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Managing currency risk: An application of copula-based multivariate dynamic models

by

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Abstract

In order to access foreign markets, global investors often need to post collateral in currencies that are different from their benchmark currency, resulting in undesired foreign exchange risk. Investors might therefore be tempted to strictly post the minimum margins required, however doing so maximizes the probability of margin calls and their associated operational risk and cost. This thesis finds the posted margins that represent the best equilibrium between foreign exchange risk and probability of a margin call. In order to do so, a robust dynamic model of the asset prices and exchange rates dynamics is proposed, and an optimization problem seeking the optimal equilibrium is solved. The proposed model overperformed both a naive and a Gaussian alternative model on a dataset of five futures contracts in four different currencies over a period of nine years for the metrics of interest.

Keywords: Copula-based multivariate dynamic models, foreign exchange risk, multidimensional constrained optimization, collateral management.

Résumé

Afin de pouvoir accéder aux marchés des capitaux étrangers, les investisseurs institutionnels doivent souvent déposer en collatéral chez leurs contreparties des montants en devises autres que la devise de référence de leurs portefeuilles, ce qui résulte en un risque de taux de change indésirable. Ces investisseurs peuvent alors être temptés de déposer strictement les montants minimaux requis dans les comptes de marges, toutefois cette politique maximise la probabilité d'appels de marge et leurs coûts et risques opérationnels associés. Cet ouvrage trouve les montants collatéraux qui représentent le meilleur équilibre entre les coûts découlant du risque de change et ceux associés aux appels de marges. Pour se faire, un modèle robuste de la dynamique des prix d'actifs et des taux de changes est proposé, et le problème d'optimisation cherchant l'équilibre optimal est résolu. Le modèle proposé surperforme des modèles naïf et Gaussien pour un ensemble de données de cinq contrats à termes en quatre devises sur une période de neuf ans pour les métriques visées.

Mots clés: Modèles dynamiques multivariés, copules paramétriques multidimensionnelles, risque de taux de change, optimisation contrainte, gestion de collatéral.

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Proper recognition needs also be directed to my parents whose ongoing support has been fundamental to my academic success culminating in this thesis.

1 Introduction

With the globalization of financial markets, investors have access to a larger investment universe giving them the opportunity to better diversify their portfolio. A subclass of financial transactions (short selling and entering into derivative contracts for example) requires the posting of margins, that is, the deposit of collateral to attenuate the credit risk for the counterparty. If such a transaction is performed in a foreign market, the required collateral will most likely be in a currency different from the portfolio's benchmark currency, creating foreign exchange risk for the global investor. This investor might therefore want to maintain the posted collateral to the strict minimum. This policy will however maximize the probability that the counterparty, either the investor's broker or exchange, will require the posting of additional collateral following an adverse price change, so-called a margin call.

Margin calls sometimes result in undesired operational costs for investors, they are therefore to be avoided if possible. Not only counterparties may impose penalties on margin calls, but the regulatory framework agreed upon by members of the Basel Committee on Banking Supervision in the Basel III accords impose tougher capital buffers and liquidity coverage on derivatives transactions (BCBS, 2011a,b).

The purpose of this work is to use recent developments in the field of econometrics to come up with a rigorous solution to the problem of multi-currency collateral management. In addition, the methodology proposed here can be extended to any financial or risk management applications where an optimization problem needs to be constrained by practical considerations (transaction costs for example).

The remainder of this thesis is organized as follows. Section 2 reviews the existing literature on which this work builds upon. Section 3 exhibits the framework used to model the dynamics of the asset prices and exchange rates. Section 4 overviews the optimization problem at hand. Section 5 presents the results of a backtest of the proposed solution performed on a real data set. Section 6 concludes and proposes avenues of future research.

2 Literature review

This section surveys the existing literature upon which this work builds and explains how these different contributions come together to compose the proposed solution. First, it is necessary to pick a model for the joint dynamics of the financial time series. Seminal papers of univariate and multivariate time series modeling are therefore reviewed. Second, the development of copula theory provides a more flexible and powerful approach to capture the dependence between multiple time series, so the exercise is repeated for the literature on the theory of dependence copulas. Third, before blindly applying an arbitrary model to a problem, one has to gauge its validity from both a qualitative and quantitative point of view. Statistically, it means testing the model's goodness of fit with respect to the data. The literature on the empirical testing of the correctness of statistical models is therefore discussed. Fourth, in order to access and control risk in a quantitative manner, it is necessary to choose the risk measures to be monitored. The history as well as the strengths and weakness of the risk measures used in this work is summarized. Finally, papers addressing constrained optimizations problems similar to ours are surveyed.

2.1 Univariate processes

Since the beginning of the quantitative study of finance, many models have been proposed to explain or reproduce the dynamics of financial time series. Due to its convenient mathematical properties, Brownian motion has been the most often used model to this day. First developed by the botanist Robert Brown to describe the jiggling of pollen grains in water (Brown, 1828), the first application of Brownian motion to model the returns of financial time series is credited to the French mathematician Louis Bachelier (Bachelier, 1900). The defining characteristic of Brownian processes is that their increments are independent and normally distributed. In mathematical terms, for a Brownian process B_t,

$$B_t - B_s \;\sim\; N(\mu, \sigma)$$

, where $N(\mu, \sigma)$ is a normal distribution with mean μ and standard deviation σ. The use of the Brownian process became mainstream in the financial literature with the works of Merton (1969), who posited that it is in fact the natural logarithm of the returns of financial assets that follow a Brownian process.

Despite its widespread use, the Brownian model is often criticized for failing to capture some features of the dynamics of asset prices (refer to Mandelbrot (1963) and Haug and Taleb (2011) for a sample of those critics). Indeed, the empirical study of financial time series reveals that asset price returns are leptokurtic and heteroskedastic, thus leading market participants to underestimate the risk of financial assets. Not only does the distribution of returns often have fat tails, that is, realizations far from the mean occurring more frequently than the normal model would predict, but their volatility is not constant and periods of high volatility tend to cluster together, a phenomenon dubbed volatility clustering. The reader is directed to Guillaume et al. (1997), Pagan (1996), Vries and Leuven (1994), Cont (2001) for empirical surveys of these stylized facts about financial time series.

In response to the cited caveats of Brownian motion, AutoRegressive Conditionnal Heteroskedasticity (ARCH) models were tools developed to characterize time series exhibiting time-varying volatility. The term was introduced by Engle (1982) who proposed the following dynamic for the variance $h_{i,t}$ of an i^{th} financial time series at time t:

$$h_{i,t} = \kappa_{h,i} + \sum_{j=1}^{q} \alpha_{i,j} \; \epsilon_{i,t-j}^2 \; h_{i,t-j},$$

where $\epsilon_{i,t-j}$ are standardized residuals belonging to a distribution with mean 0 and standard deviation 1 and $\kappa_{h,i}$ and $\alpha_{i,j}$ are constants. For $q = 1$, it was proved in the original paper (Engle, 1982) that the above process has finite variance if $\alpha_{i,1} < 1$ and finite kurtosis if $3\alpha_{i,1}^2 < 1$. Clearly, the variance, and hence the volatility, of the time series for a given period is a direct function of the variance in the previous period, thus accounting for volatility clustering. Furthermore, the process generates data with fatter tails than the normal density. Tim Bollerslev, a graduate student of Engle, extended his model with the generalized ARCH (GARCH) (Bollerslev, 1986). Thanks to its parsimonious notation (see equations (3) and (4) in Section 3), this model has become the most popular ARCH model in practice, including this work.

2.2 Multivariate processes

Obviously proper financial modeling requires the consideration of dependence between time series. For this reason the development of multivariate models quickly followed the one of univariate models. It is now widely accepted (Cont, 2001, Peng and Ng, 2012) that robust risk management requires the inclusion of higher-order moments and co-moments in the time series of financial assets. Not only do univariate distributions capturing higher moments need to be picked to model single financial time series, but a model of dependence between the different time series allowing for higher co-moments is primordial. Indeed, traditional linear correlation between the returns of financial assets fails to capture the "correlation breakdown" or "assets boom alone but bust together" asymmetry observed in the markets.

Models have been proposed to extend the ARCH-type framework described in the previous section to multivariate settings. They all face the challenge of balancing sophistication with parsimony, of being flexible enough to capture co-moment dynamics while avoiding the curse of dimensionality. The objective is to model the dynamic of the random vector process r_t with dimensions $N \times 1$:

$$r_t = H_t^{1/2}\ \eta_t,$$

where H_t is the $N \times N$ conditional covariance matrix of r_t and η_t is the $N \times 1$ matrix of standardized residuals. The models distinguish themselves from each other by providing different specifications of the matrix process H_t.

The VEC-GARCH of Bollerslev et al. (1988) is one of the earliest such models. It is a direct generalization of the univariate GARCH model, where every variance and covariance is conditional on lagged variances and covariances:

$$\mathrm{vech}(H_t) = \kappa + \sum_{j=1}^{q} A_j\ \mathrm{vech}(r_{t-j} \cdot r'_{t-j}) + \sum_{j=1}^{p} B_j\ \mathrm{vech}(H_{t-j}),$$

where vech$(\cdot)$ is an operator that stacks the columns of the lower triangular part of its argument square matrix, κ is an $N(N+1)/2 \times 1$ vector, and Aj and Bj are $N(N+1)/2 \times N(N+1)/2$ parameter matrices. While providing flexibility, this model's total number of parameters, $(p+q)(N(N+1)/2)^2 + N(N+1)/2$, grows large very quickly as the number of dimensions increases, making it computationally intractable.

Engle and Kroner (1995) propose a restricted version of the VEC model, dubbed the Baba-Engle-Kraft-Kroner (BEKK) model. Its advantage is that the conditional covariance matrix stays positive definite by construction, which is not the case for the VEC model. The BEKK model is defined by

$$H_t = \kappa\kappa' + \sum_{j=1}^{q}\sum_{k=1}^{K} A'_{kj} r_{t-j} r'_{t-j} A_{kj} + \sum_{j=1}^{p}\sum_{k=1}^{K} B'_{kj} H_{t-j} B_{kj},$$

where A, B and κ are $N \times N$ parameter matrices and κ is lower triangular. Note that the number of parameters, $(p+q)KN^2 + N(N+1)/2$, although reduced, is still high.

Many other proposed simplified models attempt to reduce the number of parameters and/or keep the covariance matrix positive definite. A significant subset of these models adopt a factor decomposition approach. For a comprehensive survey of multivariate GARCH models, refer to Silvennoinen and Teräsvirta (2008) or Laurent et al. (2006). Due to their lack of tractability in higher dimensions, we decided to omit multivariate GARCH models in the application part of this work.

2.3 Copulas

Copula theory allows to model dependence in a parsimonious yet rigorous fashion. A copula, in combination with the univariate marginal distributions, is sufficient to fully specify a multivariate distribution function, as proved by Sklar (1959). The copula C underlying the random variables X_1, X_2, ..., X_D is the joint cumulative distribution function of the transformed variables $F_1(x_1), F_2(x_2), \ldots, F_D(x_D)$, where $F_i(x) = \mathbb{P}[X_i \leq x]$ are the marginal cumulative distribution functions, and

$$C(u_1,\ u_2,\ \ldots\ , u_D) = \mathbb{P}[F_1(X_1) \leq u_1,\ F_2(X_2) \leq u_2,\ \ldots\ , F_D(X_D) \leq u_D].$$

The theory of copulas can be traced back to Fréchet (1951) who studied the properties of bivariate density functions with uniform margins. However, their name was coined by Sklar (1959), borrowing the term from the field of linguistics where it designates a word used to link the subject of a sentence with a predicate. Sklar also produced the fundamental theorem mentioned above that bears his name.

The literature of the albeit young field of copulas is already too large (and growing) to be covered in this work in any way that is not superficial. Figure 1, borrowed from Genest et al. (2009a), displays the evolution of the number of documents on copula theory in the last part of the twentieth century. An historical perspective can be found in Sklar (1996) and Jaworski et al. (2010), while Joe (1997) and Nelsen (2006) are the references for a rigorous introduction of copula theory. The biggest strength of copulas is the ability to model the marginal distributions of random variables and their dependence independently. For illustration purposes, figure 2 exhibits a Monte Carlo simulation of a bivariate Clayton copula with mixture of two normals and Student marginal distributions. Note that the marginal distributions need not be the same, and the asymmetry of dependence between positive and negative realizations of the marginal variables.

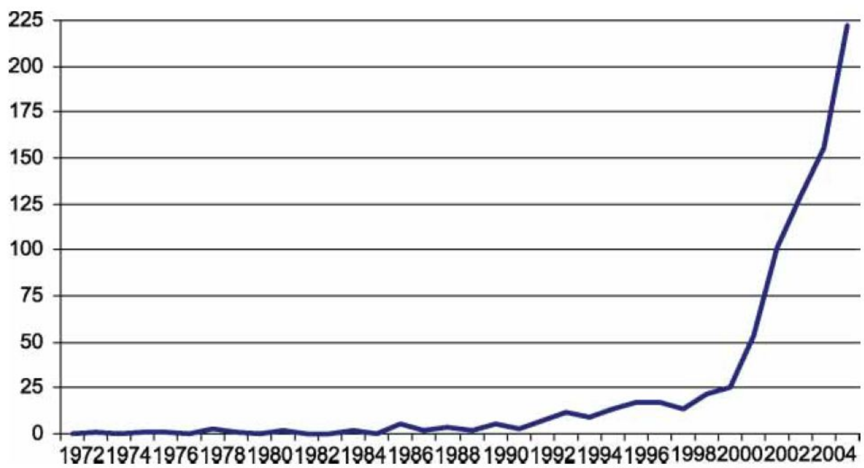

Figure 1: Number of documents on copula theory, 1971-2005 (Genest et al., 2009a)

Paul Embrecht and fellow researchers are credited for making the application of copula theory mainstream in the field of finance (Embrechts et al., 1999, McNeil et al., 2005). His group showed the pitfall of correlation and proposed the use of copulas as a more rigorous way to manage financial risk. Not long after, a paper by Li (2001), where the use of Gaussian copula models for the pricing of collateralized debt obligations was introduced, became widely read amongst both academics and practitioners. The reader is pointed

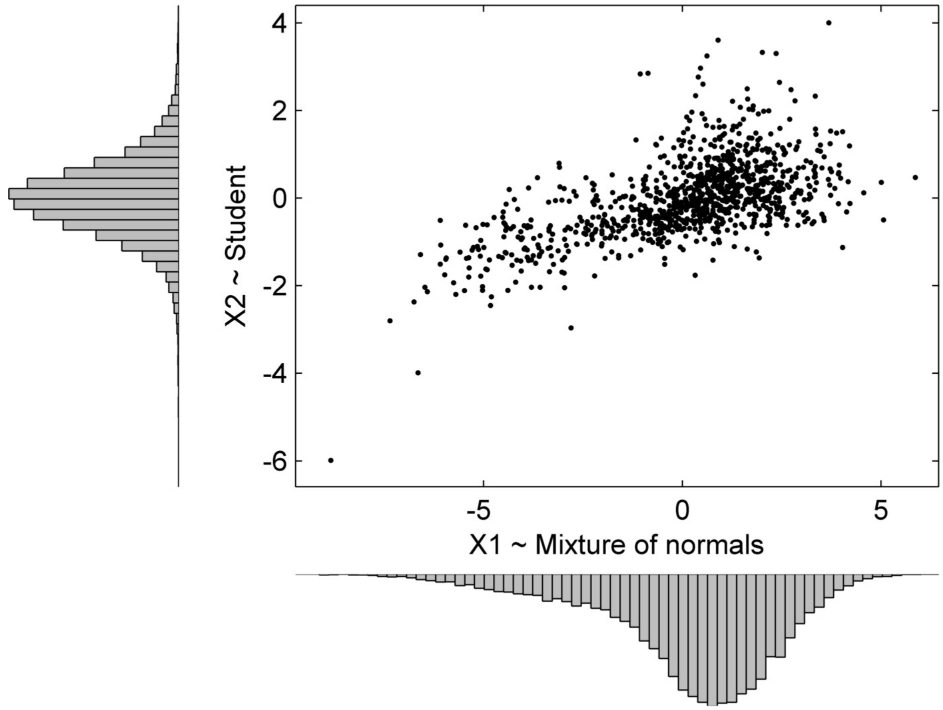

Figure 2: Bivariate Clayton copula modeling with non-normal margins

to Mikosch (2006) for a critical review of the use of copula in finance and to Genest et al. (2009a) for a bibliometrical survey.

Closer to our domain of interest, applications of copula theory in the field of econometrics is even younger than in risk management. This work owes much to the findings of Chen Xiaohong and Fan Yanqin (Chen and Fan, 2006), Andrew Patton (Patton and Kearney, 2000, Patton, 2006) and Bruno Rémillard (Rémillard, 2010). The basis of these articles is that the dependence between the error terms of multivariate time series is described not in term of traditional correlation but by a given copula.

2.4 Goodness-of-fit tests

Before the practical use of a model, it is primordial to access its validity with respect to the dataset. There exists many frequentist tests to determine the probability with which an univariate sample belongs to an arbitrary parametric distribution: the Shapiro-Wilk test, Kolmogorov-Smirnov test, Lilliefors test and the Anderson-Darling test to name a few. A performance review of these tests can be found in Razali and Wah (2011).

When it comes to non-normal multivariate distributions and copulas, few such tests existed until recently. In 1979, Efron (1979) introduced the resampling bootstrap technique, which takes advantage of the ever growing available computing power. A derivative of this technique is parametric bootstrapping, which makes goodness-of-fit testing possible with any multivariate distributions.

Unfortunately, copulas have often been used prior to the existence of proper goodness-of-fit tests, that is, without properly testing whether the data being modeled belongs to the chosen copula or not in a statistically significant manner. One of the first instance of copulas goodness-of-fit test for multivariate time series can be found in Chen and Fan (2006), however this test can only rank the appropriateness of different copulas relative to each other, not in a statistically absolute sense. Fortunately, Genest et al. (2009b) and Rémillard (2010) applied the parametric bootstrapping technique to copulas allowing one to do so in an intuitive, powerful manner. Its use with copulas evolved naturally from previous applications to univariate and multivariate distributions. The application of this test prevents the use of a copula where inappropriate.

2.5 Risk measures

Quantitative risk management requires the selection of existing risk measures or the definition of new ones. In order to minimize risk, one first has to define what risk is, and risk measures translate the vague concept of risk into measurable quantities. This work uses three different measures of risk: the tracking error, the Value-at-Risk (VaR) and Tail Conditional Expectation (TCE).

Tracking error is simply the absolute divergence of a portfolio with its benchmark or, more specifically, the absolute difference between the returns of the portfolio of interest and those of an arbitrary chosen benchmark. It is almost exclusively used when the explicit objective of a portfolio is to track an arbitrary predetermined benchmark.

VaR is perhaps the most widely measure of financial risk. In financial terms, VaR is the threshold loss such that the probability that the return on the portfolio over a predetermined time horizon is under this value for the chosen probability level. In statistical terms, VaR is the negative of an arbitrary quantile of the distribution of returns of a given portfolio over an arbitrary time horizon. Let $\mathbb{Q}_\alpha(X)$, $\alpha \in [0, 1]$, be a quantile of X such that $P(X \leq \mathbb{Q}_\alpha(X)) = \alpha$. VaR is then defined by

$$\mathrm{VaR}^\alpha(X) = -\mathbb{Q}_\alpha(X).$$

The mathematics of VaR were first defined by Roy (1952) and Markowitz (1952) in the context of equity portfolio construction. However, the term was popularized in the 1990s by RiskMetrics, then a group within J.P. Morgan. An often mentioned criticism of VaR is that it does not take into account the characteristic of the distribution of losses beyond the chosen threshold. Despite this shortcoming, VaR has arguably become the most used risk measure in market finance.

As time passes and with the recurrence of financial crises, the field of risk management matures and gains in sophistication. Artzner et al. (1999) formulated the four conditions that a risk measure must meet to be *coherent*. These are monotonicity, sub-additivity, positive homogeneity and translation invariance (refer to appendix A for a formal description). VaR does not satisfy the sub-additivity criteria, which leads investors using it to underdiversify. As an alternative, the authors propose TCE, a measure satisfying the four criterion. Simply put, TCE is the expected loss incurred in the $\alpha\%$ worst cases of the portfolio:

$$\mathrm{TCE}^{\alpha}(X) = \mathbb{E}[X|X \leq \mathbb{Q}_{\alpha}(X)].$$

This measure captures the shape of the loss distribution in the tails, therefore accounting for adverse extreme outliers ("black swans").

2.6 Similar constrained optimization problems

The tools developed in this work can also be applied to related problems: balancing tracking error and transaction costs (see Chan and Ramkumar (2011)), foreign currency risk and hedging costs (see Campbell et al. (2010)), sales and marketing costs, etc. In the case of balancing foreign exchange risk on posted collateral and the costs associated to margin calls, global investors have historically used heuristics or more sophisticated policies to mitigate the two inconveniences. For literature on the subject, one can refer to Miller and Orr (1966), Higson et al. (2010) for solutions to the similar problem of optimal inventory management, to Cotter (2001), Lam et al. (2004), Kao and Lin (2010), Longin (1999) for solutions to the problem of setting the margins requirements by a central counterparties (e.g. clearing houses) and to Fujii et al. (2010) for an example where the market participant has the choice of collateral currency.

3 Copula-based multivariate dynamic models

In order to develop a cash management strategy which strikes an optimal balance between the opposing goals of minimizing foreign exchange risk and minimizing the cost associated with margin calls, it is first necessary to choose a model of underlying asset prices and exchange rates movements. It is now widely accepted that proper modeling is robust to volatility clustering and excess kurtosis. That is, the volatility in financial time series is not constant across time but periods of relatively high volatility tend to cluster together. Furthermore, the probability of extreme events in the joint distributions of multivariate financial time series innovations is higher than predicted under a Gaussian model. The latter phenomenon is also known as "fat tails". Refer to Pagan (1996), Cont (2001) for evidence of stylized facts of financial time series and more precisely to Vries and Leuven (1994), Guillaume et al. (1997) for evidence of such facts in exchange rate returns.

We propose the use of a dynamic multivariate discrete stochastic volatility model with a copula-based dependence structure. This model has been successfully applied to exchange rate returns in the literature (Chen and Fan, 2006, Rémillard, 2010, Patton, 2006). The multivariate time series $\mathbf{X}_t$, $t \geq 1$ has D dimensions and is given by

$$X_{i,t} = \mu_t(\boldsymbol{\theta}_i) + h_t(\boldsymbol{\theta}_i)^{1/2}\ \epsilon_{i,t}, \tag{1}$$

where $i = 1, \ldots, D$ and innovations $\epsilon_{1,t}, \ldots, \epsilon_{D,t}$ are $i.i.d.$ with copula distribution function C.

We know from Sklar theorem (Sklar, 1959) that, given that K is continuous, there exists a unique copula C such that

$$K(x_1, \ldots, x_D) = C_{\boldsymbol{\theta}}(F_1(x_1), \ldots, F_D(x_D)), \tag{2}$$

where the F_i are the cumulative distribution functions of the marginal distributions X_i and $C_{\boldsymbol{\theta}}$ is the copula function with parameter(s) $\boldsymbol{\theta}$.

An interesting property of copulas is that the dependence between the variables is encapsulated in the copula function and is independent of the marginal distribution functions chosen.

We model the marginal distributions of the financial time series with AR(k)-GARCH(p,q) models (Bollerslev, 1986). We chose this model because we believe it offers the best equilibrium between sophistication and parsimony. Their formulation is given by

$$\mu_t(\boldsymbol{\theta}_i) = \kappa_{\mu,i} + \sum_{j=1}^{k} \gamma_{i,j}\ x_{i,t-j}, \tag{3}$$

and

$$h_t(\boldsymbol{\theta}_i) = \kappa_{h,i} + \sum_{j=1}^{q} \alpha_{i,j}\ \epsilon_{i,t-j}^2 + \sum_{j=1}^{p} \beta_{i,j}\ h_{t-j}(\boldsymbol{\theta}_i). \tag{4}$$

We tested the goodness-of-fit of common elliptical (Gaussian and Student) and Archimedean (Clayton, Frank, Gumbel) copulas on the standardized residuals of the AR-GARCH margins processes emerging from the dataset described in section 5.1. Their distributions are given below.

The Gaussian copula with dependence parameter matrix Σ is given by

$$C_\Sigma(\mathbf{u}) = \Phi_\Sigma\left(\Phi^{-1}(u_1),\ \dots\ \Phi^{-1}(u_D)\right), \tag{5}$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of a standard normal, $\Phi_\Sigma(\cdot)$ is the joint cumulative distribution function of a multivariate normal distribution with zero means and covariance matrix Σ. Its density is given by

$$c_\Sigma(\mathbf{u}) = |\Sigma|^{-\frac{1}{2}}\ \exp\left(-\frac{1}{2}\begin{pmatrix}\Phi^{-1}(u_1)\\ \vdots \\ \Phi^{-1}(u_D)\end{pmatrix}^T (\Sigma^{-1} - \mathbf{I})\begin{pmatrix}\Phi^{-1}(u_1)\\ \vdots \\ \Phi^{-1}(u_D)\end{pmatrix}\right).$$

The D-dimensional Student copula distribution as given by (Demarta and McNeil, 2005):

$$C_{\Sigma,\nu}(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_D)} \frac{\Gamma\left(\frac{\nu+D}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^D|\Sigma|}} \left(1 + \frac{\mathbf{x}'\Sigma^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+D}{2}} d\mathbf{x}, \tag{6}$$

where $t_\nu^{-1}()$ is the quantile function of a standard univariate Student distribution with ν degrees of freedom. Its density can be derived to be

$$c_{\Sigma,\nu}(\mathbf{u}) = \frac{f_{\Sigma,\nu}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_D))}{\prod_{i=1}^{D} f_\nu(t_\nu^{-1}(u_i))}, \tag{7}$$

where $f_{\Sigma,\nu}$ is the joint density of a D-dimensional random vector from a multivariate Student distribution with ν degrees of freedom and covariance matrix Σ and f_ν is the density of a univariate Student distribution with ν degrees of freedom. This copula not only captures excess kurtosis but was also shown to accurately model the dependence between financial time series in recent literature (Chen and Fan, 2006, Fischer et al., 2009).

Archimedean copulas are characterized by a single dependence parameter and the following representation:

$$C(u_1,\ u_2,\ \ldots\ , u_D) = \psi(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_D)),$$

where $\psi(\cdot)$ is the *generator* of the copula. The generators for the three Archimedean copulas tested in the applied section of this thesis are displayed in table 1.

Family	**Generator $\psi(\mathbf{t})$**	**Parameter**	**G-distribution**
Clayton	$(1+t)^{-1/\theta}$	$0 < \theta < \infty$	Gamma$(1/\theta, 1)$
Frank	$-\frac{1}{\theta}\log(1-(1-e^{-\theta})e^{-t})$	$0 < \theta < \infty$	Log series with $\alpha = (1-e^{-\theta})$
Gumbel	$\exp(-t^{1/\theta})$	$1 \le \theta < \infty$	Stable$(1/\theta, 1, (\cos(\pi/(2\theta)))^\theta, 0)$

Table 1: Generators of selected Archimedean copulas. G-distribution refers to the distribution that has as Laplace transform the generator $\psi(\cdot)$.

The parameters of the marginal processes and dependence copula can be estimated with the maximum likelihood estimation (MLE) method. An alternative two-stage estimation method for the Student copula proposed by McNeil et al. (2005) is also described in Appendix B.

Chen and Fan (2006) showed the important result that applying the copula dependence structure from equation (2) to the innovations $\epsilon_{i,t}$ from equation (1) rather than to the realizations $X_{i,t}$ yields the same results in terms of estimation of the parameters of the copula. Rémillard (2010) showed that the empirical copula and most dependence measures are unaffected as well.

3.1 Goodness-of-fit tests

Responsible modeling requires proper testing as to whether the modeled data do in fact belong to the chosen model. When it comes to dynamic models, many authors either omit goodness-of-fit (GOF) tests, or use only relative tests that ranks the fit of different models. The problem with the latter approach is that picking the model with the best fit out of a set of incorrect models will still yield an incorrect model. Fortunately, absolute GOF tests for dynamic models based on parametric bootstrapping have recently been made available in the literature (Genest et al., 2009b, Rémillard, 2011).

The null hypothesis of the GOF test for a general dynamic univariate process X can be stated as follows:

> H_0: The conditional distribution of X_t given $\mathcal{F}_{t-1}$ is $F_{t,\theta}$, for some parameter $\theta \subseteq \mathcal{O}$.

Under the null hypothesis, it can be shown that $U_1 = F_{1,\theta}(X_1), \ldots, U_T = F_{T,\theta}$ are i.i.d. uniform variates on $(0, 1)$. A general recipe to use parametric bootstrapping for the hypothesis is as follow:

(i) Estimate the parameter θ on the process $X_1, \ \ldots \ , X_T$ by $\hat{\theta}$.

(ii) Compute a distance statistic S_T between the uniform distribution function and the distribution function F_T of the pseudo-observations $u_1 = F_{1,\hat{\theta}}(X_1), \ldots, u_T = F_{T,\hat{\theta}}(X_T)$. A good candidate is the Cramér-von Mises criterion:

$$S_T = \int_0^1 \{F_T(u) - u\}^2 \, du.$$

(iii) Generate a large number $k = 1, \ldots, N$ of random sequences $X_1^{(k)}, \ldots, X_T^{(k)}$ from the dynamic model with parameters $\hat{\theta}$.

(iv) For each k from step (iii):

  (a) Estimate the parameter θ by $\theta^{(k)}$ for the sample $X_1^{(k)}, \ldots, X_T^{(k)}$.

  (b) Compute the same distance statistic $S_T^{(k)}$ as in step (ii) for the sample $X_1^{(k)}, \ldots, X_T^{(k)}$.

(v) The p-value of the test is approximated by the fraction of values $S_T^{(k)}$ greater than the S_T computed in step (ii):

$$p = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(S_T^{(k)} > S_T\right).$$

As for the null hypothesis of the GOF test of a copula-based multivariate dynamic model, it is of the form

> H_0: The copula associated with the innovations $\epsilon_t = (\epsilon_{1,t}, \ldots, \epsilon_{D,t})$, $t = 1, \ldots, T$, belongs to a parametric family $C_{\boldsymbol{\Theta}}$.

The procedure for the parametric bootstrap is:

(i) Estimate the parameters of each univariate marginal process and compute the associated standardized residuals $e_t = (e_{1,t}, \ldots, e_{D,t})$, $t = 1, \ldots, T$.

(ii) Compute the normalized ranks $u_{i,t},\ i = 1,\ \dots\ , D,\ t = 1,\ \dots\ , T$ of the standardized residuals resulting from step (i):

$$u_{i,t} = \frac{1}{T+1} \sum_{k=1}^{T} \mathbb{1}(e_{i,t} \geq e_{i,k}).$$

(iii) Estimate the dependence parameter $\mathbf{\Theta}$ of the parametric copula by $\hat{\mathbf{\Theta}}$, using the normalized ranks resulting from step (ii).

(iv) Compute a distance statistic S_T between the empirical copula C_T of the normalized ranks and the parametric copula $C_{\hat{\mathbf{\Theta}}}$, where the empirical copula is given by

$$C_T(x_1,\ \dots\ , x_D) = \frac{1}{T} \sum_{t=1}^{T} \prod_{i=1}^{D} \mathbb{1}(u_{i,t} \leq x_i).$$

A good candidate for S_T is the Cramér-von Mises statistic

$$S_T = \frac{1}{T} \sum_{t=1}^{T} \left\{ C_T(u_{1,t},\ \dots\ , u_{D,t}) - C_\theta(u_{1,t},\ \dots\ , u_{D,t}) \right\}^2 .$$

(v) For some large integer N, repeat the following steps for each k in $[1, \dots, N]$:

(a) Generate random vectors $\mathbf{U}_1^{(k)}, \dots, \mathbf{U}_T^{(k)}$ with distribution $C_{\hat{\mathbf{\Theta}}}$. Most existing statistical packages does not currently support the generation of multivariate Archimedean copulas random variables; see Appendix D for correct methodology.

(b) Repeat steps (ii) to (iv) on trajectories generated in (a) to obtain $S_T^{(k)},\ k = 1,\ \dots\ , N$.

(vi) The approximate p-value for the test is given by

$$p = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1} \left(S_T^{(k)} > S_T \right).$$

Using the above procedure to test for the goodness-of-fit of elliptical copulas can prove tedious because there exists no explicit form for their cumulative distribution functions and Monte Carlo integration of equations (5) or (6) requires excessive computational resources for any non-small number of dimensions. An ingenious alternative is described by Genest et al. (2009b). The method uses a critical property of Rosenblatt's probability integral transform[1] $\mathcal{T}$ (Rosenblatt, 1952), namely that a multivariate vector $\mathbf{U} \in [0,1]$ has distribution function C if and only if the Rosenblatt transform of $\mathbf{U}$ has the independence copula as distribution function $C_\perp$:

$$\mathbf{U} \sim C \;\Leftrightarrow\; \mathcal{T}(\mathbf{U}) \sim C_\perp$$

The algorithm of the parametric bootstrap test applied to the elliptical copulas is as follow:

1. Estimate the parameters of each univariate marginal process and compute the associated standardized residuals $e_t = (e_{1,t}, \ldots, e_{D,t})$, $t = 1, \ldots, T$.

2. Compute the normalized ranks $\mathbf{u}_t = (u_{1,t}, \ldots, u_{D,t})$, where for $i = 1, \ldots, D$, and $t = 1, \ldots, T$,

$$u_{i,t} = \frac{1}{T+1} \sum_{k=1}^{T} \mathbb{1}(e_{i,t} \geq e_{i,k}).$$

[1]Refer to Appendix C for the definition of the transformation.

3. Estimate the dependence parameter $\mathbf{\Theta}$ of the parametric copula by $\hat{\mathbf{\Theta}}$, using the normalized ranks resulting from step (2).

4. Compute Rosenblatt transforms $\mathbf{v}_t = (v_{1,t}, \ldots, v_{D,t}) = \mathcal{T}_{\hat{\mathbf{\Theta}}}(\mathbf{u}_t)$, $t = 1, \ldots, T$, as shown in Appendix C.

5. Compute a distance statistic S_T between the empirical copula C_T of the Rosenblatt transforms and the independence copula $C_\perp$. The empirical copula is given by

$$C_T(x_1,\ \ldots\ , x_D) = \frac{1}{T} \sum_{t=1}^{T} \prod_{i=1}^{D} \mathbb{1}(v_{i,t} \leq x_i).$$

The Cramér-von Mises criterion in this case is given by:

$$\begin{aligned} S_T &= T \int_{[0,1]^D} \{F_T(\mathbf{v}) - C_\perp(\mathbf{v})\}^2 d\mathbf{v} \\ &= \frac{T}{3^D} - \frac{1}{2^{D-1}} \sum_{t=1}^{T} \prod_{i=1}^{D} \left(1 - v_{i,t}^2\right) + \frac{1}{T} \sum_{t=1}^{T} \sum_{k=1}^{T} \prod_{i=1}^{D} (1 - \max(v_{i,t}, v_{i,k})). \end{aligned}$$

6. For some large integer N, repeat the following steps for each k in $[1, \ldots, N]$:

   (a) Generate random vectors $\mathbf{U}_1^{(k)}, \ldots, \mathbf{U}_T^{(k)}$ with distribution $C_{\hat{\mathbf{\Theta}}}$.

   (b) Repeat steps (2) to (4) on trajectories generated in (a) to obtain $S_T^{(k)},\ k = 1,\ \ldots\ , N$.

7. The approximate p-value for the test is given by

$$p = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}\left(S_T^{(k)} > S_T\right).$$

4 Constrained optimization

One difficulty in the problem at hand is to find a way to balance foreign exchange risk on one hand and the costs associated to margin calls on the other. This exercise proves difficult given that these two forces have different units. We posited that the costs associated to margin calls are an increasing function of their frequency, and chose to minimize the foreign exchange risk on the posted collateral conditionally to an arbitrary upper bound on the probability of a margin call. Chan and Ramkumar (2011) offers an elegant framework to a problem similar in nature to ours. Their goal is to balance trading costs associated with the rebalancing of a given portfolio and tracking error. Their solution is to minimize the trading costs subject to an imposed upper ceiling on the forecasted tracking error. Our objective function described below is inspired from this approach.

We formulate the issue at hand as a one-period constrained optimization problem. The first step consists in choosing a risk measure to assess the foreign exchange risk. This work uses the expected tracking error (ETE), the Value-at-Risk (VaR) and the Tail Conditional Expectation (TCE). The method consists in finding the cash balances that minimize the (foreign exchange) risk measure subject to an arbitrary tolerance for the probability of a margin call. A mathematical formulation of the optimization objective is

$$\min_{\lambda_{1,t},\ldots,\lambda_{D,t}} R^{\alpha}\left(\sum_{i=1}^{D}\left(\lambda_{i,t}-\lambda_{i,t}^{*}\right)\cdot Y_{i,t+1}\right), \tag{8}$$

where

$$R^{\alpha}(X)=\begin{cases}\mathbb{E}\left[|X|\right] & \text{for expected tracking error,}\\ -\mathbb{Q}_{\alpha}(X) & \text{for Value-at-Risk,}\\ -\mathbb{E}[X|X\leq\mathbb{Q}_{\alpha}(X)] & \text{for Tail Conditional Expectation.}\end{cases} \tag{9}$$

In words, equation (8) finds the collateral in each currency $\lambda_{1,t}, \ldots, \lambda_{D,t}$ above the minimum margins requirements imposed by the counterparty $\lambda^*_{1,t}$, ..., $\lambda^*_{D,t}$ that minimizes one of the risk measures $R^\alpha(\cdot)$ described in equation (9) against the changes in the exchange rates $Y_{1,t+1}$, ..., $Y_{D,t+1}$. The first constraint of the optimization is

$$\lambda^*_{i,t} \leq \lambda_{i,t} \leq \infty, \; i = 1, \ldots, D,$$

that is, each posted collateral $\lambda_{i,t}$ must be between the minimum margin requirement $\lambda^*_{i,t}$ and infinity. The second constraint of the optimization is

$$\mathbb{P}\left(\left(\prod_{i=1}^{D} \mathbb{1}\left(\lambda_{i,t} + \mathrm{PnL}_{i,t+1} \geq \lambda^*_{i,t}\right)\right) = 0\right) \leq P_{\text{tol}},$$

where

$$\mathrm{PnL}_{i,t+1} = \sum_{j=1}^{n_{i,t}} {}_i\omega_{j,t} \cdot {}_iW_{j,t+1}.$$

Here, P_{tol} is an arbitrary tolerance for the probability of a margin call, $n_{i,t}$ is the number of different assets of the i^{th} currency held, ${}_i\omega_{j,t}$ is the number of the j^{th} asset of the i^{th} currency held and ${}_iW_{j,t+1}$ is the change in the j^{th} asset of the i^{th} currency between time t and $t+1$. To make things clear, the only random variables in the model are the $Y_{i,t}$ and the ${}_iW_{j,t+1}$. Furthermore, these random variables are not independent of each other. That is, the spot exchange rates are not independent of the futures contracts prices and vice versa.

The proposed methodology for the optimization is as follow: First, univariate process are fitted to each time-series of log returns of exchange rates and future prices. This step often involves the calibration of parameters with the maximum likelihood estimation method. Second, goodness-of-fit tests of the selected model are performed on each time-series. If the test result is negative for a given time-series of log returns, then another univariate model is chosen for that specific time-series and the process is started anew from the first step. Again, different time-series can have different univariate models. Third, a parametric multivariate copula is fitted to the normalized ranks of the standardized residuals obtained in the first step. Fourth, a goodness-of-fit test as put forward in section 3.1 is performed on the copula-based dynamic model with respect to the sample of time-series. If the test result is negative, another copula is chosen and the process is restarted from the third step. At this point, we know that the model is appropriate for the data (at least in the statistical sense of the term). Here is where Monte Carlo methods come into play. Fifth, a large number N of D-dimensional $U[0,1]$ realizations is drawn from the parametric copula, were D is the number of time-series modeled. Most modern statistical software packages provide tools to perform this step for elliptical copulas. Methods used to generate random drawings from selected multivariate Archimedean copulas are found in Appendix D. Sixth, the simulated standardized residuals of each univariate process are obtained by entering the uniform drawings from the previous step into their respective inverse cumulative distribution function. Seventh, the simulated realizations of the log returns are obtained by combining of the random standardized residuals and the computed deterministic part of each univariate process. The final step consists in using constrained numerical optimization to find the values of the control variables that optimize the objective function while satisfying the constraint function. For us, the control variables are the amount of collateral held in each currency, the objective function is the risk measure (tracking error, Var and TCE) and the constraint function is the probability of a margin call. Notice that both the objective and constraint are functions of the control variables (the amount of collateral held in each foreign currency) and the simulated realizations from the Monte Carlo simulation (the log returns of exchange rates and future prices). The objective and constraint functions being nonlinear, we obtained the best results using the interior point optimization method, closely followed by the active set algorithm.

5 Backtesting

5.1 Dataset

In order to demonstrate the validity of the model proposed in the previous sections, we compare its performance over a data set against alternative strategies described in Section 5.2. The data set consists of daily holdings of five futures contracts denominated in non-USD currencies from November 2003 to March 2012, for a total of 1941 observations. Figure 3 exhibits the future contracts prices and the number of each contract held at any moment. These holdings are exogeneous to our control; our aim is to find the optimal cash margins given certain holdings of contracts denominated in foreign currencies. Figure 4 displays the exchange rates of the currencies of the futures contracts. The reader can notice the increase in volatility for both the future prices and exchange rates during the banking crisis of 2008-2009, with sharp declines of the Euro, the Australian and Canadian dollars relative to the US dollar. The Japanese Yen, often considered a refuge currency, gained during this difficult period.

A graphical test proposed by Gnanadesikan (1977) to test joint normality was run on the data. If $\mathbf{X} = X_1, \ \ldots \ , X_d$ is from a multivariate Gaussian distribution, then

$$(\mathbf{X}_i - \bar{\mathbf{X}})' \ \Sigma \ (\mathbf{X}_i - \bar{\mathbf{X}}), \ i = 1, \ \ldots \ , N$$

where Σ is the covariance matrix of $\mathbf{X}$ have a χ^2_d-distribution. A QQ plot can then be used as a quick, "litmus" test of joint normality, as seen in Figure (5). Evidently joint normality has to be rejected.

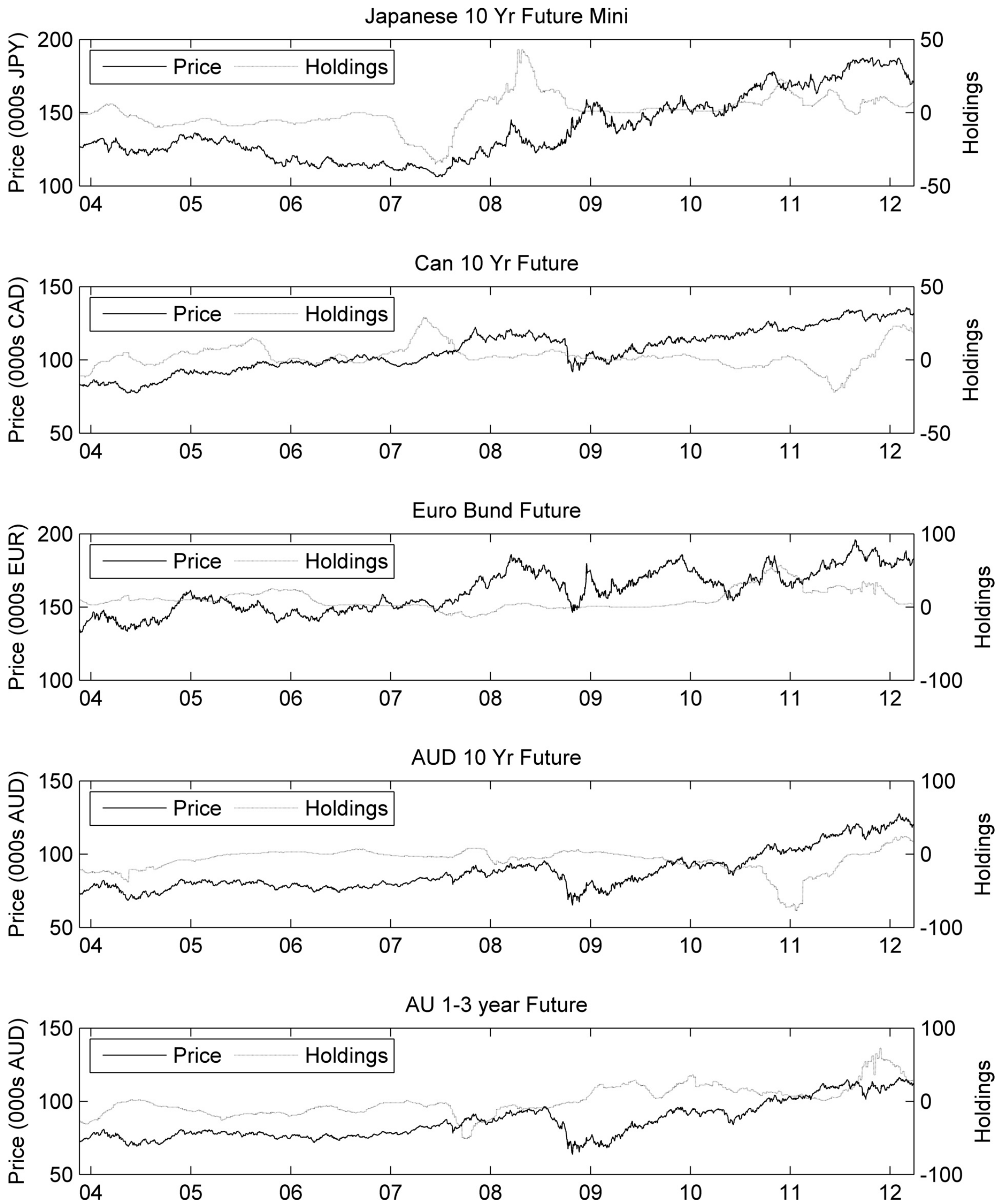

Figure 3: Futures contracts prices and holdings

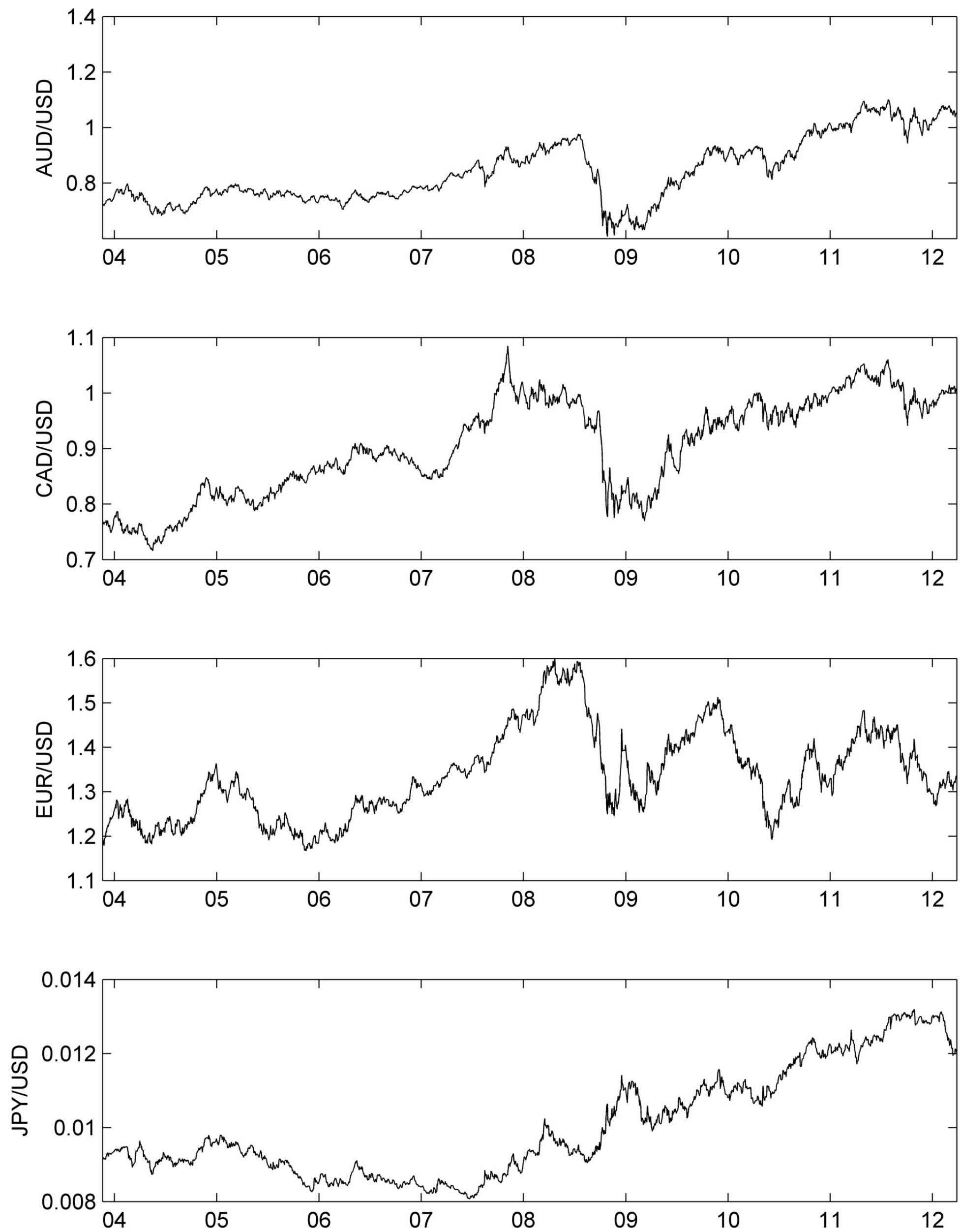

Figure 4: Exchange rates of interest

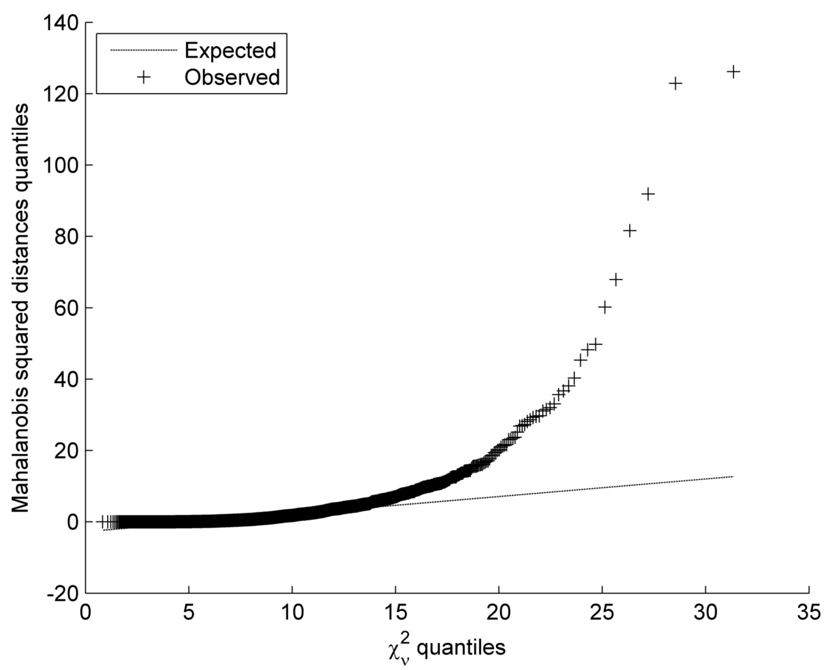

Figure 5: Graphical test of joint normality - QQ plot

5.2 Alternative strategies

In order to access the validity of our model, we backtested it on the dataset presented in the previous section. We also backtested two other strategies on the same dataset to contrast their performances.

The first, "naive" strategy simply consists in keeping twice the compulsory collateral in the cash accounts at all times. That is, the collateral posted in the accounts of the different currencies is brought back to twice the mandatory minimum every day according to that day's movements in futures contracts prices and holdings.

The second strategy consists in modeling the returns of the futures contracts and exchange rates with a static multivariate normal distribution as in the Markowitz framework. Doing so allows us to gauge the accuracy added by factoring in stochastic volatility and higher co-moments as done by the model proposed in Section 3.

5.3 Calibration

A buffer of 500 business days (approximately 2 years) is used at the beginning of the sample to calibrate the marginal processes and the dependence copula parameters. The means and covariance matrix for the static multivariate strategy is also computed with this buffer. Both models are recalibrated every day using all data from the beginning of the sample (i.e. with an extending window).

The tolerance for the probability of a margin call P_{tol} was arbitrarily set to 0.05, which was also the level chosen for the quantile α of the VaR and TCE measures.

Once at the beginning and then every 250 business days, with an extending window, the goodness-of-fit tests introduced in Section 3.1 were performed on both the marginal processes and the dependence copulas. The results of the goodness-of-fit tests for the AR(1)-GARCH(1,1) with Gaussian residuals, AR(2)-GARCH(2,2) with Gaussian residuals, AR(1)-GARCH(1,1) with Student residuals and different dependence copulas are displayed in tables 2, 3, 4 and 5 respectively. Clearly, a combination of AR(1)-GARCH(1,1) with Student residuals and the Student copula is the only appropriate model from a statistical point of view for the time-series at hand.

5.4 Results

Figure 6 shows the difference between the optimal posted collateral if the time series are modeled with the copula-based multivariate model or with a static multivariate Gaussian distribution, for each currency. The units on the y-axis are percentage of minimum margin requirements. The optimization objective was to maximize the Tail Conditional Expectation of the value of the posted collateral in USD. Notice the supplementary collateral posted during the 2008-2009 crisis.

Figure 7 displays the daily change in the value of the posted collateral in USD for each of the three strategies. The optimization objective was to minimize the tracking error. Notice the increased volatility for the year following the collapse of Lehman Brothers in September 2008 that marked the start of the 2008-2009 financial crisis, and the large drawdown that occurred in May 2010 “flash crash”. Figure 8 displays an histogram of the daily change in the value of the posted collateral in USD for the static multivariate Gaussian model and the copula-based multivariate dynamic model strategies. Note the slightly higher unwanted kurtosis for the Gaussian model.

Nov03 till	**Feb06**	**Mar07**	**Apr08**	**Apr09**	**May10**	**Jun11**
Japanese 10 Yr Future Mini	0.02	0.07	0.00	0.00	0.00	0.00
Can 10 Yr Future	0.10	0.52	0.62	0.02	0.02	0.00
Euro Bund Future	0.00	0.00	0.00	0.00	0.00	0.00
AUD 10 Yr Future	0.00	0.00	0.00	0.00	0.00	0.00
AU 1-3 year Future	0.05	0.05	0.00	0.00	0.00	0.00
AUD/USD	0.30	0.14	0.00	0.00	0.00	0.00
CAD/USD	0.09	0.23	0.08	0.00	0.00	0.00
EUR/USD	0.00	0.00	0.00	0.00	0.00	0.00
JPY/USD	0.05	0.01	0.00	0.00	0.00	0.00

Table 2: p-values from the goodness-of-fit tests of AR(1)-GARCH(1,1) with Gaussian residuals on the marginal processes. The number of bootstrapped samples is $N = 100$.

Nov03 till	**Feb06**	**Mar07**	**Apr08**	**Apr09**	**May10**	**Jun11**
Japanese 10 Yr Future Mini	0.06	0.08	0.01	0.00	0.00	0.00
Can 10 Yr Future	0.12	0.45	0.46	0.02	0.00	0.00
Euro Bund Future	0.00	0.01	0.00	0.00	0.00	0.00
AUD 10 Yr Future	0.00	0.01	0.00	0.00	0.00	0.00
AU 1-3 year Future	0.06	0.05	0.00	0.00	0.00	0.00
AUD/USD	0.36	0.14	0.00	0.00	0.00	0.00
CAD/USD	0.08	0.36	0.04	0.00	0.00	0.00
EUR/USD	0.01	0.01	0.00	0.00	0.01	0.00
JPY/USD	0.09	0.03	0.00	0.00	0.00	0.00

Table 3: p-values from the goodness-of-fit tests of AR(2)-GARCH(2,2) with Gaussian residuals on the marginal processes. The number of bootstrapped samples is $N = 100$

Nov03 till	**Feb06**	**Mar07**	**Apr08**	**Apr09**	**May10**	**Jun11**
Japanese 10 Yr Future Mini	0.42	0.64	0.59	0.50	0.56	0.50
Can 10 Yr Future	0.51	0.79	0.75	0.72	0.69	0.79
Euro Bund Future	0.42	0.47	0.37	0.32	0.37	0.41
AUD 10 Yr Future	0.62	0.52	0.56	0.56	0.58	0.52
AU 1-3 year Future	0.53	0.61	0.53	0.52	0.50	0.49
AUD/USD	0.67	0.63	0.51	0.48	0.45	0.45
CAD/USD	0.53	0.68	0.69	0.50	0.60	0.45
EUR/USD	0.39	0.47	0.42	0.34	0.54	0.54
JPY/USD	0.45	0.64	0.53	0.64	0.62	0.48

Table 4: p-values from the goodness-of-fit tests of AR(1)-GARCH(1,1) with Student residuals on the marginal processes. The number of bootstrapped samples is $N = 100$.

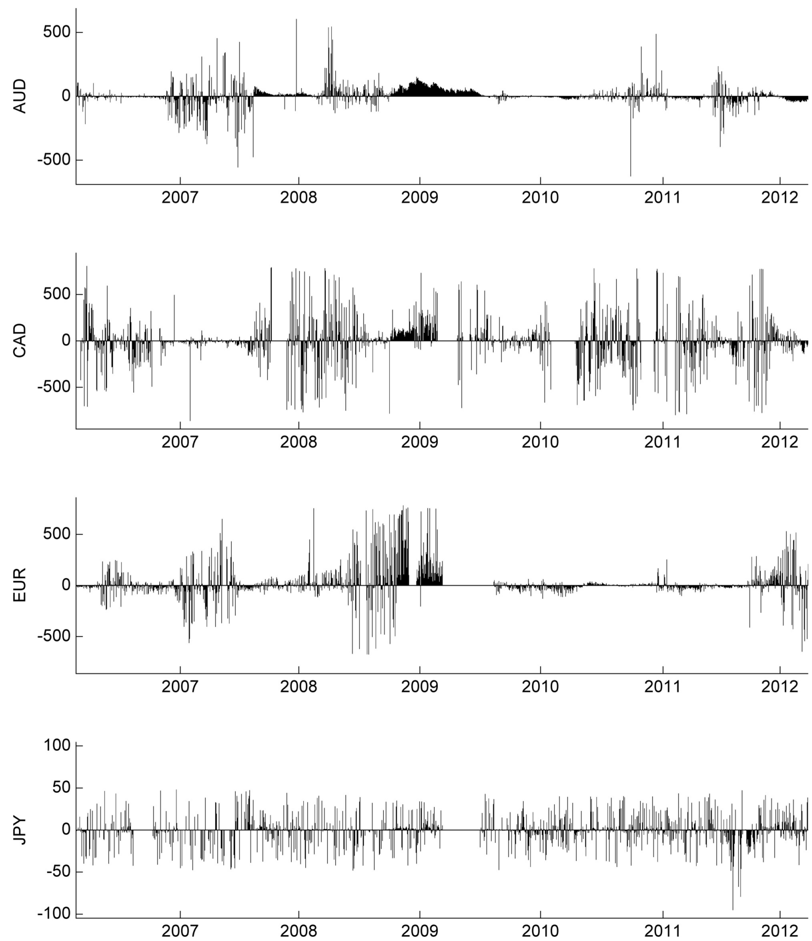

Figure 6: Difference between the optimal posted collateral if the time series are modeled with the copula-based multivariate model or with a static multivariate Gaussian distribution, for each currency. The units on the y-axis are percentage of minimum margin requirements.The optimization objective was to maximize the Tail Conditional Expectation of the value of the posted collateral.

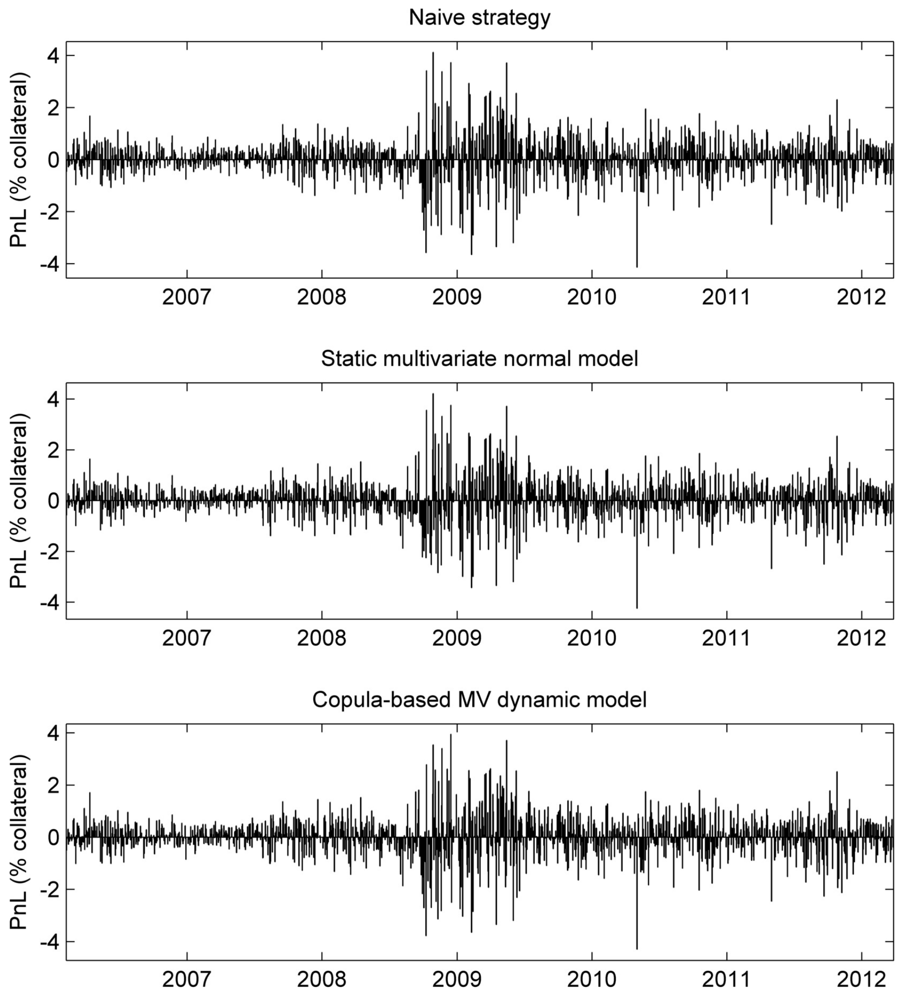

Figure 7: Daily changes in the value of the posted collateral in USD due to changes in the exchange rates for the three strategies. The optimization objective was to minimize the tracking error.

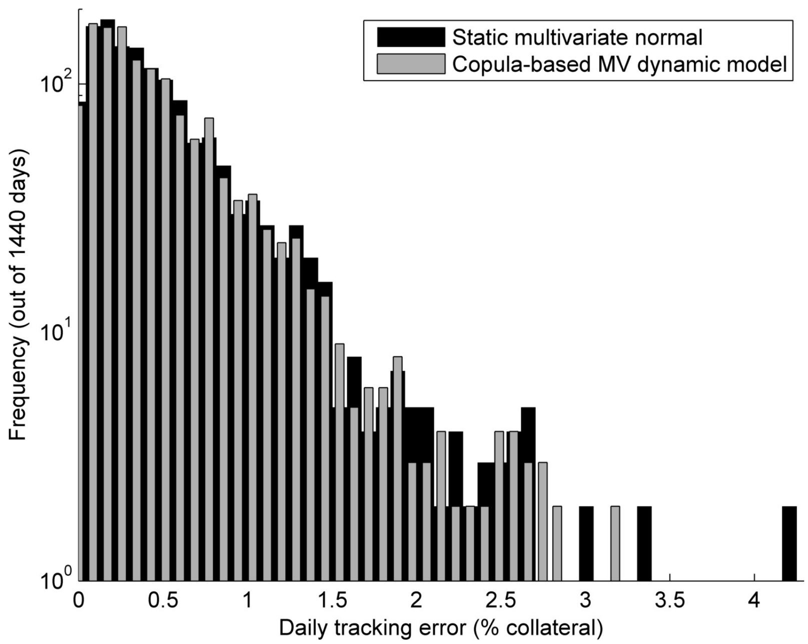

Figure 8: Frequency of daily tracking errors caused by exchange rates movements for the multivariate Gaussian and the copula-based multivariate dynamic models. The optimization objective was to minimize the tracking error.

Nov03 till	Feb06	Mar07	Apr08	Apr09	May10	Jun11
MV Gaussian	0.00	0.00	0.00	0.00	0.00	0.00
Gaussian copula	0.10	0.02	0.02	0.01	0.00	0.00
Student copula	0.59	0.28	0.38	0.21	0.09	0.19
Clayton copula	0.00	0.00	0.00	0.00	0.00	0.00
Frank copula	0.00	0.00	0.00	0.00	0.00	0.00
Gumbel copula	0.00	0.00	0.00	0.00	0.00	0.00

Table 5: p-values from the goodness-of-fit tests of copulas. The marginal processes were modeled with AR(1)-GARCH(1,1) with Student residuals. The number of bootstrapped samples is $N = 100$.

Tables 6, 7 and 8 exhibit the results of the backtests over the entire sample for the three risk measures of interest. The copula-based multivariate dynamic approach is the only one that did not breach the imposed upper limit on the frequency of margin calls, while maintaining similar risk measures realizations.

The quality of a risk management strategy is known during rough, volatile markets environments. We thus observed how our proposed model fared during the 2008-2009 financial crisis. We chose the period form March 17^{th} 2008, the fall of Bear Stearns, followed not long after by Lehman Brothers and American Insurance Group (AIG), to February 17^{th} 2009, when the American Recovery and Reinvestment Act was passed. As we now know this date did not mark the end of the crisis, but it does represent a milestone when volatility in the futures and exchange rates markets declined. Tables 9, 10 and 11 exhibit the results of the backtests over this critical period. During the crisis, the copula-based multivariate dynamic model clearly outperformed the two other strategies. Though it did breach the limits on the frequency of margin calls, it did so in a way much less drastic than the naive and Gaussian strategies, while realizing similar risk statistics.

	Naive strategy	MV Gaussian model	Copula-based MV dynamic model
Avg. daily tracking error (% collateral)	0.54	0.55	0.55
Number of margin calls (out of 1440 days)	104	94	71
Realized frequency of margin call	0.07	0.07	0.05

Table 6: Results of the backtests for the three strategies for the complete dataset period (November 2003 to March 2012) when the optimization objective is set to minimize the tracking error.

	Naive strategy	MV Gaussian model	Copula-based MV dynamic model
Realized daily VaR (% collateral)	1.19	1.21	1.24
Number of margin calls (out of 1440 days)	104	105	78
Realized frequency of margin call	0.07	0.07	0.05

Table 7: Results of the backtests for the three strategies for the complete dataset period (November 2003 to March 2012) when the optimization objective is set to minimize the Value-at-Risk.

	Naive strategy	MV Gaussian model	Copula-based MV dynamic model
Avg. daily tail loss (% collateral)	-1.83	-1.85	-1.85
Number of margin calls (out of 1440 days)	104	100	71
Realized frequency of margin call	0.07	0.07	0.05

Table 8: Results of the backtests for the three strategies for the complete dataset period (November 2003 to March 2012) when the optimization objective is set to maximize the Tail Conditional Expectation

	Naive strategy	MV Gaussian model	Copula-based MV dynamic model
Avg. daily tracking error (% collateral)	0.75	0.77	0.79
Number of margin calls (out of 220 days)	38	45	16
Realized frequency of margin call	0.17	0.20	0.07

Table 9: Results of the backtests for the three strategies for the crisis subperiod (March 2008 to February 2009) when the optimization objective is set to minimize the tracking error.

	Naive strategy	MV Gaussian model	Copula-based MV dynamic model
Realized daily VaR (% collateral)	2.09	2.05	2.11
Number of margin calls (out of 220 days)	38	46	18
Realized frequency of margin call	0.17	0.21	0.08

Table 10: Results of the backtests for the three strategies for the crisis subperiod (March 2008 to February 2009) when the optimization objective is set to minimize the Value-at-Risk.

	Naive strategy	MV Gaussian model	Copula-based MV dynamic model
Avg. daily tail loss (% collateral)	-2.81	-2.69	-2.78
Number of margin calls (out of 220 days)	38	45	18
Realized frequency of margin call	0.17	0.20	0.08

Table 11: Results of the backtests for the three strategies for the crisis subperiod (March 2008 to February 2009) when the optimization objective is set to maximize the Tail Conditional Expectation.

6 Conclusion

We met three objectives in this work; first, we proposed a model that better fits high-dimensional multivariate financial time-series than the classical Gaussian model with the copula-based multivariate dynamic model. Second, we used recent advances in absolute goodness-of-fit tests to show the appropriateness of the chosen model from a statistical point of view. Third, we used the proposed model to solve a problem encountered by managers whose portfolio contains multiple assets on different exchanges and/or in different currencies, that is, how to balance the opposing nuisances of margin calls and foreign exchange risk on collateral posted in a currency different from the participant's benchmark currency. The solution lies in calibrating the model proposed in this work (or another robust model encapsulating stochastic volatility, excess kurtosis and higher co-moments) to the financial time series of interest. Then the desired constrained optimization is solved using Monte Carlo methods.

Two potential avenues of future study are improvements in model sophistication and the development of rigorous models for problems with high numbers of random variables. On the first front, models including time-varying copula parameters, regime-switching and/or correlation asymmetry may yield a better fit to financial time series. On the second front, portfolios often have a large number of assets in many different currencies, however the use of copula becomes exponentially harder when the number of dimension is above 10. Methods for dimensionality reduction that conserve higher moments in the factors or efficiency improvements in algorithms for copula use would help toward accomplishing this objective.

A Risk measure coherence

Let $\mathcal{G}$ be the set of all risks, and ρ be a mapping of $\mathcal{G}$ into $\mathbb{R}$. ρ is said to be a coherent risk measure if it satisfies the following conditions (Artzner et al., 1999):

(i) *Monotonicity:* for all X and $Y \in \mathcal{G}$ and $X \leq Y$, $\rho(Y) \leq \rho(X)$

(ii) *Sub-additivity:* for all X and $Y \in \mathcal{G}$, $\rho(X+Y) \leq \rho(X) + \rho(Y)$

(iii) *Positive homogeneity:* for all $\lambda \in \mathbb{R}^+$ and $X \in \mathcal{G}$, $\rho(\lambda X) = \lambda \rho(X)$

(iv) *Translation invariance:* for all $X \in \mathcal{G}$ and all $\alpha \in \mathbb{R}$, $\rho(X+\alpha) = \rho(X) - \alpha$

B Parameters estimation for Student copula

Estimating the parameters of the t copula can be computationally tedious. McNeil et al. (2005) offers an efficient hybrid two-stages method. In the first stage, estimate the i^{th}-j^{th} component of the correlation matrix Σ by

$$\Sigma_{ij} = \sin\left(\frac{\pi \tau_{ij}}{2}\right)$$

where τ_{ij} is Kendall tau rank correlation coefficient between time series i and j.

If Σ is not positive-semidefinite, it can be rescaled by performing the following steps (Rousseeuw and Molenberghs, 1993):

(i) Replace negative entries in the diagonal eigenvalues matrix D of Σ by zero.

(ii) Set $\hat{\Sigma} = V \; D \; V^{\top}$ where V is a matrix with the eigenvectors of Σ as columns and D is the eigenvalues described in step (i).

(iii) Compute scaling column vector $\vec{t}$ where $\vec{t_i} = \hat{\Sigma}_{i,i}^{-\frac{1}{2}}$.

(iv) Compute scaling matrix $T = \vec{t}\,\vec{t}^{\top}$.

(v) Set $\Sigma_{i,j} = \hat{\Sigma}_{i,j} \cdot T_{i,j}$.

This method is sometimes dubbed *Spectral Decomposition* in the literature.

In the second stage, the number of degrees of freedom ν can be estimated using the MLE method by maximizing

$$L(\nu) = \sum_{t=1}^{T} \log(c_{\nu,\Sigma}(\mathbf{u}_t)) \tag{10}$$

where $c_{\nu,\Sigma}$ is the density given by equation (7) and $\mathbf{u}_t$ are the normalized ranks of the D-dimensional innovations of the t^{th} draw.

C Rosenblatt's transform

Rosenblatt (1952) introduces the transformation of a continuous D-variate distribution $C(u_1, \ldots, u_D)$, $u_i \in (0,1)$ into the uniform distribution on the D-dimensional hypercube.

Let $\mathbf{U} = (U_1, \ldots, U_D)$ be a random vector with distribution function $C(U_1 = u_1, \ldots, U_2 = u_D)$. Consider the transformation $\mathcal{T}(u_1, \ldots, u_D) = (e_1, \ldots, e_D)$ given by $e_1 = u_1$ and, for $i = 2, \ \ldots \ , D$,

$$e_i = \frac{\frac{\partial^{i-1}}{\partial u_1 \ldots \partial u_{i-1}} C(u_1, \ldots, u_i, 1, \ldots, 1)}{\frac{\partial^{i-1}}{\partial u_1 \ldots \partial u_{i-1}} C(u_1, \ldots, u_{i-1}, 1, \ldots, 1)}$$

Algorithms 1 and 2 show how to compute the Rosenblatt transforms of a sample of quantiles for the Gaussian and Student copulas respectively given the parameter(s) of the elliptical copula. $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of an univariate standard normal, $\Phi(\cdot)$ is the cumulative distribution function of an univariate standard normal, $F_\nu^{-1}(\cdot)$ is the inverse cumulative distribution function of an univariate Student distribution with mean 0, variance 1 and number of degrees of freedom ν, diag$(\cdot)$ is an operator that returns a column vector composed out of the diagonal entries of a square matrix and $F_\nu(\cdot)$ is the cumulative distribution function of an univariate Student distribution with mean 0, variance 1 and number of degrees of freedom ν.

Algorithm 1 Compute Rosenblatt transforms for Gaussian copula C_Σ

Require: quantile vectors $\mathbf{u}_1, \ldots, \mathbf{u}_D$, $D \times D$ covariance matrix Σ

$\mathbf{e}_1 \leftarrow \mathbf{u}_1$
for $i = 2$ to D **do**
  $\Sigma_{i\times i} \leftarrow i \times i$ covariance submatrix of the i^{th} first elements
  $\vec{c}_{1\times i-1} \leftarrow$ vector of covariances between elements i and $1, \ldots, i-1$
  $B \leftarrow \vec{c}_{1\times i-1} \cdot \Sigma_{i\times i}^{-1}$
  $X_{n\times i-1} \leftarrow \Phi^{-1}(\mathbf{u}_1, \ldots, \mathbf{u}_{i-1})$
  $\vec{y}_{n\times 1} \leftarrow \Phi^{-1}(\mathbf{u}_i)$
  $\Omega \leftarrow 1 - B\ \Sigma_{i\times i}\ B^\intercal$
  $\mu \leftarrow X_{n\times i-1} \times B^\intercal$
  $\mathbf{e}_i \leftarrow \Phi((\vec{y}_{n\times 1} - \mu)/\sqrt{\Omega})$
end for
return $\mathbf{e}_1, \ldots, \mathbf{e}_D$

Algorithm 2 Compute Rosenblatt transforms for Student copula $C_{\Sigma,\nu}$

Require: quantile vectors $\mathbf{u}_1, \dots, \mathbf{u}_D$, $D \times D$ covariance matrix Σ, number of degrees of freedom ν
  $\mathbf{e}_1 \leftarrow \mathbf{u}_1$
  for $i = 2$ to D **do**
    $\Sigma_{i\times i} \leftarrow i \times i$ covariance submatrix of the i^{th} first elements
    $\vec{c}_{1\times i-1} \leftarrow$ vector of covariances between elements i and $1, \dots, i-1$
    $B \leftarrow \vec{c}_{1\times i-1} \cdots \Sigma_{i\times i}^{-1}$
    $X_{n\times i-1} \leftarrow F_\nu^{-1}(\mathbf{u}_1, \dots, \mathbf{u}_{i-1})$
    $\vec{y}_{n\times 1} \leftarrow F_\nu^{-1}(\mathbf{u}_i)$
    $\Omega \leftarrow 1 - B\ \Sigma_{i\times i}\ B^\intercal$
    $\mu \leftarrow X_{n\times i-1}\ B^\intercal$
    $Z \leftarrow \text{diag}(\nu + X_{n\times i-1}\ \vec{c}_{1\times i-1}^{\,-1}\ X_{n\times i-1}^\intercal)/(\nu + i - 1)$
    $\mathbf{e}_i \leftarrow F_\nu((\vec{y}_{n\times 1} - \mu)/\sqrt{\Omega\ Z})$
  end for
  return $\mathbf{e}_1, \dots, \mathbf{e}_D$

D Multivariate Archimedean copulas random number generation

Generation of multivariate Archimedean copulas has historically been difficult, mainly because the most commonly used method, based on conditional distributions, requires differentiation of the copula's cumulative distribution function for each dimensions of the problem. Marshall and Olkin (1988) proposes an approach based on the Laplace transform that circumvents the differentiation issue. The procedure below outlines the steps needed to be followed in order to generate multivariate Archimedean copulas random variables according to the Marshall-Olkin method (Melchiori, 2006):

(i) Simulate D independent uniform variables $u_1, \dots, u_D \sim U[0,1]$

(ii) Simulate a variable Y with distribution function G such that the Laplace transform of G is the copula's generator (the distributions G for each Archimedean copulas considered is given in Table 1 while techniques producing random numbers for these distributions are proposed below)

(iii) Set $s_i = -\log(u_i)/Y$ for $i = 1, \ldots, D$

(iv) Set $X_i = \psi^{-1}(s_i)$ for $i = 1, \ldots, D$

$X_1, \ldots, X_D$ belongs to the chosen Archimedean copula.

Stable random variables $\text{Stable}(\alpha, \beta, \gamma, \delta)$ can be generated with the following recipe:

(i) Simulate an uniform variable $\Theta \sim U[-1/2, 1/2]$

(ii) Simulate an exponentially distributed variable W with mean 1 independently of Θ

(iii) Set $\theta_0 = \arctan(\beta \tan(\pi\alpha/2))/\alpha$

(iv) Compute $Y \sim \text{Stable}(\alpha, \beta, 1, 0)$:

$$Z = \begin{cases} \frac{\sin(\alpha)(\theta_0+\Theta)}{(\cos(\alpha\theta_0)\cos(\Theta))^{1/\alpha}} \left[\frac{\cos(\alpha\theta_0+(\alpha-1)\Theta)}{W}\right]^{\frac{1-\alpha}{\alpha}} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi}\left[\left(\frac{2}{\pi} + \beta\Theta\right)\tan(\Theta) - \beta\log\left(\frac{\frac{\pi}{2}W\cos(\Theta)}{\frac{\pi}{2}+\beta\Theta}\right)\right] & \text{if } \alpha = 1 \end{cases}$$

(v) Compute $Z \sim \text{Stable}(\alpha, \beta, \gamma, \delta)$:

$$Z = \begin{cases} \gamma Z + \delta & \text{if } \alpha \neq 1 \\ \gamma Z + \left(\delta + \beta\frac{2}{\pi}\gamma\log(\gamma)\right) & \text{if } \alpha = 1 \end{cases}$$

Logarithmic series-distributed random variables can be generated using Kemp's second accelerated generator (Devroye, 1986):

Algorithm 3 Kemp's 2^{nd} accelerated generator of Logarithmic Distribution

Require: parameter α

$c \leftarrow \log(1-\alpha)$

Simulate $V \sim U[0,1]$

if $V \geq \alpha$ **then**

$X \leftarrow 1$

else

Simulate $U \sim U[0,1]$

$q \leftarrow 1 - e^{cU}$

switch

case $V \leq q^2$

$X \leftarrow \text{floor}\left[1 + \frac{\log(V)}{\log(q)}\right]$

case $q^2 < V \leq q$

$X \leftarrow 1$

case $V > q$

$X \leftarrow 2$

end if

return Logarithmic series-distributed random variable X

References

Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.

Louis Bachelier. Théorie de la spéculation. *Annales Scientifiques de l'École Normale Supérieure*, 3(17):21–86, 1900.

BCBS. Basel III: A global regulatory framework for more resilient banks and banking systems. Basel Committee on Banking Supervision, 2011a.

BCBS. Basel III: International framework for liquidity risk measurement, standards and monitoring. Basel Committee on Banking Supervision, 2011b.

Tim Bollerslev. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31(3):307–327, 1986.

Tim Bollerslev, Robert F Engle, and Jeffrey M Wooldridge. A capital asset pricing model with time-varying covariances. *Journal of Political Economy*, 96(1):116–31, 1988.

Robert Brown. A brief account of microscopical observations made in the months of june, july and august, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *Edinburgh New Philosophical Journal*, pages 358–371, 1828.

John Y. Campbell, Karine Serfaty-De Medeiros, and Luis M. Viceira. Global currency hedging. *Journal of Finance*, 65(1):87–121, 2010.

Lydia J. Chan and Sunder R. Ramkumar. Tracking error rebalancing. *Journal of Portfolio Management*, 37(4):54–66, 2011.

Xiaohong Chen and Yanqin Fan. Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification. *Journal of Econometrics*, 135(1-2):125–154, 2006.

Rama Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1(2):223–236, 2001.

John Cotter. Margin exceedences for european stock index futures using extreme value theory. *Journal of Banking & Finance*, 25(8):1475–1502, 2001.

Stefano Demarta and Alexander J. McNeil. The t copula and related copulas. *International Statistical Review*, 73(1):111–129, 2005.

Luc Devroye. *Non-uniform random variate generation.* Springer-Verlag, 1986.

Brad Efron. Bootstrap methods: another look at the jackknife. *The Annals of Statistics*, 7:1–26, 1979.

Paul Embrechts, Alexander McNeil, and Daniel Straumann. Correlation and dependence in risk management: Properties and pitfalls. In *Risk management: Value at Risk and beyond*, pages 176–223. Cambridge University Press, 1999.

Robert Engle. Autoregressive conditional heteroskedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, 50(4):987–1008, 1982.

Robert F. Engle and Kenneth F. Kroner. Multivariate simultaneous generalized ARCH. *Econometric Theory*, 11(1):122–150, 1995.

Matthias Fischer, Christian Köck, Stephan Schlter, and Florian Weigert. An empirical analysis of multivariate copula models. *Quantitative Finance*, 9 (7):839–854, 2009.

Maurice Fréchet. Sur les tableaux dont les marges et des bornes sont données. *Ann. Univ. Lyon, Science*, 4:13–84, 1951.

Masaaki Fujii, Yasufumi Shimada, and Akihiko Takahashi. Collateral posting and choice of collateral currency. *CIRJE Discussion paper*, 2010.

Christian Genest, Michel Gendron, and Michaël Bourdeau-Brien. The advent of copulas in finance. *The European Journal of Finance*, 15(7-8):609–618, 2009a.

Christian Genest, Bruno Rémillard, and David Beaudoin. Goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and Economics*, 44:199–213, 2009b.

R. Gnanadesikan. *Methods of Statistical Data Analysis for Mulitivariate Observations*. John Wiley, NY, 1977.

Dominique M. Guillaume, Michel M. Dacorogna, Rakhal R. Davé, Ulrich A. Müller, Richard B. Olsen, and Olivier V. Pictet. From the birds eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets. *Finance and Stochastics*, 1:95–129, 1997.

Espen Gaarder Haug and Nassim Nicholas Taleb. Option traders use (very) sophisticated heuristics, never the black-scholes-merton formula. *Journal of Economic Behavior and Organization*, 77(2):97–106, 2011.

Andrew Higson, Shinozawad Yoshikatsu, and Mark Tippett. Organization size and the optimal investment in cash. *IMA Journal of Management Mathematics*, 21(1):27–38, 2010.

Piotr Jaworski, Fabrizio Durante, Wolfgang Karl Härdle, and Tomasz Rychlik. *Copula Theory and Its Applications: Proceedings of the Workshop Held in Warsaw, 25-26 September 2009*. Lecture Notes in Statistics. Springer, 2010.

Harry Joe. *Multivariate models and dependence concepts.* Monographs on Statistics and Applied Probability. Chapman & Hall, London, 1997.

Tzu-chuan Kao and Chu-hsiung Lin. Setting margin levels in futures markets: An extreme value method. *Nonlinear Analysis: Real World Applications*, 11(3):1704–1713, 2010.

Kin Lam, Chor-Yiu Sin Sin, and Rico Leung. A theoretical framework to evaluate different margin-setting methodologies. *Journal of Futures Markets*, 24(2):117–145, 2004.

Sébastien Laurent, Luc Bauwens, and Jeroen V. K. Rombouts. Multivariate garch models: a survey. *Journal of Applied Econometrics*, 21(1):79–109, 2006.

David X. Li. On default correlation: a copula function approach. *Journal of Fixed Income*, 9:43–54, 2001.

François Longin. Optimal margin level in futures markets: extreme price movements. *Journal of Futures Markets*, 19(2):127–152, 1999.

Benoit Mandelbrot. The variation of certain speculative prices. *The Journal of Business*, 36, 1963.

Harry Markowitz. Portfolio selection. *Journal of Finance*, 7(1):77–91, 1952.

Albert W. Marshall and Ingram Olkin. Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403):834–841, 1988.

A.J. McNeil, R. Frey, and P. Embrechts. *Quantitative risk management: concepts, techniques and tools.* Princeton series in finance. Princeton University Press, 2005.

Mario R. Melchiori. Tools for sampling multivariate archimedean copulas. Technical report, April 2006.

Robert C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.

Thomas Mikosch. Copulas: tales and facts. *Extremes*, 9(1):320, 2006.

Merton H. Miller and Daniel Orr. A model of the demand for money by firms. *The Quarterly Journal of Economics*, 80(3):413–435, 1966.

Roger B. Nelsen. *An introduction to copulas, second edition*. Springer series in statistics. Springer, New-York, 2006.

Adrian Rodney Pagan. The econometrics of financial markets. *Journal of Empirical Finance*, 3(1):15–102, 1996.

Andrew Patton and Colm Kearney. Multivariate garch modeling of exchange rate volatility transmission in the european monetary system. *The Financial Review*, 35(1):25–46, 2000.

Andrew J. Patton. Modelling asymmetric exchange rate dependence. *International Economic Review*, 47(2):527–556, 2006.

Yue Peng and Wing Ng. Analysing financial contagion and asymmetric market dependence with volatility indices via copulas. *Annals of Finance*, 8: 49–74, 2012.

Nornadiah Modh Razali and Yap Bee Wah. Power comparisons of shapiro-wilk, kolmogorov-smirnov, lilliefors and anderson-darling tests. *Journal of Statistical Modeling and Analytics*, 2(1):21–33, 2011.

Bruno Rémillard. Goodness-of-fit tests for copulas of multivariate time series. Technical report, SSRN Working Paper Series No. 1729982, 2010.

Bruno Rémillard. Validity of the parametric bootstrap for goodness-of-fit testing in dynamic models. Technical report, SSRN Working Paper Series No. 1966476, 2011.

Murray Rosenblatt. Remarks on a multivariate transformation. *The Annals of Mathematical Statistics*, 23(3):470–472, 1952.

Peter J. Rousseeuw and Geert Molenberghs. Transformation of non positive semidefinite correlation matrices. *Communications in Statistics - Theory and Methods*, 22(4):965–984, 1993.

Arthur D Roy. Safety first and the holding of assets. *Econometrica*, 20(3): 431–439, 1952.

Annastiina Silvennoinen and Timo Teräsvirta. Multivariate garch models. CREATES Research Papers 2008-06, January 2008.

Abe Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959.

Abe Sklar. Random variables, distribution functions, and copulas - a personal look backward and forward. *IMS Lecture Notes - Monograph Series*, 28: 1–14, 1996.

Casper G. de Vries and K.U. Leuven. Stylized facts of nominal exchange rate returns. Working papers, 1994.