Hidden Limit Orders and Liquidity in Limit Order Markets.[1]

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Abstract

Many limit order markets allow limit order traders to submit "hidden" orders (also called "iceberg" or "undisclosed" orders). Liquidity suppliers thus have the possibility to post a quote and either display none or only a fraction of their order's quantity to the market. Some recent empirical studies show that such orders represent a large proportion of the market liquidity. These orders are indeed supposed to limit the liquidity suppliers' risk of adverse selection and information disclosure. However, these orders could also be used by informed agents to trade a large volume without disseminating their private information. In this paper, we propose a theoretical sequential and discrete model of trading in a limit order book to investigate the impact of hidden orders on market performance and agents' welfare, when a limit order trader possesses a private information on the realization of the value of the security with some probability. We show that submitting hidden orders may indeed be part of the informed agent's equilibrium camouflage strategy. However, counter-intuitively, the informed agent may be better off in the transparent market. The move to an opaque market indeed increases the total depth at the best quotes in the limit order book, but decreases the visible depth. Thus it becomes more difficult, for the informed agent, to get a large transaction volume in this market while replicating the behavior of an uninformed limit order trader. We show that this effect can counterbalance what she earns by submitting hidden orders. Conversely, the Bayesian and strategic uninformed market order trader beneficiates from a larger liquidity supply in the opaque market.

Keywords: Limit Order Book, Transparency, Hidden orders, Informed limit order trading.

1 Introduction

The recent development of electronic trading systems[1] has drawn attention to "order driven markets". In those markets, liquidity is provided by investors submitting limit orders[2]. These orders queue in the limit order book, and are then matched and executed against market orders submitted by other investors, who act as liquidity demanders. Regulators of those markets have adopted heterogenous rules regarding the disclosure of information concerning the full limit order book. In order to facilitate liquidity demand, most of these systems provide to their members some information on the characteristics of at least the five best limits on each side of the limit order book. However, some markets allow agents to submit "hidden" limit orders (also called "iceberg" or "undisclosed" orders). Liquidity suppliers thus have the possibility to post a quote and either display only a fraction of their order to the market (as in Euronext for instance), or fully hide the quantity of their limit order and display the price only (as in the Australian Stock Exchange). This feature introduces some opacity in the limit order book's information disclosure, since market participants in those markets are unable to observe the total depth supplied at the best quotes of the limit order book.

Recent empirical studies have highlighted the importance of hidden depth of limit order markets. Since researchers have had access to proprietary data from exchanges that have enabled them to reconstitute the full limit order book, including its hidden part, they have indeed shown that hidden orders represent a large proportion of the market liquidity. Hasbrouck and Saar [2002] report that hidden orders account for 12% of all order executions in Island. On the Nasdaq, Tuttle [2002] finds that hidden liquidity represents 20% of the inside depth in Nasdaq 100 stocks. Even more strikingly, in Euronext Paris, the hidden depth accounts for 45% of the total depth available at the best five quotes, and 55% of the total depth at the best limit according to D'Hondt, De Winne and François-Heude

[1] For instance Euronext, or ECNs such as Inet ATS (formally Island) or Instinet.

[2] Those markets differ from the more traditional "quote-driven markets" (like the NYSE or the LSE) where dealers first set their quotes, and then investors (brokers) willing to trade the security submit an order at the best price.

[2004]. These findings have drawn attention to the question of knowing who gains and looses from the authorization of hidden orders. Providing an answer to this question would help institutions and regulators in designing the optimal limit order market organization. However, there exists no theoretical research on hidden orders, and our paper tries to fill this gap.

There are two different arguments justifying the usage of hidden orders[3]. First, hidden orders would reduce the exposure of liquidity suppliers to the adverse selection risk. Aitken et al. [2001] indeed suggest that this risk would be less severe for undisclosed than for disclosed limit orders, "since picking off is more complicated than simple hitting the stale limit order for a known total number of disclosed shares". Their empirical findings, as those of Pardo and Pascual [2003], provide support to this economic intuition. Second, Harris [1998] argues that hidden orders may be used by large traders as a defensive strategy against quote-matchers, by limiting the price impact of their order, and thus their execution cost. By providing to limit order traders a trade-off between liquidity and transparency, that would allow them to reduce their risk exposure, hidden orders would thus encourage liquidity provision.

However, this economic rationale does not take into account the fact that more opacity in the information content of the limit order book may not serve only the interests of a liquidity trader. Recently, Kaniel and Liu [2001] have indeed shown that an informed agent ("she") could profitably submit limit orders, which is also corroborated by the empirical findings of Aitken et al. [2001] and Amand and Martell [2001]. On the one hand, this informed agent is induced to submit large limit orders in order to increase her expected trading profit. But on the other hand, if limit orders are fully disclosed to market participants, such a behavior could easily reveal her presence and the direction of her information to the market, thus, her limit order would be less likely to get executed. Submitting a large hidden order could thus provide to an informed agent the opportunity to overcome this trade-off between transaction volume and probability of execution. In this case, does the authorization of hidden orders still improve the liquidity of limit order markets? To provide an answer to this question,

[3]See Pardo and Pascual [2003] or D'Hondt, De Winne and François-Heude [2004] for a complete description of the rationale of hidden orders.

we propose a theoretical sequential and discrete model of trading in a limit order book , where a limit order trader possesses a private information on the realization of the value of the security.

The informed limit order trader's expected transaction volume depends on the uninformed market order trader's reaction ("he"), conditional on the limit order book he observes. When he observes the depth at the best quotes, he updates his beliefs on the value of the security by taking into account two elements. First, the direction of the limit order he observes is informative. A sell order signals him that the security is likely to be overvalued. The impact of this signal on the uninformed agent's beliefs depends on the probability to face an informed limit order trader, π. Second, the size of the limit order allows him to better assess his adverse selection risk, since he takes into account the fact that an informed agent would ideally be more willing to submit a large limit order to increase her expected profits. Here is where the informed agent's limit order submission strategy plays a crucial role. We show that for extreme levels of the adverse selection parameter π, the uninformed agent's decision does not rely much on the observation of the second signal on market depth. Whatever the informed agent's equilibrium strategy, the adverse selection risk faced by the market order trader is indeed either too low to prevent him from submitting a large order, or too large to induce him to submit a market order. Consequently, there is no rationale, for the informed agent, to submit an hidden order. In the opposite case, when the level of adverse selection is intermediate, the uninformed agent takes into account the information revealed by the depth he observes at the best quotes. In our model, the informed limit order trader has then a camouflage's equilibrium strategy: she tries to mimic the uninformed limit order trader's behavior in order not to be detected. As in the static model of Kyle [1985], her optimal strategy is indeed to replicate the trading behavior of a noise trader, since revealing her presence would deter the uninformed agent from submitting a market order. We show that in some cases, the submission of an hidden limit order allows the informed agent to get a large transaction volume, without signalling her presence.

Finally, we derive some comparative statics. An increase in the level of pre-trade transparency in the limit order book has two opposite effects on the depth supplied by the uninformed liquidity trader at the best quotes. On the one hand, the visible depth at the best quotes increases in the transparent market. Consequently, the informed agent can more easily submit large unhidden orders without being detected, in this market. On the other hand, the total depth decreases. Thus, the informed agent's camouflage's strategy becomes more difficult to implement, all the more as she cannot submit an hidden order to moderate the impact of a large limit order on the uninformed agent's beliefs. Both effects interact so that, counter-intuitively, the informed agent may be better off in the transparent market. Besides, our theoretical conclusions are in line with some of the results of recent empirical papers on hidden orders, and enable us to derive some new testable implications.

The paper is organized as follows. Section 2 introduces our theoretical model. In Section 3, we solve for the equilibrium's strategies. Section 4 finally derives some comparative statics, which are used to reconcile our theoretical findings with the empirical literature, and to draw some new empirical predictions of our model. Section 5 concludes. The proofs that do not appear in the text are collected in the appendix.

2 The model

In this section, we introduce a simple model of trading, in a market for a single risky security $\tilde{v}$.

2.1 Timing and market structure

Risky asset - The liquidation value of the security is a random variable $\tilde{v}$. The realization of the random variable becomes publicly known at the end of the trading session. For simplicity, we assume that its final value can take two values with equal probability $\frac{1}{2}$, (v_H, v_L) with $v_H = v_0 + \sigma$ and $v_L = v_0 - \sigma$,.where v_0 denotes the unconditional expected value of the asset, i.e. $v_0 = E(\tilde{v})$. The distribution of the liquidation value of the security is common knowledge to all market participants.

Timing - Trading occurs in a sequential game with three periods. At date 0, limit orders to buy or to sell the asset stand in the initial limit order book, waiting for execution. At date 1, one agent submits a limit order to buy or to sell a quantity of one or two units of the risky asset. At date 2, after having observed the depth at the best quotes in the order book, one agent may submit a market order to buy or to sell one or two units of the security, that is immediately executed against the opposite order with the first best limit price standing in the limit order book. If the size of the market order exceeds the total depth supplied in the book at the best price, the market order is first executed against the whole quantity available in the book at the best price and then walks down (or up) the book at the second best price, up to full execution (the case of marketable limit orders is discussed below). Price then time priority are enforced. At date 3, the game ends, and the value of the security is revealed.

Price grid - Limit orders of liquidity suppliers are submitted on a price grid $\{p_k, k \in [|-n, n|]\}$. We assume that the minimum tick size between two consecutive prices, is Δ, so that $p_{k+1} = p_k + \Delta$. For simplicity, we assume that $p_0 = v_0$ and that $\Delta < \sigma$. In this case, orders submitted at prices p_k for $1 \leq k \leq n$ (resp. $-n \leq k \leq -1$) represent sell (resp. buy) orders. Therefore, to avoid confusion, we define p_k for $k \geq 1$ (resp. $k \leq -1$) as an ask price A_k (resp. a bid price B_k), and we note $A_k = v_0 + k\Delta$ (resp. $B_k = v_0 - k\Delta$). Because of the symmetry of the model, we concentrate our analysis on the ask side.

Initial limit order book - To understand how limit order traders choose their order size and the quantity they display to the market, we fix the prices, and we focus on the quantity submitted and displayed at these prices. We do so by assuming that at date 0, the book is filled in with a sell order at price A_2. This ask price A_2 can be viewed as the minimum price at which investors currently in the market are willing to sell two units of the security, given their (unknown) characteristics and their expectations on the final value of the security. Because of the priority rules defined above, for a limit order trader entering the market at date 1, only the limit orders undercutting the current quotes by (at least) one tick have a non-zero probability of being executed.

Market transparency - We shall distinguish two different trading systems: (i) the transparent limit order market and (ii) the opaque limit order market. In the transparent limit order market, agents are able to observe the total depth supplied at the best quotes in the limit order book. In the opaque limit order market however, liquidity suppliers have the opportunity to hide a part of the total quantity of their limit order. Thus, before submitting market orders, agents are only able to observe the visible depth at the best quotes.

Figure 1 depicts the timing of the game and of the limit order book information disclosure.

2.2 Agents and information structure

We assume that agents differ in their motive for trading and in their degree of impatience.

Limit order traders - At date 1, one agent submits a limit order. This agent can either be willing to trade to speculate on her private information, or for liquidity reasons.

1) A non-strategic uninformed agent (with probability $1-\pi$)

With probability $1-\pi$, an agent ("it") is randomly selected to submit a limit order. This agent acquires no information about the realized value of the security, but trades for liquidity motives. With an equal probability $\frac{1}{2}$, it is a net seller or a net buyer, and we assume that it submits a limit order undercutting the current quotes by one tick[4]. Let D^t be the "total depth", i.e. the quantity submitted at price A_1 at date 1, D^v the "visible depth", i.e. the quantity displayed to the market and *observed* by market participants, and D^h the "hidden depth", i.e. the quantity *undisclosed* to market participants. Notice that by definition, $D^t = D^v + D^h$. The uninformed agent may either submit a large (unhidden) order of size 2 units, so that $D^t = D^v = 2$ and $D^h = 0$ (with probability l), or a small (unhidden) order of size 1 unit, such that $D^t = D^v = 1$ and $D^h = 0$ (with probability s). In the opaque market, it may also submit a large order $D^t = 2$, but hide a fraction of it and only display one unit to the market, so that $D^v = 1$ and $D^h = 1$ (with probability h).

[4] Allowing agents to undercut the current spread by more than one tick would not change the nature of the results, so for simplicity, we focus on this simple case.

We assume that this agent is non-strategic[5], and that its trading strategy is exogenous.

2) An informed agent (with probability π)

We assume that with probability π, there is an "information acquisition" at that date, so that one agent ("she") obtains private information about the realized value of the security before trading occurs. For clarity, let us assume that she submits a limit order undercutting the current spread by one tick[6]. Let λ be the probability at which she submits a large order, so that $D^t = D^v = 2$ and $D^h = 0$, ξ the probability at which she submits a small order, such that $D^t = D^v = 1$ and $D^h = 0$, and χ the probability at which she submits an hidden order in the opaque market, such that $D^t = 2$, but $D^v = 1$ and $D^h = 1$. Since we decided to focus on the ask side, we assume that this agent receives the signal $\tilde{v} = v_L$. Let TS be the informed limit order's trade size. This trade size is zero if her order is not executed, one if she submits a small limit order which is fully executed, or if she submits a large order which is only partially executed, and two if she submits a large order, hidden or fully disclosed, that is fully executed in the limit order book. The informed agent wants to maximize her expected profit at date 1, which is as follows:

$$
\begin{aligned}
E\Pi_i(\lambda,\chi,\xi) &= \xi\,(A_1 - v_L)\Pr(TS=1|D^v=1) \\
&\quad + \lambda\,(A_1 - v_L)\left(\Pr(TS=1|D^v=2) + 2\Pr(TS=2|D^v=2)\right) \\
&\quad + \chi\,(A_1 - v_L)\left(\Pr(TS=1|D^v=1) + 2\Pr(TS=2|D^v=1)\right)
\end{aligned}
$$

The expected trade size, which depends on market order trading in the limit order book, is described below.

Table 1 summarizes the probabilities to face any of the three types of limit orders

[5] As we have seen in the Introduction, an uninformed agent could use hidden orders to decrease its adverse selection risk, or to avoid quote-matching. Since these motives are completely different from those of the informed agent, for clarity we do not model such a complex game of trading, but we assume that its strategies are exogenous.

[6] Recall that due to time priority, a limit order at price A_2 would have a zero-probability to get executed. Besides, we show below in Section 2.4 that she has no incentives to submit a limit order at a lower price A_0.

described above, depending on the type of the limit order trader.

TABLE 1

Strategy	**Limit order's size**			**Probabilities**	
	submitted	**hidden**	**visible**	**Informed** (π)	**Uninformed** $(1-\pi)$
Small limit order	$D^t = 1$	$D^h = 0$	$D^v = 1$	ξ	s
Large hidden order	$D^t = 2$	$D^v = 1$	$D^v = 1$	χ	h
Large unhidden order	$D^t = 2$	$D^h = 0$	$D^v = 2$	λ	l

Consequently, observing a large limit order, such that $D^v = 2$, perfectly reveals the presence of a large limit order in the limit order book, i.e. that $D^t = 2$. Observing a small limit order $D^v = 1$ however is a noisy signal of the total depth available at price A_1, since in this case, there may be hidden depth at this quote, i.e. $D^h = 1$, or not, i.e. $D^h = 0$. Given the probabilities given in Table 1, the probability to have hidden depth at the best quote when there is one unit visible at price A_1 is as follows:

$$\Pr\left(D^t = 2 | D^v = 1\right) = 1 - \Pr\left(D^t = 1 | D^v = 1\right) = \frac{\pi\chi + (1-\pi)\, h}{\pi\,(\chi + \xi) + (1-\pi)\,(h+s)}$$

This probability can be interpreted in terms of correlation between the total depth and the visible depth.

One strategic market order trader - At date 2, one strategic agent ("he") potentially submits a market order. As for the uninformed limit order trader, we assume that this agent is uninformed and trades for liquidity reasons[7], and is willing to buy or sell the asset with equal probabilities. We assume that he is ready to pay higher transaction costs in order to get his order executed, therefore, he only trades using market orders at date 2. To represent this impatience, we assume that he has a marginal private value from trading $\beta \in]\Delta, \sigma[$, which is strictly positive in the case of a buyer, that may either be due to an inventory cost or to a difference in the valuation of the security. A buyer can submit a market order $M \in \{0, 1, 2\}$ to buy the security at the best ask quote. If he does not trade, his expected

[7] We could have introduced an informed market order trader in our model. However, the presence of such an agent would only impact the expected profits of the informed limit order trader, but not her strategy, since she cannot trade against another informed agent when both know that the value of the security is low.

profit is zero. Observing a limit order of size D^v at price A_1, the expected profit of a buyer submitting a market order $M \in \{1,2\}$ can then be written:

$$E\Pi_u\left(M|D^v\right) = \sum_{m=1}^{M}\left(E\left(\tilde{v} + \beta - A\left(m\right)|D^v\right)\right)$$

where $A(m)$ represents the ask price paid for the m^{th} unit exchanged.

Unlike the uninformed limit order trader, we assume that this impatient uninformed agent is rational Bayesian. His market order submission strategy crucially relies on the observation of the depth displayed at the best quotes in the order book at the end of date 1. The limit order book is indeed not only informative on the value of the security, but also on his expected transaction price. For instance, if he observes a large limit order standing in the book at price A_1, i.e. $D^v = 2$, the market order trader expects to pay this price A_1 whatever his order size. Still, observing this large limit order is informative on the value of the security, due to the presence of an informed liquidity supplier. His expected profit is thus as follows:

$$E\Pi_u\left(M|D^v = 2\right) = \left(E\left(\tilde{v}|D^v = 2\right) + \beta - A_1\right) \times M$$

When the market order trader observes a small limit order at price A_1 however, i.e. $D^v = 1$, he must not only revise his beliefs on the value of the security due to adverse selection, but also form beliefs relative to the price he is going to pay for the second marginal unit of his order. If he submits a small market order, he easily evaluate his trading costs.

$$E\Pi_u\left(M = 1|D^v = 1\right) = E\left(\tilde{v}|D^v = 1\right) + \beta - A_1$$

If he submits a large market order, his expected profit is as follows:

$$E\Pi_u\left(M = 2|D^v = 1\right) = 2E\left(\tilde{v}|D^v = 1\right) + 2\beta - A_1 - E\left(A\left(2\right)|D^v = 1\right)$$

The market order trader pays $A(1) = A_1$ for the first unit of his large order, since the visible depth is $D^v = 1$. The price he will pay for his second marginal unit, $A(2)$, however depends on whether there is hidden depth at the best quotes, or not. If the limit order he observes is actually an hidden order (i.e. if $D^v = 1$ and $D^h = 1$), then the second marginal unit of his order will be executed at price A_1, otherwise (i.e. if $D^v = 1$ and $D^h = 0$), it will be executed at price A_2. Consequently, he must also form beliefs on his expected trading price.

We note $\mu_M(D^v)$ the probability at which he submits an order of size M when he observes the depth D^v. Agents' decision planning on the ask side of the limit order book is depicted in Figure 2.

2.3 Equilibrium's definition

Our model actually represents a simple signaling game between an agent whose type is "informed" with an ex ante probability π, moving first, and an uninformed market order trader, moving second. In this dynamic game of incomplete information, we look for a perfect Bayesian equilibrium. At this equilibrium, both the informed agent and the uninformed market order trader are indifferent between all the strategies they play with a strictly positive probability[8].

Definition 1 *An equilibrium in the opaque market is defined by (i) a limit order submission strategy for the informed trader* (λ, χ, ξ)*; (ii-a) an expectation on the value of the risky asset* $E(\tilde{v}|D^v)$ *and (ii-b) a market order submission strategy* $(\mu_0(D^v), \mu_1(D^v), \mu_2(D^v))$ *for the uninformed agent, both conditional on the depth* $D^v \in \{-2, -1\}$ *he observes; such that whatever* $D^v \in \{-1, -2\}$*:*

a. $E(\tilde{v}|D^v)$ *follows from Bayes' rule and the informed trader's strategy* (λ, χ, ξ)*; b.* $(\mu_0(D^v), \mu_1(D^v), \mu_2(D^v))$ *maximizes the uninformed agent's expected profits, given his expectation* $E(\tilde{v}|D^v)$*; c. Given the uninformed agent's strategy,* (λ, χ, ξ) *maximizes the informed agent's expected profits.*

Notice that, for clarity of the presentation, we replaced the beliefs of the second mover on the probability to face an agent whose type is "informed" by his expectations on the

[8]For clarity, we will not present, in our Propositions, the equilibrium's beliefs of the uninformed market order trader.

value of the asset in our definition of the equilibrium[9]. An equilibrium in the transparent market is defined similarly. The difference lies in the fact that in this case $h = \chi = 0$, i.e. the strategies' space of the liquidity suppliers is restricted.

2.4 Parametrization of the model

Informed agent's strategies' space - We first show that submitting a sell limit order at a lower price than A_1 is under-optimal for the informed agent. Assume that she submits an order for D_0 shares at price A_0. Since the uninformed limit order trader never submits an order at this price, the informed agent reveals her presence, thus the uninformed market order trader perfectly infers that the value of the security is low, i.e. $E(\tilde{v}|D_0) = v_0 - \sigma$. Consequently, his expected profits from buying M units of the asset at this price are as follows:

$$E\Pi_u (M|D_0 \text{ units at } A_0) = (\beta - \sigma) \times M$$

Since we assumed that $\beta < \sigma$, the uninformed agent's optimal strategy is to submit no market order, so that the informed agent's expected profits corresponding to such a strategy would be zero.

Benchmark model with no information - We now present a benchmark model of trading in an opaque limit order market, when there is no asymmetric information, i.e. $\pi = 0$. Observing the size of the limit order displayed at price A_1, D^v, is not informative on the value of the security, because there is no adverse selection. Consequently, when he observes a large limit order, the agent's expected profits from trading M units of the security are simply:

$$E\Pi_u (M|D^v = 2) = (v_0 + \beta - A_1) \times M$$

For clarity, let us define:

$$\gamma \equiv v_0 + \beta - A_1 = \beta - \Delta$$

We refer to this parameter γ as to his "gains from trade" at price A_1, since the market

[9] We are allowed to do so because there is a one-to-one relation between his beliefs and his expectations.

order trader obtains a surplus γ for each unit traded at this price. If this parameter was negative, he would never submit a market order, whatever the size of the depth available at price A_1, since $E\Pi_u(M|D^v = 2) = \gamma \times M$. We therefore assumed that $\beta \geq \Delta$, so that $\gamma \geq 0$.

However, when $D^v = 1$, the uninformed market order trader's expected cost of trading is random, since there may be, or not, hidden depth at the best quote. Thus, the price the agent expects to pay for his second unit is A_1 if the limit order is actually an hidden order (i.e. if $D^t = 2$), and $A_1 + \Delta$ otherwise (i.e. if $D^t = 1$). His expected profit from submitting a large market order, while observing a limit order of size $D^v = 1$, is thus:

$$E\Pi_u(M = 2|D^v = 1) = E\Pi_u(M = 1|D^v = 1) + \gamma - \frac{s}{s+h}\Delta$$

The impact of the trade-off that the uninformed agent faces, between earning gains from trade and paying an extra cost Δ for the second unit traded in case his market order walks up the book, on his reaction, is described in Lemma 1.

Lemma 1 *When there is no asymmetric information ($\pi = 0$), the optimal strategy of the uninformed agent at date 2 when he observes a small limit order $D^v = 1$ in the limit order book is to submit a large market order if $\gamma > \left(\frac{s}{s+h}\right)\Delta$ and a small market order if $\gamma \leq \left(\frac{s}{s+h}\right)\Delta$.*

The proof is straightforward, comparing the expected profits of the uninformed agent for each limit order observed $D^v \in \{-1, -2\}$, and each market order submission $M \in \{0, 1, 2\}$. If the condition $\gamma \leq \left(\frac{s}{s+h}\right)\Delta$ holds, then even when there is no adverse selection, the uninformed agent never submits a larger market order than the quantity he observes at price A_1. His gains from trade are indeed too low in this case to cover the extra-cost of Δ he needs to pay for his second marginal unit, if the limit order is unhidden. At the equilibrium, there is consequently no rationale, for the informed agent, for submitting an hidden order rather than a small limit order, since both types of order provide the same signal to market participants, and since the second unit of her limit order would never be executed. We therefore restrict our attention to the case where $\gamma > \left(\frac{s}{s+h}\right)\Delta$.

Benchmark model with full information - To further parametrize our model, let us finally turn to the opposite case, where there is always information acquisition at date 0, i.e. $\pi = 1$. Then the direction of a limit order, namely a sell order, perfectly reveals to the uninformed agent that the value of the security is low, $\tilde{v} = v_0 - \sigma$. The visible depth thus only impacts his expected transaction price. When there is no uncertainty on his trading cost, i.e. when he observes $D^v = 2$ or when he observes $D^v = 1$ but submits a small market order $M = 1$, then the uninformed agent's expected profit is $(\gamma - \sigma)$ per unit traded. Consequently, if his gains from trade, γ, are sufficiently high, namely $\gamma \geq \sigma$, then though adverse selection, he is willing to trade at least one unit, two in the case where $D^v = 2$, even out of the equilibrium path. Thus, in equilibrium, the informed agent would have no incentives to submit an hidden order rather than a large limit order. On the one hand, both types of limit orders have the same information content on the value of the security. On the other hand, the probability to get two units executed is lower for hidden orders, since the uninformed agent may be reluctant to submit a large market order when he observes $D^v = 1$, by fear of paying an extra cost of Δ for his second unit. Therefore, we restrict our attention to the case where $\gamma < \sigma$, so that the uninformed agent is not willing to trade at any cost.

Assuming that $\gamma \in]\frac{s}{s+h}\Delta, \sigma - \Delta[$ creates a trade-off for the informed limit order trader. On the one hand, she wants to submit large limit orders, in order to increase her trade size. But on the other hand, she cares about the impact of her strategy on the uninformed market order trader's beliefs, since these beliefs influence her probability of execution. Submitting an hidden order could enable her to submit a large order, without signalling her presence to the uninformed market order trader.

2.5 Discussion: Marketable limit orders

In the model described above, for simplicity, we assumed that the uninformed agent could only submit market orders. It could however be argued that when this agent has the possibility to submit a marketable limit order, then he would rather use such a strategy than submit a large market order, in order to avoid to bear uncertainty on his trading price.

The difference between a large market order and a large marketable limit order is inexistent when the total depth is large, i.e. when $D^t = 2$. When the total depth at price A_1 is only one unit however, i.e. when $D^h = 0$, the large market order gets fully executed, but it walks up the book, so that its second unit is executed at price A_2. The marketable buy limit order (MLO) however gets only partially executed at the best standing quote, i.e. A_1, then its remaining part becomes a buy limit order, with a bid price which is equal to A_1. There exists uncertainty in the execution conditions of both types of orders. On the one hand, with a market order, the trader can be certain to get his order fully executed, but not about his trade price, so that his expected profit, as seen above, is as follows:

$$E\Pi_u\left(M = 2|D^v = 1\right) = 2E\left(\tilde{v}|D^v = 1\right) + 2\beta - A_1 - E\left(A\left(2\right)|D^v = 1\right)$$

On the other hand, with a marketable limit order, he can be sure about his trade price, but not about his trade size, so that his expected profit becomes:

$$E\Pi_u\left(M = MLO|D^v = 1\right) = \left(E\left(\tilde{v}|D^v = 1\right) + \beta - A_1\right) E\left(D^t|D^v = 1\right)$$

Assume that there is no adverse selection. Then if $\gamma \geq \Delta$, submitting large market order when $D^v = 1$ stays a strictly dominant strategy for the uninformed agent, since:

$$E\Pi_u\left(M = MLO|D^v = 1\right) = E\Pi_u\left(M = 2|D^v = 1\right) + \left(\Delta - \gamma + v_0 - E\left(\tilde{v}|D^v = 1\right)\right) \times \Pr\left(D^t = 1|D^v = 1\right)$$

In this case indeed, the agent's gains from trade exceed the potential extra-cost he incurs when there is no hidden depth at price A_1. In the opposite case however, if $\gamma < \Delta$, then submitting a marketable limit order becomes a strictly dominant strategy, for the uninformed agent, so that there is no need, for liquidity suppliers, to submit hidden limit orders. If in the presence of an informed limit order trader, this type of orders is still optimal for the uninformed trader, then her incentives to submit an hidden order may also vanish. We address this point here by expanding the market order trader's strategies space to allow him to submit a marketable limit order to buy or sell two units of the security.

We claim that excluding a marketable limit order submission strategy for the uninformed trader is a conservative approach in our model[10]. First, if $\gamma \geq \Delta$, the agent's gains from

[10] A complete proof of this claim is available upon request.

trade are so high that in the absence of adverse selection, he would be willing to trade at price A_2. Thus, he may submit a large market order when he observes $D^v = 1$, rather than a marketable limit order, in order be sure to get his order fully executed. We show that this is indeed the case *in equilibrium*. Consequently, the equilibria described below are robust to the introduction of marketable limit orders when $\gamma \geq \Delta$, since this strategy is dominated in equilibrium.

In the opposite case, if $\gamma < \Delta$, then submitting a marketable limit order dominates submitting a large market order when $D^v = 1$, even out of the equilibrium path. The opportunity to submit a marketable limit order enables the trader to avoid getting his second unit executed at a price at which he would certainly bear a loss. Whether the uninformed agent submits a large market order rather than a marketable limit order, or reciprocally, does however not impact the informed agent's expected profits from trading when she submits an hidden order. In any of both cases, she is indeed ensured to get her hidden limit order fully executed. However, since submitting a marketable limit order is less costly, for the uninformed agent, than submitting a large market order, he is ready to bear a higher adverse selection risk. Therefore, his total order size is two units when he observes $D^v = 1$ for higher levels of adverse selection than in the absence of marketable limit orders. This in turn increases the informed agent's incentives to submit hidden orders, since those become more likely to get executed.

3 Equilibrium's strategies and expected profits

In this section, we solve for the equilibrium of the sequential trading game in the opaque market . We first compute the impatient uninformed agent's beliefs and reaction at date 2 for each possible observation of the depth at price A_1 in the limit order book. Given his reaction, we then find the informed agent's optimal limit order submission strategy at date 1.

3.1 Beliefs of the uninformed agent at date 2

At date $t = 2$, the behavior of the uninformed impatient agent depends on his expectations on the value of the asset, that he updates after observing the limit order displayed in the limit order book at price A_1 at date 1. At this stage, as illustrated in Table 1 above, there are three possible states of the best quotes in the limit order book. The uninformed agent therefore updates his beliefs, conditional on the limit order observed, D^v. The parameter π measures the degree of adverse selection faced by an uninformed agent at date 2. Following Bayes' rule:

$$
\begin{aligned}
E\left(\tilde{v}|D^v=1\right) &= v_0 - \sigma \frac{\pi\left(\xi+\chi\right)}{\pi\left(\xi+\chi\right)+\left(1-\pi\right)\left(s+h\right)} \\
E\left(\tilde{v}|D^v=2\right) &= v_0 - \sigma \frac{\pi\lambda}{\pi\lambda+\left(1-\pi\right)l}
\end{aligned}
$$

Observing the limit order book allows the impatient uninformed agent to update his beliefs on the true value of the security, but only in case the probability that the limit order has been submitted by an informed agent, π, is strictly positive. If there is a sell limit order standing in the book at price A_1, and if the informed agent may have submitted it, then it reveals to the uninformed agent that the realized value of the security is more likely to be low, so that the expected value of the asset conditional on observing a sell limit order is inferior to its unconditional value. The magnitude of the beliefs' update increases with π and with the asset volatility, σ.

The informed agent's strategy (λ, χ, ξ), as compared to the patient uninformed agent behavior (l, h, s), influences the uninformed agent's beliefs. Suppose for instance that she always submits small orders, i.e. $\xi = 1$. Then, for the uninformed agent, observing a large order $D^v = 2$ is not informative on the value of the security. Suppose in an opposite case that the patient uninformed agent never submits large orders, i.e. $l = 0$. Then observing a large order $D^v = 2$ perfectly reveals the presence of the informed agent, and her private information that the value of the security is low. The conditional expectation of the value of the security is indeed in this case $E\left(\tilde{v}|D^v=2\right) = v_0 - \sigma = v_L$. The magnitude of the beliefs' update therefore also relies on the informed agent's strategy and on the trading noise created by the patient uninformed agent's behavior. We will show in Section 3 that in equilibrium, the magnitude of the beliefs' update is higher when the uninformed agent observes a large order, $D^v = 2$, than when he observes a small order, $D^v = 1$.

3.2 Reaction of the uninformed agent at date 2

Given the beliefs' update described above, we study the optimal reaction of an uninformed impatient agent at date 2, which is conditional on the size of the limit order standing in the book at price A_1 he observes, for two reasons. First, this order is informative on the price he is going to pay for each unit of his order. Second, the limit order book at the best quotes contains information on the value of the security.

Reaction of an uninformed buyer at date 2 when he observes $D^v = 2$

In the trade-off he faces between his gains from trade, and his implicit costs of trading that are due to adverse selection, observing a large order allows the uninformed agent to be sure that he pays a relatively low price $A(1) = A(2) = A_1$ for each unit traded, which induces him to trade. But he also updates his beliefs on the value of the security. Because of adverse selection, he may not be ready to trade if he fears to face an informed limit order trader too frequently, while observing a large limit order. His optimal reaction is described in Lemma 2.

Lemma 2 *The uninformed market order trader's strictly dominant strategy is to submit a large market order $M = 2$ if $\lambda < \frac{\gamma}{(\sigma-\gamma)}\frac{(1-\pi)}{\pi}l$. If $\lambda > \frac{\gamma}{(\sigma-\gamma)}\frac{(1-\pi)}{\pi}l$, his strictly dominant strategy is to submit no market order $M = 0$. Finally, if $\lambda = \frac{\gamma}{(\sigma-\gamma)}\frac{(1-\pi)}{\pi}l$, he is indifferent between any size of market order, $M \in \{0, 1, 2\}$.*

For simplicity, we will assume that when the uninformed agent is indifferent between different trading strategies, he chooses to submit a large market order.

The reaction of the impatient uninformed agent at date 2 depends on the probability to face an informed limit order, given the book, thus on the strategy of large order submission of the informed agent at date 1, λ. A threshold value for this strategy naturally emerges. If the informed agent submits a large limit order with a small probability, then the uninformed agent submits a large market order when he observes a book filled in with two units at the best ask price because he has few risks to face an agent informed that the value of the security is low, and vice-versa for a large λ. The cutoff value is obtained for $\pi\lambda = \frac{\gamma}{(\sigma-\gamma)}(1-\pi)\,l$, which shows that the uninformed agent's reaction relies on two elements, adverse selection and gains from trade. The adverse selection component appears as he compares the probability that an informed agent submits a large limit order, $\pi\lambda$, with the probability to observe a large uninformed limit order, $(1-\pi)\,l$. When the probability that the patient uninformed agent submits a large unhidden order, l, increases, or when π decreases, the adverse selection risk faced by the market order trader decreases, therefore, he is induced to submit larger orders. At the same time, the market order trader takes into account his private benefit from trading, in the component $\frac{\gamma}{\sigma-\gamma}$, that is increasing in his gains from trade at price A_1, namely γ. The larger this parameter, the more the uninformed agent is likely to submit a large market order.

Reaction of an uninformed buyer at date 2 when he observes $D^v = 1$

Similarly, the uninformed agent's reaction when he observes a small limit order at the best ask in the limit order book is a trade-off between his desire to trade and his risk of adverse selection. This trade-off is however more complex than in the case where he observes a large limit order, since he is likely to face an hidden order. This possibility may induce him to submit a larger market order than the limit order he currently observes. By submitting a large market order when $D^v = 1$, he earns γ, which corresponds to his gains from trade. But he faces two different risks, which are related respectively to his expected trading price relative to A_1, i.e. $E\left(A\left(2\right)|D^v = 1\right) - A_1$, and to his implicit net profit from trading at price A_1, i.e. $E\left(\tilde{v}|D^v = 1\right) + \gamma - A_1$. First, if the limit order is fully displayed, such that $D^t = D^v = 1$, then his second unit is executed at price A_2. His expected net profits thus depend on his gains from trade relative to the tick size, Δ. Second, if there is an hidden order at the best quote, such that $D^h = 1$, then he does not pay this extra cost, but his adverse selection risk may increase. Indeed, the following necessary and sufficient condition holds:

$$E\left(\tilde{v}|D^v{=}\,1 \cap D^h{=}\,1\right) \leq E\left(\tilde{v}|D^v{=}\,1 \cap D^h{=}\,0\right) \Leftrightarrow \chi s \geq h\xi$$

Even out of the equilibrium path, if the informed agent strategy of hidden order submission relative to her small limit order submission is higher than the equivalent proportion for the uninformed limit order trader, then revealing an hidden order may be such a bad news for the uninformed agent that his gains from trade are not sufficient to cover the loss he incurs when he buys the asset at the price. Consequently, the uninformed agent's expected net profits also depend on the probability to face an informed hidden order when he observes a small limit order in the book. Lemma 3 describes the optimal reaction of the uninformed agent, depending on the strategy of the informed agent.

Lemma 3 *The uninformed market order trader's strictly dominant strategy is to submit a large market order $M = 2$ if (Condition 1) holds. If (Condition 2) holds, his strictly dominant strategy is to submit no market order $M = 0$. If none condition holds, his strictly dominant strategy is to submit a small market order $M = 1$.*

$$\chi + \left(\frac{\sigma-\gamma+\Delta}{\sigma-\gamma}\right)\xi \;<\; \frac{\gamma}{(\sigma-\gamma)}\frac{(1-\pi)}{\pi}\left(h+\left(1-\frac{\Delta}{\gamma}\right)s\right) \qquad \textit{(Condition 1)}$$

$$\xi+\chi \;>\; \frac{\gamma}{\sigma-\gamma}\frac{(1-\pi)}{\pi}(s+h) \qquad \textit{(Condition 2)}$$

Two conditions emerge. First, the condition $\lambda > \frac{\gamma}{\sigma-\gamma}\frac{(1-\pi)}{\pi}l$ we had found for an observation of $D^v = 2$ here naturally translates into Condition 2 when $D^v = 1$, which corresponds to the limit value between submitting a market order or not trading for the impatient agent. When the probability at which the informed agents submits limit orders such that $D^v = 1$, i.e. $\chi + \xi$, is too large, then the uninformed agent is deterred from submitting a market order. On the other hand, if this probability is sufficiently small, the uninformed agent always submits a large market order when $D^v = 1$. A second threshold indeed arises, that determines whether the uninformed agent is better off submitting a larger market order than the quantity displayed in the limit order book at the best ask price, or simply submitting a small order. As described in Section 2, we have parametrized the model so that the uninformed agent would submit a large market order when observing a small limit order, in the absence of asymmetric information. This ensures that $h + \left(1 - \frac{\Delta}{\gamma}\right) s > 0$. If an informed agent potentially submits a limit order however, the uninformed agent has to reevaluate his expected trading price as in the benchmark case with no adverse selection.

But now, he also has to update his beliefs on his expected implicit net profit from trading, which decreases which adverse selection, so that when π increases, the strategies' space such that Condition 1 holds shrinks. Both beliefs depend on the informed agent's strategy of hidden order submission, χ, relative to her strategy of small order submission, ξ, but also relative to the exogenous strategy of the uninformed limit order trader, h and s.

3.3 Informed agent's optimal limit order submission strategy

When the uninformed market order trader observes a limit order at price A_1, he updates his beliefs on the value of the security by taking into account two elements. First, the direction of the limit order, a sell order in our case, signals him that the security is likely to be overvalued. The impact of this signal on the uninformed agent's beliefs depends on the probability of information acquisition, π. Second, the size of the limit order allows him to better assess his adverse selection risk. Here is where the informed agent's limit order submission strategy plays a crucial role.

Actually, for a fixed level of adverse selection, there exists a multiplicity of equilibria. For clarity, we decide to focus on the simplest and most meaningful equilibrium, i.e. where the informed agent always submits as few hidden orders as possible[11].

Equilibria with no signaling

For extreme values of the parameter π, the uninformed agent's reaction is independent on the informed agent's limit order submission equilibrium strategy. At the equilibrium, the market order trader indeed focuses on the information content of a sell limit order, rather than on its size. Assume first that adverse selection is very low, then the uninformed agent always submits a large market order, whatever the limit order he observes and whatever the informed agent's strategy. Therefore, even if she always submits and displays a large order, the informed agent can expect to see it getting fully executed. Formally, Proposition 1 shows that there indeed exists such an equilibrium in pure strategies when $\pi \leq \frac{\gamma l}{\sigma-\gamma+\gamma l}$.

[11] Hidden orders may be more costly, in terms of monitoring, than fully-displayed orders for the informed agent.

Proposition 1 *If $\pi \leq \frac{\gamma l}{\sigma-\gamma+\gamma l}$, there exists an equilibrium in pure strategies, in which:*

1. The informed agent submits large unhidden orders with a probability $\lambda^* = 1$.

2. The uninformed agent updates his beliefs and reacts by submitting a large market order whatever the limit order D^v he observes.

The condition $\pi \leq \frac{\gamma l}{\sigma-\gamma+\gamma l}$ indeed ensures that $\frac{\gamma(1-\pi)}{\pi(\sigma-\gamma)} l \leq 1$, which, according to Lemma 2, guarantees that the uninformed agent is willing to trade two units when he observes a large limit order, whatever the informed agent's strategy. It thus determines the cutoff value of adverse selection such that an equilibrium in pure strategies, where the informed agent submits and displays a large unhidden order, exists.

In the opposite case, assume that adverse selection is very high, for instance, $\pi = 1$. If the informed agent submits a small or a large hidden order with a strictly positive probability, then the uninformed market order trader submits no order when he observes $D^v = 1$, as stated in Lemma 3. Similarly, according to Lemma 2, if she submits a large disclosed limit order with a strictly positive probability, she cannot get her large order executed since the uninformed market order trader again submits no order when he observes $D^v = 2$ in this case. Consequently, if the informed agent submits $D^v = 1$ or $D^v = 2$ with strictly positive probabilities, there is no trade in equilibrium. Assume now that she plays a pure strategy, say $\lambda^* = 1$ for instance. Then her expected profit would be zero, since the large limit order she submits is never executed, whereas small or hidden limit orders would get executed by the uninformed agent since $\chi^* + \xi^* = 0$. But then, there exists no equilibrium in pure strategies, since she would have incentives to deviate from this pure strategy to submit a small or a large hidden limit order. Proposition 2 similarly shows that for $\pi > \frac{\gamma}{\sigma}$, this type of equilibrium with no execution arises .

Proposition 2 *If $\pi > \frac{\gamma}{\sigma}$, there exists multiplicity of equilibria, in which the informed agent's strategies $(\lambda^*, \chi^*, \xi^*)$ belong to a continuum, and the uninformed agent submits no market order whatever the limit order D^v he observes.*

The condition $\pi > \frac{\gamma}{\sigma}$ indeed implies that the uninformed agent so strongly updates his beliefs after observing a sell limit order, that whatever the informed agent's equilibrium limit order submission strategy, the expected value of the security conditional on the limit order he observes, plus the gains from trade γ, are strictly negative. Therefore, he does not submit any market order.

In both cases, we notice that there is no economic rationale, for the informed agent, to use hidden orders. This finding is not surprising. Intuitively indeed, the informed agent may be induced to submit an hidden order in order to moderate the impact on her large order submission on the uninformed agent's beliefs. But when adverse selection is low, the information content of a limit order is also low, therefore, the uninformed agent does not take into account the signal he receives when observing the depth at price A_1 to modify his trading strategy. Thus the informed agent does not need to use such a strategy in order to get her large limit order executed. Conversely, when adverse selection is high, the uninformed agent who observes a sell limit order, whatever its size, perceives a rather precise signal that the security is currently overvalued. The impact of this signal induces him not to buy the security. Since mainly the direction of the limit order submitted at price A_1 matters for the uninformed agent, no equilibrium's limit order submission strategy used by the informed agent, even including hidden orders, could influence the market order trader's decision not to trade. In both cases, at this equilibrium, the uninformed market order trader does not take much into account the information content of the size of the undercutting limit order on the value of the security, but mainly its direction. Therefore, the informed agent is not induced to use a hidden order to manipulate his beliefs.

Equilibria with signaling

For intermediate levels of π however, the uninformed agent gives credit to his observation of the depth at the best price in his trading decision. Market depth at the best quotes is indeed perceived, by this agent, as a signal on his adverse selection risk. The informed agent's limit order strategy therefore influences his reaction. While choosing the optimal size of her limit order, the informed limit order trader faces a trade-off. On the one hand, if she submits a small limit order, her maximum transaction volume is only one unit, so she is induced to submit a larger limit order. But on the other hand, if she displays a large order too frequently, she cannot get her order fully-executed, because she would then impact the beliefs of the uninformed market order trader too strongly. She may therefore be willing to submit a large hidden order, to mitigate the impact of her order on the uninformed agent's beliefs on the value of the security, without restricting her trade size to one single unit. Proposition 3 shows that this is indeed the case, when $\pi \in [\frac{\gamma l}{\sigma-\gamma+\gamma l}, \frac{\gamma-\Delta s}{\sigma-\Delta s}[$[12].

Proposition 3 *If $\pi \in]\frac{\gamma l}{\sigma-\gamma+\gamma l}, \frac{\gamma-\Delta s}{\sigma-\Delta s}[$, there an equilibrium in mixed strategies, in which:*

1. The informed agent submits large unhidden orders with a probability $\lambda^* = \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$, and large hidden limit orders with probability $\chi^* = 1 - \lambda^*$;

2. The uninformed agent updates his beliefs and reacts by submitting a large market order whatever the limit order at the best quotes he observes.

Focusing on this mixed equilibrium allows us to better understand how adverse selection and gains from trade interact in the uninformed agent's trading decision. It is indeed such that the uninformed agent's net profit from trading, defined above, perfectly matches its costs when he observes a large limit order, $D^v = 2$, so that he is indifferent between submitting a market order of size 0, 1 or 2 whatever the limit order he observes. We indeed notice that conditional on this order, the expected value of the security for the uninformed agent, $E(\tilde{v}|D^v = 2)$, equals $v_0 - \gamma$.

In the case corresponding to Proposition 3, there really exists an economic rationale for the use of hidden limit orders by the informed agent. They indeed mitigate the market order trader's inference ability on the value of the security, while enabling the informed agent to expect a transaction volume that would be equivalent to the transaction volume expected for a large limit order submission, i.e. two units . In order to induce the uninformed agent to submit a large market order when he observes either $D^v = 2$ or $D^v = 1$, the informed agent submits a large unhidden (resp. hidden) order with a relatively low probability λ (resp. χ). The threshold $\frac{\gamma-\Delta s}{\sigma-\Delta s}$ corresponds to the maximum value of adverse selection, such that the informed agent is able to always submit a large limit order, hidden (i.e. such

[12]Notice that for $\pi \leq \frac{\gamma(1-l)-\Delta s}{\sigma-\gamma+\gamma(1-l)-\Delta s}$, there also exists an equilibrium in pure strategies, in which the informed agent always submits an hidden order.

that $D^v = D^h = 1$) or not (i.e. such that $D^t = D^v = 2$), and get her order fully-executed whatever the visible depth she displays. For higher levels of adverse selection indeed, if the informed agent wants to get two units executed either when she displays her large order, or when she hides it, she must also submit small orders with a strictly positive probability, since the sum of strategies λ and χ that would respect the conditions of Lemmas 2 and 3 is strictly inferior to one. But then, in order not to have incentives to deviate from her mixed strategy, the informed agent must get the same expected profits whatever the limit order she submits, and consequently, the only equilibrium that exists is such that her expected transaction volume is one unit whatever her limit order.

Proposition 4 *If $\pi \in [\frac{\gamma-\Delta s}{\sigma-\Delta s}, \frac{\gamma}{\sigma}]$, there exists an equilibrium in mixed strategies, in which:*

1. The informed agent submits large unhidden orders with a probability $\lambda^* = \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$, and displays small limit orders with probability $\xi^* = 1 - \lambda^*$;

2. The uninformed agent reacts by submitting a small limit order whatever the limit order at the best quotes he observes.

Notice that in this case again,there exists no economic rationale for the use of hidden orders by the informed agent, since only the probability of observing a limit order $D^v = 1$ matters for the uninformed agent, and since a large hidden order is never fully executed. Besides, at the cutoff value $\pi = \frac{\gamma}{\sigma}$, there is a pooling equilibrium, meaning that the uninformed agent's expectations on the value of the security conditional on the book are identical when he observes a small order and when he observes a large order, i.e. $E(\tilde{v}|D^v = 1) = E(\tilde{v}|D^v = 2)$. Whatever adverse selection indeed, by mimicing the patient uninformed agent's behavior and choosing $\lambda^* = l$, the informed agent perfectly hides in the trading noise, so that the beliefs of the impatient uninformed agent at date 2 are only affected by the direction of the limit order observed at the end of date 1, but not by its exact size. This characteristics illustrates the role of the threshold value $\frac{\gamma}{\sigma}$ for the adverse selection degree, π. For π strictly higher than this value, whatever the limit order book he observes, the uninformed agent is not willing to exchange, because with an expected value of the asset strictly inferior to $v_0 - \gamma$, his expected profit from exchange would be strictly negative.

3.4 Equilibria in the transparent market

In the opaque limit order market, liquidity suppliers have the opportunity to hide a part of the total quantity of their limit order. We assumed that the patient uninformed agent submits a large hidden order with a probability h, and we denoted χ this probability for the informed agent. In the transparent limit order market however, agents are able to observe the full quantity supplied at the best quotes in the limit order book. It can be easily shown that Propositions 1 and 2 are also valid in the transparent market, imposing that $\chi = 0$, since there is no economic rationale using hidden orders for the informed agent. The equilibria described in both propositions also hold when $\gamma \leq \Delta$[13], which only impacts the uninformed agent's reaction when he observes a small limit order at price A_1, for $\pi \leq \frac{\gamma l}{\sigma-\gamma+\lambda}$. Besides, it can, be easily shown that Proposition 4 applies in the transparent market for a larger set of parameter π's values, namely $\pi \in]\frac{\gamma l}{\sigma-\gamma+\lambda}, \frac{\gamma}{\sigma}]$.

3.5 Equilibrium's expected profits

The profit function of the uninformed market order trader, when he observes a limit order D^v, is defined in Section 2. Let us note $E\Pi_u^* (D^v)$ his equilibrium expected profit, given his optimal reaction and his beliefs, when he observes D^v. We compute his ex ante expected profits as follows:

$$E\Pi_u^{ante} = (\pi\lambda + (1-\pi)\, l)\, E\Pi_u^* (D^v = 2) + (\pi (1-\lambda) + (1-\pi)(1-l))\, E\Pi_u^* (D^v = 1)$$

The equilibrium ex ante expected profits of the informed agent rely on two elements. First, she has to enter the market, which she does ex ante with a probability π. Second, her expected profits are linear in the equilibrium trade size of her order, TS^*, and the probability to face a buyer, $\frac{1}{2}$. Her expected trade size depends on one hand on the size of

[13]In Section 2, we indeed calibrate our model so that, in the opaque market, $\gamma > \frac{s}{s+h}\Delta$, which translates into $\gamma > \Delta$ in the transparent market.

the limit order submitted by the informed agent, and on the other hand, on the uninformed agent's reaction at date 2. At the equilibrium, both elements are determined by adverse selection.

$$E\Pi_i^{ante}\left(\pi|\tilde{v}=v_L\right)=\pi\times\left(A_1-v_L\right)\times TS^*\left(\pi\right)\times\frac{1}{2}$$

As we have seen, depending on the level of adverse selection, we find different types of equilibria. For each equilibrium, we compute the strategic agents' expected profits as described above. Details of the computation of the uninformed agent's expected profits are provided in the Appendix. Table 2 summarizes the results of Propositions 1 to 4.

TABLE 2: EQUILIBRIUM'S EXPECTED PROFITS FOR DIFFERENT LEVELS OF ADVERSE SELECTION

Adverse selection π	$]0,\frac{\gamma l}{\sigma-\gamma+\gamma l}]$	$]\frac{\gamma l}{\sigma-\gamma+\gamma l},\frac{\gamma-\Delta s}{\sigma-\Delta s}[$	$[\frac{\gamma-\Delta s}{\sigma-\Delta s},\frac{\gamma}{\sigma}]$	$]\frac{\gamma}{\sigma},1]$
Order submission strategies				
Informed agent	$\lambda^*=1$	$\lambda^*=\frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)}l$ $\chi^*=1-\lambda^*$	$\lambda^*=\frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)}l$ $\xi^*=1-\lambda^*$	λ^*,ξ^*
Uninformed agent	$M\left(D^v=2\right)=2$ $M\left(D^v=1\right)=2$		$M\left(D^v=2\right)=1$ $M\left(D^v=1\right)=1$	$M\left(D^v=2\right)=0$ $M\left(D^v=1\right)=0$
Equilibrium expected profits				
Informed agent	$\pi\left(\Delta+\sigma\right)$		$\frac{1}{2}\pi\left(\Delta+\sigma\right)$	0
Uninformed agent	$\left(2\gamma-\Delta s\right)-\left(2\sigma-\Delta s\right)\pi$		$\gamma-\sigma\pi$	0

4 Comparative statics and related empirical predictions

In this section, we derive some comparative statics. We focus on the case where $\pi\leq\frac{\gamma}{\sigma}$, so that there is liquidity demand in equilibrium. We first analyze the determinants of hidden orders submission, then information content of limit orders, and finally we compare strategic agents' profits and market performance according to market transparency. The objective of this Section is to compare our conclusions with recent empirical findings, and to suggest some new empirical predictions.

In the previous section, we decided to focus on equilibria with no hidden order submission in Propositions 1 and 4, in order to better understand how and why submitting an hidden order could be profitable for the informed agent. These are however not the only existing equilibria. It can indeed be easily shown that for $\pi \in]0, \frac{\gamma}{\sigma}]$, the "camouflage strategy", i.e. submitting a large order with probability $\lambda^* = \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$, and an hidden order with probability $\chi^* = 1 - \lambda^*$, also constitutes an equilibrium strategy for the informed agent. In this equilibrium with hidden order submission, the optimal strategy of the uninformed agent is the same as the equilibrium strategy described in both propositions. Since our primary interest is to focus on the impact of hidden orders, we now restrict our attention to this equilibrium strategy to derive some comparative statics.

4.1 Hidden orders' submission

Our first concern is to figure out when in which market conditions it is more likely to have hidden depth at the best quotes. Corollary 1 analyses this probability $\Pr\left(D^h = 1 | D^v = 1\right)$.

Corollary 1 *The frequency at which there is hidden depth at the best quotes, conditional on the visible depth being small, increases with adverse selection π and with the asset volatility σ.*

Since both the adverse selection level and the asset volatility impact the uninformed agent's adverse selection risk, when they increase, the informed agent needs to use a hiding strategy more often in order not to deter the market order trader from demanding liquidity. But what is the impact of hidden order submission on market efficiency? To provide an answer to this question, the empiricist needs to condition his analysis on the same information set as market participants. Those indeed do not have access to the full order book, only to its visible part. However, in most of existing limit order markets, an hidden order at the best quote may be "observed" by the market as a whole, if a market order trader submits an order in the opposite direction which size exceeds the depth displayed in the book at this quote. By convention, we say that such a behavior aims at "testing" the presence of an hidden order, and that if there is indeed hidden depth at the best quotes, this hidden depth has been "revealed". We first provide some original predictions on the conditions for the presence of an hidden order to be "tested".

Corollary 2 *The frequency at which the uninformed agent submits a larger order than the quantity displayed at the best quotes, i.e.* $\Pr(M = 2|D^v = 1)$*, decreases with the probability of informed trading, with the tick size and with the volatility of the asset.*

In our model, it may be costly, for the uninformed market order trader, to submit a large order when he observes a small limit order. At the equilibrium, he "tests" the presence of an hidden order when he observes a small limit order if and only if $\pi < \frac{\gamma - \Delta s}{\sigma - \Delta s}$. Therefore, when the adverse selection level π, when the asset volatility σ, or when the tick size Δ increase, this condition may become obsolete.■ This prediction is partly consistent with the findings of Aitken et al. [2001], who find a negative correlation between the hidden order use and the relative tick size.

Combining corollaries 1 and 2 is shows that there is no monotonic relationship between hidden order submission and hidden order revelation. The frequency at which an hidden order is revealed is indeed such that:

$$\Pr\left(M = 2|D^h = 1\right) = \Pr(M = 2|D^v = 1) \times \Pr\left(D^h = 1|D^v = 1\right)$$

The impact of the asset volatility or of adverse selection on this probability is consequently ambiguous.

4.2 The information content of limit orders

Our model shows that the informed agent may benefit from the submission of an hidden order. Such a strategy allows her to get a large transaction volume, without revealing her presence. This is indeed the case because the uninformed market order trader updates his beliefs less strongly when he observes a small than a large limit order at price A_1, as shown in Corollary 3.

Corollary 3 *In equilibrium, the price impact of the observation of a large limit order is larger than the price impact of the observation of a small limit order, i.e.* $|E(\tilde{v}/D^v = 2) - v_0| \geq |E(\tilde{v}/D^v = 1) - v_0|$.

Let us now turn to the impact of hidden order revelation on the uninformed agent's beliefs, and let us define the expected value of the security, conditional on an hidden order to have been submitted and revealed, as $E\left(\tilde{v}/D^h=1\right)$. Our model enables us to shed lights on the existing empirical debate relative to hidden orders. On one hand, Tuttle [2003] finds a significant temporary price impact of the reserve size (i.e. the hidden depth) at the inside market. Our model shows that the hidden depth indeed contains information about future price movements, in line with Tuttle [2003]'s findings. Since in equilibrium, an hidden order is likely to have been submitted by an informed agent, such an order is informative on the value of the security, as would be any limit order submitted by such an agent. But on the other hand, according to Pardo and Pascual [2003], revealing an hidden order has no permanent price impact, relative to the price impact of an equally sized and matched ordinary trade. In our model, the magnitude of their beliefs' update when the uninformed agent detects the presence of a large hidden order, relative to what it would be if he observed a fully displayed limit order of the same size, is ambiguous.

Corollary 4 *The information content of a revealed hidden order is smaller than the information content of a non-hidden order of the same size, i.e.* $\left|E\left(\tilde{v}/D^t=2\cap D^h=1\right)-v_0\right|\leq\left|E\left(\tilde{v}/D^t=2\cap D^h=0\right)-v_0\right|$, *if and only if except if* $\pi\in]0,\frac{\gamma-\Delta s}{\sigma-\Delta s}[$.

Depending on market conditions, which indeed determine the informed agent's incentives to submit hidden orders relative to large orders, revealing an hidden order could thus either be good news or not, which may explain why empirical findings on this point do not converge.

Besides, in our model, the characteristics of the equilibrium are sensitive to the level of adverse selection, π. This feature enables us to draw some original empirical prediction on the proportion of hidden orders depending on adverse selection, and thus, relates to the emerging literature on the computation of this probability of informed trading (PIN).

Corollary 5 *The informational content of an hidden order revelation,* $\left|E\left(\tilde{v}|D^t=2\cap D^h=1\right)-v_0\right|$, *increases with the probability of informed trading* π.

When π increases, it may not be more difficult, for the informed agent, not to reveal her presence when she submits and displays large orders, if she uses a camouflage strategy. But she has then to submit hidden orders more frequently, which alters the information content of the limit order observed at the best quotes.

4.3 The impact of market transparency

Impact of market transparency on market liquidity

We first analyze the impact of the move to a transparent market on market liquidity. Since we have assumed that the prices were fixed to focus on the size of the limit order submitted and displayed at price A_1, the best indicator of market liquidity in our model is the depth at the best quotes. Actually, there exist two measures of this depth, namely the visible depth, D^v, and the total depth, D^t. Although we assume that the uninformed limit order trader's strategy is exogenous, the change in market model necessarily impacts the probabilities at which it submits a large unhidden limit order, or a small limit order, simply because it has to transfer the orders it submits as hidden orders in the opaque market to either one of both types of fully-displayed limit orders in the transparent market. Let us denote l^o (resp. l^t) and s^o (resp. s^t) the proportion of large and small uninformed limit orders in the opaque (resp. transparent) market. An increase in the level of pre-trade transparency in the limit order book has two opposite effects on the depth supplied by the uninformed liquidity trader at the best quotes. On one hand, the proportion of the uninformed large orders disclosed at these quotes increases, because this agent cannot hide its large orders in the transparent market. Mathematically, this increase in the visible depth in the transparent market translates into the condition $l^o \leq l^t$. On the other hand, the total depth at the best quotes decreases in the transparent market, because the uninformed agent may transfer a fraction of its formal hidden orders to small rather than large fully-disclosed limit orders. We must therefore have $s^o \leq s^t$. What is then the impact of such a move on the overall market depth, i.e. on the size of limit orders submitted both by the informed agent the uninformed agent?

Corollary 6 *Visible depth D^v increases in the move to a transparent market . The impact of the change of market transparency on the total depth D^t is however ambiguous. The total depth D^t is lower in the transparent market if and only if* $\pi < \frac{\left(l^o+h^o-l^t\right)(\sigma-\gamma)+\gamma l^o}{((1-l^t+h^o)(\sigma-\gamma)+\sigma l^o)}$.

Because the informed agent's strategy is to mimic the uninformed limit order trader's behavior, in equilibrium, the proportion of informed large orders increases with l, thus, both the uninformed and the informed proportion of large orders is higher in the transparent market, so is thus the visible depth. The total depth however is composed of the visible plus the hidden depth, i.e. $D^t = D^v + D^h$. Since D^h jumps to zero in the transparent market, the total depth may only be higher than in the opaque market if the increase in the visible depth, discussed above, is sufficiently high to compensate the absence of hidden depth. When this is the case, surprisingly, market liquidity may be reduced in the opaque market.

Impact of market transparency on strategic agents' profits

An increase in the level of pre-trade transparency in the limit order book has two opposite effects on the informed agent's expected profits.

On one hand, because of the increase in the visible depth in the transparent market, the informed agent may submit large unhidden orders with probability one, without being detected, for larger levels of adverse selection in the transparent market[14]. On the other hand, the total depth at the best quotes decreases in the transparent market, therefore, the informed agent's camouflage's strategy induces her to submit small limit orders more frequently in the transparent market, thus reducing her expected transaction size, all the more as she cannot submit an hidden order. The overall impact of a change in the level of the ex ante transparency of the limit order book depends on the relative strength of both effects, which relies on the sign of $\gamma s^t - \Delta s^o$. If it is positive, both strategic agents are better off in the opaque market. In this market indeed, the informed agents benefits from the use of hidden orders, and the uninformed market order trader from the increase in the total market depth.

[14]The cutoff value for such an equilibrium's strategy, namely $\frac{\gamma l}{\sigma-\gamma+\gamma l}$, is indeed increasing in l.

Corollary 7 *The informed agent is strictly better off in the transparent market or indifferent between both market organizations if and only if $\Delta s^o \geq \gamma s^t$. The uninformed market order trader is always better off in the opaque market.*

Counter-intuitively however, in the opposite case where $\gamma s^t \leq \Delta s^o$ (although by assumption $\gamma > \frac{s^o}{s^o+h^o}\Delta$), the benefits, for the informed agent, of an increase in the visible depth offsets the effect of the decrease in the total depth, so that she is at least as well off in the transparent market as in the opaque market. Although she cannot submit an hidden order in this market, the increase in the proportion of large uninformed limit orders makes it easier, for her, not to signal her presence, while still submitting large limit orders. Therefore, we may observe lower trading costs for large market orders in the transparent market if the liquidity supply does not decrease drastically in the move to the transparent market. For the uninformed market order trader, the decrease in the total depth in the opaque market overcomes the increase the visible depth, so that he is better off in the opaque market.

Impact of market transparency on market efficiency

We finally compare how the uninformed agent updates his beliefs on the value of the security, in both market organizations. In many papers analyzing market transparency, liquidity improvement in a market is often counter-balanced by a decrease in market efficiency. To analyze this, we compute the difference in the evaluation of the value of the asset, conditioning on the market participants' information set. Let $E_x\left(\tilde{v}|D^v\right)$ denote the conditional expected value of the security, when the market is transparent $(x = t)$ or opaque $(x = o)$. Surprisingly here, a decrease in the ex ante transparency of the market increases market efficiency.

Corollary 8 *Whatever the level of adverse selection π, the expected value of the security, conditional on the observation of the visible depth D^v, is closer to its true value v_L in the opaque market. Indeed, $E_t\left(\tilde{v}|D^v\right) - E_o\left(\tilde{v}|D^v\right) \geq 0$.*

In the opaque market indeed, the informed agent is induced to submit large orders, hidden or not, more frequently. Besides, the decrease in the proportion of uninformed large orders in this market makes it more difficult, for the uninformed market order trader, not to reveal her presence.

5 Conclusion

In this paper, we present a simple theoretical model of trading in a limit order book, to investigate the impact of the authorization of hidden orders on agents' expected profits, when an informed agent potentially submits limit orders. At the equilibrium, for intermediate levels of adverse selection, the informed limit order trader has a camouflage's equilibrium strategy: she tries to mimic the uninformed limit order trader's behavior in order not to be detected. We show that in some cases, the submission of an hidden limit order allows the informed agent to get a large transaction volume, without signalling her presence. But counter-intuitively, she may however be worse off in the opaque market, due to the decrease in the total depth supplied by the uninformed limit order trader. Our model enables us to derive some new empirical predictions, that could be tested in future research.

6 References

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7 Proofs

Reaction of the uninformed agent

Proof. [Lemma 2] When a buyer impatient trader observes a large limit order $D^v = 2$, he may either submit no order, or a buy order of size 1, or a buy order of size 2. For each of these three strategies, his expected profits are then:

$$
\begin{aligned}
E\Pi_u(0|D^v=2) &= 0 \\
E\Pi_u(1|D^v=2) &= E(\tilde{v}|D^v=2) - A_1 + \beta = \gamma - \sigma\frac{\pi\lambda}{\pi\lambda + (1-\pi)l} \\
E\Pi_u(2|D^v=2) &= 2E\Pi_u(1|D^v=2)
\end{aligned}
$$

The proof is straightforward. ■

Proof. [Lemma 3] When a buyer impatient trader observes a deep book $D^v = 1$, he may either submit no order, or a buy order of size 1, or a buy order of size 2, or a marketable limit order of size 2. For each of these three strategies, his expected profits are then:

$$
\begin{aligned}
E\Pi_u(0|D^v=1) &= 0 \\
E\Pi_u(1|D^v=1) &= (E(\tilde{v}|D^v=1) - A_1) + \beta \\
&= \gamma - \sigma\frac{\pi(\xi+\chi)}{\pi(\xi+\chi) + (1-\pi)(s+h)} \\
E\Pi_u(2|D^v=1) &= 2(E(\tilde{v}|D^v=1) - A_1 + \beta) - (A_2 - A_1)\Pr\left(D^t = 1|D^v = 1\right) \\
&= 2\left(\gamma - \sigma\frac{\pi(\xi+\chi)}{\pi(\xi+\chi) + (1-\pi)(s+h)}\right) - \Delta\left(\frac{\pi\xi + (1-\pi)s}{\pi(\xi+\chi) + (1-\pi)(s+h)}\right)
\end{aligned}
$$

Since we assumed that $0 < \gamma < \sigma$, comparing the expected payoffs for all pairs of strategies, we find the following conditions of the uninformed agent's reaction. The condition $x \succ y$ means that strategy x strictly dominates strategy y.

$$1 \succ 0 \Leftrightarrow \xi + \chi < \frac{\gamma}{\sigma - \gamma}\frac{(1-\pi)}{\pi}(s+h)$$

$$2 \succ 0 \Leftrightarrow \chi + \left(1 + \frac{\Delta}{2(\sigma-\gamma)}\right)\xi < \frac{\gamma}{\sigma-\gamma}\frac{(1-\pi)}{\pi}\left(h + \left(1 - \frac{\Delta}{2\gamma}\right)s\right)$$

$$2 \succ 1 \Leftrightarrow \chi + \left(1 + \frac{\Delta}{(\sigma-\gamma)}\right)\xi < \frac{\gamma}{(\sigma-\gamma)}\frac{(1-\pi)}{\pi}\left(h + \left(1 - \frac{\Delta}{\gamma}\right)s\right)$$

We notice that in some cases, the respect of one of these conditions guarantees that the other two hold. For instance, if $2 \succ 1$, then $1 \succ 0$ must hold. If $0 \succ 1$, then $1 \succ 2$. This yields to Lemma 3. ■

Equilibrium's strategies

Proof. [Proposition 1] The condition $\pi \leq \frac{\gamma l}{\sigma-\gamma+\gamma l}$ is equivalent to $\frac{\gamma}{(\sigma-\gamma)}\frac{(1-\pi)}{\pi}l \geq 1$. According to Lemma 2, this ensures that whatever the informed agent's large order submission strategy λ, the uninformed agent's optimal reaction when he observes $D^v = 2$ is to submit a large market order. Consequently, there exists an equilibrium in pure strategies, in which the informed agent always submits and displays a large limit order, $\lambda^* = 1$. Since Lemma 3's Condition 1 holds, and since $\left(h + \left(\frac{\gamma-\Delta}{\gamma}\right)s\right) > 0$ by assumption, at this equilibrium, the uninformed agent submits a large market order whatever the limit order he observes. ■

If $\pi > \frac{\gamma l}{\sigma-\gamma+\gamma l}$, as stated in Lemma 2, then the uninformed agent's reaction when he observes $D^v = 2$, depends on the location of λ relative to $\frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$, since this cutoff value is now strictly inferior to 1. There can be no equilibrium in pure strategies in which the informed agent would play $\lambda = 1$. By definition, in equilibrium in mixed strategies, the expected profits of the strategies the informed agent plays with a strictly positive probability must be equal. Otherwise, she would have incentives to deviate and to play the most profitable strategy more frequently.

Proof. [Proposition 2] If $\pi > \frac{\gamma}{\sigma}$, then $\frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l < 1 - \frac{(1-\pi)}{\pi}\frac{\gamma}{\sigma-\gamma}(1-l)$. Consequently, no equilibrium strategy used by the informed agent could induce the uninformed agent to submit a market order. Assume first that the informed agent submits a large unhidden order with a probability $\lambda \leq \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$. According to Lemma 2, her large limit order gets fully-executed. But in this case, it also implies that $\lambda < 1 - \frac{(1-\pi)}{\pi}\frac{\gamma}{\sigma-\gamma}(1-l)$, so that according to Lemma 3's Condition 2, her limit orders do not get executed when she displays only one unit of them. Therefore, there can be no equilibrium in mixed strategies such that $\lambda \leq \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$ when $\pi > \frac{\gamma}{\sigma}$. Similarly, there can be no equilibrium such that she submits a large unhidden order with a probability $\lambda > 1 - \frac{(1-\pi)}{\pi}\frac{\gamma}{\sigma-\gamma}(1-l)$, in which case she would be induced to deviate and display small limit orders more frequently. Finally, for $\pi > \frac{\gamma}{\sigma}$, there exists an equilibrium if and only if the informed agent submits and displays large unhidden orders with a probability $\lambda^* \in]\frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)}l, \frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)}l + \frac{\pi\sigma-\gamma}{\pi(\sigma-\gamma)}[$ and displays small orders with a probability $\chi^* + \xi^* = 1 - \lambda^*$. The uninformed agent's optimal reaction is to submit no market order. This leads to Proposition 2. ■

If $\pi \in]\frac{\gamma l}{\sigma-\gamma+\gamma l}, \frac{\gamma}{\sigma}]$, then on one hand there exists no equilibrium in pure strategies such that the informed agent always submit large limit orders. On the other hand, since $1 - \frac{(1-\pi)}{\pi}\frac{\gamma}{\sigma-\gamma}(1-l) < \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$, there exist a set of strategies, for the informed agent, such that the uninformed agent would be ready to demand liquidity.

Proof. [Proposition 3] Let us first look for equilibria with large execution. If such an equilibrium exists, three conditions must simultaneously be fulfilled. The optimal reaction of the uninformed agent must be to submit a large market order when he observes a large limit order, and when he observes a small limit order. According to Lemma 2 and to Lemma 3's Condition 1, both conditions respectively translates into:

$$\begin{aligned} \lambda &\leq \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l \\ \chi + \left(\frac{\sigma-\gamma+\Delta}{\sigma-\gamma}\right)\xi &< \frac{(1-\pi)}{\pi}\frac{\gamma}{(\sigma-\gamma)}\left(h + \left(\frac{\gamma-\Delta}{\gamma}\right)s\right) \end{aligned}$$

Besides, the informed agent has to submit no small limit order, i.e. $\xi^* = 0$, otherwise, she would have a maximum transaction size of one unit when she displays a small limit order, but a transaction size of two units when she displays a large limit order. This system of inequalities has a solution if and only if $\pi < \frac{\gamma-\Delta s}{\sigma-\Delta s}$. Notice that $\frac{\gamma l}{\sigma-\gamma+\gamma l} < \frac{\gamma-\Delta s}{\sigma-\Delta s} < \frac{\gamma}{\sigma}$. Consequently, for $\pi \in [\frac{\gamma l}{\sigma-\gamma+\gamma l}, \frac{\gamma-\Delta s}{\sigma-\Delta s}[$, there exists a continuum of equilibria in mixed strategies, characterized by a large transaction volume. In Proposition 3, we focus on one of these equilibria, namely $\lambda^* = \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$ and $\chi^* = 1-\lambda^*$. ■

Finally, what happens if $\pi \in [\frac{\gamma-\Delta s}{\sigma-\Delta s}, \frac{\gamma}{\sigma}[$?

Proof. [Proposition 4] Since there is also no equilibrium with large execution, we therefore look for the equilibria in mixed strategies with low execution. If such an equilibrium exists, two conditions must simultaneously be fulfilled. The optimal reaction of the uninformed agent must be to submit a small market order when he observes a large limit order, and when he observes a small limit order. According to Lemma 2 and to Lemma 3, both conditions respectively translates into:

$$\begin{aligned}
\lambda &= \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l \\
\chi + \left(\frac{\sigma-\gamma+\Delta}{\sigma-\gamma}\right)\xi &\geq \frac{(1-\pi)}{\pi}\frac{\gamma}{(\sigma-\gamma)}\left(h+\left(\frac{\gamma-\Delta}{\gamma}\right)s\right) \\
\xi+\chi &\leq \frac{(1-\pi)}{\pi}\frac{\gamma}{\sigma-\gamma}(s+h)
\end{aligned}$$

An equilibrium in mixed strategies with low execution exists if $\pi \leq \min\left(\frac{(\gamma+\Delta s)}{(\sigma+\Delta s-\Delta\xi)}, \frac{\gamma}{\sigma}\right)$. Since $\frac{\gamma}{\sigma} < \frac{(\gamma+\Delta s)}{(\sigma+\Delta s-\Delta\xi)}$, if $\frac{\gamma-\Delta s}{\sigma-\Delta s} \leq \pi < \frac{\gamma}{\sigma}$, there exists an equilibrium in mixed strategies, with low execution, in which the informed agent submits a large unhidden order with a probability $\lambda^* = \frac{\gamma}{\sigma-\gamma}\frac{1-\pi}{\pi}l$ and a small order with a probability $\xi^* = 1-\lambda^*$. The uninformed agent reacts by submitting a small market order whatever the limit order he observes. ■

Computation of the expected profits of the uninformed agent

Proof. [Table 2] We compute the ex ante expected profits of the uninformed agent starting from the definition of his ex ante expected profits $E\Pi_u^{ante}$ and replacing by the expected profits of the uninformed agent computed in the proofs of Lemmas 2 and 3:

$$\begin{aligned}
E\Pi_u^{ante} &= (\pi\lambda+(1-\pi)l)\times\begin{pmatrix} 2\left(\gamma-\sigma\frac{\pi\lambda}{\pi\lambda+(1-\pi)l}\right)\mathbb{k}_{M^*(D^v=2)=2} \\ +\left(\gamma-\sigma\frac{\pi\lambda}{\pi\lambda+(1-\pi)l}\right)\mathbb{k}_{M^*(D^v=2)=1}\end{pmatrix} \\
&\quad+(\pi(1-\lambda)+(1-\pi)(1-l))\times \\
&\begin{pmatrix} \left(2\left(\gamma-\sigma\frac{\pi(\xi+\chi)}{\pi(\xi+\chi)+(1-\pi)(s+h)}\right)-\Delta\left(\frac{\pi\xi+(1-\pi)s}{\pi(\xi+\chi)+(1-\pi)(s+h)}\right)\right)\mathbb{k}_{M^*(D^v=1)=2} \\ +\left(\gamma-\sigma\frac{\pi(\xi+\chi)}{\pi(\xi+\chi)+(1-\pi)(s+h)}\right)\mathbb{k}_{M^*(D^v=1)=1}\end{pmatrix}
\end{aligned}$$

Since the strategies of the informed agent, and the reaction of the uninformed agent, depend on the adverse selection level, the equilibrium expected profits of both agents rely on adverse selection. For each type of equilibrium described in Propositions 1 to 4, we replace $(\lambda^*, \chi^*, \xi^*)$ and the indicator variables $\mathbb{k}_{M^*(D^v)}$ by their equilibrium values in this equation. This leads to Table 2. ■

Comparative statics

Recall that we focus on the case $\pi \leq \frac{\gamma}{\sigma}$, and on the equilibrium $\lambda^* = \frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)} l = 1 - \chi^*$.

Proof. [Corollary 1] Let us first analyze the probability to have hidden depth at the best quotes.

$$\Pr\left(D^h = 1 | D^v = 1\right) = \frac{\pi\chi + (1-\pi) h}{\pi(\chi + \xi) + (1-\pi)(h+s)} = \frac{(1 - s + \pi s)(\sigma - \gamma) - \sigma(1-\pi) l}{(\sigma - \gamma) - \sigma(1-\pi) l}$$

Thus:

$$\begin{aligned}
\frac{\partial \Pr\left(D^h = 1 | D^v = 1\right)}{\partial \pi} &= \frac{(\sigma-\gamma)^2 s}{((\sigma-\gamma) - \sigma(1-\pi) l)^2} > 0 \\
\frac{\partial \Pr\left(D^h = 1 | D^v = 1\right)}{\partial \sigma} &= \frac{\gamma(1-\pi)^2 ls}{((\sigma-\gamma) - \sigma(1-\pi) l)^2} > 0
\end{aligned}$$

■

Proof. [Corollary 3] Let us first compute these expectations.

$$\begin{aligned}
\left|E\left(\tilde{v}/D^v = 1\right) - v_0\right| &= \left|-\sigma \tfrac{\pi(\xi+\chi)}{\pi(\xi+\chi)+(1-\pi)(s+h)}\right| = \left|-\sigma \tfrac{\pi(\sigma-\gamma)-(1-\pi)\gamma l}{\pi(\sigma-\gamma)+(1-\pi)(\sigma-\gamma-\sigma l)}\right| \\
\left|E\left(\tilde{v}/D^v = 2\right) - v_0\right| &= \left|-\sigma \tfrac{\pi\lambda}{\pi\lambda+(1-\pi)l}\right| = \gamma
\end{aligned}$$

Consequently,

$$\left|E\left(\tilde{v}/D^v = 1\right) - v_0\right| - \left|E\left(\tilde{v}/D^v = 2\right) - v_0\right| = \tfrac{(\sigma-\gamma)(\pi\sigma-\gamma)}{\pi\sigma-\gamma+(1-l)(1-\pi)\sigma}$$

The difference in the absolute values is negative for $\pi \leq \frac{\gamma}{\sigma}$. ■

Proof. [Corollary 4] Let us first compute these expectations.

$$\begin{aligned}\left|E\left(\tilde{v}/D^{v}=1\cap D^{h}=1\right)-v_{0}\right| &= \left|-\sigma\frac{\pi\chi}{\pi\chi+(1-\pi)h}\right| = \left|-\sigma\frac{\pi(\sigma-\gamma)-(1-\pi)\gamma l}{\pi(\sigma-\gamma)+(1-\pi)((\sigma-\gamma)(1-s)-\sigma l)}\right| \\ \left|E\left(\tilde{v}/D^{v}=2\cap D^{h}=0\right)-v_{0}\right| &\equiv \left|E\left(\tilde{v}/D^{v}=2\right)-v_{0}\right|\end{aligned}$$

Consequently,

$$\left|E\left(\tilde{v}/D^{v}=2\cap D^{h}=0\right)-v_{0}\right|-\left|E\left(\tilde{v}/D^{v}=1\cap D^{h}=1\right)-v_{0}\right|=\frac{(\sigma-\gamma)((\gamma s-\sigma)\pi+(\gamma-\gamma s))}{\pi\sigma-\gamma+(1-l)(1-\pi)\sigma}$$

The difference is positive on $\pi\in]0,\frac{\gamma-\Delta s}{\sigma-\Delta s}[$, then is negative on $[\frac{\gamma-\Delta s}{\sigma-\Delta s},\frac{\gamma}{\sigma}]$. ■

Proof. [Corollary 5] The informational content of an hidden order revelation, $\left|E\left(\tilde{v}|D^{t}=2\cap D^{h}=1\right)-v_{0}\right|$ increases with π.

$$\begin{aligned}\frac{\partial\left|E\left(\tilde{v}/D^{v}=1\cap D^{h}=1\right)-v_{0}\right|}{\partial\pi} &= \frac{\partial\left|-\sigma\frac{\pi(\sigma-\gamma)-(1-\pi)\gamma l}{\pi(\sigma-\gamma)+(1-\pi)((\sigma-\gamma)(1-s)-\sigma l)}\right|}{\partial\pi} \\ &= \sigma\frac{h\left(\sigma-\gamma\right)^{2}}{\left(\pi\left(\sigma-\gamma\right)+\left(1-\pi\right)\left(\left(\sigma-\gamma\right)\left(1-s\right)-\sigma l\right)\right)^{2}}>0\end{aligned}$$

■

Proof. [Corollary 6] Since $\lambda^{t*}-\lambda^{o*}=\frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)}\left(l^{t}-l^{o}\right)$ which is positive since $l^{o}\leq l^{t}$, in equilibrium, $\lambda^{*o}\leq\lambda^{*t}$, so that $\pi\left(\lambda^{*t}\text{-}\lambda^{*o}\right)+(1-\pi)\left(l^{t}-l^{o}\right)\geq 0$.

Since we focus on the equilibrium such that $\xi=0$ in the opaque market, we have $\xi^{*o}\leq\xi^{*t}$. Therefore,

- If $\pi\leq\frac{\gamma l^{t}}{\sigma-\gamma+\gamma l^{t}}$, $\pi\left(\xi^{*t}-\xi^{*o}\right)+(1-\pi)\left(s^{t}-s^{o}\right)=(1-\pi)\left(s^{*t}-s^{*o}\right)\geq 0$;

- If $\frac{\gamma l^{t}}{\sigma-\gamma+\gamma l^{t}}<\pi\leq\frac{\gamma}{\sigma}$,

$$\pi\left(\xi^{*t}-\xi^{*o}\right)+(1-\pi)\left(s^{t}-s^{o}\right)=l^{o}+h^{o}-l^{t}-\frac{\left(1-l^{t}+h^{o}+l^{o}\right)\sigma\pi-\left(1-l^{t}+h^{o}\right)\gamma\pi-\gamma l^{o}}{(\sigma-\gamma)}$$

Consequently,

$$\pi\left(\xi^{*t}-\xi^{*o}\right)+(1-\pi)\left(s^{t}-s^{o}\right)\geq 0\Leftrightarrow\pi\leq\frac{\left(l^{o}+h^{o}-l^{t}\right)(\sigma-\gamma)+\gamma l^{o}}{\left((1-l^{t}+h^{o})(\sigma-\gamma)+\sigma l^{o}\right)}$$

■

Proof. [Corollary 7] If $\Delta s^o \geq \gamma s^t$, then $\frac{\gamma l^t}{\sigma-\gamma+\gamma l^t} \geq \frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}$. Although the informed agent can reach large execution by using a hiding strategy when $\pi \in [\frac{\gamma l^o}{\sigma-\gamma+\gamma l^o}, \frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}[$ in the opaque market, she can obtain the same expected profits by simply submitting a large unhidden order in the transparent market. Besides, there exists an interval $[\frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}, \frac{\gamma l^t}{\sigma-\gamma+\gamma l^t}[$ such that, for π belonging to that interval, she is able to get a large transaction volume in the transparent market, but not in the opaque market.

For the uninformed market order trader, we compute the difference $E\Pi_t^* - E\Pi_o^*$ for each $\pi-$interval. In case $\Delta s^o \geq \gamma s^t$, then $E\Pi_t^* - E\Pi_o^* = (1-\pi)\,\Delta\left(s^o - s^t\right)$ if $\pi \in]0, \frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}[$, $E\Pi_t^* - E\Pi_o^* = \left(\gamma - \Delta s^t\right) - \left(\sigma - \Delta s^t\right)\pi$ if $\pi \in [\frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}, \frac{\gamma l^t}{\sigma-\gamma+\gamma l^t}[$ and $E\Pi_t^* - E\Pi_o^* = 0$ if $\pi \in [\frac{\gamma l^t}{\sigma-\gamma+\gamma l^t}, \frac{\gamma}{\sigma}]$. Since $s^t \geq s^o$, this difference is always negative. If $\pi \in [\frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}, \frac{\gamma l^t}{\sigma-\gamma+\gamma l^t}[$ in particular, $\pi \geq \frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}$ implies that $\pi \geq \frac{\gamma-\Delta s^t}{\sigma-\Delta s^t}$, so in this interval also, $E\Pi_t^* - E\Pi_o^* \leq 0$. In case $\Delta s^o < \gamma s^t$, then $E\Pi_t^* - E\Pi_o^* = (1-\pi)\,\Delta\left(s^o - s^t\right)$ if $\pi \in]0, \frac{\gamma l^t}{\sigma-\gamma+\gamma l^t}]$, $E\Pi_t^* - E\Pi_o^* = -\left(\gamma - \Delta s^o\right) + \left(\sigma - \Delta s^o\right)\pi$ if $\pi \in]\frac{\gamma l^t}{\sigma-\gamma+\gamma l^t}, \frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}[$ and $E\Pi_t^* - E\Pi_o^* = 0$ if $\pi \in [\frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}, \frac{\gamma}{\sigma}]$. Since $s^t \geq s^o$, this difference is always negative. If $\pi \in]\frac{\gamma l^t}{\sigma-\gamma+\gamma l^t}, \frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}[$ in particular, $\pi \leq \frac{\gamma-\Delta s^o}{\sigma-\Delta s^o}$ implies that $-\left(\gamma - \Delta s^o\right) + \left(\sigma - \Delta s^o\right)\pi \leq 0$, so in this interval also, $E\Pi_t^* - E\Pi_o^* \leq 0$. ■

Proof. [Corollary 8] Let us first compute, for each potential undercutting limit order observed, the difference in the expectations between the transparent and the opaque market.

$$E_t\left(\tilde{v}|D^v = 1\right) - E_o\left(\tilde{v}|D^v = 1\right) = \frac{\sigma\pi\left(1-\pi\right)}{\left[\pi\left(1-\lambda^o\right) + \left(1-\pi\right)\left(1-l^o\right)\right] \times \left[\pi\left(1-\lambda^t\right) + \left(1-\pi\right)\left(1-l^t\right)\right]} \times \left(\left(1-\lambda^o\right)\left(1-l^t\right) - \left(1-l^o\right)\left(1-\lambda^t\right)\right) \qquad (1)$$

$$E_t\left(\tilde{v}|D^v = 2\right) - E_o\left(\tilde{v}|D^v = 2\right) = \frac{\sigma\pi\left(1-\pi\right)}{\left[\pi\lambda^o + \left(1-\pi\right)l^o\right] \times \left[\pi\lambda^t + \left(1-\pi\right)l^t\right]} \times \left(\lambda^o l^t - l^o\lambda^t\right)$$

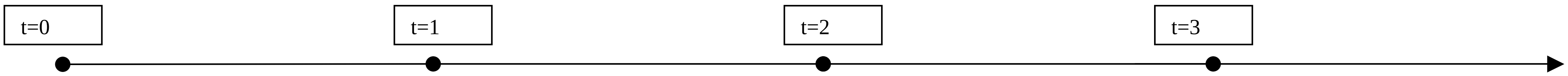
Since the first term of each formula above, the fraction, is always positive, we focus on the sign of the second term to find the sign of the difference. We input strategies into this second term for each of these three intervals. As shown in Table 3, since $l^t - l^o \geq 0$, we find that differences are always positive.

TABLE 3

Large order submission strategies	$\lambda^t = \frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)} l^t, \lambda^o = \frac{(1-\pi)\gamma}{\pi(\sigma-\gamma)} l^o$
$\text{sign}(E_t\left(\tilde{v}|D^v = 1\right) - E_o\left(\tilde{v}|D^v = 1\right))$	$\text{sign}\frac{\pi(\sigma-\gamma)+(1-\pi)\gamma}{\pi(\sigma-\gamma)}\left(l^o - l^t\right)$
$\text{sign}(E_t\left(\tilde{v}|D^v = 2\right) - E_o\left(\tilde{v}|D^v = 2\right))$	0

■

FIGURE 1 : TIMING OF THE GAME AND OF INFORMATION DISCLOSURE



t=0

- The final value of the asset is realized.
- Limit orders to buy and to sell the security stand in the limit order book.
- The initial best limits in the order book are (A_2,B_{-2}).

t=1

- Agents observe the initial best limits in the order book (A_2,B_2).
- One agent submits a limit order, undercutting the initial quotes by one tick: a sell (resp. Buy) order may be submitted at price A_1 (resp. B_{-1}).

t=2

- Agents observe the new best quotes in the limit order book.
- One agent potentially submits a market order to buy or sell the security.

t=3

- The final value of the asset is revealed.

FIGURE 1

FIGURE 2 : AGENTS' DECISION PLANNING ON THE ASK SIDE OF THE LIMIT ORDER BOOK