. Note that one does not replicate the value of $R^{(3)}_{0,T}$ at period T, but instead one wants to imitate its distribution properties like its expectation, volatility, skewness, kurtosis, as well as dependence measures with respect to $R^{(1)}_{0,T}$ such as Pearson and Spearman correlations to name a few.

The payoff's return function g is easily shown to be calculable using the marginal distribution functions F_1, F_2 and F_3 of $S^{(1)}_T$, $S^{(2)}_T$, $S^{(3)}_T$, and the copulas $\mathcal{C}_{1,2}$ and $\mathcal{C}_{1,3}$ associated respectively with the joints returns $\left(R^{(1)}_{0,T}, R^{(2)}_{0,T}\right)$ and $\left(R^{(1)}_{0,T}, R^{(3)}_{0,T}\right)$. For details on its derivations see Kat and Palaro (2005). The exact expression for g is given by

$$g(x,y) = Q\left\{x, P\left(R^{(2)}_{0,T} \le y | R^{(1)}_{0,T} = x\right)\right\}, \tag{3.1}$$

where $Q(x,\alpha)$ is the order α quantile of the conditional law of $R^{(3)}_{0,T}$ given $R^{(1)}_{0,T} = x$, i.e., for any $\alpha \in (0,1)$, $q(x,\alpha)$ satisfies

$$P\left\{R^{(3)}_{0,T} \le Q(x,\alpha) | R^{(1)}_{0,T} = x\right\} = \alpha.$$

Using properties of copulas, e.g. Nelsen (1999), the conditional distributions can be expressed in terms of the margins and the associated copulas.

$$P\left(R_{0,T}^{(2)} \leq y | R_{0,T}^{(1)} = x\right) = \frac{\partial}{\partial u}\mathcal{C}_{1,2}(u,v)\bigg|_{u=F_1(x),v=F_2(y)}.$$

Once the function has been calculated all that remains is to find the trading strategy that will allow to replicate the function. In essence, we can view the function as an option that cannot be traded, so we need to replicate the payoff of the option with the greatest possible precision by trading the underlying securities.

3.3 Replication and the shortcomings of the Kat-Palaro approach

There are three steps in the replication procedure.

- Modelling part :
  - Estimation of the parameters of the marginal distribution functions F_1, F_2 and F_3 of $S_T^{(1)}, S_T^{(2)}, S_T^{(3)}$, and the copulas $\mathcal{C}_{1,2}$ and $\mathcal{C}_{1,3}$ associated respectively with the joints returns $\left(R_{0,T}^{(1)}, R_{0,T}^{(2)}\right)$ and $\left(R_{0,T}^{(1)}, R_{0,T}^{(3)}\right)$.
- Calculate the payoff function g.
- Replication part :
  - Choose an appropriate replication method ;
  - Find the initial amount v_0 to be invested in the portfolio and find an hedging strategy φ.

3.3.1 Modeling issues

The correct calculation of the payoff function relies therefore on the precise modeling of the statistical properties of our three assets. The marginal distributions F_1, F_2 and F_3 must be capable of capturing the necessary skewness and kurtosis, and a proper empirical test must be implemented in order to select the two copulas $\mathcal{C}_{1,2}$ and $\mathcal{C}_{1,3}$. Any mis-specification of the statistical properties will induce an error in the calculation of the payoff function g, which, in turn, will not capture the statistical properties of $R_{0,T}^{(3)}$. Kat and Palaro (2005) use the Gaussian, Student and Johnson distributions to model the monthly returns of the three assets and five copula functions (Gaussian, Student, Frank, Gumbel and symmetrized Joe-Clayton) to model the dependence. The estimation and choice of marginal distribution and copula is performed using the Inference for Margins (IFM) method.

There are two significant shortcomings related to the modeling approach proposed by Kat and Palaro (2005). The first issue relates to the aggregation properties of the distributions and copula functions, and represents a fundamental flaw in the modeling approach. The second issue is also not trivial and relates to the choice of estimation technique.

The main flaw in the Kat and Palaro (2005) model has to do with the distribution of the returns $R_1, \ldots, R_T$ versus the distribution of $R_{0,T}$. In their paper, Kat and Palaro (2005) start by fixing the law of the monthly returns, distribution functions F_1, F_2, F_3 and the copulas $\mathcal{C}_{1,2}, \mathcal{C}_{1,3}$, and then solve for the corresponding daily hedging strategy for assets $S^{(1)}$ and $S^{(2)}$. The compatibility problem between the law of the daily returns and monthly returns is not addressed by the authors. According to Sklar's theorem (Sklar, 1959), the law of the bivariate vector $R_{0,T}$ is determined by F_1, F_2 and $\mathcal{C}_{1,2}$. However, the joint law of the returns $(R_t)_{t=1}^T$ must be compatible with the relation

$$R_{0,T} = \sum_{t=1}^{T} R_t. \tag{3.2}$$

Let's consider, for the sake of simplicity, that returns are independent and identically distributed. In the bivariate Gaussian case, it is easy to find the law of the returns $(R_t)_{t=1}^T$ given the law of $R_{0,T}$. In fact, even if the marginal distribution of $R_{0,T}^{(1)}$ and $R_{0,T}^{(2)}$ are Gaussian and their copula $\mathcal{C}_{1,2}$ is not Gaussian, the margins of $R_t^{(1)}$ and $R_t^{(2)}$ are

Gaussian. However, there is no known way to find out what the common copula of the R_t's should be so that the copula of the sum match the copula $\mathcal{C}_{1,2}$. Although copula provide us with much flexibility in terms of modeling the dependence, there is however no proof to this day that the statistical properties of copula functions are divisible. This compatibility condition is not a trivial matter. In fact, if for any T, the relation (3.2) is satisfied with independent and identically distributed returns $(R_t)_{t=1}^T$, then the law of $R_{0,T}$ must be infinitely divisible. Such laws can be characterized completely (see Barndorff-Nielsen et al. (2001) or Sato (1999)). For example, it is known that the univariate Student distribution is infinitely divisible, but the common law of the associated returns $(R_t)_{t=1}^T$ satisfying (3.2) is not known. Note that Johnson's law, proposed in Kat and Palaro (2005), is not infinitely divisible. Therefore, it should not serve as a model for the distribution of $R_{0,T}^{(1)}$ or $R_{0,T}^{(2)}$ if the daily returns are assumed to be independent.

The lesser of the two problems pertains to the choice of estimation technique. IFM is a two-stage estimation process : first the marginal distributions are estimated and then these distributions are used in order to calculate the parameters of the copula. Kim et al. (2007) show that an inappropriate choice of models for the margins may have detrimental effects on the estimation of the dependence parameter per se. A much more robust method consists of separating the estimation for the margins and the dependence. Ideally the estimation of the dependence should rely on normalized ranks and be independent of the marginal distributions. For a detailed description see Genest et al. (1995).

Overcoming the aggregation problems

In order to deal with the compatibility restriction, instead of estimating the law of the monthly returns $R_{0,T}$ for assets $S^{(1)}$ and $S^{(2)}$, it is preferable to take the opposite point of view, by first determining a model for the daily returns $(R_t)_{t=1}^T$, and then solving for the associated law for the composed bivariate return $R_{0,T}$. The important issue is select bivariate laws whose aggregation properties are known. A good candidate for the law of the returns R_t is a mixture of bivariate Gaussian distributions. It is easy to check that the law of $R_{0,T}$ will then be also a Gaussian mixture. Properties of Gaussian mixtures, as well as estimation and goodness-of-fit are treated in Papageorgiou et al. (2007). We do not need to concern ourselves with the distribution of asset $S^{(3)}$ since it is not used in the trading strategy.

A concern in the modeling of the daily returns can be presence of serial correlation in the daily time series. One interesting extension of Papageorgiou et al. (2007) would be to be to the model joint returns of assets $S^{(1)}$ and $S^{(2)}$ as a mixture of bivariate Gaussian distribution with a Markovian dependence in the mixtures. One could also consider mixture of bivariate GARCH processes. The aggregation properties and estimation of multi-variate mixtures of GARCH processes have been studied by Hafner and Rombouts (2007).

3.3.2 Hedging issues

Having modeled the return distributions and dependence structures, we can then calculate the payoff function g. The final step is to find a dynamic trading strategy that allows us to best approximate this function. The hedging strategy proposed by Kat and Palaro (2005) is quite simple. They use a trinomial approach proposed by He (1990) even though the law of the (daily) returns

$$R_t = \left\{ \log\left(S_t^{(1)}/S_{t-1}^{(1)}\right), \log\left(S_t^{(2)}/S_{t-1}^{(2)}\right) \right\}^\top$$

is not necessarily Gaussian.

In their calculations they implemented the technique of Boyle and Lin (1997), a trinomial approach that incorporates transactions costs. This approach is clearly inefficient, specially since the distributions of the traded assets $S^{(1)}$ and $S^{(2)}$, and the hedge fund $S^{(3)}$ are clearly not Gaussian. In order to get rid of this inconsistency which is common in option pricing, Papageorgiou et al. (2007) propose an alternative methodology adapted from American option pricing techniques. The authors extend the results of Schweizer (1995) by selecting the portfolio (v_0, φ) such as to minimize the (square) root mean square hedging error (RMSHE)

$$\sqrt{E\left[\beta_T^2\left\{V_T(v_0,\varphi) - C_T\right\}^2\right]},$$

where β_T is the discount factor and φ is a dynamic replication strategy. The value, at period t, of the portfolio defined by the initial value v_0 and strategy φ is denoted by $V_t(v_0, \varphi)$. Note that there is no "risk-neutral" evaluation involved, all calculations are carried out under the objective probability measure.

Optimal hedging

Suppose that $(\Omega, P, \mathcal{F})$ is a probability space with filtration $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$, under which the stochastic processes are defined. Assume that the price process S_t is d-dimensional, i.e. $S_t = \left(S_t^{(1)}, \ldots, S_t^{(d)}\right)$.

A dynamic replicating strategy can be described by a (deterministic) initial value v_0 and a sequence of random weight vectors $\varphi = (\varphi_t)_{t=0}^T$, where for any $j = 1, \ldots, d$, $\varphi_t^{(j)}$ denotes the number of parts of assets $S^{(j)}$ invested during period $(t-1, t]$. Because φ_t may depend only on the values values $S_0, \ldots, S_{t-1}$, the stochastic process φ_t is assumed to be predictable. Initially, $\varphi_0 = \varphi_1$, and the portfolio initial value is v_0. It follows that the amount initially invested in the non risky asset is $v_0 - \sum_{j=1}^d \varphi_1^{(j)} S_0^{(j)} = v_0 - \varphi_1^\top S_0$.

Since the hedging strategy must be self-financing, it follows that for all $t = 1, \ldots, T$,

$$\beta_t V_t(v_0, \varphi) - \beta_{t-1} V_{t-1}(v_0, \varphi) = \varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1}). \tag{3.3}$$

Using the self-financing condition (3.3), it follows that

$$\beta_T V_T = \beta_T V_T(v_0, \varphi) = v_0 + \sum_{t=1}^{T} \varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1}). \tag{3.4}$$

The replication strategy problem for a given payoff C is thus equivalent to finding the strategy (v_0, φ) so that the hedging error

$$G_T(v_0, \varphi) = \beta_T V_T(v_0, \varphi) - \beta_T C \tag{3.5}$$

is as small as possible. Here, the RMSHE measures the quality of replication. It is therefore natural to suppose that the prices $S_t^{(j)}$ have finite second moments. We further assume that the hedging strategy φ satisfies a similar property, namely that for any $t = 1, \ldots, T$, $\varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1})$ have finite second moments. Note that these two technical conditions were also made by Schweizer (1995).

For simplicity, set $\Delta_t = S_t - E(S_t | \mathcal{F}_{t-1})$, $t = 1, \ldots, T$. Under the above moment conditions, the conditional covariance matrix Σ_t of Δ_t exists and is given by

$$\Sigma_t = E\left\{ \Delta_t \Delta_t^\top | \mathcal{F}_{t-1} \right\}, \; 1 \leq t \leq T. \tag{3.6}$$

In Schweizer (1995), the author treats the case $d = 1$ and assumes a restrictive boundedness condition. Here, in contrast, we treat the general d-dimensional case and we only suppose that Σ_t is invertible for all $t = 1, \ldots, T$. This was implicitly part of the boundedness condition of Schweizer (1995).

If Σ_t is not invertible for some t, there would exists a $\varphi_t \in \mathcal{F}_{t-1}$ such that $\varphi_t^\top S_t = \varphi_t^\top E(S_t | \mathcal{F}_{t-1})$, that is, $\varphi_t^\top S_t$ is predictable. Our assumption can be interpreted as saying that the genuine dimension of the assets is d.

Difference between optimal hedging and hedging under Black-Scholes setting

To compare the two methods, simply take $T = 1$, $\beta_T = 1$, and $d = 1$. In this case, the solution for optimal hedging yields $\varphi^\star = \text{Cov}\{\Delta S_1, C(S_1)\}/\text{Var}(\Delta S_1)$, where

$\Delta S_1 = S_1 - S_0$, and $v_0^\star = E\{C(S_1)\} - \varphi^\star E(\Delta S_1)$.

For the Black-Scholes setting, $v_0^{BS} = E\left\{C\left(S_0 e^{\sigma Z - \sigma^2/2}\right)\right\}$ and $\varphi^{BS} = E\left\{e^{\sigma Z - \sigma^2/2} C'\left(S_0 e^{\sigma Z - \sigma^2/2}\right)\right\}$, with $\sigma^2 = \mathrm{Var}\left\{\log(S_1/S_0)\right\}$, where $Z \sim N(0,1)$, provided C is differentiable. See, e.g., Broadie and Glasserman (1996).

In general, $\varphi^\star \neq \varphi^{BS}$ and $v_0^\star \neq v_0^{BS}$, so

$$E\left[\{V_1(v_0^\star, \varphi^\star) - C(S_1)\}^2\right] < E\left[\left\{V_1(v_0^{BS}, \varphi^{BS}) - C(S_1)\right\}^2\right].$$

For an analysis of the (discrete) hedging error in a Black-Scholes setting, see, e.g., Wilmott (2006).

Hedging Error Comparison

To illustrate the advantage of the optimal hedging strategy proposed in Papageorgiou et al. (2007), we compare the mean hedging error and the RMSHE as defined in equation (3.5) for the optimal hedging and for the Kat-Palaro approach. For this example, we specify assets $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ as follows :

- Asset $S^{(1)}$ is a proxy for the typical institutional Canadian pension fund as described in *Benefits Canada Review (May 2007)*
- Asset $S^{(2)}$ is a diversified portfolio of typical market exposures, specifically global equity indices, credit indices and commodity indices
- Asset $S^{(3)}$ that is being replicated is chosen to be gaussian distribution with an annual volatility of 12%.

We model bivariate daily and monthly distributions of assets $S^{(1)}$ and $S^{(2)}$ over the period from 2000 to 2007 using normal mixtures, as detailed in Papageorgiou et al. (2007). This leads to 7 regimes for the daily mixture and 2 regimes for the monthly mixture. We do not specify the required dependence between $S^{(3)}$ and $S^{(1)}$, instead we run the hedging comparison for different levels of dependence between the two assets.

More precisely, we allow Kendall's Tau to vary from -0.9 to 0.9 for three different copulas (Gaussian, Clayton and Frank) and measure the impact of this dependency variable between $S^{(1)}$ and $S^{(3)}$ on hedging error measures. To compare the optimal hedging replication method and the Kat-Palaro method, 10 000 scenarios of 22 daily returns (1 trading month) were simulated for the assets $S^{(1)}$ and $S^{(2)}$. For each scenario, the terminal value V_T of the portfolio was computed and the hedging error is calculated. The plots of the hedging errors are presented below.

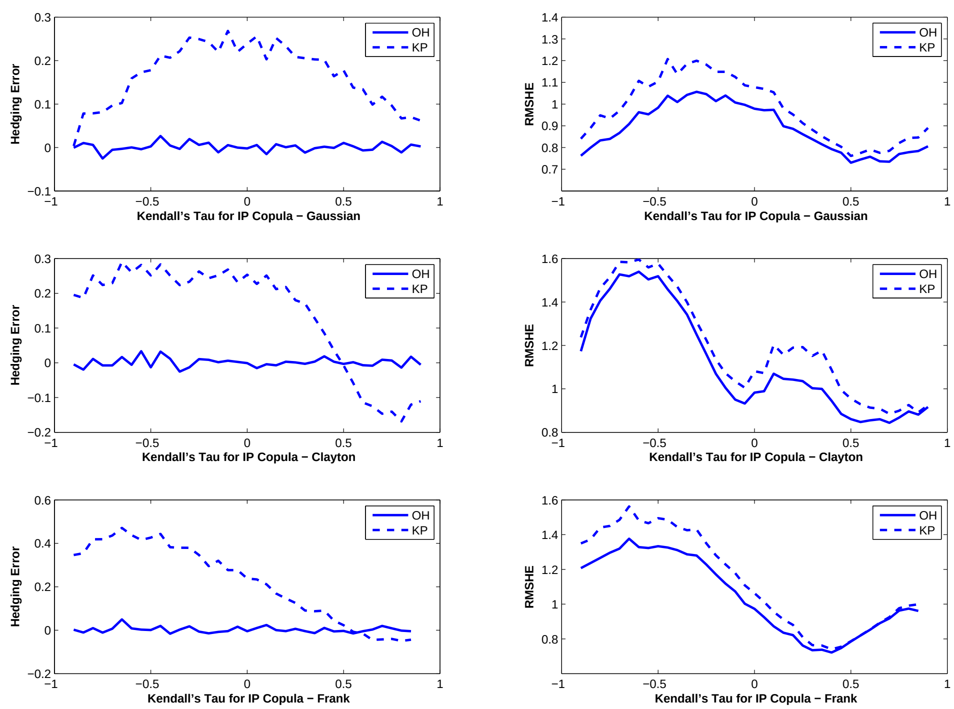


FIGURE 3.1 – Hedging Error Measures

The results lend strong support to the hedging approach put forth in Papageorgiou et al. (2007). Hedging Errors for the "Optimal Hedging" algorithm are centered on 0 with a low sensitivity to Kendall's Tau as well as to the type of copula. The Kat-Palaro algorithm is considerably more sensitive to the level of dependence (Kendall's tau) and copula family. This is a direct result of their approach being nested in the

Black-Scholes setting and can lead to large hedging errors. It is also important to note that the Optimal Hedging approach systematical produces smaller Root Mean Square Hedging Errors (RMSHE) providing further validation of the Papageorgiou et al. (2007) approach.

3.4 Conclusion

In the paper, we have discussed some of the challenges that one is confronted with in implementing the bivariate Payoff Distribution Model proposed by Kat and Palaro (2005). We exposed some of the flaws in the modeling and the dynamic trading strategy, and proposed some techniques for overcoming these inconsistencies. Finally, we showed that the hedging algorithm proposed in Papageorgiou et al. (2007) provides a more precise replication of the payoff function that the Black-Scholes approach put forth by Kat and Palaro (2005)

What remains to be seen is how well these statistical replication techniques fare in practice. Desjardins Global Asset Management should soon be able to provide some insight into this issue. They have been working with the authors of Papageorgiou et al. (2007) and have recently launched the first statistical replication fund that is open to investors.

Bibliography

Amin, G. and Kat, H. (2003). Hedge fund performanc 1990-2000 : Do the “money machines” really add value. *Journal of Financial and Quantitative Analysis*, 38(2) :251–275.

Barndorff-Nielsen, O., Mikosch, T., and Resnick, S., editors (2001). *Levy Processes : Theory and Applications*. Birkhäuser, Boston.

Boyle, P. and Lin, X. (1997). Valuation of options on several risky assets when there are trabnsactions costs. In Boyle, P., Pennacchi, G., and Richken, P., editors, *Advances in Futures and Options Research*, volume 9, pages 111–129. JAI Press.

Broadie, M. and Glasserman, P. (1996). Estimating security price derivatives using simulation. *Management Science*, 42 :260–285.

Dybvig, P. (1988). Distributional analysis of portfolio choice. *The Journal of Business*, 61(3) :369–393.

Genest, C., Ghoudi, K., and Rivest, L.-P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82 :543–552.

Hafner, C. and Rombouts, J. (2007). Estimation of temporally aggregated multivariate GARCH models. *Journal of Statistical Computation and Simulation*, 77 :629–650.

He, H. (1990). Convergence from discrete to continuous-time contingent claims price. *Review of Financial Studies*, 3 :523–546.

Kat, H. and Palaro, H. (2005). Who needs hedge funds ? A copula-based approach to hedge fund return replication. Technical report, Cass Business School, City University.

Kim, G., Silvapulle, M. J., and Silvapulle, P. (2007). Comparison of semiparametric and parametric methods for estimating copulas. *Comm. Statist. Simulation Comput.*, 51 :2836–2850.

Nelsen, R. B. (1999). *An introduction to copulas*, volume 139 of *Lecture Notes in Statistics*. Springer-Verlag, New York.

Papageorgiou, N., Rémillard, B., and Hocquard, A. (2007). Replicating the properties of hedge fund returns. Technical Report 1, DGAM-HEC Alternative Invesments Research.

Sato, K.-I. (1999). *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

Schweizer, M. (1995). Variance-optimal hedging in discrete time. *Math. Oper. Res.*, 20(1) :1–32.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8 :229–231.

Wilmott, P. (2006). *Paul Wilmott on Quantitative Finance*, volume 3. John Wiley & Sons.

Chapitre 4

The Payoff Distribution Model : An Application to Dynamic Portfolio Insurance

4.1 Introduction

The recent market meltdown has put the spotlight back on the dangers of financial leverage and the importance of careful and flexible risk management techniques. Many financial institutions and asset management firms suffered unprecedented losses during the financial crisis, impacting their balance sheet and jeopardizing the viability of many structured products, such as equity linked notes and guaranteed principal notes. The vast majority of institutions employ leverage and manage their market exposures (de-leveraging) on these products using some form of portfolio insurance strategies. Recent events have highlighted some of the important limitations of the traditional dynamic portfolio insurance techniques used to manage downside risk. These approaches include the stop-loss insurance, option based replication insurance, and constant proportion portfolio insurance (CPPI). However, given the often prohibitive costs and institutional constraints in purchasing OTC portfolio insurance, not to mention the increasing concern about counterparty risk, these dynamic portfolio insurance methodologies often present the only viable risk management option for fund managers.

The earliest portfolio insurance model, proposed by Brennan and Schwartz (1979) and Rubinstein and Leland (1981), consisted of overlaying a synthetic put option on the existing portfolio, and delta managing the overall exposure using the Black and Scholes (1973) option pricing formula. Although theoretically sound, this approach is subject to significant error when confronted to the reality of non-continuous trading, transaction costs and the time-varying nature of volatility. A further approach to dynamic risk management, specifically Constant Proportion Portfolio Insurance (CPPI), was proposed by Black and Jones (1987) and Black and Perold (1992). The CPPI strategy requires that exposure to the risky asset is a linear function of a cushion, defined as the excess wealth above a specific floor limit. The exposure is then determined by multiplying the cushion by a predetermined multiple. The initial cushion, multiple, floor and tolerance can be chosen according to the investor's own objectives and preferences.

The 1987 stock market crash provided a clear evidence as to the limitations and dangers inherent in these dynamic risk management strategies. Lack of liquidity and suspension of trading in certain markets left many orders unexecuted and the underlying portfolios exposed to massive gap risk. This motivated more recent research by Cont and Tankov (2007), who build on the work of Liu et al. (2003) and Bertrand and Prigent (2003) and study the impact that jumps in prices and volatility have on investment strategies such as CPPI. Liu et al. (2003) provide analytical solutions to the optimal portfolio problem and prove that event risk dramatically affects the optimal strategy. Cont and Tankov (2007) develop analytically tractable expressions for the probability of hitting the floor, the expected loss and the distribution of losses but also use a criterion for adjusting the multiplier based on the investorŠs risk aversion. More recently, Annaert et al. (2009) evaluate the performance of the stop-loss, synthetic put and constant proportion portfolio insurance techniques based on a block-bootstrap simulation and compare them using the stochastic dominance criteria. The main drawback of their approach is the arbitrary assumption that the CPPI multiplier is time invariant. Moreover their bootstrap methodology results in a positive expected return for the underlying portfolio, which is not consistent with guaranteed capital program testing.

We present a novel approach to dynamic portfolio insurance that overcomes many of the limitations of the earlier techniques. Our approach is based on the Payoff Distribution Model (PDM) proposed by Dybvig (1988) and incorporates recent extensions by Papageorgiou et al. (2008). The underlying principle of the PDM is quite simple : it aims to see whether the statistical properties of a fund or asset can be generated more efficiently using a systematic trading strategy on a liquid assets (or portfolio of liquid assets). This approach was at first conceived as a tool for performance evaluation, and it was shown by Dybvig (1988) and by Amin and Kat (2003) that the marginal return distributions of mutual fund and hedge fund managers could be successfully replicated using the PDM. The methodology was later extended to a bi-variate setting by Papageorgiou et al. (2008), who also propose an optimal hedging strategy. However, beyond it's applications as a performance measure to evaluate the ex-post distribution of an asset/fund, the PDM offers a unique framework that can be used to generate funds with "target" distributions that are tailored to an investor's specific needs. In this paper we extend this latter application of the PDM to funds with embedded risk controls. We propose an innovative methodology to manage the downside risk of such funds by targeting a distribution that incorporates the desired risk profile. Specifically, we generate a fund that is characterized by a Left Truncated Gaussian distribution and then demonstrate, using different performance and risk measures, that this approach to managing market exposure leads to a better risk control at a lower cost than more popular dynamic portfolio insurance strategies.

The paper will be structured as follows. In section 4.2, we present an overview of the Payoff Distribution model. Next, we detail our portfolio insurance methodology by introducing the Truncated Gaussian distribution family. In section 4.4 we discuss the benchmark models and performance measures. Section 4.5 presents the empirical results of our study and section 4.6 concludes.

4.2 The Payoff Distribution Model (PDM)

The Payoff distribution model was introduced by Dybvig (1988) to price and evaluate the distribution of consumption of a given portfolio. The main idea was to propose a new performance measure that allowed preferences to depend on all the moments of a distribution, providing a richer framework than the traditional mean-variance approach. For example, in evaluating the performance of a US equity mutual fund, the PDM can be used to price the payoff function that links the return distributions of the fund and that of an equity index such as the S&P500. This allows us to evaluate, using all the information available in the return distributions, whether the performance of the fund is superior to that of the S&P500.

4.2.1 Tailor-made Funds

The most innovative and interesting application of the Payoff Distribution Model is as a tool to generate funds with pre-specified monthly statistical properties. The PDM allows us to deduce and price the payoff function that must be applied to the distribution of an asset (S&P500 or other) in order to generate the desired distributional properties. The payoffs are replicated by implementing a dynamic delta management strategy on the underlying asset. Typically, one seeks to generate monthly properties, hence the maturity of the payoff function is one month. Over several months of generating the payoff, the properties of the resulting monthly returns will match those of the specified target density. By targeting a defined monthly distribution, the aim is to control the whole risk profile of the fund, specifically the volatility, the asymmetry, as well as the potential monthly draw-down. These controls are embedded in a unique risk model, hence eliminating the need for any risk management overlay. This methodology clearly requires a liquid underlying asset to manage the exposure, or at least a liquid proxy that should not be exposed to excessive basis risk.

The steps required to generate a fund with a target distribution are as follows :

- Define the underlying asset or fund and its tradable proxies if needed.
- Identify the desired statistical properties of the target fund (select the density function and the necessary parameters).
- Estimate the daily process of the underlying asset and infer its monthly distribution.
- Derive the monthly payoff function of the targeted distribution.
- Price the replication strategy and derive the hedging strategy over the month. In essence, the dynamic trading strategy distorts the distribution of the underlying asset so as to generate the desired payoff.

Details regarding the derivation of the hedging strategy are provided in appendix 4.6.

4.2.2 The Payoff Function

In Amin and Kat (2003), the authors show that given an underlying asset S_{Under} with monthly returns R_{Under} and a targeted distribution to deliver F_{Target}, it is possible to "generate" the statistical properties of the returns at time T (end of month). Specifically, there exists a function g such that the distribution of $g\left(R_{Under}\right)$ is the same as the distribution F_{Target}. This payoff's return function g is easily shown to be calculable using the distribution function F_{Under} of the underlying asset and the marginal distribution function of the targeted distribution F_{Target}.

The exact expression for g is given by

$$g(x) = Q\left\{P\left(R_{Under} \leq x\right)\right\} \; ; \forall x \in \mathbb{R} \tag{4.1}$$

where $Q(\alpha)$ is the order α quantile of the distribution F_{Target}.
An other notation for g is :

$$g(x) = F_{Target}^{-1}\left(F_{Under}(x)\right) \; ; \forall x \in \mathbb{R} \tag{4.2}$$

This payoff function g falls in the same category of more classical known payoffs such as put and call options except than instead of being written on the underlying price, g is written on the underlying monthly return. This implies a more adapted payoff to integrate the whole risk profile of the underlying returns density.

4.3 Extensions of the PDM to Risk Management

The ability to generate any type of distribution (Gaussian or other) using the PDM provides us with a very flexible setting for fund management. In order to address the need for managing downside risk and incorporate dynamic portfolio insurance principles, we opt to target a Left Truncated Gaussian distribution. The properties of the Left Truncated Gaussian distribution are presented below.

4.3.1 Truncated Distributions

A truncated distribution is a conditional distribution that is derived from a more general probability distribution. Let X a random variable with probability density function $f(x)$ and cumulative distribution function $F(x)$ with infinite support. The idea underlying the truncation is to identify the probability density of x after restricting the support with two constants such that $a < X \leq b$.

Then

$$f_{X|a<X\leq b}(x) = \frac{g(x)}{F(b) - F(a)} = Tr(x) \tag{4.3}$$

with

$$g(x) = \begin{cases} f(x) & a < X \leq b \\ 0 & \text{Otherwise.} \end{cases} \tag{4.4}$$

The truncated distribution $Tr(x)$ is a probability density function and integrates to one :

$$\int_a^b Tr(x)dx = \int_a^b f_{X|a<X\leq b}(x)dx = \frac{1}{F(b) - F(a)} \int_a^b g(x)dx = 1 \tag{4.5}$$

Left-side Truncation

A truncated distribution with only a left-side truncation is then written :

$$f_{X|X>a}(x) = \frac{g(x)}{1 - F(a)} \tag{4.6}$$

with

$$g(x) = \begin{cases} f(x) & a < X \\ 0 & \text{Otherwise.} \end{cases} \tag{4.7}$$

Truncated Gaussian distribution

Let X be $N(\mu, \sigma^2)$ and Y a truncated normal $TrN(\mu, \sigma^2, a, b)$ random variable. Then :

$$f\left(y, \mu, \sigma^2, a, b\right) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} exp\left(\frac{-(y-\mu)^2}{2\sigma^2}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} I_{[a,b]}(y) \tag{4.8}$$

with Φ the standard normal cumulative distribution function, ϕ the standard normal probability density function and

$$I_{[a,b]}(y) = \begin{cases} 1 & a < y \leq b \\ 0 & \text{Otherwise.} \end{cases} \tag{4.9}$$

Left Truncated Gaussian distribution

Let X be $N(\mu, \sigma^2)$ and Y a truncated normal $LTrN(\mu, \sigma^2, a)$ random variable. Then :

$$f\left(y, \mu, \sigma^2, a, b\right) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} exp\left(\frac{-(y-\mu)^2}{2\sigma^2}\right)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} I_a(y) \tag{4.10}$$

with Φ the standard normal cumulative distribution function, ϕ the standard normal probability density function and

$$I_{a,}(y) = \begin{cases} 1 & a < y \\ 0 & \text{Otherwise.} \end{cases} \tag{4.11}$$

For details on these results see Johnson and Balakrishnan (1996). The formulas for the cumulative density functions and the probability density functions are presented in appendix 4.6. The formulas for the first four moments are presented in appendix 4.6. Note that if we decide to left side truncate a Gaussian distribution, the resulting distribution will have a higher mean, lower volatility and be positively skewed that the original distribution. All these features make the Left Truncated Gaussian distribution an interesting choice of target distribution from an investor's perspective. Figure 4.1 illustrates a Left Gaussian Truncated pdf and cdf with parameters $\mu = 0$, $\sigma = 3\%$ and the left truncation point $a = -4\%$.

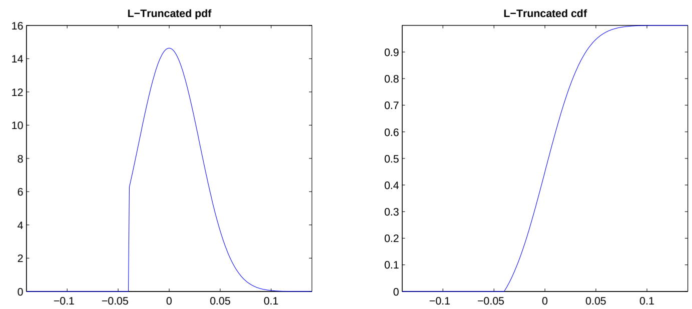


FIGURE 4.1 – Left Truncated Gaussian Distribution

4.3.2 Payoff Function g and hedging

The targeted distribution to deliver F_{Target} is a Left Truncated Gaussian distribution, with mean μ_T, standard deviation σ_T and left-side floor a. The payoff function g can be expressed :

$$g(x) = \mu_T + \sigma_T * \Phi^{-1}\left[\Phi\left(\frac{a-\mu_T}{\sigma_T}\right) + F_{Under}(x)\left[1 - \Phi\left(\frac{a-\mu_T}{\sigma_T}\right)\right]\right] \tag{4.12}$$

with F_{Under} the monthly distribution of the underlying asset and x the associated monthly return.

When F_{Under} is a Gaussian distribution $N(\mu_R, \sigma_R)$, g can be expressed :

$$g(x) = \mu_T + \sigma_T * \Phi^{-1}\left[\Phi\left(\frac{a-\mu_T}{\sigma_T}\right) + \Phi\left(\frac{x-\mu_R}{\sigma_R}\right)\left[1 - \Phi\left(\frac{a-\mu_T}{\sigma_T}\right)\right]\right] \tag{4.13}$$

with Φ the standard normal cumulative distribution function and Φ^{-1} the inverse.

Once the target density is defined, we derive the optimal hedging strategy that replicates the payoff function g. This can be performed in a Black-Scholes setting as done by Amin and Kat (2003). However, in order to resolve the Black-Scholes option replication bias, we price and derive the replication strategy by minimizing the root mean square hedging error using a Monte Carlo approach under the real probability measure, as described in appendix 4.6. For more detail on the implementation of the hedging methodology, and for a comparison of the Black-Scholes hedging strategy and the Optimal hedging strategy in a Gaussian framework see Hocquard et al. (2008).

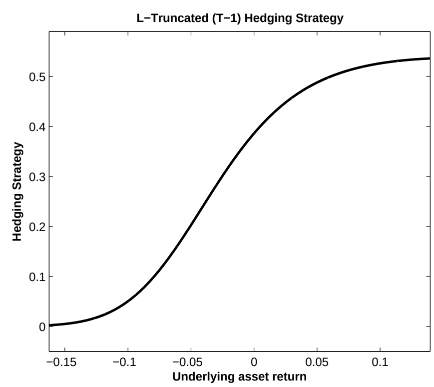


FIGURE 4.2 – Left Truncated Gaussian Hedging Strategy

Figure 4.2 plots the $(T-1)$ hedging strategy of the Left Truncated Gaussian with zero mean, a target monthly volatility of 3%, a downside protection at -4% written on an underlying asset with zero mean and 5% monthly volatility. The delta is similar to a call option delta on returns. Since a long position in a risky asset combined with a put option written on this asset is equivalent to a long position in a call option, the Left Truncated Gaussian payoff respects this intuition and allows for a better control of the risk factors of the underlying asset.

4.4 Methodology

In order to highlight the advantages of the Left Truncated Gaussian distribution, we contrast our methodology with three commonly used portfolio insurance strategies : a stop loss strategy, a synthetic put strategy and a constant proportion portfolio insurance (CPPI) strategy. We use several performance measure, notably the Sharpe ratio, Omega ratio and Cornish-Fisher VaR, to evaluate the effectiveness and cost of these different dynamic portfolio insurance strategies.

To evaluate the effectiveness of the different approaches, we assume a very simple scenario. An investor has access to a risky asset S and a non-risky asset B paying interest r. The investor wants his portfolio Π to be exposed to S for a time horizon T, but manages his downside risk using different methods. We denote ω_t the weight of the portfolio invested in the risky asset S at time t. $(1-\omega_t)$ will be invested in the non-risky asset B_t. If $(1-\omega_t) < 0$ the investment in the risky asset S_t will be leveraged an the investor should borrow in B_t. In order to illustrate the $(T-1)$ hedging strategies for each methodology, a plot is presented targeting a downside protection at -4% written on an underlying asset with 5% monthly volatility. Section 4.4.1 and 4.4.2 provide a brief review of the three benchmark models and the performance measures, respectively. All empirical results are provided in Section 4.5.

4.4.1 Portfolio Insurance Strategies

Stop Loss

The stop loss strategy is the easiest way to protect a portfolio against major losses. The portfolio Π is fully invested in S at time $t = 0$, and the investor selects a floor F

to be the stop loss level. This strategy consists, at any time t $(t = 0, ..., T-1)$:

$$\begin{aligned}
&\Pi_0 = S_0 \to \omega_0 = 1 \\
&\quad \text{while } \Pi_t \geq e^{-r(T-t)}F \\
&\Pi_t = S_t \to \omega_t = 1 \\
&\quad \text{if } \Pi_k < e^{-r(T-k)}F \text{ for } k = 1, ..., T-1 \\
&\Pi_t = B_t \to \omega_t = 0 \text{ for } t = k, ..., T
\end{aligned} \tag{4.14}$$

Then :

$$\Pi_t = \omega_t S_t + (1 - \omega_t) B_t \tag{4.15}$$

If the portfolio value is higher that the discounted floor, the investor remains fully invested in the risky asset, otherwise the risky asset is sold and the portfolio is fully invested in the non-risky asset until the end of the investment horizon T.

Advantages

- The portfolio is totally unexposed to the risky asset once the floor is reached, preserving the portfolio against a larger drop in S.
- No dynamic trading in involved, which minimizes the transaction costs during the investment horizon.

Disadvantages

- The investor cannot profit from any upward move in the risky asset after $\omega_t = 0$.
- The investor is exposed to substantial losses since the portfolio is fully exposed $(\omega_t = 1)$ until the floor is reached.
- The investor will have to liquidate all the positions in the risky asset at once, exposing himself to large transaction costs and liquidity constraints.

In fact the stop loss strategy could be viewed as an "asset-or-nothing call", typical binary option, paying one unit of asset if above the strike at maturity.

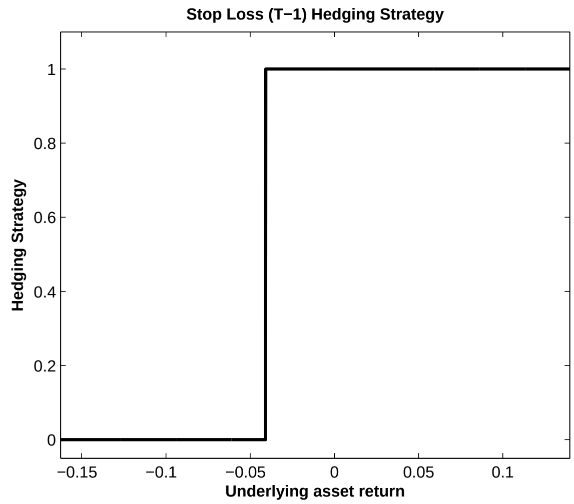


FIGURE 4.3 – Stop Loss Hedging Strategy

BS Synthetic Put

The synthetic put strategy is a dynamic trading strategy that attempts to replicate a long put position Q with strike level K. The hedge ratios Δ^{Put} can be computed at every time t according to the portfolio value S_t, portfolio volatility σ_t, interest rate level and time to horizon. In a Black Scholes framework, the formula for the put is (non-dividend underlying assumed) :

$$\begin{aligned} Q_t &= -S_t \Phi(-d_{1,t}) + K e^{-r(T-t)} \Phi(-d_{2,t}) \\ d_{1,t} &= \frac{ln(S_t/K) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_{2,t} &= d_{1,t} - \sigma\sqrt{T-t} \end{aligned} \tag{4.16}$$

$$\Delta_t^{Put} = \Phi(d_{1,t}) - 1$$

Then a protective put investment is :

$$S_t + Q_t = S_t \Phi(d_{1,t}) + K e^{-r(T-t)} \Phi(-d_{2,t}) \tag{4.17}$$

such as at any time t in $0, ..., T-1$, the proportion invested in the risky asset S is :

$$\omega_t = \frac{S_t(1 + \Delta_t^{Put})}{S_t + Q_t} \tag{4.18}$$

and $(1-\omega_t)$ will be invested in the non-risky asset B, and Q_t the price of the put option at time t.

Then :

$$\Pi_t = \omega_t S_t + (1-\omega_t)B_t \tag{4.19}$$

As the value of the portfolio approaches the strike price, the impact of the put increases on the overall strategy and the investor transfers an increasing proportion of his portfolio from the risky asset to risk-free asset. If the portfolio put is deep out-of-the money, the portfolio is then fully invested in the risky asset. At the other end of the scale, if the put is deep in-the-money, the investor will be fully invested in the risk-free asset.

Advantages

- There is no binary decisions in contrast to the stop loss strategy. Except deep-in-the money put scenario, the portfolio is always at least partially invested in the risky asset and could benefit from upward movements in S.
- The dynamic trading strategy allows the investor to react frequently according to the evolution of S.

Disadvantages

- The strategy requires a good approximation of the volatility in the BS framework.
- Depending of the volatility level, the put convexity can be very high, meaning high gamma, implying large adjustments and potentially large transaction costs.

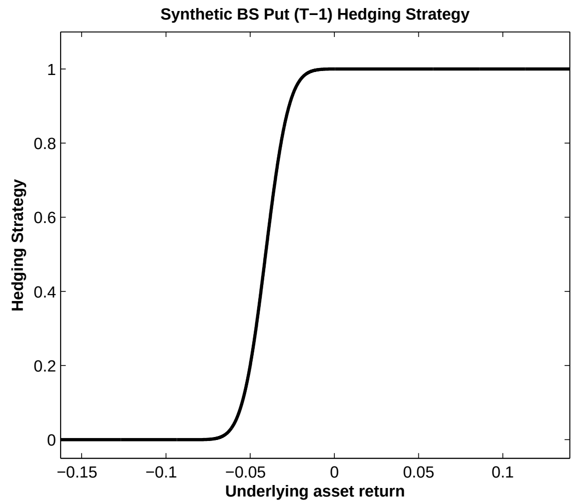


FIGURE 4.4 – Synthetic BS Put Hedging Strategy

Constant Proportion Portfolio Insurance

This strategy provides a cushion to the risky asset, adjusted by a multiplier. The cushion is computed by subtracting a floor value F_t from the portfolio value Π_t. The multiplier represents the sensitivity of the CPPI strategy to the risky asset movements, and can be interpreted as the risk aversion sensitivity factor. To stay consistent with the different methodologies, we impose a no-short sale constraint and a leverage constraint on the CPPI strategy. The exposure at time t in the risky asset S_t according to the CPPI is :

$$\omega_t = max\left[min\left[\frac{m\left(S_t - Fe^{-r(T-t)}\right)}{S_t}, Cap\right], 0\right] \tag{4.20}$$

and $(1-\omega_t)$ will be invested in the non-risky asset B, with m the multiplier and Cap a cap factor on leverage. We impose a long position in S_t with the $max(.,0)$ constraint. The cushion is the value$\left(S_t - Fe^{-r(T-t)}\right)$ with the associated weight $\frac{m\left(S_t - Fe^{-r(T-t)}\right)}{S_t}$

Then :

$$\Pi_t = \omega_t S_t + (1-\omega_t)B_t \tag{4.21}$$

When the portfolio value decreases, the cushion decreases and the investor transfers part of his portfolio from the risky asset to the non-risky asset at the "speed" m.

Advantages

- The CPPI strategy is simple and does not require estimation of volatility or price process.

Disadvantages

- The CPPI strategy is very sensitive to the multiplier m value, and there is no rule of selection for m.
- This strategy can lead to large adjustments in the portfolio, and hence large transaction costs and market impact.

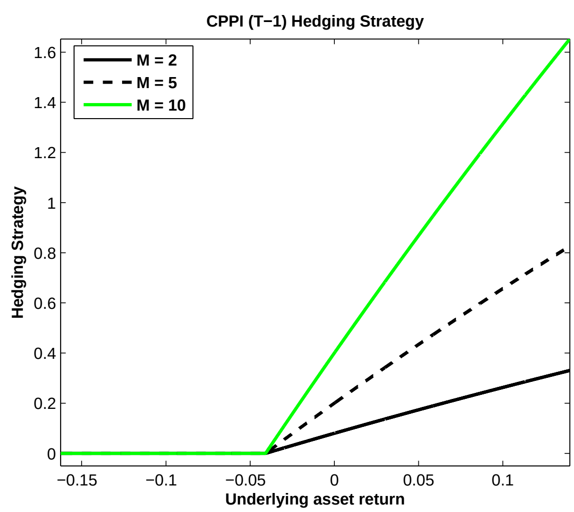


FIGURE 4.5 – CPPI Hedging Strategy for different multiplier values

4.4.2 Performance and Risk Measures

In order to compare the different portfolio insurance strategies, we compute a number of performance measures and risk measures.

We define $R_i = ln\left(\frac{\Pi_{i,T}}{\Pi_{i,0}}\right)$ the i-th portfolio monthly return of a time series of length N and R_f the monthly risk free rate.

To compare the performance of each portfolio insurance strategy, we use :

– Sharpe Ratio

– Omega Ratio

To compare the risk management of each portfolio insurance strategy, we use :

– 5% - Cornish Fisher Value at Risk

– Maximum drawdown

– Floor Ratio : $\frac{\sum_{i=1}^{N} I_{R_i<Floor}(R_i)}{N}$

– Floor Shortfall : $E\left[R_i|R_i < Floor\right]$

– Floor Maximum breakdown : $min\left(R_i|R_i < Floor\right)$

$$I_{R_i<Floor}(R_i) = \begin{cases} 1 & \text{if } R_i < Floor \\ 0 & \text{Otherwise.} \end{cases} \tag{4.22}$$

The Sharpe Ratio (SR)

The Sharpe ratio introduced by Sharpe (1966) is the most commonly used ratio in the industry. The main advantage of this measure is that it is easy to compute and interpret. The underlying assumption is that any asset class can be fully described in terms of risk-return relationship by the expected excess return and the variance of the asset class. All assets evolve in a Gaussian world in which risk is fully characterized by the volatility (no asymmetry and kurtosis).

The Sharpe ratio (SR) can be expressed as :

$$SR = \frac{(E\left[R_i\right] - R_f)}{\sigma_\Pi} \tag{4.23}$$

where σ_Π is the standard deviation of the portfolio returns.

The Omega Ratio (Ω)

The Omega ratio introduced by Keating and Shadwick (2002) relaxes the hypothesis that returns follow a Gaussian distribution. In fact, it is a well accepted fact that returns are not normally distributed. This measure leads to a full characterization of the risk reward properties of the distribution by measuring the overall impact of all moments.

Omega ratio (Ω) can be expressed as :

$$\Omega(L) = \frac{\int_L^{+\infty} \left[1 - F(x)\right] dx}{\int_{-\infty}^{L} F(x) dx} \tag{4.24}$$

where F the portfolio's return distribution and L a threshold selected by the investor (could be R_f).

Omega could also be written in terms of returns R_i :

$$\Omega(L) = \frac{E\left[max(R_i - L, 0)\right]}{E\left[max(L - R_i, 0)\right]} \tag{4.25}$$

Cornish Fisher Value at Risk

We use the modified Cornish-Fisher VaR through the use of a Cornish Fisher expansion to come up with a risk measure that takes the higher moments of non-normal distributions. The Cornish Fisher expansion approximates quantiles of a random variable based on its first five cumulants.

Cumulants κ_r of a random variable X can be expressed in terms of its mean $\mu = E(X)$ and central moments $\mu_r = E[(X-\mu)^r]$ such as :

$$\begin{aligned}
\kappa_1 &= \mu \\
\kappa_2 &= \mu_2 \\
\kappa_3 &= \mu_3 \\
\kappa_4 &= \mu_4 - 3\mu_2^2 \\
\kappa_5 &= \mu_5 - 10\mu_3\mu_2
\end{aligned}$$

Suppose that X has mean 0 and standard deviation 1. The q-quantile $\Phi_X^{-1}(q)$ of X based upon its cumulants is :

$$\begin{aligned}
\Phi_X^{-1}(q) \approx\ & \Phi_Z^{-1}(q) + \frac{\Phi_Z^{-1}(q)^2 - 1}{6}\kappa_3 + \frac{\Phi_Z^{-1}(q)^3 - 3\Phi_Z^{-1}(q)}{24}\kappa_4 \\
& - \frac{2\Phi_Z^{-1}(q)^3 - 5\Phi_Z^{-1}(q)}{36}\kappa_3^2 + \frac{\Phi_Z^{-1}(q)^4 - 6\Phi_Z^{-1}(q)^2 + 3}{120}\kappa_5 \\
& - \frac{\Phi_Z^{-1}(q)^4 - 5\Phi_Z^{-1}(q)^2 + 2}{24}\kappa_3\kappa_4 + \frac{12\Phi_Z^{-1}(q)^4 - 53\Phi_Z^{-1}(q)^2 + 17}{324}\kappa_3^3
\end{aligned}$$

Then one can easily express the q-quantile x^* of $X^* = \frac{X-\mu}{\sigma}$ where μ and σ are respectively the mean and the standard deviation of X. For more details on the calculation one can refer to Zangari (1996) and Favre and Galeano (2002). The Cornish-Fisher expansion also avoids computationally intensive techniques such as re-sampling or Monte-Carlo simulation to compute the Value at Risk.

4.5 Empirical Results

In order to evaluate the performance of the Left Truncated Gaussian distribution we run several out-of-sample tests, adjusting both the level and maturity of the desired insurance. Specifically, we will consider insurance horizons of both 1 month and 6 months, and provide portfolio insurance at the 5% and 10% levels. Hedging will be applied on a daily basis for the Left Truncated Gaussian as well as the benchmark strategies. We also present results for the CPPI using a monthly re-balancing which is more consistent with the industry standard (daily re-balancing is prohibitively expensive given the relative size of the trades).

The risky asset will be the front-month $S\&P500$ futures contract from January 1988 and December 2008. We use the $1-month$ BBA Libor as the non-risky asset. All prices are close prices extracted from the Bloomberg database. The experiments will be applied out of sample, using a rolling 251 days window for underlying return's process modeling. To illustrate the embedded cost of such strategies in a "bull" market versus the effectiveness in a "bear" market, we split the data in two samples : $1988-1998$ and $1998-2008$.

To implement a realistic environment, we propose two layers of hedging costs :

- Transaction costs : $10bps$ applied on portfolio adjustment size.
- Financing Spread : the spread between lending and borrowing a dollar amount for a hedging strategy is $50bps$ per annum.

The cost C function can therefore be expressed as :

$$C_t = |W_t - W_{t-1}| * S_t * 0.001 + I_{W_t>1} * |W_t - 1| * \left(e^{0.05/360} - 1\right) \tag{4.26}$$

Payoff Distribution Model

The target monthly distribution is a Left Truncated Gaussian distribution, which allows for volatility, asymmetry and downside risk control. We test for two different target volatility : 8% and 12% monthly annualized volatility. The underlying process of the daily returns of the S&P500 is modeled as a Gaussian distribution and simulated $100,000$ times for each day step. The monthly law is then inferred from the daily process. For the sake of simplicity and comparability across the methodologies, we make the assumption that the the return distribution of the S&P500 is Gaussian, however the PDM can accommodate any form of underlying distribution. Using a less restrictive assumption about the returns would only strengthen the results.

BS Synthetic Put

The classical put option will be evaluated under a Black-Scholes framework, as an industry standard for option valuation. We use the daily standard deviation on the past 251 days (rolling window) as the volatility input for the Black-Scholes formula.

CPPI

The CPPI approach needs to fix a value for the multiplier m. Since there is no methodology to evaluate this parameter, instead of fixing the multiplier constant arbitrarily, the value for m is computed each month by fitting the $CCPI_{t=0}$ exposure to the BS delta value (ω_0^{BS}), such as :

$$m = 100 * \frac{\omega_0^{BS}}{(100 - Fe^{-rT})} \tag{4.27}$$

with 100 the standardized initial monthly value for the hedged portfolio, F the selected floor value and r the risk free rate.

4.5.1 Numerical Results

All the results presented in the following section are out-of-sample.

Experiment 1

The downside protection is set at -5% per month. Results are presented for the two sub-periods, 1988-1998 and 1998-2008.

TABLE 4.I – Monthly downside protection at -5% $1988 - 1998$

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	0.0123	0.0095	0.0095	0.0059	0.0108	0.0059	0.0072
Std. dev.	0.0382	0.0383	0.0357	0.0395	0.0329	0.0205	0.0277
Skewness	-0.8925	-0.3642	-0.4406	0.5858	-0.7972	-0.5481	-0.3551
Kurtosis	5.7848	2.8542	3.4131	4.5841	5.2215	3.5761	3.2019
Minimum	-0.1631	-0.0891	-0.1027	-0.1086	-0.1327	-0.0593	-0.0659
Maximum	0.1072	0.1072	0.1052	0.1586	0.0960	0.0609	0.0844
Sharpe Ratio	1.1106	0.8626	0.9170	0.5163	1.1321	0.9975	0.8958
Omega Ratio	2.2838	1.8471	1.9439	1.4831	2.2979	2.0666	1.9217
VaR @95%	-0.0642	-0.0593	-0.0554	-0.0521	-0.0526	-0.0319	-0.0428
Max DD	0.1654	0.1514	0.1585	0.2057	0.1368	0.0929	0.1306
Fl. Ratio (%)	4.5455	10.6061	6.0606	4.5455	3.0303	1.5152	3.0303
Floor Shortfall	-0.0843	-0.0638	-0.0719	-0.0695	-0.0848	-0.0549	-0.0600
Fl. Max breakdown	-0.1631	-0.0891	-0.1027	-0.1086	-0.1327	-0.0593	-0.0659
Trans. Costs (bps)	0	2.1798	5.9741	25.3541	0.1592	2.0639	6.1538
Lev. Costs (bps)	0	0	0	0.4296	0	0	0.0996

Table 4.II – Monthly downside protection at -5% $1998-2008$

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	-0.0027	-0.0040	-0.0060	-0.0104	-0.0017	-0.0013	-0.0031
Std. dev.	0.0449	0.0412	0.0395	0.0367	0.0362	0.0203	0.0278
Skewness	-0.8266	-0.1168	-0.3043	0.5649	-0.6967	-1.0374	-0.6589
Kurtosis	4.7925	2.5548	2.9135	3.0267	4.0719	5.5761	3.7602
Minimum	-0.1894	-0.1036	-0.1280	-0.0987	-0.1411	-0.0956	-0.1100
Maximum	0.0993	0.0993	0.0887	0.1052	0.0823	0.0397	0.0577
Sharpe Ratio	-0.2089	-0.3385	-0.5233	-0.9861	-0.1654	-0.2241	-0.3829
Omega Ratio	0.8510	0.7852	0.6845	0.5067	0.8832	0.8447	0.7531
VaR @95%	-0.0882	-0.0747	-0.0744	-0.0618	-0.0684	-0.0408	-0.0529
Max DD	0.4642	0.5559	0.5573	0.7142	0.3916	0.2436	0.3626
Fl. Ratio (%)	13.3333	21.6667	15.0000	10.0000	10.8333	0.8333	5.8333
Floor Shortfall	-0.0860	-0.0635	-0.0712	-0.0621	-0.0724	-0.0956	-0.0634
Fl. Max breakdown	-0.1894	-0.1036	-0.1280	-0.0987	-0.1411	-0.0956	-0.1100
Trans. Costs (bps)	0	4.2618	10.1051	28.4649	0.1792	1.9053	5.4877
Lev. Costs (bps)	0	0	0	0.3377	0	0	0.0417

Overall, the Payoff Distribution Model delivers a portfolio with a better risk profile. The PDM funds exhibit lower volatilities than the other portfolio insurance strategies, because of the 8% and 12% volatility targets. Note that the realized volatilities (out-of-sample) are very close to the targeted volatilities. The Payoff Distribution approach tends to adjust the leverage for the prevailing market conditions, as illustrated in figure 4.8. All of the portfolio insurance methodologies deliver a lower return than the S&P500 in the $88-98$ period. This is not surprising as there is an implicit cost to any insurance program. In the case of dynamic hedging, that cost will be reflected in the performance during upward trending markets. The cost is comparable across the different approaches, with the exception of the CPPI with daily hedging, which underperforms significantly due to the important transaction costs (over 40 bps per month).

During the bear market period, the Omega ratio is highest for the two PDM strategies and the CPPI with monthly re-balancing, and are comparable to the performance of the market. The value-at-risk estimates are however lower for the PDM strategies.

The two PDM models also outperform the monthly CPPI in terms of respecting the maximum drawdown and the other risk parameters.

For illustration purpose we present the evolution of the different strategies over the $1998-2008$ period, as well as the monthly return densities and fund exposures. We also plot in appendix figure 4.15 the evolution of the CPPI multiplier over the $98-08$ period.

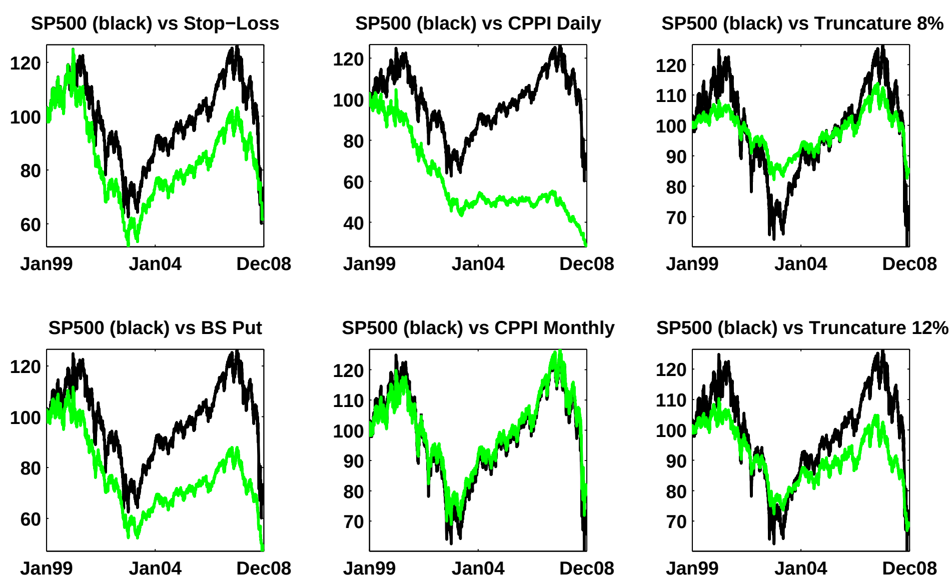


FIGURE 4.6 – Hedged Portfolios versus $S\&P500$

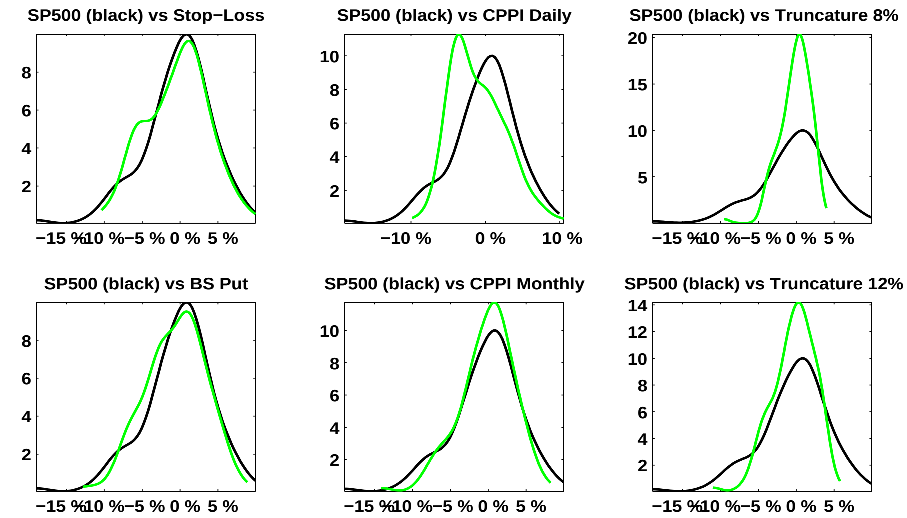


FIGURE 4.7 – Hedged Portfolios Monthly Returns Kernel Densities

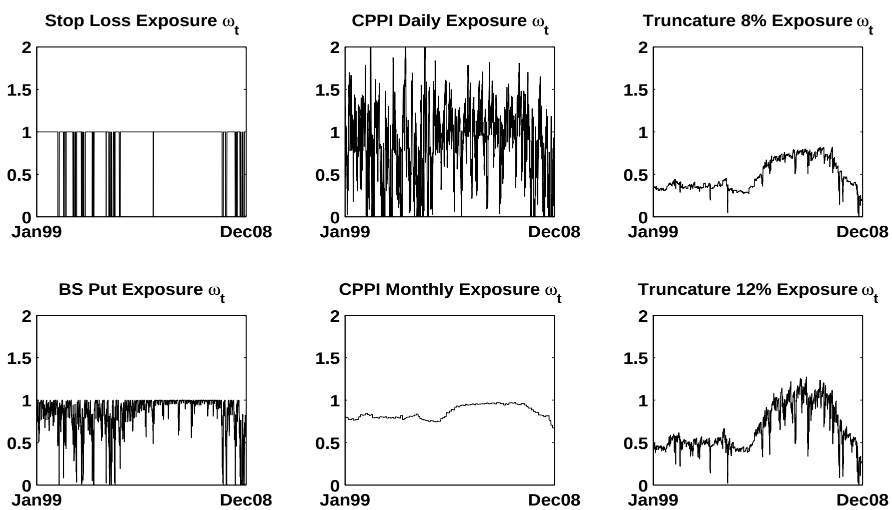


FIGURE 4.8 – Hedged Portfolios Exposure

Experiment 2

The downside protection is now set at -10% with a 6-months horizon.

Table 4.III – Monthly properties for Hedged Campaigns 1988 – 1998

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	0.0123	0.0101	0.0096	0.0068	0.0090	0.0061	0.0086
Std. dev.	0.0382	0.0364	0.0350	0.0364	0.0349	0.0198	0.0301
Skewness	-0.8925	-0.9271	-0.6396	0.3710	-0.8783	-0.5413	-0.5437
Kurtosis	5.7848	6.4913	4.8138	4.0835	6.2274	4.1804	4.1497
Minimum	-0.1631	-0.1631	-0.1272	-0.0787	-0.1525	-0.0695	-0.1012
Maximum	0.1072	0.1072	0.1071	0.1451	0.1069	0.0599	0.0919
Sharpe Ratio	1.1106	0.9566	0.9541	0.6486	0.8943	1.0742	0.9839
Omega Ratio	2.2838	2.0968	2.0452	1.6507	1.9957	2.2223	2.0794
VaR @95%	-0.0642	-0.0672	-0.0572	-0.0536	-0.0637	-0.0301	-0.0473
Max DD	0.1654	0.1661	0.1611	0.2569	0.1872	0.0743	0.1193
Fl. Ratio (%)	0	9.0909	9.0909	13.6364	9.0909	0	0
Floor Shortfall	0	-0.1478	-0.1347	-0.1288	-0.1511	0	0
Fl. Max breakdown	0	-0.1816	-0.1548	-0.1690	-0.1704	0	0
Trans. Costs (bps)	0	0.3006	2.5213	12.1335	0.5397	1.4000	2.5252
Lev. Costs (bps)	0	0	0	0.2290	0.0000	0	0.2316

Table 4.IV – Monthly properties for Hedged Campaigns 1998 – 2008

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	-0.0027	-0.0019	-0.0047	-0.0092	-0.0025	-0.0005	-0.0014
Std. dev.	0.0449	0.0357	0.0367	0.0335	0.0290	0.0173	0.0260
Skewness	-0.8266	-1.0806	-0.8827	-0.4072	-0.8042	-0.3001	-0.3546
Kurtosis	4.7925	5.6058	5.3579	3.3629	4.1911	2.2972	2.3912
Minimum	-0.1894	-0.1376	-0.1631	-0.1041	-0.0986	-0.0422	-0.0626
Maximum	0.0993	0.0784	0.0670	0.0713	0.0618	0.0351	0.0473
Sharpe Ratio	-0.2089	-0.1865	-0.4451	-0.9493	-0.2940	-0.0964	-0.1859
Omega Ratio	0.8510	0.8530	0.7111	0.4627	0.7842	0.9348	0.8766
VaR @95%	-0.0882	-0.0754	-0.0775	-0.0698	-0.0585	-0.0310	-0.0477
Max DD	0.4642	0.5140	0.5770	0.6837	0.4647	0.2358	0.3438
Fl. Ratio (%)	20.0000	40.0000	40.0000	50.0000	5.0000	0	10.0000
Floor Shortfall	-0.1909	-0.1154	-0.1473	-0.1384	-0.1121	0	-0.1282
Fl. Max breakdown	-0.3530	-0.1374	-0.2193	-0.1940	-0.1121	0	-0.1561
Trans. Costs (bps)	0	1.3111	6.6194	17.5245	1.5319	1.6464	2.8916
Lev. Costs (bps)	0	0	0	0.2681	0	0	0.1204

For a rolling 6 months campaign, findings are similar to the previous experiments. The 6 months downside protection is breached 10% of the time for the 12% Truncation in the $98-08$ period, in comparison to 50% of the time for the CPPI and 40% for the Stop Loss methodology. The Stop Loss model is unadapted for long horizon downside hedging, since it cannot recover losses that occur at the beginning of the period.

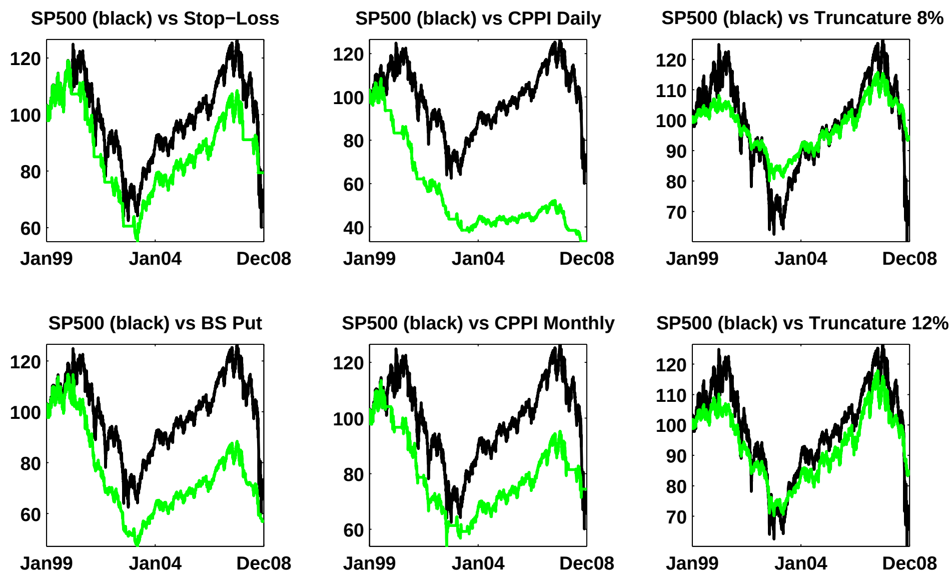


FIGURE 4.9 – Hedged Campaigns versus $S\&P500$

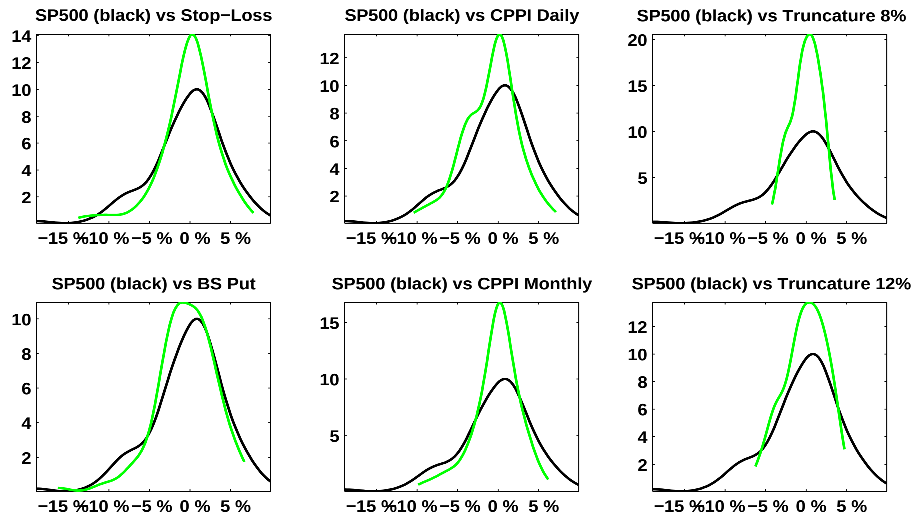


FIGURE 4.10 – Hedged Portfolios Monthly Returns Kernel Densities

One could argue that these results are highly dependent on the starting point of the experiment, so the next table presents the average results for all possible 6 months campaigns. That is, we run overlapping windows so each calendar month represents a start date. Results are presented as 6 months cumulative returns.

TABLE 4.V – 6 Months cumulative return properties for Hedged Campaigns 1988−1998

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	0.0683	0.0624	0.0582	0.0498	0.0609	0.0384	0.0539
Std. dev.	0.0753	0.0841	0.0872	0.0854	0.0818	0.0479	0.0729
Skewness	-0.1994	-0.4906	-0.5630	-0.2872	-0.3152	-0.0449	-0.0657
Kurtosis	2.5583	2.9263	3.0954	2.6621	2.6318	2.4465	2.4553
Minimum	-0.1101	-0.1606	-0.1763	-0.2019	-0.1374	-0.0630	-0.0993
Maximum	0.2305	0.2305	0.2302	0.2268	0.2301	0.1416	0.2094
Sharpe Ratio	3.1400	2.5706	2.3102	2.0182	2.5778	2.7750	2.5600
Omega Ratio	9.0352	5.7622	4.8360	4.1591	5.9528	7.1995	6.1147
Fl. Ratio (%)	1.5748	7.0866	5.5118	3.9370	3.9370	0	0
Floor Shortfall	-0.1086	-0.1226	-0.1490	-0.1386	-0.1236	0	0
Fl. Max breakdown	-0.1101	-0.1606	-0.1763	-0.2019	-0.1374	0	0

TABLE 4.VI – 6 Months cumulative returns properties for Hedged Campaigns 1998 – 2008

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	-0.0095	-0.0032	-0.0150	-0.0323	-0.0011	0.0005	-0.0027
Std. dev.	0.1136	0.0884	0.1004	0.1020	0.0843	0.0466	0.0705
Skewness	-1.2768	-0.0311	-0.2478	-0.5099	-0.1352	-0.4793	-0.4498
Kurtosis	5.1231	1.6937	1.7787	2.5827	1.9440	2.3755	2.4114
Minimum	-0.4475	-0.1373	-0.2187	-0.2929	-0.1718	-0.1118	-0.1862
Maximum	0.1810	0.1800	0.1729	0.1515	0.1720	0.0897	0.1364
Sharpe Ratio	-0.2901	-0.1254	-0.5189	-1.0989	-0.0448	0.0389	-0.1318
Omega Ratio	0.7999	0.9211	0.7026	0.4310	0.9705	1.0270	0.9134
Fl. Ratio (%)	18.2609	33.9130	31.3043	23.4783	17.3913	1.7391	10.4348
Floor Shortfall	-0.1953	-0.1110	-0.1442	-0.1801	-0.1233	-0.1114	-0.1322
Fl. Max breakdown	-0.4475	-0.1373	-0.2187	-0.2929	-0.1718	-0.1118	-0.1862

The Left Truncated strategy outperform the underlying $S\&P500$ buy and hold strategy. The downside protection for the was breached only 2% and 10% for the 8% and 12% PDM campaigns respectively, in comparison to almost 20% – 30% for the other hedge programs in the period 98 – 08.

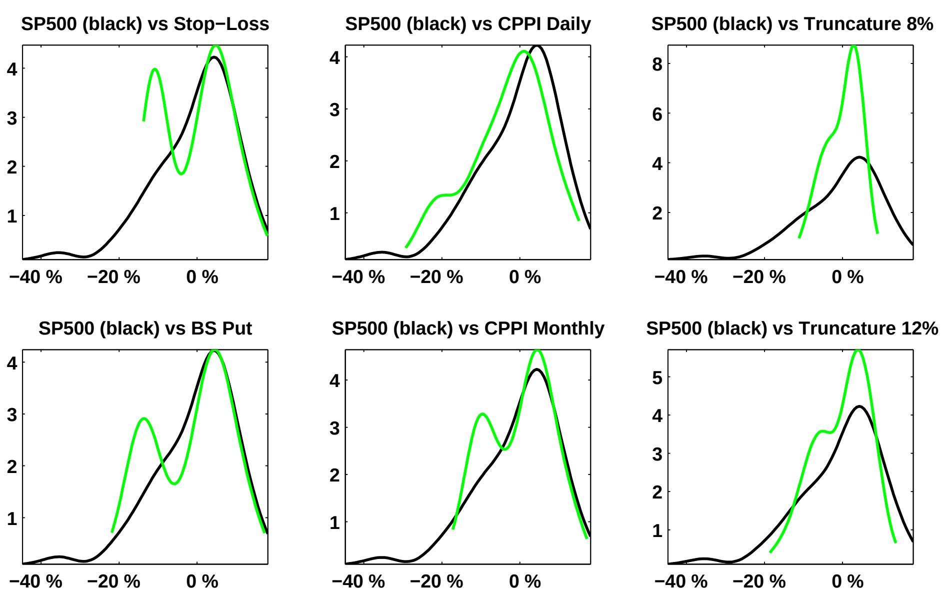


FIGURE 4.11 – Hedged Portfolios Campaigns Returns Kernel Densities

The complete out-of-sample test scenarios are presented in appendix 4.6. The Left

Truncated PDM strategy is the less exposed to liquidity constraints. In the case of a severe corrections, investors trying to cover his losses by liquidating his positions could be confronted with a serious liquidity crunch, due to a lack of buyers or market depth provided by market makers. In this context, the PDM with Left Truncation is more dynamic and less exposed to liquidity risk. The Payoff Distribution also allows for a volatility control of the hedged portfolio, even in a high volatile market condition such as in recent past months.

4.6 Conclusion

In this paper, we propose a new approach to dynamic portfolio insurance. We extend Dybvig (1988) Payoff Distribution Model to include downside risk protection. By targeting a Left Truncated Gaussian distribution using the PDM, an investor can customize his return distribution and prevent significant drawdowns. This embedded portfolio insurance technique does not require the fund manager to overlay any further risk management structures. We demonstrate the effectiveness of the approach by comparing it to the more traditional dynamic portfolio insurance approaches, specifically Constant Proportion Portfolio Insurance, a Stop loss strategy or a synthetic put. The results clearly indicate that the PDM provides a more reliable framework for portfolio insurance, without sacrificing the performance of the fund.

Bibliography

Amin, G. and Kat, H. (2003). Hedge fund performanc 1990-2000 : Do the "money machines" really add value. *Journal of Financial and Quantitative Analysis*, 38(2) :251–275.

Annaert, J., Osselaer, S. V., and Verstraete, B. (2009). Performance evaluation of portfolio insurance strategies using stochastic dominance criteria. *Journal of Banking & Finance*, 33(2) :272–280.

Bertrand, P. and Prigent, J.-L. (2003). Portfolio insurance strategies : A comparison of standard methods when the volatility of the stock is stochastic. *International Journal Of Business*, 8(4).

Black, F. and Jones, R. (1987). Symplifying portfolio insurance. *Journal of Portfolio Management*, 14 :48–51.

Black, F. and Perold, A. (1992). Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control*, 16 :403–426.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81 :637–654.

Brennan, M. J. and Schwartz, E. S. (1979). Alternative investment strategies for the issuers of equity-linked life insurance policies with an asset value guarantee. *Journal of Business*, 52 :63–93.

Cont, R. and Tankov, P. (2007). Constant proportion portfolio insurance in presence of jumps in asset prices. Pre- and post-print documents, HAL.

Del Moral, P., Rémillard, B., and Rubenthaler, S. (2006). Monte Carlo approximations of American options. Technical report, GERAD.

Dybvig, P. (1988). Distributional analysis of portfolio choice. *The Journal of Business*, 61(3) :369–393.

Favre, L. and Galeano, J.-A. (2002). Mean-modified value-at-risk optimization with hedge funds. *Journal of Alternative Investments*, Fall.

Hocquard, A., Papageorgiou, N., and Rémillard, B. (2008). Optimal hedging strategies with an application to hedge fund replication. *Wilmott Magazine*, (Jan-Feb) :62–66.

Johnson, K. and Balakrishnan (1996). Continuous univariate distributions : N.l. johnson, s. kotz and n. balakrishnan vol. 1, 2nd edition. john wiley, new york, 1994, pp. xix + 756, price :. *Computational Statistics & Data Analysis*, 21(1) :119–119.

Keating, C. and Shadwick, W. F. (2002). A universal performance measure. Technical report, The Finance Development Centre, London.

Liu, J., Longstaff, F., and Pan, J. (2003). Dynamic asset allocation with event risk. *Journal of Finance*, 58(1) :231–259.

Papageorgiou, N., Rémillard, B., and Hocquard, A. (2008). Replicating the properties of hedge fund returns. *Journal of Alternative Invesments*, 11 :8–38.

Rubinstein, M. and Leland, H. (1981). Replicating option with positions in stock and cash. *Financial Analyst Journal*, 46 :61–70.

Schweizer, M. (1995). Variance-optimal hedging in discrete time. *Math. Oper. Res.*, 20(1) :1–32.

Shah, S. and Jaiswal, M. (1966). Estimation of parameters of doubly truncated normal distribution from first four sample moments. *Ann. Inst. Statist Math*, 88 :107–111.

Sharpe, W. (1966). Mutual fund performance. *Journal of Business*, 39 :119–138.

Zangari, P. A. (1996). Var methodology for portfolios that include options. *RiskMetrics Monitor*, pages 4–12.

Appendix A : Experiments Results

TABLE 4.VII – Monthly downside protection at −10% : Monthly Properties 1988−1998

	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	0.0123	0.0121	0.0115	0.0091	0.0119	0.0070	0.0098
Std. dev.	0.0382	0.0387	0.0384	0.0393	0.0373	0.0203	0.0308
Skewness	-0.8925	-0.9653	-0.9594	0.1455	-0.8803	-0.5659	-0.6017
Kurtosis	5.7848	6.0069	5.9800	3.6867	5.7927	3.8596	3.7102
Minimum	-0.1631	-0.1631	-0.1623	-0.0952	-0.1583	-0.0673	-0.0959
Maximum	0.1072	0.1072	0.1071	0.1434	0.1066	0.0604	0.0858
Sharpe Ratio	1.1106	1.0842	1.0391	0.7995	1.1068	1.1968	1.0967
Omega Ratio	2.2838	2.2496	2.1793	1.8191	2.2804	2.3785	2.2138
VaR @95%	-0.0642	-0.0679	-0.0676	-0.0569	-0.0628	-0.0300	-0.0471
Max DD	0.1654	0.1692	0.1690	0.1815	0.1608	0.0719	0.1204
Fl. Ratio (%)	1.5152	1.5152	1.5152	0	1.5152	0	0
Floor Shortfall	-0.1345	-0.1434	-0.1419	0	-0.1311	0	0
Fl. Max breakdown	-0.1631	-0.1631	-0.1623	0	-0.1583	0	0
Trans. Costs (bps)	0	0.2255	1.2642	14.0707	0.1011	1.1229	1.9060
Lev. Costs (bps)	0	0	0	0.2870	0	0.0000	0.2338

TABLE 4.VIII – Monthly downside protection at −10% : Monthly Properties 1998−2008

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	-0.0027	-0.0041	-0.0053	-0.0087	-0.0025	-0.0002	-0.0013
Std. dev.	0.0449	0.0474	0.0472	0.0428	0.0428	0.0192	0.0312
Skewness	-0.8266	-0.8487	-0.8727	-0.0543	-0.7740	-0.6994	-1.3309
Kurtosis	4.7925	3.9754	4.1258	3.0051	4.4888	3.9789	7.6132
Minimum	-0.1894	-0.1525	-0.1605	-0.1376	-0.1746	-0.0767	-0.1632
Maximum	0.0993	0.0993	0.0980	0.1055	0.0965	0.0431	0.0606
Sharpe Ratio	-0.2089	-0.3019	-0.3874	-0.7073	-0.2003	-0.0317	-0.1401
Omega Ratio	0.8510	0.7888	0.7383	0.5981	0.8579	0.9773	0.8982
VaR @95%	-0.0882	-0.0953	-0.0962	-0.0789	-0.0828	-0.0347	-0.0719
Max DD	0.4642	0.5027	0.5543	0.6775	0.4502	0.2058	0.3322
Fl. Ratio (%)	2.5000	6.6667	4.1667	1.6667	1.6667	0	0.8333
Floor Shortfall	-0.1355	-0.1220	-0.1344	-0.1242	-0.1420	0	-0.1632
Fl. Max breakdown	-0.1894	-0.1525	-0.1605	-0.1376	-0.1746	0	-0.1632
Trans. Costs (bps)	0	1.2292	3.3663	17.6007	0.1353	1.1886	1.8275
Lev. Costs (bps)	0	0	0	0.2452	0	0	0.1154

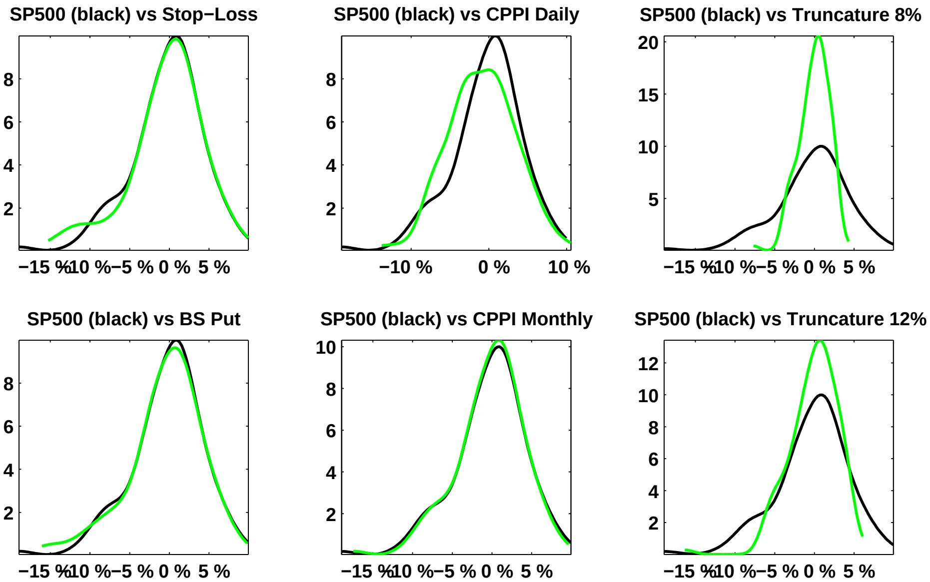


FIGURE 4.12 – Hedged Portfolios versus $S\&P500$

Table 4.IX – 6−*Months* downside protection at −5% : Monthly Properties 1988−1998

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	0.0123	0.0053	0.0092	0.0051	0.0074	0.0055	0.0076
Std. dev.	0.0382	0.0303	0.0313	0.0356	0.0304	0.0187	0.0263
Skewness	-0.8925	-0.0240	-0.0605	0.9252	-0.4994	-0.3373	-0.1685
Kurtosis	5.7848	3.9585	3.0656	6.0834	4.6639	3.3706	3.0737
Minimum	-0.1631	-0.0736	-0.0659	-0.0768	-0.1107	-0.0458	-0.0518
Maximum	0.1072	0.1072	0.1055	0.1697	0.0977	0.0575	0.0840
Sharpe Ratio	1.1106	0.6083	1.0161	0.4938	0.8466	1.0266	1.0074
Omega Ratio	2.2838	1.6602	2.0875	1.5088	1.9654	2.1421	2.0975
VaR @95%	-0.0642	-0.0490	-0.0444	-0.0542	-0.0496	-0.0281	-0.0385
Max DD	0.1654	0.1920	0.1390	0.1765	0.1592	0.0831	0.1174
Fl. Ratio (%)	4.5455	40.9091	13.6364	31.8182	13.6364	9.0909	9.0909
Floor Shortfall	-0.0916	-0.0648	-0.0741	-0.0718	-0.0997	-0.0549	-0.0758
Fl. Max breakdown	-0.0916	-0.0819	-0.1036	-0.1177	-0.1258	-0.0583	-0.0869
Trans. Costs (bps)	0	1.3530	5.5710	17.6921	1.1879	2.3624	4.5218
Lev. Costs (bps)	0	0	0	0.3269	0	0.0000	0.1634

Table 4.X – 6−*Months* downside protection at −5% : Monthly Properties 1998−2008

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	-0.0027	-0.0022	-0.0057	-0.0075	-0.0020	-0.0018	-0.0032
Std. dev.	0.0449	0.0251	0.0268	0.0265	0.0203	0.0177	0.0248
Skewness	-0.8266	-0.5174	-0.2082	-0.1616	-0.9624	-1.2079	-0.7779
Kurtosis	4.7925	4.5080	3.1787	5.5907	6.0186	6.7151	4.1377
Minimum	-0.1894	-0.0896	-0.0788	-0.1093	-0.0861	-0.0889	-0.0991
Maximum	0.0993	0.0631	0.0576	0.0731	0.0484	0.0280	0.0408
Sharpe Ratio	-0.2089	-0.3082	-0.7376	-0.9765	-0.3388	-0.3470	-0.4459
Omega Ratio	0.8510	0.7532	0.5884	0.3807	0.7209	0.7668	0.7173
VaR @95%	-0.0882	-0.0509	-0.0513	-0.0621	-0.0452	-0.0395	-0.0497
Max DD	0.4642	0.3803	0.5255	0.5925	0.3610	0.2401	0.3535
Fl. Ratio (%)	35.0000	65.0000	55.0000	75.0000	30.0000	20.0000	35.0000
Floor Shortfall	-0.1441	-0.0588	-0.0985	-0.0747	-0.0673	-0.0816	-0.0934
Fl. Max breakdown	-0.3530	-0.0765	-0.1638	-0.1587	-0.0925	-0.1545	-0.2198
Trans. Costs (bps)	0	2.1307	11.9065	17.1488	2.1788	3.0330	5.6543
Lev. Costs (bps)	0	0	0	0.2427	0	0	0.0651

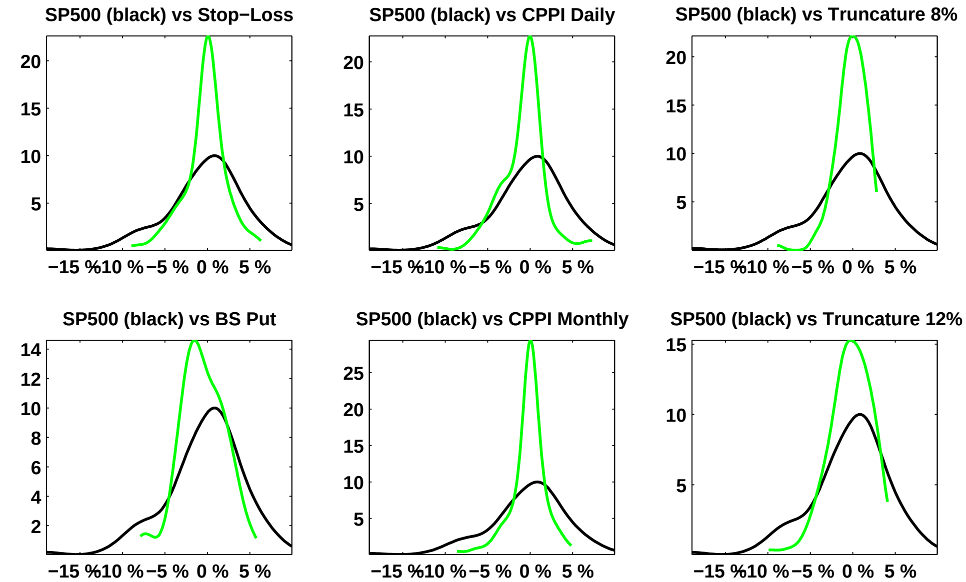


FIGURE 4.13 – Hedged Portfolios Monthly Returns Kernel Densities

TABLE 4.XI – $6 - Months$ downside protection at -5% : 6 Months cumulative returns properties $1988 - 1998$

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	0.0683	0.0409	0.0487	0.0328	0.0525	0.0334	0.0440
Std. dev.	0.0753	0.0876	0.0823	0.0947	0.0760	0.0493	0.0717
Skewness	-0.1994	0.1599	-0.0984	-0.3353	0.0492	0.0324	0.1491
Kurtosis	2.5583	1.8626	2.2420	3.1732	2.3351	2.3693	2.4312
Minimum	-0.1101	-0.0964	-0.1081	-0.3145	-0.1060	-0.0614	-0.0919
Maximum	0.2305	0.2305	0.2243	0.2175	0.2239	0.1409	0.2089
Sharpe Ratio	3.1400	1.6176	2.0484	1.1991	2.3941	2.3485	2.1260
Omega Ratio	9.0352	2.9634	4.1658	2.3495	5.8709	5.4006	4.7330
Fl. Ratio (%)	6.2992	32.2835	16.5354	20.4724	7.0866	3.9370	11.0236
Floor Shortfall	-0.0839	-0.0628	-0.0806	-0.0965	-0.0860	-0.0579	-0.0727
Fl. Max breakdown	-0.1101	-0.0964	-0.1081	-0.3145	-0.1060	-0.0614	-0.0919

TABLE 4.XII – $6 - Months$ downside protection at -5% : 6 Months cumulative returns properties $1998 - 2008$

Measure	$S\&P500$	Stop-Loss	BS Put	CPPI D	CPPI M	Tr. 8%	Tr. 12%
Mean	-0.0095	0.0014	-0.0181	-0.0358	0.0007	-0.0026	-0.0093
Std. dev.	0.1136	0.0691	0.0804	0.0870	0.0678	0.0433	0.0620
Skewness	-1.2768	0.6492	0.2816	-0.2058	0.1480	-0.0926	-0.0708
Kurtosis	5.1231	2.1774	1.8710	2.8864	2.2040	1.9047	2.1711
Minimum	-0.4475	-0.1033	-0.1616	-0.2756	-0.1499	-0.0890	-0.1452
Maximum	0.1810	0.1800	0.1523	0.1647	0.1519	0.0893	0.1310
Sharpe Ratio	-0.2901	0.0681	-0.7814	-1.4270	0.0354	-0.2117	-0.5175
Omega Ratio	0.7999	1.0450	0.5985	0.3498	1.0246	0.8675	0.6989
Fl. Ratio (%)	29.5652	52.1739	45.2174	42.6087	28.6957	20.0000	29.5652
Floor Shortfall	-0.1499	-0.0574	-0.0947	-0.1153	-0.0793	-0.0635	-0.0837
Fl. Max breakdown	-0.4475	-0.1033	-0.1616	-0.2756	-0.1499	-0.0890	-0.1452

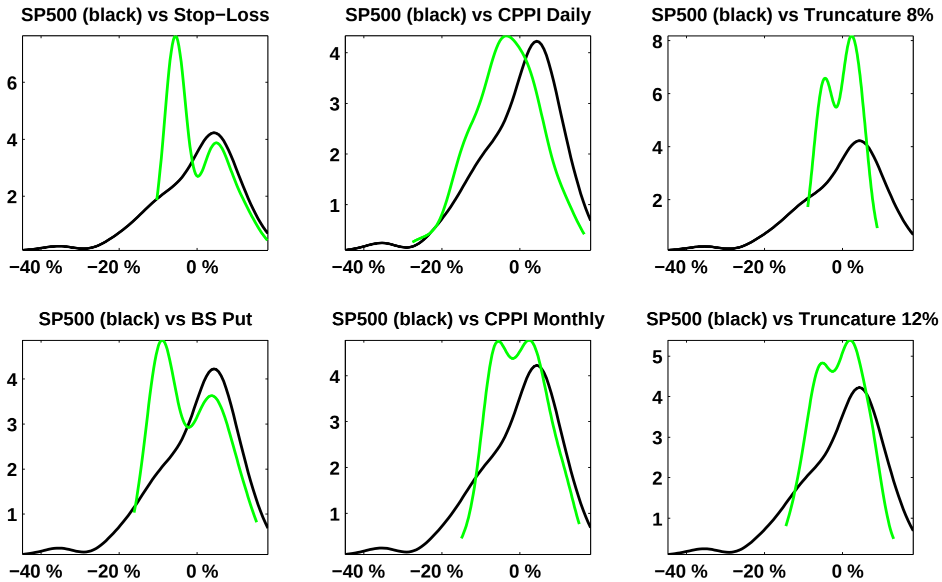


FIGURE 4.14 – Hedged Portfolios Campaigns Returns Kernel Densities

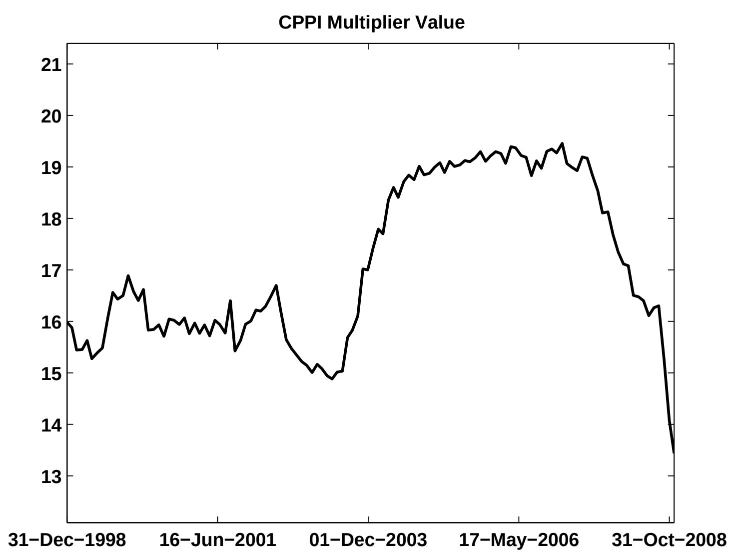


FIGURE 4.15 – CPPI Monthly Multiplier 1998 – 2008 with 5% Floor

Appendix B : Optimal hedging Strategy

In this section we describe the methodology used to derive the optimal hedging strategy. Having solved for the payoff function $g(R_T)$, we need to find an optimal dynamic trading strategy that will replicate the payoff function. We do so by selecting the portfolio (V_0, φ) such as to minimize the expected square hedging error

$$E\left[\beta_T^2 \left\{V_T(V_0, \varphi) - C_T\right\}^2\right],$$

where β_T is the discount factor and $C_T = 100 \exp^{g(R_T)}$ is the payoff at maturity.

In order to achieve this, we develop extensions of the results of Schweizer (1995). Suppose that $(\Omega, P, \mathcal{F})$ is a probability space with filtration $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$, under which the stochastic processes are defined. Assume that the price process S_t is d-dimensional, i.e. $S_t = \left(S_t^{(1)}, \ldots, S_t^{(d)}\right)$.

A dynamic replicating strategy can be described by a initial value V_0 and a sequence of random weight vectors $\varphi = (\varphi_t)_{t=0}^T$, where for any $j = 1, \ldots, d$, $\varphi_t^{(j)}$ denotes the number of parts of assets $S^{(j)}$ invested during period $(t-1, t]$. Because φ_t may depend only on the values values $S_0, \ldots, S_{t-1}$, the stochastic process φ_t is assumed to be predictable. Initially, $\varphi_0 = \varphi_1$, and the portfolio initial value is V_0. It follows that the amount initially invested in the non risky asset is $V_0 - \sum_{j=1}^d \varphi_1^{(j)} S_0^{(j)} = V_0 - \varphi_1^\top S_0$.

Since the hedging strategy must be self-financing, it follows that for all $t = 1, \ldots, T$,

$$\beta_t V_t(V_0, \varphi) - \beta_{t-1} V_{t-1}(V_0, \varphi) = \varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1}). \tag{4.28}$$

Using the self-financing condition (4.28), it follows that

$$\beta_T V_T = \beta_T V_T(V_0, \varphi) = V_0 + \sum_{t=1}^T \varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1}). \tag{4.29}$$

The replication strategy problem for a given payoff C is thus equivalent to finding the strategy (V_0, φ) so that the hedging error

$$G_T(V_0, \varphi) = \beta_T V_T(V_0, \varphi) - \beta_T C \tag{4.30}$$

is as small as possible. Here, the RMSHE (root mean square hedging error) measures the quality of replication. It is therefore natural to suppose that the prices $S_t^{(j)}$ have finite second moments. We further assume that the hedging strategy φ satisfies a similar property, namely that for any $t = 1, \ldots, T$, $\varphi_t^\top (\beta_t S_t - \beta_{t-1} S_{t-1})$ have finite second moments. Note that these two technical conditions were also made by Schweizer (1995).

For simplicity, set $\Delta_t = S_t - E(S_t|\mathcal{F}_{t-1})$, $t = 1, \ldots, T$. Under the above moment conditions, the conditional covariance matrix Σ_t of Δ_t exists and is given by

$$\Sigma_t = E\left\{\Delta_t \Delta_t^\top | \mathcal{F}_{t-1}\right\}, \ 1 \le t \le T. \tag{4.31}$$

In Schweizer (1995), the author treats the case $d = 1$ and assumes a restrictive boundedness condition. Here, in contrast, we treat the general d-dimensional case and we only suppose that Σ_t is invertible for all $t = 1, \ldots, T$. This was implicitly part of the boundedness condition of Schweizer (1995).

If Σ_t is not invertible for some t, there would exists a $\varphi_t \in \mathcal{F}_{t-1}$ such that $\varphi_t^\top S_t = \varphi_t^\top E(S_t|\mathcal{F}_{t-1})$, that is, $\varphi_t^\top S_t$ is predictable. Our assumption can be interpreted as saying that the genuine dimension of the assets is d.

Theorem 3 *Suppose that Σ_t is invertible for all $t = 1, \ldots, T$.*

Then the risk $E\{G^2(v_0, \varphi)\}$ is minimized by choosing recursively $\varphi_T, \ldots, \varphi_1$ satisfying

$$\varphi_t = (\Sigma_t)^{-1} E\left(\{S_t - E(S_t|\mathcal{F}_{t-1})\} C_t | \mathcal{F}_{t-1}\right), \ t = T, \ldots, 1, \tag{4.32}$$

where $C_T, \dots, C_0$ are defined recursively by setting $C_T = C$ and

$$\beta_{t-1} C_{t-1} = \beta_t E(C_t | \mathcal{F}_{t-1}) - {\varphi_t}^\top E(\beta_t S_t - \beta_{t-1} S_{t-1} | \mathcal{F}_{t-1}), \tag{4.33}$$

for $t = T, \dots, 1$.

Moreover the optimal value of v_0 is C_0, and

$$E(G^2) = \sum_{t=1}^{T} E\left(\beta_t^2 G_t^2\right),$$

where $G_t = {\varphi_t}^\top \{S_t - E(S_t|\mathcal{F}_{t-1})\} - \{C_t - E(C_t|\mathcal{F}_{t-1})\}$, $1 \le t \le T$.

Remark 4.6.1 *Because of the relation (4.33) and the fact that $v_0 = C_0$, one can interpret C_t as the value to be invested at time t to replicate the payoff C at period T. In an option context, C_t would be the "value" of the option at time t.*

Example 4.6.1 (The Markovian case) *If the price process S is Markovian, i.e., the law of S_t given $\mathcal{F}_{t-1}$ is $\nu_t(S_{t-1}, dx)$, and if the terminal payoff $C_T = C$ only depends on the terminal prices, that is $C = f_T(S_T)$, then the Markov property, together with Theorem 3, yield that $C_t = f_t(S_t)$ and $\varphi_t = \psi_t(S_{t-1})$, where*

$$\begin{aligned}
L_{1t}(s) &= E(S_t|S_{t-1} = s) = \int x \nu_t(s, dx), \\
L_{2t}(s) &= E(S_t S_t^\top | S_{t-1} = s) = \int x x^\top \nu_t(s, dx), \\
A_t(s) &= L_{2t}(s) - L_{1t}(s) L_{1t}(s)^\top, \\
\psi_t(s) &= A_t(s)^{-1} E\left[\{S_t - L_{1t}(s)\} f_t(S_t) | S_{t-1} = s\right] \\
&= A_t(s)^{-1} \int (x - L_{1t}(s)) f_t(x) \nu_t(s, dx), \\
U_t(s, x) &= 1 - (L_{1t}(s) - \beta_{t-1} s/\beta_t)^\top A_t(s)^{-1} (x - L_{1t}(s)), \\
f_{t-1}(s) &= \frac{\beta_t}{\beta_{t-1}} E\{U_t(s, S_t) f_t(S_t) | S_{t-1} = s\} \\
&= \frac{\beta_t}{\beta_{t-1}} \int U_t(s, x) f_t(x) \nu_t(s, dx).
\end{aligned}$$

Note that $E(S_t|\mathcal{F}_{t-1}) = L_{1t}(S_{t-1})$ *and* $\Sigma_t = A_t(S_{t-1})$. *Explicit calculations can be done when the returns are assumed to be a finite Markov chain. In most models, one can write* $S_t = \omega_t(S_{t-1}, \xi_t)$ *where* ξ_t *is independent of* $\mathcal{F}_{t-1}$ *and has law* P_t. *When* μ_t *has an infinite support, there are ways to approximate* ψ_t *and* f_t. *In the Markovian case, one can use the methodology developed by Del Moral et al. (2006) to calculate both the* φ_t*'s and the* C_t*'s. The algorithm for implementing the dynamic trading strategy is based on Monte Carlo simulations and linear interpolation.*

Appendix C : Truncated Gaussian Distribution

C.1 : Cdf and Pdf

Two sides Truncation

Let X be $N(\mu, \sigma^2)$ and Y a truncated normal $TrN(\mu, \sigma^2, a, b)$ random variable. With ϕ the standard normal probability density function, we can write :

The truncated normal pdf :

$$f(y, \mu, \sigma^2, a, b) = \frac{\phi\left(\frac{y-\mu}{\sigma}\right)}{\sigma\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} I_{[a,b]}(y) \tag{4.34}$$

The truncated normal cdf :

$$F(y, \mu, \sigma^2, a, b) = \frac{\Phi\left(\frac{y-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} I_{[a,b]}(y) \tag{4.35}$$

Left-side Truncation

A truncated normal distribution with only a left-side truncation is then written :

$$f(y, \mu, \sigma^2, a) = \frac{\phi\left(\frac{y-\mu}{\sigma}\right)}{\sigma\left[1 - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} I_{(y>a)}(y) \tag{4.36}$$

The truncated normal cdf :

$$F(y, \mu, \sigma^2, a) = \frac{\Phi\left(\frac{y-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} I_{(y>a)}(y) \tag{4.37}$$

C.2 : Four moments

Two sides Truncation

The expressions for the mean and variance respectively are :

$$E(Y) = \mu + \sigma \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} \right] \tag{4.38}$$

$$VAR(Y) = \sigma^2 \left[1 + \frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] - \sigma^2 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right)}{\left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} \right]^2 \tag{4.39}$$

The skewness and kurtosis of a truncated normal distribution with arbitrary mean and variance can be obtained from cumulants based on the moment generating function, detailed in Shah and Jaiswal (1966).

Skewness is defined by $Sk = \mu_3/\mu_2^{3/2}$ with μ_i central moments.

$$Sk = -\frac{1}{V^{3/2}} \left(2(z_b - z_a)^3 + (3bz_b - 3az_a - 1)(z_b - z_a) + b^2 z_b - a^2 z_a \right) \tag{4.40}$$

Kurtosis is defined by $Ku = \mu_4/\mu_2^2$ with μ_i central moments.

$$\begin{aligned} Ku = \quad & \tfrac{1}{V^2}(-3(z_b - z_a)^4 - 6(bz_b - az_a)(z_b - z_a)^2 - 2(z_b - z_a)^2 \\ & -4(b^2 z_b - a^2 z_a)(z_b - z_a) - 3(bz_b - az_a) - (b^3 z_b - a^3 z_a) + 3) \end{aligned}$$

Where :

$$V = 1 - (bz_b - az_a) - (z_b - z_a)^2 \,;\; z_a = \frac{\phi(a)}{\Phi(b) - \Phi(a)} \,;\; z_b = \frac{\phi(b)}{\Phi(b) - \Phi(a)} \tag{4.41}$$

With ϕ denotes the standard normal distribution function and Φ denotes the density.

Left-side Truncation

$$E(Y) = \mu + \sigma \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)}{\left[1 - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} \right] \tag{4.42}$$

$$VAR(Y) = \sigma^2 \left[1 + \frac{\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} \right] - \sigma^2 \left[\frac{\phi\left(\frac{a-\mu}{\sigma}\right)}{\left[1 - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]} \right]^2 \tag{4.43}$$

$$\begin{aligned} Sk &= \frac{1}{V_a^{3/2}} \left(2z_a^3 - 3az_a^2 + (a^2 - 1)z_a \right) \\ Ku &= \frac{1}{V_a^2} \left(-3z_a^2 + 6az_a^3 - 2(2a^2 - 1)z_a^2 + (a^3 + 3a)z_a + 3 \right) \end{aligned}$$

Where :

$$V_a = 1 + az_a - z_a^2 \,;\; z_a = \frac{\phi(a)}{1 - \Phi(a)} \tag{4.44}$$

Chapitre 5

Option Pricing and Dynamic Hedging for Regime-Switching Geometric Random Walks Models

5.1 Introduction

In complete, frictionless capital markets with no transaction costs and where the underlying securities follow geometric Brownian motions, the Black-Scholes formula (Black and Scholes, 1973) provides an elegant and tractable solution for pricing derivative securities. Unfortunately the actual financial markets are far more complex and empirical testing of the Black-Scholes model have highlighted its' many shortcomings. It is well documented (Fama, 1965, Mandelbrot, 1963, Schwert, 1989) that the observed properties of financial time series are not consistent with the underlying assumptions of the Black-Scholes framework. Time-varying volatility, the presence of higher-order moments and serial correlation are now well established characteristics of asset returns. Moreover, liquidity constraints, market frictions, transaction costs and discrete-time hedging lead to sub-optimal replication of the option's payoff function (Duffie and Huang, 1985, Huang, 1985). Furthermore, Boyle and Emanuel (1980), Gilster (1990), Mello and Neuhaus (1998) and Buraschi and Jackwerth (2001) demonstrate that unrealistic assumptions about continuous-time hedging can lead to large hedging errors and residual hedging risk.

Over the past decade, several studies have proposed discrete time hedging models based on different objective functions, see for example Owen (2002), Potters et al. (2001) and Pochart and Bouchaud (2004). The idea of dynamic hedging, as detailed in Cox and Ross (1976) and Harrison and Kreps (1979), is to find a self-financing optimal investment strategy that replicates a terminal payoff of the option. In this paper we build on the previous work of Föllmer and Schweizer (1990), Schweizer (1992, 1995), Papageorgiou et al. (2008) and Rémillard and Rubenthaler (2009) to derive an optimal discrete time hedging strategy based on the mean-square hedging error function for asset returns that follow a regime-switching random walk. Our hedging methodology is therefore robust to serially-correlated and non-Gaussian returns. Previous attempts to incorporate conditional returns in option pricing include GARCH models (see Christoffersen and Jacobs (2004) for a complete review), stochastic volatility models (Hull and White, 1987, Wiggins, 1987, Heston, 1993) and jump models (Kou (2002) and Kou and Wang (2004) to cite a few). These approaches have generally been successful at reproducing market prices, however none of them offer an effective, let alone optimal, hedging strategy.

Regime-switching models, popularized by Hamilton (1990) and Kim et al. (2008), have many characteristics that lend themselves nicely to financial time-series modeling. These models are easy to interpret, allow for time-dependent parameters and the aggregate returns conserve their non-Gaussian properties. Regime-switching models have previously been used by Bansal and Zhou (2002) to capture interest rate dynamics and by So et al. (1998) and Fong and See (2001) to model volatility. However, very few papers have attempted to apply regime-switching models to option pricing and hedging. In the case of American options, Buffington and Elliott (2002) and Guo and Zhang

(2004) develop pricing models and Garcia et al. (2003) and Chabi-Yo et al. (2008) propose a deep analysis of HMM processes applied to options characteristics, but none of these studies are extended to the hedging properties. The aim of this paper is to demonstrate how to implement optimal hedging strategies and obtain derivatives prices when the underlying assets returns are modeled as regime-switching random walks. The model that we propose is a discretized version of the continuous time regime-switching model, and is sometimes referred to as a transmutation-diffusion model (Freidlin and Lee, 1996) in the probability literature. The Baum-Welch algorithm (Baum et al., 1970) and the EM algorithm (Dempster et al., 1977) both provide efficient estimation procedures. For more details and results on estimation and convergence of estimators, see, e.g., Cappé et al. (2005). We also propose a new goodness-of-fit test, based on the work of Genest and Rémillard (2008), for selecting the optimal number of regimes.

The rest of the paper is structured as follows. Section 5.2 presents the models and its properties and describes the goodness-of-fit test. To simplify the presentation, the model estimation is deferred to Appendix 5.6 and the testing is presented in Appendix 5.6. In Section 5.3, we describe the optimal dynamic discrete time hedging model adapted to regime-switching processes. We show how to implement the proposed dynamic hedging algorithm for European option payoffs when the underlying asset returns are modeled by Gaussian regime-switching random walks. We illustrate the benefits of such processes in Section 5.4 and propose numerical applications to option pricing and hedging in Section 5.5.

5.2 Regime-switching geometric random walk models

A regime-switching geometric random walk model S is a process such that the associated (d-dimensional) log-returns $R_t = \log(S_t/S_{t-1})$ form a regime-switching random walk.

The non-observable regimes τ_t, with values in $\{1,\ldots,l\}$, form a Markov chain with transition matrix Q, stationary distribution ν, and given $\tau_1 = i_1, \ldots, \tau_n = i_n$, $R_1, \ldots, R_n$ are independent, with densities $f_{i_1}, \ldots, f_{i_n}$. As a result, the law of R_t is a mixture with densities

$$f(x) = \sum_{i=1}^{l} \nu_i f_i(x).$$

In general, $(S_t)_{k\geq 0}$ is not a Markov process. Nonetheless, the process $(S_t, \tau_t)_{t\geq 0}$ is Markovian. Note that these models are particular cases of Hidden Markov Models (HMM).

Several reasons justify this choice of model. First, even in the case of Gaussian densities, the law of the returns can be modeled adequately, provided the number of regimes is large enough. Note that we do not restrict ourselves to only 2 or 3 regimes as is often the case in the economic and financial literature. As a result of the serial dependence in the regimes, the returns of the assets also exhibit serial dependence, which is consistent with what is observed in financial time series. Finally, the conditional distribution is not constant, leading to conditional volatility as well as conditional asymmetry and kurtosis. The Black-Scholes-Merton model is a particular case when there is only one regime and that the density is Gaussian.

5.2.1 Properties of regime-switching random walks models

Regime prediction

In this section, we show how to find $\eta_t(i) = P(\tau_t = i|R_1 = y_1, \ldots, R_t = y_t)$. For more details, see, e.g., Baum et al. (1970).

- Choose an a priori distribution q_0 for the regimes ; for example, one could take a uniform distribution on $\{1, \ldots, l\}$.
- For any $t \geq 1$, once $R_t = y_t$ is observed, compute, for every $i = 1, \ldots, l$,

$$q_t(i) \quad = \quad f_i(y_t) \sum_{j=1}^{l} q_{t-1}(j) Q_{ji}, \tag{5.1}$$

  and

$$\eta_t(i) = \frac{q_t(i)}{Z_t}, \tag{5.2}$$

  where $Z_t = \sum_{j=1}^{r} q_t(j)$.

Remark 5.2.1 *The choice of q_0 is not so important in long term, as long as all regimes have positive probability. Next, note that $q_t(i) = E\left\{\mathbb{I}(\tau_t = i)\prod_{k=1}^{t} f_{\tau_k}(y_k)\right\}$, so Z_t is the joint density of $(R_1, \ldots, R_t)$ at $(y_1, \ldots, y_t)$.*

Moments

One easy way to measure serial dependence is to look at the auto-covariance. The following result provides the necessary formulas for the first and second moments of the distribution.

Proposition 3 *Suppose that the mean and covariance matrix of each density f_i is given by μ_i and A_i respectively, for all $i \in \{1,\ldots,l\}$. Then, for all $k,t \geq 1$, one has*

$$\begin{aligned}
E(R_t) &= \mu = \sum_{i=1}^{l} \nu_i \mu_i, && (5.3)\\
\mathrm{Cov}(R_t, R_t) &= A = \sum_{i=1}^{l} \nu_i A_i + \sum_{i=1}^{l} \nu_i \mu_i \mu_i^\top - \mu\mu^\top, && (5.4)\\
\mathrm{Cov}(R_t, R_{t+k}) &= \sum_{i=1}^{l}\sum_{j=1}^{l} \nu_i (Q^k)_{ij} \mu_i \mu_j^\top - \mu\mu^\top. && (5.5)\\
&= \sum_{i=1}^{l}\sum_{j=1}^{l} \nu_i \mu_i \mu_j^\top \left\{(Q^k)_{ij} - \nu_j\right\}. && (5.6)
\end{aligned}$$

If Q is ergodic, then there exist a positive constant C and $a \in (0,1)$ so that for all $k \geq 1$, $\max_{1\leq i,j\leq l} \left|(Q^k)_{ij} - \nu_j\right| \leq Ca^k$. It follows from (5.6) that $\mathrm{Cov}(R_t, R_{t+k})$ converges exponentially fast to 0 as $k \to \infty$.

Conditional distributions

Recall from Remark 5.2.1 that the joint density $f_{1:k}$ of $R_1,\ldots,R_t$ can be expressed as Z_t.

Next, for any $k \geq 2$, the conditional density of R_t given $R_1,\ldots,R_{t-1}$, denoted by $f_{t:1}$, can be expressed as a mixture, viz.

$$\begin{aligned}
f_{t:1}(x_t|x_1,\ldots,x_{t-1}) &= f_{1:k}(x_1,\ldots,x_t)\Big/ f_{1:k-1}(x_1,\ldots,x_{t-1})\\
&= \frac{\sum_{i=1}^{l}\sum_{j=1}^{l} q_{t-1}(i) Q_{ij} f_j(x_t)}{\sum_{i=1}^{l} q_{t-1}(i)} && (5.7)\\
&= \sum_{j=1}^{l} W_{j,k-1} f_j(x_t), && (5.8)
\end{aligned}$$

where

$$W_{j,k-1} = \frac{\sum_{i=1}^{l} q_{t-1}(i) Q_{ij}}{\sum_{i=1}^{l} q_{t-1}(i)}, \quad j \in \{1,\ldots,l\}. \tag{5.9}$$

Since $q_0(j) = \nu_j$, then $W_{j,0} = \nu_j$, for all $j \in \{1,\ldots,l\}$.

Forecasting properties

First, for any nice function g, it is easy to check that

$$E\{g(R_{t+1})|\mathcal{F}_t\} = \sum_{i=1}^{l} E\{g(R_{t+1})|R_t, \tau_t = i\}\eta_t(i) = \sum_{i=1}^{l}\sum_{j=1}^{l} \eta_t(i)Q_{ij}\int g(x)f_j(x)dx. \tag{5.10}$$

Formula (5.10) entails that the conditional law of R_{t+1} given $R_1, \ldots, R_t$ has density

$$f_{t+1:k}(x) = \sum_{i=1}^{l}\sum_{j=1}^{l} \eta_t(i)Q_{ij}f_j(x). \tag{5.11}$$

Similarly, using the Markov property, it is easy to check that all $\ell \geq 1$,

$$E\{g(R_{t+\ell})|\mathcal{F}_t\} = \sum_{i=1}^{l}\sum_{j=1}^{l} \eta_t(i)\left(Q^\ell\right)_{ij}\int g(x)f_j(x)dx. \tag{5.12}$$

That is, the conditional law of $R_{t+\ell}$ given $R_1, \ldots, R_t$ has density

$$f_{t+\ell:k}(x) = \sum_{i=1}^{l}\sum_{j=1}^{l} \eta_t(i)\left(Q^\ell\right)_{ij} f_j(x), \tag{5.13}$$

which is a mixture with the same densities $(f_j)_{j=1}^l$ and weights $\sum_{i=1}^l \eta_t(i)\left(Q^\ell\right)_{ij}$ for regime j, $j \in \{1, \ldots r\}$. In particular, the prediction for $R_{t+\ell}$ is

$$\sum_{i=1}^{l}\sum_{j=1}^{l} \eta_t(i)\left(Q^\ell\right)_{ij}\mu_j.$$

Confidence intervals for the prediction can be constructed using the quantiles of the density $f_{t+1:k}$ given by (5.13).

Next, if the Markov chain $(\tau_t)_{t\geq 1}$, with transition matrix Q, is ergodic, then the conditional law of $R_{t+\ell}$ given $R_1, \ldots, R_t$, converges to the stationary distribution

$$f(x) = \sum_{i=1}^{l} \nu_i f_i(x).$$

That is, for long time predictions, the behavior of the variable becomes independent of its past.

5.2.2 Goodness-of-fit

Having selected a model and estimated its parameters (see Appendix 5.6), one must next test the adequacy of the fitted model. This is generally done by using a test based on the likelihood, however, as expresed in Hamilton (1990), hypothesis testing using MLE methods can be problematic due to singularities and unidentifiable parameters. Cappé et al. (2005) show that goodness-of-fit tests based on likelihood ratio are not recommended for regime-switching models. Using score functions, Hamilton (1996) suggests some tests of goodness-of-fit which are not necessarily consistent because they are not based on distribution functions. Building on the famous Rosenblatt's transform (Rosenblatt, 1952) and the idea of Durbin (1973), Diebold et al. (1998) proposed to apply the conditional distribution functions to data. However, because parameters are estimated and the limiting distribution depends in general on these unknown parameters, the methodology proposed by Diebold et al. (1998) is useless. When testing goodness-of-fit for parametric families, one can use a parametric bootstrap for estimating P-values, even when the limiting distribution of the test statistics depends on unknown parameters. That was extended recently in Genest and Rémillard (2008) for semi-parametric models. Furthermore, in Genest et al. (2009), it was shown that tests based on the Rosenblatt's transform were quite powerful for testing goodness-of-fit for copulas, a class of semi-parametric models. The new goodness-of-fit test is described in Appendix 5.6.

5.3 Optimal discrete time hedging

We recall the main properties of the optimal hedging methodology then we detail the implementation issues when adapted to regime-switching models. See also Appendix 5.6 for additional details.

5.3.1 Optimal hedging

For any d-dimensional vector x, let $D(x)$ be the diagonal matrix with diagonal elements $x_1, \ldots, x_d$, and further let $\mathbf{e}(x)$ denote the vector with components e^{x_j}, $j = 1, \ldots, d$. Next, for every $i \in \{1, \ldots, d\}$, set

$$\kappa(i) = \int \left(\mathbf{e}^{y-r\mathbf{1}}\right) f_i(dy), \qquad B(i) = \int \left(\mathbf{e}^{y-r\mathbf{1}}\right)\left(\mathbf{e}^{y-r\mathbf{1}}\right)^\top f_i(dy).$$

Assume that $B(i)$ is invertible[1]. If ϕ_t denotes the number of shares of the d risky assets in the portfolio at the beginning of period $t-1$, and V_t is the value of the portfolio at period t, then the optimal choice of V_0 and $\phi_1, \ldots, \phi_T$ that minimize the mean square hedging error for a payoff $\Phi(S_T)$ at maturity T is $V_0 = C_0(S_0, \tau_0)$ and

$$\phi_t = \alpha_t(S_{t-1}, \tau_{t-1}) - V_{t-1} D^{-1}(S_{t-1}) \rho_{t+1}(\tau_{t-1}), \tag{5.14}$$

where $\rho_{T+1}(i) = \left\{\sum_{j=1}^l Q_{ij} B(j)\right\}^{-1} \left\{\sum_{j=1}^l Q_{ij}\kappa(j)\right\}$, and for all $t = T, \ldots, 1$ and every $i \in \{1, \ldots, l\}$,

$$\gamma_t(i) = \sum_{j=1}^l Q_{ij}\gamma_{t+1}(j)\left\{1 - \rho_{t+1}(i)^\top \kappa(j)\right\}, \tag{5.15}$$

$$\rho_t(i) = \left\{\sum_{j=1}^l Q_{ij}\gamma_t(j) B(j)\right\}^{-1} \left\{\sum_{j=1}^l Q_{ij}\gamma_t(j)\kappa(j)\right\}, \tag{5.16}$$

$$\begin{aligned} C_{t-1}(s,i) &= \frac{e^{-r}}{\gamma_t(i)} \sum_{j=1}^l Q_{ij}\gamma_t(j) \\ &\quad \times \int C_t\left\{D(s)\mathbf{e}^y, j\right\}\left\{1 - \rho_{t+1}(i)^\top \left(\mathbf{e}^{y-r\mathbf{1}} - \mathbf{1}\right)\right\} f_j(dy), \end{aligned} \tag{5.17}$$

$$\begin{aligned} \alpha_t(s,i) &= e^{-r} D^{-1}(s) \left\{\sum_{j=1}^l Q_{ij}\gamma_{t+1}(j) B(j)\right\}^{-1} \sum_{j=1}^l Q_{ij}\gamma_{t+1}(j) \\ &\quad \times \int C_t\left\{D(s)\mathbf{e}^y, j\right\}\left(\mathbf{e}^{y-r\mathbf{1}} - \mathbf{1}\right) f_j(dy). \end{aligned} \tag{5.18}$$

Note that (5.15) and (5.16) can be evaluated explicitly off-line in general. However, this is not the case for (5.17) and (5.18), even if they are expressed in terms of

1. That is equivalent to supposing that the genuine dimension of R_t is d.

expectations. Therefore, one has to rely on approximations for their evaluation. This can be achieved in several ways, one of which is the Simulation/Interpolation method proposed in Papageorgiou et al. (2008). This approach is described briefly in Appendix 5.6. Another approach is the linear approximation methods used in most dynamical programming problems. One major problem with these kinds of approximations using interpolations is the dimension. As d increases it becomes much more difficult to get good approximations as the number of points required for interpolation increases exponentially. Finally, in order to implement the optimal strategy, it follows from (5.14) that we must be able to predict the non-observable regimes. Section 5.2.1 describes the methodology for predicting these regimes.

Remark 5.3.1 *Note that V_0 is chosen so that the expected hedging error is zero. Rémillard and Rubenthaler (2009) also show that $C_t(S_t, \tau_t)$ is the optimal investment at period k so that the value of the portfolio at period n is as close as possible to $\Phi(S_n)$, in terms of mean square hedging error, so C_t can be interpreted as the option price at period k. That interpretation is justified since by increasing the number of hedging periods, they showed that C_t tends to the price under a risk neutral measure.*

5.3.2 Optimal hedging strategy implementation issues

There are two main problems related to the implementation of the hedging strategy : C_t and α_t defined by expressions (5.17) and (5.18) must be approximated and regimes must be predicted. Here we chose to approximate C_t and α_t by using the Stratified Monte Carlo sampling procedure ($N = 10,000$) described in Appendix 5.6 with a grid G defined by 1000 equidistant points covering 99.9999% of each regime daily return Gaussian density. Next, we need to predict τ_0, then τ_1 based on R_1, and so on. To do so, consider n_0 past values of S, up to present time $t = 0$ and estimate τ_t by

$$\hat{\tau}_t = \arg\max_i \eta_{n_0+t}(i), \quad t = 0, \ldots, T-1. \tag{5.19}$$

The last equation indicates that the predicted regime is the regime having the largest probability given the information on the prices up to time $n_0 + t$. Then, according to (5.14), the optimal weights ϕ_t for period $[t-1, t)$, are approximated by

$$\phi_t = \alpha_t(S_{t-1}, \hat{\tau}_{t-1}) - V_{t-1} D^{-1}(S_{t-1}) \rho_{t+1}(\hat{\tau}_{t-1}), \quad t = 1, \ldots, T, \tag{5.20}$$

and V_0 is approximated by $C_0(S_0, \hat{\tau}_0)$. In particular the initial number of shares of the risky assets ϕ_1 is

$$\phi_1 = \alpha_1(S_{01}, \hat{\tau}_0) - V_0 D^{-1}(S_0) \rho_2(\hat{\tau}_0), \tag{5.21}$$

while one invests an amount $V_0 - \phi_1^\top S_0$ in the non risky asset. Next, as S_1 is observed, one can compute V_1, then predict τ_1 and evaluate ϕ_2, and so on.

5.4 Implementation of regime-switching models

To illustrate the methodology, we examine the daily log-returns of the S&P 500 from January 1st 1989 to December 31st 2009 (5086 observations). This time series includes periods of high and low volatility, as shown in Figure 5.1 and its some descriptive statistics are given in Table 5.I. Therefore, it is natural to try to model these data using a regime-switching model. For the sake of simplicity, we choose the regime densities to be Gaussian, which facilitates estimation of parameters. In this paper, we refer to the Gaussian mixture model as GM, to Gaussian regime-switching model as GRS and to Black-Scholes model as B&S.

Table 5.I – Descriptive statistics for the S&P 500 daily returns.

Mean	Volatility	Skewness	Kurtosis
0.0002	0.0116	-0.1985	12.2536

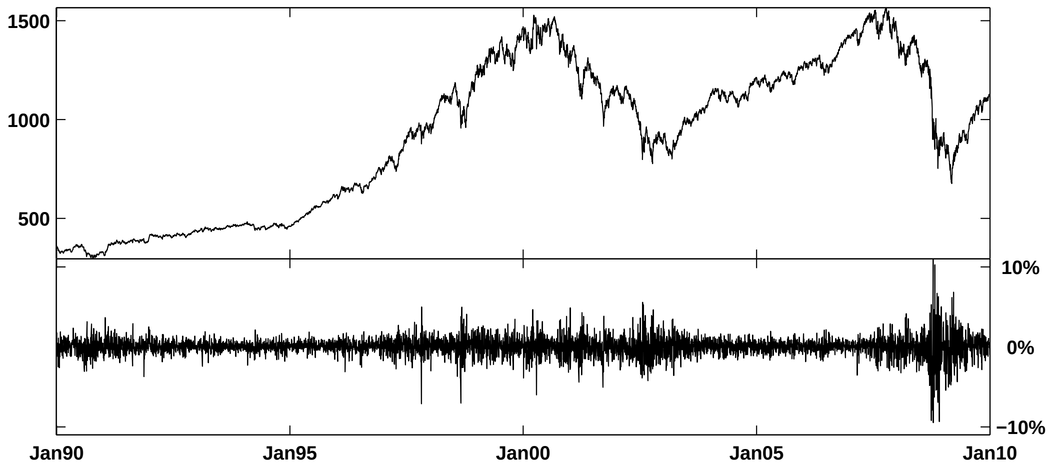


Figure 5.1 – S&P 500 over the period 12/31/1989 to 12/31/2009.

The time series is modeled using a GRS model. For more details on the estimation using the EM algorithm, see Appendix 5.6. According to Table 5.II, one should choose a regime-switching model with 3 regimes, since it is the smallest number of regimes for which the P-value of the goodness-of-fit test is larger than 5%.

Table 5.II – P-value and LLH for the goodness-of-fit tests using 1000 replications.

Number of regimes	1	2	**3**	4
P-value	0	0	**9%**	3%
Log-likelihood	10129.56	10274.14	**10295.22**	10291.37

We also provide the log-likelihood values for each number of regimes. The estimated parameters for the Gaussian densities appear in Table 5.III : Mean and covariance matrix of each density f_i are noted μ_i and A_i. The table also contains the long term regime probability ν_i of each regime, together with the conditional probability $\eta_n(i)$ for the current regime, given all past information. The estimated transition matrix is given in Table 5.IV.

TABLE 5.III – Parameter estimations for 3 regimes.

Regime i	μ_i	A_i	ν_i	$\eta_n(i)$
1	-0.00164	0.000810	0.0737	0.0003
2	0.00010	0.000131	0.4624	0.0869
3	0.00064	0.000034	0.4639	0.9128

TABLE 5.IV – Transition matrix Q for 3 regimes.

Regime	1	2	3
1	0.9673	0.0327	0
2	0.0053	0.9834	0.0113
3	0	0.0110	0.9890

Looking at the estimated transition matrix, if the process enters the high-volatility regime, it has a probability of 96.73% of remaining in this regime and a 3.27% probability of moving to the mid-volatility regime. The η's provides information on the current state. The results indicate that there is 98.9% probability of being in the third regime (lowest volatility and highest mean) by end of December 2009. ν describes the stationary distribution of the $S\&P500$. The model captures the recent the stock market behavior, which has been characterized by positive returns and low volatility on average interrupted by periods of sustained volatility and poor returns. By the end of 2009 the market enters a period of very low volatility and strong recovery, as illustrated by the third regime.

Using the density forecast formula (5.13), one can plot the daily log-return density of the forecast for several periods after December 31st 2009. This is illustrated in Figure

5.2 for 1 day ahead, 5 days ahead and 21 days ahead (1 month). The weights of the corresponding Gaussian densities are given in Table 5.V.

TABLE 5.V – Daily forecasts for the S&P 500 returns.

	$\ell = 1$	$\ell = 5$	$\ell = 21$	$\ell = \infty$
Regime 1 weight	0.0007	0.0028	0.0136	0.0737
Regime 2 weight	0.0954	0.1276	0.2274	0.4624
Regime 3 weight	0.9039	0.8696	0.7590	0.4639
Forecasted mean	0.0006	0.0005	0.0004	0.0002
Forecasted volatility	0.0066	0.0069	0.0081	0.0116
Forecasted skewness	-0.0616	-0.0888	-0.1690	-0.2037
Forecasted kurtosis	5.0050	6.4941	9.5507	9.2478

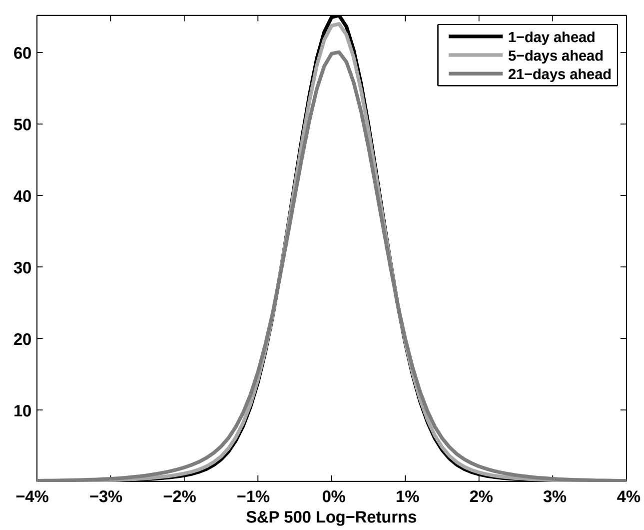


FIGURE 5.2 – Forecasted densities for the log-returns of the S&P 500

These results demonstrate how the regime-switching model converges to the stationary distribution from the current state. Because the current state is defined by a low volatility and a strong drift, the daily density exhibits little asymmetry and low kurtosis. As the state distribution converges to the stationary distribution, the daily density is characterized by a higher volatility, lower mean, and more asymmetry and kurtosis, taking into account more tail risk.

Finally, to illustrate these dynamic properties out-of-sample, we estimated both a Gaussian regime-switching process and a Gaussian mixtures process on the daily log-returns of the S&P 500 from December 1999 to December 2009. We performed each month a goodness-of-fit test on the past daily returns considering the 1989-1999 sample as the initial data set. The selected number of regimes are presented in Figure 5.3. We then forecasted the intra-month moments (Figure 5.4) using formula (5.13) and plotted the 95% confidence interval of the forecasted daily volatility on the S&P 500 daily returns (Figure 5.5).

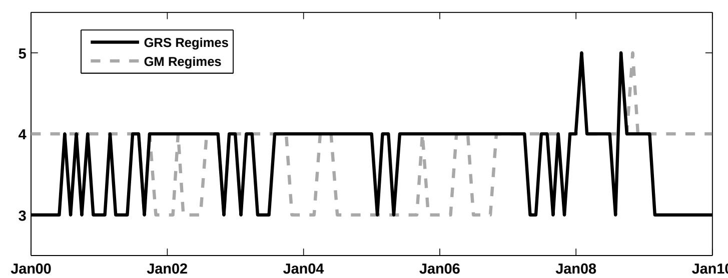


Figure 5.3 – Optimal number of regimes (goodness-of-fit)

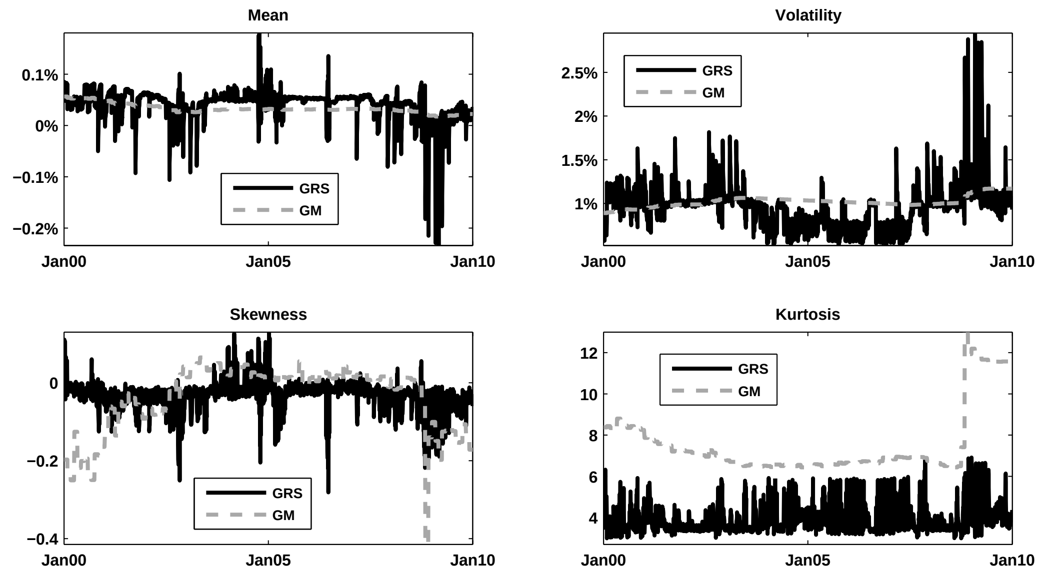


Figure 5.4 – Forecasted daily moments

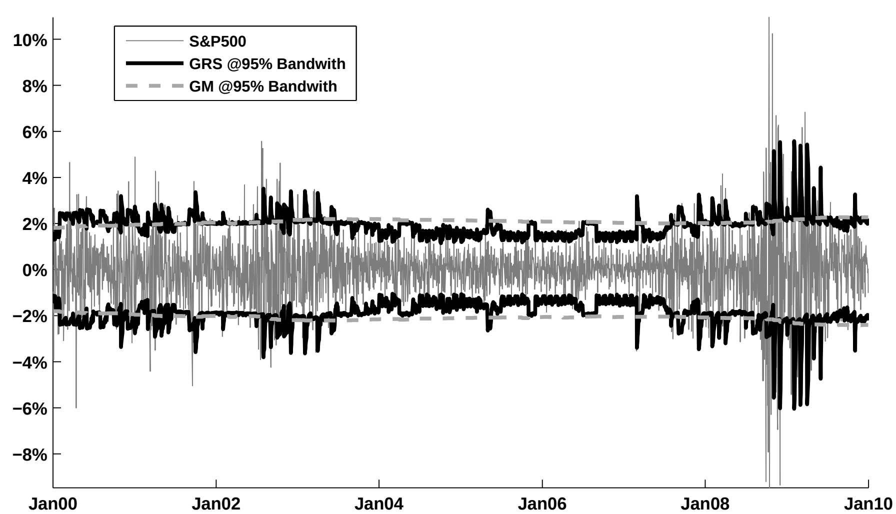


Figure 5.5 – Forecasted daily volatility on the S&P 500 daily log-returns

From Figure 5.5, we see that the Gaussian regime-switching model allows for a good prediction of volatility, specifically in 2008 during the stock market crashed. It captured both low volatility levels in 2004 and high volatility levels since 2007. That model performs better than the Gaussian mixture model due to its conditional properties. Skewness and kurtosis are also forecasted and reveal the higher asymmetry and fat tails in 2008. Both the GRS and the GM models need between 3 to 5 regimes to be well specified. Since volatility levels is an key driver for option pricing and hedging, the dynamic hedging strategy as defined by equations (5.14)–(5.18) seems quite appropriate. This will be further discussed in the next section.

5.5 Implementation of the optimal hedging strategy

To illustrate the dynamic hedging algorithm based on Gaussian regime-switching models, we price and hedge a European call and a put option for a range of maturities and moneyness. We compare the Gaussian regime-switching process (GRS) to a Gaussian mixture process (GM) and a standard Black-Scholes hedging (B&S). The return process is modeled as a 3-regimes for the GRS with parameters estimated in section 5.4 and a 4-regimes for the GM (best goodness-of-fit P-value). The stationary volatility given by the GRS is set as the Black-Scholes constant volatility. We also compute the average hedging error (MHE) and the average root mean square hedging error (RMSHE) from equation (5.25) on a sample of 100000 replications. To do so we generate 100000 series of n daily returns following a 3-regimes GRS and we compute the hedging strategy for each n-days sequence. The risk free rate is set at 3% per year and the initial stock value is standardized such as $S_0 = 100$. Because of the optimal hedging algorithm characteristics, both pricing models are done in the real probability measure, in contrast with the Black-Scholes option pricing framework.

5.5.1 Hedging error validation by Monte Carlo simulations

At-the-money pricing

Using a Gaussian regime-switching process with 3 regimes, one gets 3 option values, each specific to each particular regime. By choosing the most probable regime, as described in Section 5.3.2, one obtains the most probable price. We evaluate a call and a put option at-the-money with 21 days maturity. We provide 95% confidence intervals based on 10,000 generated pricing (except for the B&S evaluation done in closed form). The most probable regime is regime 3 with associated option prices and hedge ratios appearing in bold in table 5.VI. GRS option price is lower than the GM and B&S option price since the current state exhibits a very low volatility. The GM model can still consider the non-normality of the daily return density but the option price is computed according to the stationary density, meaning the long run volatility, so that the GM option values are close to the B&S option values. The confidence intervals quickly converge to the estimated values even with only 10,000 pricing. The hedge ratios are computed using formula (5.21).

TABLE 5.VI – ATM option prices and initial hedge ratios for GRS, GM and B&S models

	Call		Put	
	Price V_0	ϕ_1	Price V_0	ϕ_1
GRS Reg. 1	4.4988 ± 0.0050	0.5257 ± 0.0002	4.3156 ± 0.0047	-0.4750 ± 0.0002
GRS Reg. 2	2.2497 ± 0.0031	0.5229 ± 0.0001	2.0613 ± 0.0029	-0.4757 ± 0.0002
GRS Reg. 3	$\mathbf{1.2834 \pm 0.0017}$	$\mathbf{0.5389 \pm 0.0001}$	$\mathbf{1.1035 \pm 0.0019}$	$\mathbf{-0.4579 \pm 0.0001}$
GM	2.1699 ± 0.0054	0.5332 ± 0.0003	2.0133 ± 0.0044	-0.4468 ± 0.0002
B&S	2.2161	0.5237	2.0412	-0.4763

Pricing over moneyness and maturities

To compare the three methodologies, we compute the B&S implied volatility on the option prices with respect to their days-to-maturity (DTM) and strike price level. We test for a range of 20% out-of-the money to 20% in-the-money options with 21, 63, 126 and 252 days to maturity. Results are presented in Figure 5.6 and Figure 5.7.

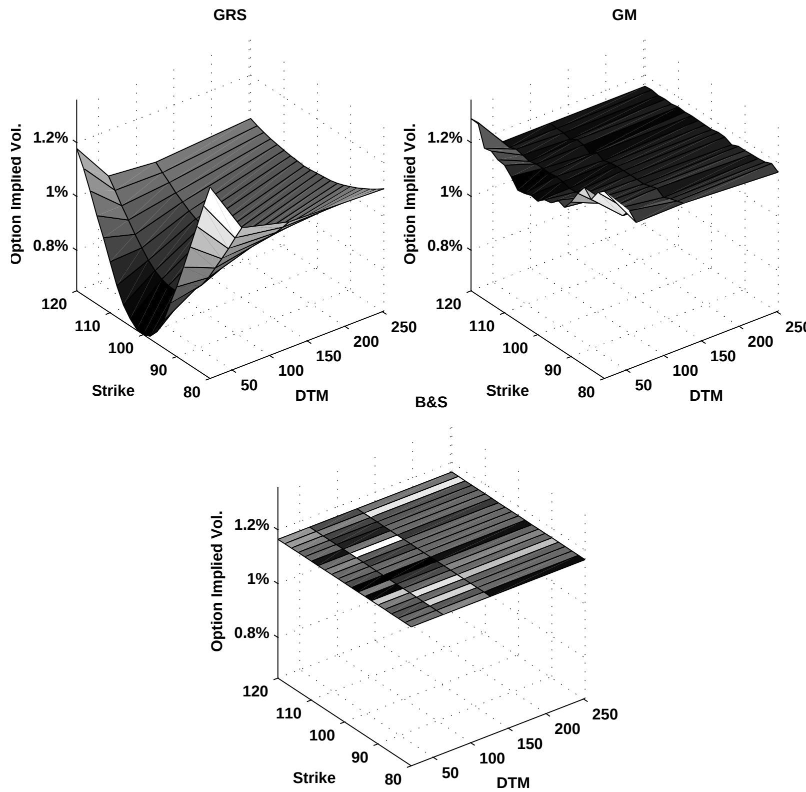


FIGURE 5.6 – Call option implied volatility

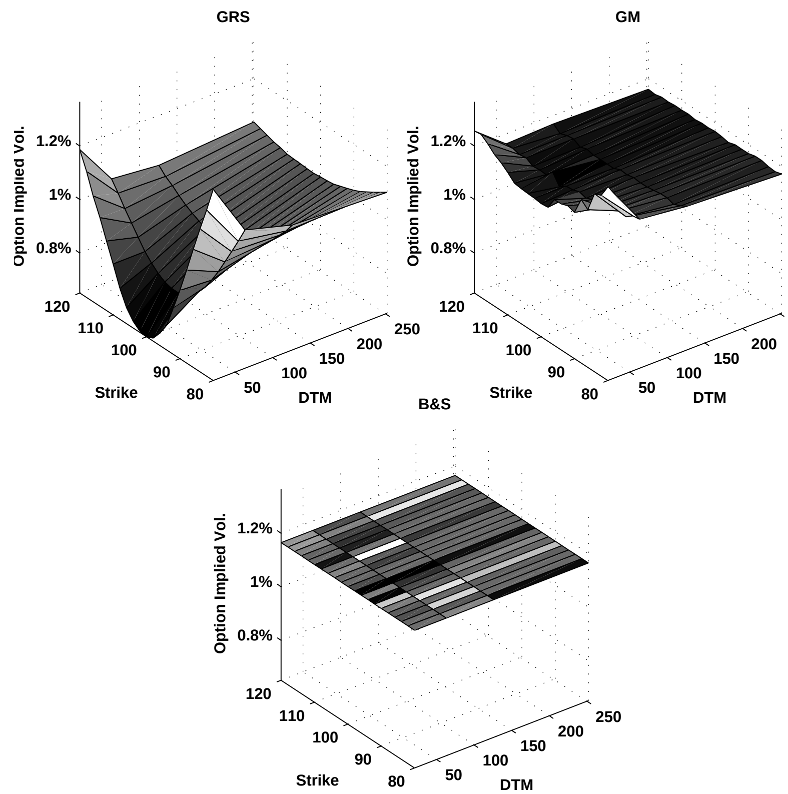


Figure 5.7 – Put option implied volatility

The GRS implied volatility exhibits the well studied “smile” effect on option prices across moneyness and the volatility term structure impact on prices over maturity. According to recent market behavior, the smile is asymmetric with higher effect on negative S&P 500 returns. The GM model capture some of the smile because of its non-Gaussian properties but cannot capture the term structure effect. As expected, B&S implied volatility remain constant across moneyness and maturities.

In-sample hedging error

We computed hedging errors for 21 days ATM options. GRS option prices lead to a lower hedging error and RMSHE, as illustrated in Table 5.VII. Then we evaluated the errors for a range of 20% out-of-the-money to 20% in-the-money options with 21 days to maturity. As shown in Figures 5.8– 5.9, the hedging error is always closer to 0 for the GRS. Pricing and hedging at-the-money options in a constant volatility Gaussian framework could lead to very large hedging errors due to the V_0 mispricing in the GM and B&S setting. The forecasting properties of the GRS model allow the option pricing and hedging to be more dynamic to volatility shifts than a constant volatility.

TABLE 5.VII – ATM option hedging error and RMSHE

	Call		Put	
Model	MHE	RMSHE	MHE	RMSHE
GRS	0.0076	0.5440	0.0097	0.5435
GM	0.7844	0.9855	0.7896	0.9892
B&S	0.8176	1.0062	0.8175	1.0062

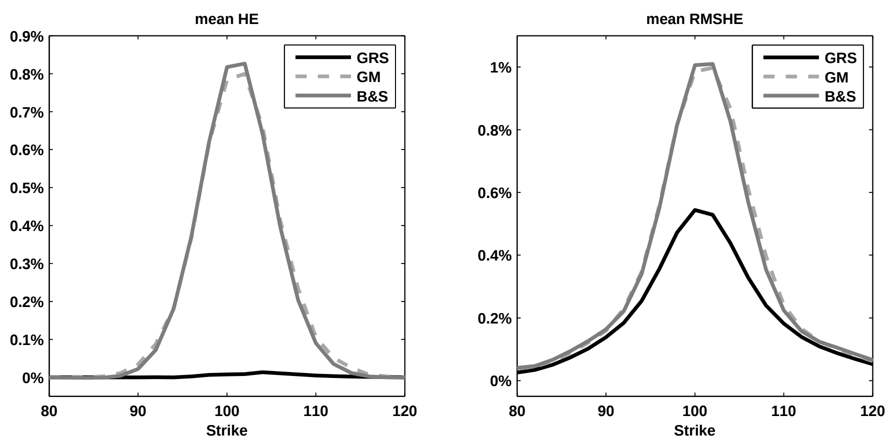


FIGURE 5.8 – Call option hedging error (HE) and RMSHE over moneyness

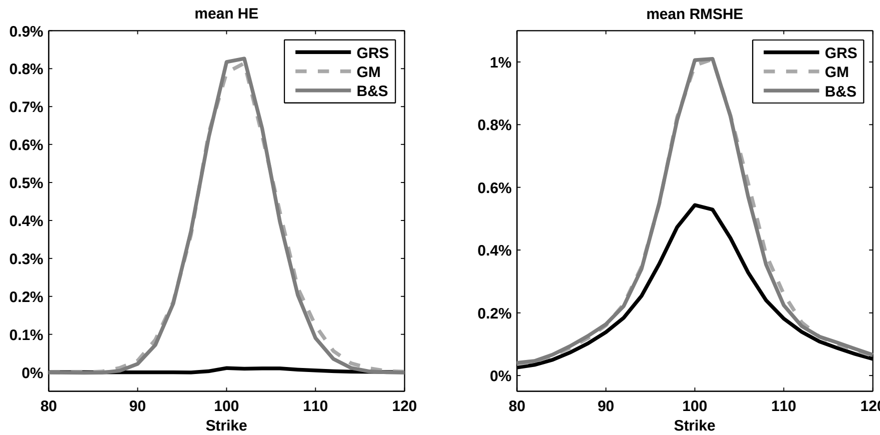


FIGURE 5.9 – Put option hedging error (HE) and RMSHE over moneyness

5.5.2 Approximation by regime-switching geometric Brownian motion

One interesting feature of the GRS model in a option pricing and hedging framework is that it can be well approximated under some conditions by a continuous time process, the so-called regime-switching geometric Brownian motion, as detailed in Rémillard and Rubenthaler (2009). Such a process is determined by a continuous time Markov chain τ_t with generator Λ [2], representing the regime at time t, and between the jumps of τ_t, the process S_t follows a geometric Brownian motion with drift $\psi(i)$ and covariance matrix $a(i)$, when $\tau_t = i$. If the discrete data corresponds to daily returns and if the densities f_i are Gaussian with mean $\mu(i)$ and covariance matrix $A(i)$, then the relation with the parameters Λ, ψ and a is the following : $\Lambda = n(Q - I)$, $\psi(i) = n(\mu(i) - \text{diag}(A(i)))$, $a(i) = nA(i)$ with $n = 252$. One could refer to Rémillard and Rubenthaler (2009) for details on the pricing and hedging algorithm. To illustrate the approximation effectiveness, we priced ATM European call and put options with 21 days maturity (dicretized in 10,000 steps) given the GRS estimations in Section 5.4. We provided 95% confidence intervals for the results based on 10^6 simulations (Table

2. Λ is defined by the relation $\Lambda_{ij} = \lim_{t \downarrow 0} P(\tau_t = j | \tau_0 = i)/t$ if $j \neq i$, and $\Lambda_{ii} = -\sum_{j \neq i} \Lambda_{ij}$.

5.VIII) and hedging errors based on 10^5 simulated path of 21 daily returns (Table 5.IX).

Table 5.VIII – ATM option prices and initial hedge ratios

	Call		Put	
Regime	Price V_0	ϕ_1	Price V_0	ϕ_1
1	4.5406 ± 0.0079	0.5288 ± 0.0002	4.3674 ± 0.0061	-0.4712 ± 0.0001
2	2.2392 ± 0.0037	0.5243 ± 0.0001	2.0644 ± 0.0033	-0.4756 ± 0.0001
3	$\mathbf{1.2882 \pm 0.0020}$	$\mathbf{0.5300 \pm 0.0001}$	$\mathbf{1.1132 \pm 0.0018}$	$\mathbf{-0.4699 \pm 0.0001}$

Table 5.IX – ATM option hedging error and RMSHE

Call		Put	
MHE	RMSHE	MHE	RMSHE
-0.0704	0.5803	-0.0701	0.5789

The continuous time approximation leads to call and put option values (prices and ϕ_1's) that are very close to the values given by the discrete time hedging algorithm. The difference is still significant in term of confidence intervals. It is the result of the continuous time approximation and appears to be significant in term of hedging error. The GRS forecasting properties remain valid when approximated by a regime-switching geometric Brownian motion. That approximation could be very useful when the dimension d of the pricing problem grows (options on d underlying assets) because it does not require any interpolation or polynomial approximation by contrast with the discrete time hedging algorithm.

5.5.3 Out-of-sample validation

To complete our validation of the Gaussian regime-switching hedging model, we propose an out-of-sample test of the pricing and hedging efficiency for both the GRS, the GM and the B&S model. From December 31st 1999 to December 31st 2009 we priced and hedged at the first trading day of each month a European call a put option expiring the last trading day of the respective month (1-month maturity). We estimated the GRS model and GM model as described in section 5.4. We present the average hedging error and RMSHE on the 120 hedged options for different moneyness across the 3 models. These results are illustrated in Figures 5.10–5.11. In particular they show that the GRS model has lower mean error and root mean square measures that for the other models in an out-of-sample experiment, for both call and put options.

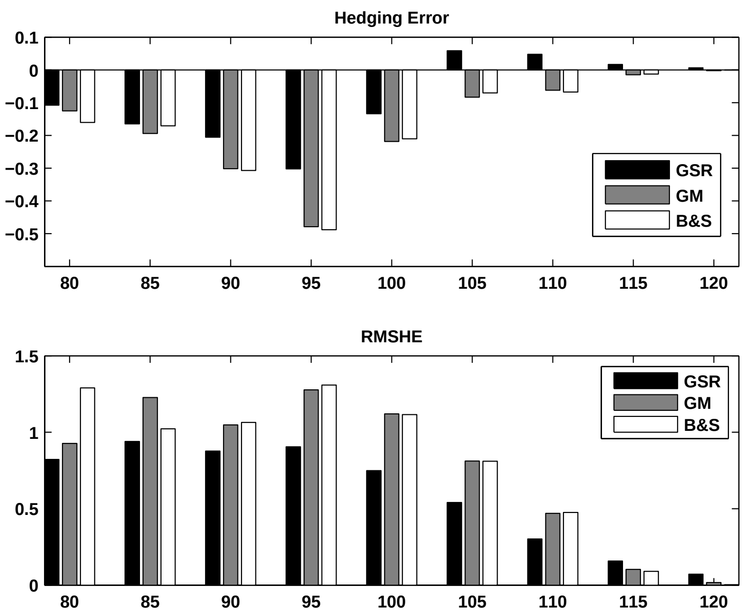


Figure 5.10 – Call hedging errors across moneyness

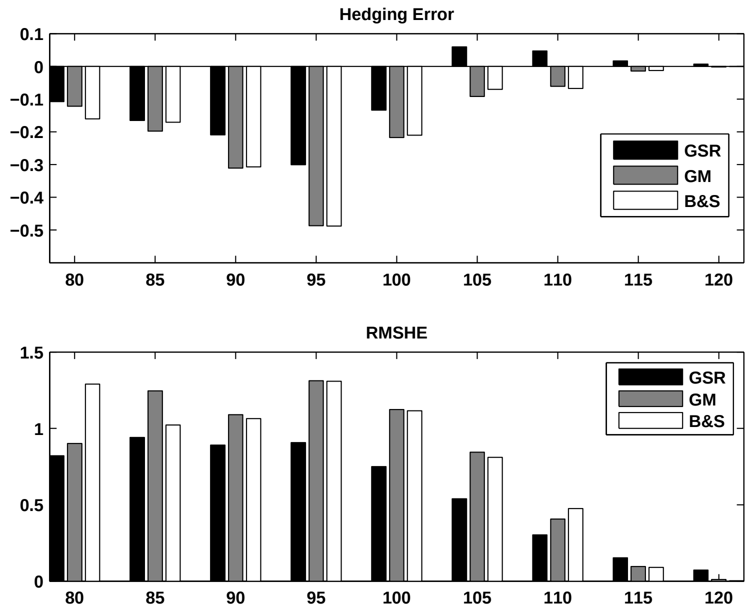


FIGURE 5.11 – Put hedging errors across moneyness

5.6 Conclusion

In this paper, we propose a discretized version of the continuous time regime-switching model, and demonstrate how to implement an optimal hedging strategies to obtain derivatives prices when the underlying assets returns are modeled as regime-switching random walks. Building mainly on the work of Hamilton (1990), we also propose a test of goodness-of-fit for Markovian regime-switching models for univariate and multivariate time series that uses the Rosenblatt's transforms. To illustrate the effectiveness of the test, we model the daily return series of the S&P 500. The results obtained from the goodness-of-fit test are consistent with the characteristics of the market evolution during high and low volatility periods. Furthermore, we develop a pricing and hedging algorithm based on the previous work of Del Moral et al. (2006), Papageorgiou et al. (2008) and Rémillard and Rubenthaler (2009) specifically adapted to regime-switching models. We compare our hedging results to a Gaussian framework and a Gaussian mixture model and prove that Gaussian regime-switching models generate lower hedging errors than constant volatility models, both in-sample and out-of-sample test. This hedging algorithm could easily be extended to American option payoffs, and adapted to conditional volatility models such as GARCH models.

Bibliography

Bansal, R. and Zhou, H. (2002). Term structure of interest rates with regime shifts. *The Journal of Finance*, 57(5) :1997–2043.

Baum, L. E., Petrie, T., Soules, G., and Weiss, N. (1970). A maximization technique occurring in the statistical analysis of probabilistic functions of Markov chains. *Ann. Math. Statist.*, 41 :164–171.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81 :637–654.

Boyle, P. P. and Emanuel, D. (1980). Discretely adjusted option hedges. *Journal of Financial Economics*, 8 :259–282.

Buffington, J. and Elliott, R. (2002). American options with regime switching. *International Journal of Theoretical and Applied Finance*, 5 :497Ű–514.

Buraschi, A. and Jackwerth, J. (2001). The price of a smile : Hedging and spanning in option markets. *The Review of Financial Studies*, 14(2) :495–527.

Cappé, O., Moulines, E., and Rydén, T. (2005). *Inference in hidden Markov models*. Springer Series in Statistics. Springer, New York.

Chabi-Yo, F., Garcia, R., and Renault, E. (2008). State dependence can explain the risk aversion puzzle. *Review of Financial Studies*, 21 :973–1011.

Christoffersen, P. and Jacobs, K. (2004). Which GARCH model for option valuation ? *Management Science*, 50(9) :1204Ű–1221.

Cox, J. and Ross, S. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3 :145Ű–166.

Del Moral, P., Rémillard, B., and Rubenthaler, S. (2006). Monte Carlo approximations of American options. Technical report, GERAD.

Dempster, A. P., Laird, N. M., and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm. *J. Roy. Statist. Soc. Ser. B*, 39 :1–38.

Diebold, F. X., Gunther, T. A., and Tay, A. S. (1998). Evaluating density forecasts with applications to financial risk management. *International Economic Review*, 39(4) :863–883.

Duffie, D. and Huang, C. (1985). Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities. *Econometrica*, 53(6) :1337–Ű1356.

Durbin, J. (1973). Weak convergence of the sample distribution function when parameters are estimated. *Ann. Statist.*, 1(2) :279–290.

Fama, E. F. (1965). The behavior of stock-market prices. *Journal of Business*, 38(1) :34–Ű105.

Föllmer, H. and Schweizer, M. (1990). Hedging of contingent claims under incomplete information. In Davis, M. and Elliott, R., editors, *Applied Stochastic Analysis*, volume 5 of *Stochastics Monographs*, pages 389–414. Gordon and Breach Science Publishers.

Fong, W. M. and See, K. H. (2001). Modelling the conditional volatility of commodity index futures as a regime switching process. *Journal of Applied Econometrics*, 16(2) :133–Ű163.

Freidlin, M. I. and Lee, T.-Y. (1996). Large deviation principle for the diffusion-transmutation processes and Dirichlet problem for PDE systems with small parameter. *Probab. Theory Related Fields*, 105(2) :227–254.

Garcia, R., Luger, R., and Renault, E. (2003). Empirical assessment of an intertemporal option pricing model with latent variables. *Journal of Econometrics*, 116 :49–83.

Genest, C. and Rémillard, B. (2008). Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. *Ann. Inst. H. Poincaré Sect. B*, 44 :1096–1127.

Genest, C., Rémillard, B., and Beaudoin, D. (2009). Omnibus goodness-of-fit tests for copulas : A review and a power study. *Insurance Math. Econom.*, 44 :199–213.

Ghoudi, K. and Rémillard, B. (1998). Empirical processes based on pseudo-observations. In *Asymptotic Methods in Probability and Statistics (Ottawa, ON, 1997)*, pages 171–197. North-Holland, Amsterdam.

Ghoudi, K. and Rémillard, B. (2004). Empirical processes based on pseudo-observations. II. The multivariate case. In *Asymptotic Methods in Stochastics*, volume 44 of *Fields Inst. Commun.*, pages 381–406. Amer. Math. Soc., Providence, RI.

Gilster, John E., J. (1990). The systematic risk of discretely rebalanced option hedges. *Journal of Financial and Quantitative Analysis*, 25(4) :507–516.

Guo, X. and Zhang, Q. (2004). Closed-form solutions for perpetual American put options with regime switching. *SIAM Journal on Applied Mathematics*, 64(6) :2034–2049.

Hamilton, J. (1990). Analysis of time series subject to changes in regime. *J. Econometrics*, 45(1-2) :39–70.

Hamilton, J. (1996). Specification testing in Markov-switching time-series models. *J. Econometrics*, 70(1) :127–157.

Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20 :381–408.

Heston, S. (1993). A closed-form solution for options with stochastic volatility with application to bond and currency options. *Review of Financial Studies*, 6(2) :327Ű–343.

Huang, C. (1985). Information structure and equilibrium asset prices. *J. Econom. Theory*, 53 :33–Ű71.

Hull, J. and White, A. (1987). The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, XLII, No.2 :281–300.

Kim, C.-J., Piger, J., and Startz, R. (2008). Estimation of Markov regime-switching regression models with endogenous switching. *Journal of Econometrics*, 143 :263–273.

Kou, S. G. (2002). A Jump-Diffusion Model for Option Pricing. *Management Science*, 48(8) :1086–1101.

Kou, S. G. and Wang, H. (2004). Option Pricing Under a Double Exponential Jump Diffusion Model. *Management Science*, 50(9) :1178–1192.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business*, pages 307–332.

Mello, A. S. and Neuhaus, H. J. (1998). A portfolio approach to risk reduction in discretely rebalanced option hedges. *Management Science*, 44(7) :921–934.

Owen, M. P. (2002). Utility based optimal hedging in incomplete markets. *Ann. Appl. Probab.*, 12(2) :691–709.

Papageorgiou, N., Rémillard, B., and Hocquard, A. (2008). Replicating the properties of hedge fund returns. *Journal of Alternative Invesments*, 11 :8–38.

Pochart, B. and Bouchaud, J.-P. (2004). Option pricing and hedging with minimum local expected shortfall. *Quant. Finance*, 4(5) :607–618.

Potters, M., Bouchaud, J.-P., and Sestovic, D. (2001). Hedged Monte-Carlo : low variance derivative pricing with objective probabilities. *Phys. A*, 289(3-4) :517–525.

Rémillard, B. (2010). Validity of the parametric bootsrap for goodness-of-fit testing in dynamic models. Technical Report G-2010-XX, GERAD.

Rémillard, B. and Rubenthaler, S. (2009). Optimal hedging in discrete and continuous time. Technical Report G-2009-77, Gerad.

Rosenblatt, M. (1952). Remarks on a multivariate transformation. *Ann. Math. Stat.*, 23 :470–Ű472.

Schweizer, M. (1992). Mean-variance hedging for general claims. *Ann. Appl. Probab.*, 2(1) :171–179.

Schweizer, M. (1995). Variance-optimal hedging in discrete time. *Math. Oper. Res.*, 20(1) :1–32.

Schwert, G. W. (1989). Why does stock market volatility change over time? *Journal of Finance*, 44 :1115–Ű1153.

So, M. K. P., Lam, K., and Li, W. K. (1998). A stochastic volatility model with Markov switching. *Journal of Business & Economic Statistics*, 16(2) :244–253.

Wiggins, J. B. (1987). Option values under stochastic volatility : Theory and empirical estimates. *Journal of Financial Economics*, 19(2) :351–372.

Appendix A : Estimation of regime-switching models

In order to apply the EM algorithm for estimating parameters, see e.g., Cappé et al. (2005), it is necessary to :

(E-Step) Compute the conditional probabilities

$$\lambda_t(i) = P(\tau_t = i | R_1, \ldots, R_n) \quad \text{and} \quad \Lambda_t(i,j) = P(\tau_t = i, \tau_{t+1} = j | R_1, \ldots, R_n),$$

for all $1 \leq k \leq n$ and any $i, j \in \{1, \ldots, l\}$.

(M-Step) Estimate the new parameters.

The E-Step is described next for any densities. The M-Step will be stated only for Gaussian densities. For more details of the EM algorithm, see, e.g., Cappé et al. (2005).

A.1 : Conditional distributions of the regimes (E-Step)

First, define, for all $i \in \{1, \ldots, l\}$,

$$\begin{aligned} \bar{q}_n(i) &= 1, \\ \bar{q}_t(i) &= \sum_{\beta=1}^{l} \bar{q}_{t+1}(\beta) Q_{i\beta} g_{t+1}(\beta), \quad 1 \leq k \leq n-1. \end{aligned}$$

Then, for all $i, j \in \{1, \ldots, l\}$, one can check that

$$\lambda_t(i) = \frac{q_t(i)\bar{q}_t(i)}{\sum_{\alpha=1}^{l} q_t(\alpha)\bar{q}_t(\alpha)}, \quad k = 1, \ldots, n, \tag{5.22}$$

$$\Lambda_t(i,j) = \frac{Q_{ij} q_t(i) \bar{q}_{t+1}(j) f_j(R_{t+1})}{\sum_{\alpha=1}^{l} \sum_{\beta=1}^{l} Q_{\alpha\beta} q_t(\alpha) \bar{q}_{t+1}(\beta) g_{t+1}(\beta)}, \quad k = 1, \ldots, n-1, \tag{5.23}$$

and $\Lambda_n(i,j) = \lambda_n(i) Q_{ij}$.

To see that (5.22) and (5.23) are consistent, note that for all $1 \leq k \leq n-1$,

$$\sum_{j=1}^{r} \Lambda_t(i,j) = \sum_{j=1}^{l} \frac{Q_{ij} q_t(i) \bar{q}_{t+1}(j) f_j(R_{t+1})}{\sum_{\alpha=1}^{l} \sum_{\beta=1}^{l} Q_{\alpha\beta} q_t(\alpha) \bar{q}_{t+1}(\beta) g_{t+1}(\beta)} = \frac{q_t(i)\bar{q}_t(i)}{\sum_{\alpha=1}^{l} q_t(\alpha)\bar{q}_t(\alpha)} = \lambda_t(i),$$

using the definition of $\bar{q}_t$, $\sum_{j=1}^l \Lambda_n(i,j) = \sum_{j=1}^l \lambda_n(i) Q_{ij} = \lambda_n(i)$. Similarly, for all $1 \le k \le n-1$,

$$\sum_{i=1}^l \Lambda_t(i,j) = \sum_{i=1}^l \frac{Q_{ij} q_t(i) \bar{q}_{t+1}(j) f_j(R_{t+1})}{\sum_{\alpha=1}^l \sum_{\beta=1}^l Q_{\alpha\beta} q_t(\alpha) \bar{q}_{t+1}(\beta) g_{t+1}(\beta)} = \frac{q_{t+1}(i) \bar{q}_{t+1}(j)}{\sum_{\alpha=1}^l q_{t+1}(\alpha) \bar{q}_{t+1}(\alpha)} = \lambda_{t+1}(i),$$

using the definition of q_{t+1}.

A.2 : Estimation for Gaussian regime-switching models (M-Step)

When the densities $f_1, \ldots, f_l$ are those of Gaussian distributions with means $(\mu_i)_{i=1}^l$, and covariance matrices $(A_i)_{i=1}^l$, then the model is called a Gaussian HMM.

The M step consists in upgrading parameters $(\nu_i)_{i=1}^l$, $(\mu_i)_{i=1}^l$, $(A_i)_{i=1}^l$ and Q by setting, for all $i, j \in \{1, \ldots, l\}$,

$$\begin{aligned}
\nu_i' &= \sum_{t=1}^n \lambda_t(i)/n, \\
\mu_i' &= \sum_{t=1}^n x_t w_t(i), \\
A_i' &= \sum_{t=1}^n (x_t - \mu_i')(x_t - \mu_i')^\top w_t(i), \\
Q_{ij}' &= \sum_{t=1}^n \Lambda_t(i,j) \Big/ \sum_{t=1}^n \lambda_t(i) = \frac{1}{n} \sum_{t=1}^n \Lambda_t(i,j) \Big/ \nu_i',
\end{aligned}$$

where $w_t(i) = \lambda_t(i) \Big/ \sum_{l=1}^n \lambda_l(i)$.

Note that ν' is not a stationary distribution for Q' since for any $j \in \{1, \ldots, l\}$,

$$\sum_{i=1}^l \nu_i' Q_{ij}' = \frac{1}{n} \sum_{t=1}^n \sum_{i=1}^l \Lambda_t(i,j) = \frac{1}{n} \sum_{t=2}^{n+1} \lambda_t(j) = \nu_j' + \frac{\lambda_{n+1}(j) - \lambda_1(j)}{n} \neq \nu_j'.$$

However,

$$\max_{1 \le j \le l} \left| \sum_{i=1}^l \nu_i' Q_{ij}' - \nu_j' \right| \le 1/n.$$

Hence, when n is large, ν' is close to the stationary distribution of Q'.

Remark 5.6.1 *In Cappé et al. (2005), it is shown that the EM estimator of* ν*, when* ν *is not the stationary distribution, is* $\nu' = \lambda_1$.

It is interesting to note that the first two sample moments are preserved in the Gaussian case, i.e., the sample mean and covariance matrix are equal to the theoretical ones when applied to the estimated parameters.

A.3 : Fitting of sample moments

The sample mean and covariance matrix are defined by

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t \quad \text{and} \quad S = \frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})(x_t - \bar{x})^\top .$$

Using formula (5.3), one gets

$$\begin{aligned} \mu' &= \sum_{i=1}^{l} \nu_i' \mu_i' = \frac{1}{n}\sum_{i=1}^{l}\sum_{t=1}^{n} \lambda_t(i) x_t \\ &= \frac{1}{n}\sum_{t=1}^{n} x_t = \bar{x}. \end{aligned}$$

Next,

$$\begin{aligned} \sum_{i=1}^{l} \nu_i' \mathcal{A}_i' &= \frac{1}{n}\sum_{i=1}^{l}\sum_{t=1}^{n} \lambda_t(i)(x_t - \mu_i')(x_t - \mu_i')^\top \\ &= \frac{1}{n}\sum_{i=1}^{l}\sum_{t=1}^{n} \lambda_t(i) x_t x_t^\top + \frac{1}{n}\sum_{i=1}^{l}\sum_{t=1}^{n} \lambda_t(i)\mu_i'(\mu_i')^\top \\ &\qquad -\frac{1}{n}\sum_{i=1}^{l}\sum_{t=1}^{n} \lambda_t(i) x_t(\mu_i')^\top - \frac{1}{n}\sum_{i=1}^{l}\sum_{t=1}^{n} \lambda_t(i)\mu_i' x_t^\top \\ &= \frac{1}{n}\sum_{t=1}^{n} x_t x_t^\top - \sum_{i=1}^{l} \nu_i' \mu_i' (\mu_i')^\top . \end{aligned}$$

Therefore, using formula (5.4), one obtains

$$\begin{aligned} A' &= \sum_{i=1}^{l} \nu_i' A_i' + \sum_{i=1}^{l} \nu_i' \mu_i' (\mu_i)'^\top - \mu' (\mu')^\top \\ &= \frac{1}{n} \sum_{t=1}^{n} x_t x_t^\top - \sum_{i=1}^{l} \nu_i' \mu_i' (\mu_i')^\top + \sum_{i=1}^{l} \nu_i' \mu_i' (\mu_i)'^\top - \bar{x}\bar{x}^\top \\ &= \frac{1}{n} \sum_{t=1}^{n} (x_t - \bar{x})(x_t - \bar{x})^\top = S. \end{aligned}$$

Appendix B : Test of goodness-of-fit and Rosenblatt's transform

We define the goodness-of-fit test, which can be performed to assess the suitability as well as to select the number l of Markov states (regimes). The proposed test, based on the work of Diebold et al. (1998), Genest and Rémillard (2008) and Genest et al. (2009), uses the Rosenblatt's transform. It will be stated in full generality, not just for Markovian regime-switching models.

B.1 : Conditional distribution functions and Rosenblatt's transform

Let $i \in \{1, \ldots, l\}$ be fixed and Y_i be a random vector with density f_i. For any $j \in \{1, \ldots, d\}$, denote by $f_{i,1:j}$ the density of $\left(Y_i^{(1)}, \ldots, Y_i^{(j)}\right)$, and by $f_{i,j}$ the density of $Y_i^{(j)}$ given $\left(Y_i^{(1)}, \ldots, Y_i^{(j-1)}\right)$. Further denote by $F_{i,j}$ the distribution function associated with density $f_{i,j}$, where $F_{i,1}$ denotes the distribution function of $Y_i^{(1)}$.

In order words, the Rosenblatt's transform

$$y \mapsto T_i(y) = (F_{i,1}(y_1), F_{i,2}(y_1, y_2), \ldots, F_{i,d}(y_1, \ldots, y_d))^\top$$

is such that $T_i(Y_i)$ is uniformly distributed in $[0,1]^d$.

For example, in a bivariate Gaussian case where f_i is the density of a bivariate Gaussian distribution with mean μ_i and covariance matrix $\Sigma_i = \begin{pmatrix} v_i^{(1)} & \rho_i \sqrt{v_i^{(1)} v_i^{(2)}} \\ \rho_i \sqrt{v_i^{(1)} v_i^{(2)}} & v_i^{(2)} \end{pmatrix}$ then $f_{i,2}$ is the density of a Gaussian distribution with mean $\mu_i^{(2)} + \beta_i \left(y_i^{(1)} - \mu_i^{(1)}\right)$ and variance $v_i^{(2)}\,(1-\rho_i^2)$, where $\beta_i = \rho_i \sqrt{v_i^{(2)}/v_i^{(1)}}$.

The aim now is to find the Rosenblatt's transform Ψ_t corresponding to the density (5.8). Using the notations introduced above, one obtains that for any $z_1, \ldots, z_d \in \mathbb{R}$,

$$\Psi_t^{(1)}(z_1) = \Psi_t^{(1)}(x_1, \ldots, x_{t-1}, z_1) = \sum_{\alpha=1}^{l} W_{\alpha,k-1} F_{\alpha,1}(z_1) \tag{5.24}$$

and for $j \in \{2, \ldots, d\}$,

$$\Psi_t^{(j)}(z_1, \ldots, z_j) = \Psi_t^{(j)}(x_1, \ldots, x_{t-1}, z_1, \ldots, z_j) = \frac{\sum_{\alpha=1}^{l} W_{\alpha,k-1} f_{\alpha,1:j-1}(z_1, \ldots, z_{j-1}) F_{\alpha,j}(z_j)}{\sum_{\alpha=1}^{l} W_{\alpha,k-1} f_{\alpha,1:j-1}(z_1, \ldots, z_{j-1})}.$$

It then follows that the $U_1 = \Psi_1(R_1), \ldots, U_n = \Psi_n(R_1, \ldots, R_n)$ are independent and uniformly distributed over $[0,1]^d$.

Suppose that $R_1, \ldots, R_n$ be a sample of size n d-dimensional vectors from a joint (continuous) distribution P. Suppose that the hypotheses to be tested are

$$\mathcal{H}_0 : P \in \mathcal{P} = \{P_\theta; \theta \in \Theta\} \qquad \text{vs} \qquad \mathcal{H}_1 : P \notin \mathcal{P}$$

For example, the parametric family $\mathcal{F}$ could be the family of univariate Gaussian regime-switching models with r regimes. Suppose also that $\Psi_1(\cdot, \theta), \ldots, \Psi_n(\cdot, \theta)$ are the associated Rosenblatt's transforms, that is, the d-dimensional vectors $U_1 = \Psi_1(R_1, \theta)$, $U_2 = \Psi_2(R_1, R_2, \theta)$, $\ldots$, $U_n = \Psi_n(R_1, \ldots, R_n, \theta)$ are uniformly distributed over $[0,1]^d$ and independent. Suppose also that θ is estimated by $\theta_n = T_n(R_1, \ldots, R_n)$.

Since θ is unknown, it must be estimated by θ_n, so the pseudo-observations $\hat{U}_1 = \Psi_1(X1, \theta_n), \ldots, \hat{U}_n = \Psi_n(R_1, \ldots, R_n, \theta_n)$ are approximately uniformly distributed over $[0,1]^d$ and are approximately independent. However, it is well-known, contrary to what is stated in Diebold et al. (1998) for example, that it does not matter if θ is replaced by θ_n. There is a huge literature on empirical processes based on pseudo-observations, and the main result is that there is always a price to pay for estimating parameters, whenever empirical processes are concerned. See, e.g., Ghoudi and Rémillard (1998, 2004).

B.2 : Goodness-of-fit test

The proposed test statistic is based on the empirical process

$$D_n(u) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(U_i \le u) = \frac{1}{n}\sum_{i=1}^{n}\prod_{t=1}^{d} \mathbb{I}(U_{ik} \le u_t), \quad u = (u_1, \ldots, u_d) \in [0,1]^d.$$

To test $\mathcal{H}_0$ against $\mathcal{H}_1$, we propose to use the Cramér-von Mises type statistic

$$\begin{aligned} S_n &= B_n(\hat{U}_1, \ldots, \hat{U}_n) \\ &= n \int_{[0,1]^d} \left\{ D_n(u) - \prod_{t=1} u_t \right\}^2 du \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{t=1}^{d} \left\{ 1 - \max\left(\hat{U}_{ik}, \hat{U}_{jk}\right) \right\} - \sum_{i=1}^{n}\prod_{t=1}^{d} \left(1 - \hat{U}_{ik}^2\right) + \frac{n}{3^d}. \end{aligned}$$

Since the $\hat{U}_i$'s are "almost" uniformly distributed on $[0,1]^d$ under the null hypothesis, large values of S_n should lead to rejection of the null hypothesis. However, in general the limiting distribution of S_n depend on the unknown parameter θ. To estimate the P-value of S_n, one can use a parametric bootstrap approach as described below. The validity of the parametric bootstrap approach has been shown for a large range of contexts in Genest and Rémillard (2008). Its validity for dynamic models is proven in Rémillard (2010).

B.3 : Description of the parametric bootstrap

a) Calculate $\theta_n = T_n(R_1, \ldots, R_n)$ and $S_n = B_n(\hat{U}_1, \ldots, \hat{U}_n)$.

b) For some large integer N (say 1000), repeat the following steps for every $k \in \{1, \ldots, N\}$:

(i) Generate a random sample $R_1^{(k)}, \ldots, R_n^{(k)}$ from distribution P_{θ_n}.

(ii) Calculate

$$\begin{aligned}
\theta_n^{(k)} &= T_n\left(R_1^{(k)},\ldots,R_n^{(k)}\right), \\
\hat{U}_i^{(k)} &= \Psi_i\left(R_1^{(k)},\ldots,R_i^{(k)},\theta_n^{(k)}\right), \quad i=1,\ldots,n, \\
S_n^{(k)} &= B_n\left(\hat{U}_1^{(k)},\ldots,\hat{U}_n^{(k)}\right)
\end{aligned}$$

An approximate P-value for the test based on the Cramér–von Mises statistic S_n is then given by

$$\frac{1}{N}\sum_{t=1}^{N}\mathbb{I}\left(S_n^{(k)} > S_n\right).$$

Based on the results of Section 5.6, the Rosenblatt's transform for a general Markovian regime-switching model are also easy to calculate, so the goodness-of-fit test can be applied to that type of model. For the selection of the number l of regimes, it makes sense to choose the first l_0 for which the P-value of the test of goodness-of-fit is larger than 5%.

Appendix C : Optimal hedging in discrete time

Denote the price process by S, i.e., S_t is the value of d underlying assets at period k and let $\mathbb{F} = \{\mathcal{F}_t, k = 0, \ldots, n\}$ be a filtration under which S is adapted. Assume that S is square integrable. Set $\Delta_t = \beta_t S_t - \beta_{t-1} S_{t-1}$, where the discounting factors β_t are predictable, i.e. β_t is $\mathcal{F}_{t-1}$-measurable for $k = 1, \ldots, n$. The aim is to find the optimal initial investment amount V_0 and the optimal predictable investment strategy $\overrightarrow{\phi} = (\phi_t)_{t=1}^n$ that minimize the expected quadratic hedging error

$$E\left[\left\{G\left(V_0, \overrightarrow{\phi}\right)\right\}^2\right], \tag{5.25}$$

where $G = G\left(V_0, \overrightarrow{\phi}\right) = \beta_n(C - V_n)$, and $\beta_t V_t = V_0 + \sum_{j=1}^k \phi_j^\top \Delta_j$, $k = 0, \ldots, n$.

To that end, set $P_{n+1} = 1$, and for $k = n, \ldots, 1$, define

$$\begin{aligned}
A_t &= E\left(\Delta_t \Delta_t^\top P_{t+1} | \mathcal{F}_{t-1}\right), \\
b_t &= A_t^{-1} E\left(\Delta_t P_{t+1} | \mathcal{F}_{t-1}\right), \\
\alpha_t &= A_t^{-1} E\left(\beta_n C \Delta_t P_{t+1} | \mathcal{F}_{t-1}\right), \\
P_t &= \prod_{j=k}^n \left(1 - b_j^\top \Delta_j\right).
\end{aligned}$$

We can now state Theorem 2.0.1 of Rémillard and Rubenthaler (2009), which is an extension of a result of Schweizer (1995).

Theorem 4 *Suppose that $E(P_t|\mathcal{F}_{t-1}) \neq 0$ P-a.s., for $k = 1, \ldots, n$. Then the solution $\left(V_0, \overrightarrow{\phi}\right)$ of the minimization problem is $V_0 = E(\beta_n C P_1)/E(P_1)$, and*

$$\phi_t = \alpha_t - \beta_{t-1} V_{t-1} b_t, \quad k = 1, \ldots, n.$$

Note that V_0 is chosen so that $E(G) = 0$. Rémillard and Rubenthaler (2009) also show that if C_t is the optimal investment at period k so that the value of the portfolio at period n is as close as possible to C, in terms of mean square hedging error, then C_t is given by the following equation :

$$\beta_t C_t = \frac{E(\beta_n C P_{t+1}|\mathcal{F}_t)}{E(P_{t+1}|\mathcal{F}_t)}, \qquad t = 0, \ldots, n. \tag{5.26}$$

C_t can be interpreted as the option price at period t.

C.1 : Monte Carlo evaluations

Expression (5.17) is of the form

$$g_t(s,i) = \sum_{j=1}^{l} Q_{ij} \int g_{t+1}\{\pi(s,y), j\} w_t(y,i,j) f_j(y) dy, \qquad t = n-1, \ldots, 0, \tag{5.27}$$

where $w_0, \ldots, w_n$ and g_n are known functions, and $\pi_k(s,y) = s_k e^{y_k - r}$, $k = 1, \ldots, d$. The methodology proposed in Del Moral et al. (2006) for American options and in Papageorgiou et al. (2008) for hedging, is basically to use at each time step a Monte Carlo method to approximate $g_t(s,i)$ for all points s in some finite grid G. Since the values of g_{t+1} are also approximated at these points, an interpolation method is necessary to evaluate it any possible point.

Algorithm 1 (Simple Monte Carlo sampling) *To estimate $g_t(s,i)$ for every $s \in G$, one can proceed as follows.*

- *Fix $i \in \{1, \ldots, l\}$.*
- *For $k = 1, \ldots, N$, repeat the following steps :*
  - *Generate $v_k \sim Q_{i\cdot}$, i.e., $v_k = j$ with probability Q_{ij}.*
  - *If $v_k = j$, generate $X_k \sim f_j$.*
- *For every $s \in G$, set*

$$\hat{g}_t(s,i) = \frac{1}{N} \sum_{k=1}^{N} \hat{g}_{t+1}\{\pi(s, X_k), v_k\} w_t(X_k, i, v_k).$$

Note that the random sequence $(X_k, v_k)_{k=1}^N$ is the same for every value of $s \in G$. In fact, it can also be the same for every time period t, by looking at expression (5.27). In algorithm 1, the proportion of regimes with value j would be approximately Q_{ij}. As it is well-known in Sampling theory, usually a stratified sampling should perform better (in term of variability). Therefore, one could replace the preceding algorithm by the following one.

Algorithm 2 (Stratified Monte Carlo sampling) *To estimate $g_t(s,i)$ for every $s \in G$, one can :*

- *Fix $i \in \{1,\ldots,l\}$.*
- *For $k = 1,\ldots,N$, repeat the following steps :*
  - *For every $j \in \{1,\ldots,l\}$, generate $X_{kj} \sim f_j$.*
- *For every $s \in G$, set*

$$\hat{g}_t(s,i) = \frac{1}{N}\sum_{j=1}^{l}\sum_{k=1}^{N} Q_{ij}\hat{g}_{t+1}\{\pi(s,X_{kj}),j\}w_t(X_{kj},i,j).$$

Chapitre 6

Conclusion

Dans cette thèse, nous avons tenté d'apporter notre contribution à une littérature vaste et compétitive. La structuration du profil de risque du portefeuille d'un investisseur suscite un vif intérêt tant dans la sphère académique que dans le milieu privé. Par les résultats prometteurs énoncés au cours de ce travail, nous espérons avoir alimenté une perspective de recherche établie autour de la définition de protocoles optimaux de couverture en temps discret. Les pistes de travail restent conséquentes. L'intégration des frais de transaction à la stratégie de réplication ainsi que la caractérisation d'une stratégie optimale de couverture adaptée aux processus de type GARCH seront des avenues de recherche que nous tenteront d'exploiter à l'avenir. A partir des premiers résultats de Black-Scholes (1973) et de Schweizer (1992, 1995) nous proposons un algorithme optimal de réplication en temps discret d'un " payoff " écrit sur le niveau du sous-jacent (option classique) ou sur le rendement périodique du sous-jacent (option de densité). Cette méthodologie améliore les résultats précédents en minimisant l'erreur quadratique de couverture tout en conservant les caractéristiques de la fonction de " payoff " désirée. Ceci a été caractérisé dans un contexte univarié et multivarié. Une modélisation appropriée du processus des rendements du sous-jacent est essentielle dans la minimisation l'erreur de réplication, particulièrement lors de tests hors échantillon. Une première approche par mixture de lois gaussiennes a permis d'illustrer la nécessité de considérer les moments d'ordre supérieur dans la définition de la loi du sous-jacent, en comparaison avec le modèle Black-Scholes. Nous avons ensuite poursuivi la démarche en proposant une caractérisation de la loi des rendements par un processus à changements de régimes de lois gaussiennes, respectant du même coup la non-normalité de la loi empirique et la structure conditionnelle des rendements discrets. Ce faisant, une étude comparative a permis d'illustrer les avantages d'une telle modélisation sur la couverture d'option d'achat et de vente dans différents scénarios de volatilité. La définition d'un " payoff " de densité d'après les résultats de Dybvig (1988) nous a également permis de proposer une méthodologie d'assurance de portefeuille alliant la protection périodique d'un niveau de pertes admissibles à un contrôle de la volatilité du portefeuille d'actifs risqués. Cette innovation permet de limiter les erreurs de couverture tout en assurant un investissement approprié conditionnel à la volatilité du sous-jacent.