Correlated Defaults and the Valuation of Defaultable Securities

Fan Yu[1]
University of California, Irvine

Current Version: April 1, 2004

[1]This paper extends parts of a previous paper entitled "Correlated Defaults in Reduced-Form Models." Helpful comments from Alexander David, Kay Giesecke, Christian Gourieroux, Ian Hawkins (WestLB), Bob Jarrow, Stuart Turnbull, and participants at the Blaise Pascal International Conference on Financial Modeling in Paris, the European Finance Association Meetings in Glasgow, and the Conference on Dependence Modelling for Credit Portfolios in Venice are acknowledged. All remaining errors are my own responsibility. Address correspondence to Fan Yu, UCI-GSM, Irvine, CA 92697-3125, E-mail: fanyu@uci.edu.

Correlated Defaults and the Valuation of Defaultable Securities

Abstract

We present an intensity-based model of correlated defaults with application to the valuation of defaultable securities. The model assumes that the conditional hazard rate of default is driven by external common factors as well as other defaults in the system. A proposed recursive procedure can be used to generate default times with a broad class of correlation structures. We compare this approach to standard reduced-form models based on conditional independence, as well as recent innovations involving copula functions. We also illustrate its use with examples on the pricing of defaultable bonds, and credit default swaps of the regular and basket type.

1 Introduction

The rapid expansion in recent years of the market for credit derivatives has led to a growing interest in the valuation of these instruments. Basket credit derivatives, which have payoffs linked to credit events in a portfolio of defaultable securities, are particularly a challenge because of their sensitivity to default correlation in the portfolio.

In this paper, we present an intensity-based approach to the modeling of correlated defaults. In the spirit of reduce-form models, default is treated as an unpredictable jump with an exogenously specified hazard rate process, or intensity.[1] In contrast to the existing literature, which focuses primarily on defaultable bonds and single-name credit derivatives, our approach is distinguished by its construction of multiple defaults with a general correlation structure, which may prove useful to the valuation of basket credit derivatives or defaultable securities subject to counterparty risk.

In a typical reduced-form model, it is customary to associate with default a stopping time τ, defined as

$$\tau = \inf\left\{ t : \int_0^t \lambda_s ds \geq E \right\}, \tag{1}$$

where λ is the default intensity assumed to depend on exogenous state variables X, and E is a unit exponential random variable assumed to be independent of X. Then $N_t = 1_{\{t \geq \tau\}}$ defines a Cox process representing default, with intensity λ.

In a setting with multiple default times $\{\tau^i\}_{i=1}^I$, one can extend the above definition using a collection of independent unit exponential random variables $\{E^i\}_{i=1}^I$. This implies that the default times are independent conditional on the information contained in X, and default correlation arises because of correlation at the intensity level. Recently, Hull and White (2001) and Schonbucher and Schubert (2001) have expressed doubt on the ability of this type of models to match the level of empirical default correlations.[2]

[1] Early examples of reduced-form models include Jarrow and Turnbull (1995), Duffie, Schroder and Skiadas (1996), Jarrow, Lando and Turnbull (1997), Lando (1998), and Duffie and Singleton (1999a).

[2] Partially alleviating this concern, Yu (2003) shows that a wide range of default correlations can arise from different assumptions about what constitute the common state variables X.

To incorporate more flexibility into the modeling of correlated defaults, we assume the existence of default intensities that depend on a larger information set, which includes not only the history of X, but also observations of past defaults. This captures in an economically intuitive way the fact that defaults are driven by macroeconomic factors as well as firm-specific links to other firms.[3]

A drawback of this assumption is the difficulty in finding a valuation formula that depends only on the default intensities, and not explicitly on the default times. When τ is constructed according to (1), Duffie and Singleton (1999a) and Lando (1998) show that defaultable claims can be priced as discounted expectation of terminal payoffs using a modified short rate adjusted for default risk. However, when one allows the intensity to depend on other defaults, this pricing formula fails due to a violation of the "no-jump condition" given by Duffie, Schroder and Skiadas (1996). In fact, it is unclear how one could construct the default times under such cases.

We provide a remedy to this problem using the so-called "total hazard construction" from Norros (1987) and Shaked and Shanthikumar (1987). Given the conditional hazard rates, or intensities, of a collection of stopping times, this is a way to generate the stopping times from a set of independent unit exponential random variables. It results from the fact that the total hazards (or integrated intensities) accumulated by the stopping times by the time they occur are independent unit exponential random variables. While the original prescription calls for stopping times with respect to their internal history, it is straightforward to extend this construction to cases of a broader appeal, where the intensities are also modulated by the exogenous process X.

In a copula-based approach that combines ingredients of a reduced-form model, Schonbucher and Schubert (2001) propose a modification of (1) with the E^i's following a joint distribution $C\left(U^1, \cdots, U^I\right)$, where $U^i = \exp\left(-E^i\right)$ and C is a copula function. We show that our approach can be thought of as an extension of the Schonbucher and Schubert construction where the copula function is indexed by the sample path of X. In other words, the Schonbucher and Schubert construction is a special case of our approach where the copula function is invariant to the history of X.

Our approach also complements Duffie and Singleton (1999b), who sim-

[3]The idea that default "contagion" might affect the valuation of credit derivatives first appeared in David and Lo (2001) and Jarrow and Yu (2001).

ulate correlated defaults using a "recursive inverse-CDF" approach, where the conditional distribution of the next default can depend on state variables and past defaults. We extend their alternative approach, termed "multi-compensator simulation," to situations of equal generality, where the intensities are affected by both past defaults and the history of the state variables.

Recently, Collin-Dufresne, Goldstein and Hugonnier (2003a) show that the Duffie and Singleton (1999a) pricing formula can be preserved even when the no-jump condition is violated, if one switches to a measure that places zero probability on default prior to the maturity of the claim.[4] While this formula has been used to derive analytic solutions for defaultable bond prices in a setting with two issuers, it is not clear whether one could use it under more general settings, say with a large number of issuers or where the defaultable claim is of the basket type. Besides stating that these are setting where our simulation-based approach may be helpful, we present a procedure where the CGH formula can be used to derive, analytically in principle, the marginal distribution of default times in the presence of an arbitrary number of correlated defaults.

We apply our methodology to the valuation of the following credit-risky securities:

1. *Defaultable bonds.* We use the total hazard construction to provide a complete solution to the case of $I = 2$ where each firm's default intensity depends on the default status of the other firm.[5] Previous literature has shown that in this case, defaultable bonds can be priced by ignoring the counterparty risk for the other firm. Using examples with more complex specifications of the default intensities, we show that this intuition holds only for the case of $I = 2$. We also discuss the calibration of the model to market prices for defaultable bonds.

2. *Credit default swaps of the regular type.* These are insurance-like securities which provide a payoff in the event of default by a reference asset. Typical CDS pricing models are only concerned with the credit risk of the reference asset, while ignoring the credit risk of the CDS buyer and seller. We use the case of $I = 3$ to illustrate the impact of the credit quality of the CDS buyer, seller, and reference asset, as well as their interaction on CDS pricing.

[4] This type of "survival measure" also appears in Schonbucher (2000).

[5] A complete solution is also given by Kusuoka (1999) and Collin-Dufresne, Goldstein and Hugonnier (2003a) using different methodologies from ours.

3. *Credit default swaps of the basket type.* These contracts provide a payoff in the event of one or more defaults in a portfolio of reference assets. By definition, basket CDS pricing is sensitive to the specification of default correlation within the basket. We present a case in which the default intensities depend on a common factor in addition to the default status of other firms. Our numerical evidence shows that the CDS premium can be sensitive to both sources of default correlation.

The remainder of this paper is organized as follows. In Section 2 we explain the construction of the default times and relate it to the existing literature on correlated default modeling. In Section 3 we present several examples to illustrate the application of our methodology to the valuation of defaultable securities. We conclude in Section 4.

2 The Model

2.1 Basic Setup

We consider an economy where information is contained in an R^d-valued process X and I individual default times $\tau = \left(\tau^1, \ldots, \tau^I\right)$. Specifically, we assume the existence of a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ satisfying the usual conditions of right-continuity and completeness with respect to P-null sets. Depending upon the specific application, the probability measure P can be interpreted as the objective measure (risk management) or the equivalent martingale measure (valuation). To be consistent with our informational assumptions, we let

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^I, \tag{2}$$

where

$$\begin{aligned} \mathcal{F}_t^X &= \sigma\left(X_s, 0 \leq s \leq t\right), \quad (3) \\ \mathcal{F}_t^i &= \sigma\left(N_s^i, 0 \leq s \leq t\right), \quad (4) \end{aligned}$$

and $N_t^i = 1_{\{t \geq \tau^i\}}$. In other words, $\mathcal{F}_t$ is the minimal sigma-field needed to support both X and τ.

Further specifying the default times, we assume that τ^i possesses a strictly positive $\mathcal{F}_t$-predictable intensity λ^i with right-continuous sample paths and satisfying technical integrability conditions. This means that

$$M_t^i = N_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds \tag{5}$$

is a $(P, \mathcal{F}_t)$-martingale. The intensity λ^i can also be interpreted as the $\mathcal{F}_t$-conditional hazard rate of τ^i, because

$$\lambda_t^i 1_{\{t<\tau^i\}} = \lim_{\Delta t\to 0+} \frac{E\left(N_{t+\Delta t}^i - N_t^i|\mathcal{F}_t\right)}{\Delta t} = \lim_{\Delta t\to 0+} \frac{P\left(t<\tau^i \le t+\Delta t|\mathcal{F}_t\right)}{\Delta t}. \tag{6}$$

Below, we use these two interpretations of the intensity interchangeably. Note that this specification of the intensity includes as a special case $\lambda_t^i = \lambda^i\left(X_t, N_t^{-i}\right)$, a function of the current value of X and N^{-i}, the default status of issuers other than i. This specialization will be used throughout our subsequent examples.

Assuming the existence of an $\mathcal{F}_t$-predictable short rate r then completes the basic setup. This includes the typical situation where r is adapted to $\mathcal{F}_t^X$, as well as the "flight to quality" case where one or more of the default times exerts an influence on r.

Remark 1 Kusuoka (1999) constructs a case with $I = 2$ and no state variables by transforming two independent Poisson processes. The transformation makes use of Girsanov's Theorem for jump processes [see Brémaud (1981)] with a judicious choice of the Radon-Nikodym derivative. Kusuoka shows that in general, the survival probability

$$P\left(\tau^i > T|\mathcal{F}_t\right) \neq 1_{\{\tau^i>t\}} E\left(\exp\left(-\int_t^T \lambda_s^i ds\right)|\mathcal{F}_t\right), \tag{7}$$

unlike the case where the default times are defined by (1) with $\mathcal{F}_t^X$-adapted intensities.

Remark 2 Duffie, Schroder and Skiadas (1996) present a "no-jump condition," which states that if the conditional expectation in (7) does not jump at τ^i, (7) would be an equality. This condition, however, is violated in the current setup, as λ^i in general depends on other default times, whose intensities will jump at τ^i.

Remark 3 Collin-Dufresne, Goldstein and Hugonnier (2003a) show that when the no-jump condition is violated, one has

$$P\left(\tau^i > T|\mathcal{F}_t\right) = 1_{\{\tau^i>t\}} E'\left(\exp\left(-\int_t^T \lambda_s^i ds\right) |\mathcal{F}_t^{'}\right), \tag{8}$$

where the expectation is taken under a new probability measure P' that assigns zero probability to $\tau^i \leq T$, and $\mathcal{F}_t^{'}$ is the completion of $\mathcal{F}_t$ with respect to the P'-null sets. Roughly speaking, this is because

$$E\left(1_{\{\tau^i>T\}}|\mathcal{F}_t\right) = 1_{\{\tau^i>t\}} E'\left(\exp\left(-\int_t^T \lambda_s^i ds\right) |\mathcal{F}_t^{'}\right), \tag{9}$$

if one assumes that

$$\frac{dP'}{dP} = 1_{\{\tau^i>T\}} \exp\left(\int_0^T \lambda_s^i ds\right). \tag{10}$$

Although the CGH formula can be used to derive analytical solutions of the survival probability for the case of $I = 2$, the procedure becomes increasingly complex as I increases, and it is not clear whether the method is tractable for the joint distribution of a large number of default times.

2.2 Recursive Construction of Default Times

The above remarks suggest that the unfinished task for the previous setup is to compute the conditional distribution of the default times using their $\mathcal{F}_t$-intensities. In this subsection we present a constructive approach, where the default times can be generated from a set of unit exponential random variables. Our approach closely follows that of Norros (1987) and Shaked and Shanthikumar (1987).

Specifically, we start with the case of no state variables. In this case the filtration $\mathcal{F}_t$ is the minimal sigma-field supporting the default times, or their "internal history." Noting the explicit dependence of λ^i on other default times, we use the notation $\lambda_i\left(t|n\right)$ to denote the intensity for firm i given the observed default times of n other firms, $t_{k_1}, \ldots, t_{k_n}$, where it is assumed that $0 = t_{k_0} < t_{k_1} < \cdots < t_{k_n} < t < \tau^i$. Then, define the total hazard accumulated by firm i by time t given n observed defaults as

$$\psi_i(t|n) = \sum_{m=1}^{n} \Lambda_i\left(t_{k_m} - t_{k_{m-1}}|m-1\right) + \Lambda_i(t - t_{k_n}|n), \tag{11}$$

where $\Lambda_i(s|m) = \int_{t_{k_m}}^{t_{k_m}+s} \lambda_i(u|m)\,du$ is the total hazard accumulated by firm i for a period of length s following the mth default. It is assumed that there is no default between t_{k_m} and t.

A well-known result [see references in Norros (1987) and Shaked and Shanthikumar (1987)] shows that the total hazards accumulated by the firms until they default are independent unit exponential random variables. This suggests the definition of an inverse function that maps a unit exponential back into the original default time. Define

$$\Lambda_i^{-1}(x|n) = \inf\{s : \Lambda_i(s|n) \geq x\},\ x \geq 0. \tag{12}$$

The following recursive procedure constructs a collection of random variables $\widehat{\tau} = \left(\widehat{\tau}^1, \ldots, \widehat{\tau}^I\right)$:

1. Draw a collection of i.i.d. unit exponentials $E = \left(E^1, \ldots, E^I\right)$.

2. Let

$$k_1 = \underset{1\leq i\leq I}{\arg\min}\left\{\Lambda_i^{-1}\left(E^i|0\right)\right\}, \tag{13}$$

and let

$$\widehat{\tau}^{k_1} = \Lambda_{k_1}^{-1}\left(E^{k_1}|0\right). \tag{14}$$

3. Assume that the values of $\left(\widehat{\tau}^{k_1}, \ldots, \widehat{\tau}^{k_{m-1}}\right)$ are already given, where $m \geq 2$. Define the set $I_{m-1} = \{k_1, \ldots, k_{m-1}\}$ and $\overline{I}_{m-1}$ as the set of firms excluding I_{m-1}. Recall that $\psi_i(t|m-1)$ is the total hazard accumulated by firm i given the first $m-1$ defaults, let

$$k_m = \underset{i\in\overline{I}_{m-1}}{\arg\min}\left\{\Lambda_i^{-1}\left(E^i - \psi_i\left(\widehat{\tau}^{k_{m-1}}|m-1\right)|m-1\right)\right\}, \tag{15}$$

and let

$$\widehat{\tau}^{k_m} = \widehat{\tau}^{k_{m-1}} + \Lambda_{k_m}^{-1}\left(E^{k_m} - \psi_{k_m}\left(\widehat{\tau}^{k_{m-1}}|m-1\right)|m-1\right). \tag{16}$$

4. If $m = I$ then stop. Otherwise, increase m by 1 and go to Step 3.

Lemma 1 *Construct $\widehat{\tau}$ according to Steps 1-4. Let $\mathcal{F}_t$ be the internal history of $\widehat{\tau}$, then $\widehat{\tau}^i$ has $\mathcal{F}_t$-intensity λ^i, for $1 \leq i \leq I$.*

Remark 4 Lemma 1 is a direct consequence of Theorem 3.6 of Shaked and Shanthikumar (1987), which states that $\widehat{\tau} \overset{st}{=} \tau$, where $\overset{st}{=}$ denotes equality in law. It formalizes the idea that multivariate distributions can also be specified using the associated conditional hazard rates.

Next, we consider the initial and more interesting case where state variables X are included. Denote by $\omega \in \mathcal{F}_\infty^X$ a complete sample path of X. Given ω, the intensity λ^i collapses into a function with explicit dependence on the default times. As a result of Lemma 1, the preceding steps can be used to construct random variable $\widehat{\tau}^i$ with $\mathcal{G}_t$-conditional hazard rate λ^i, where $\mathcal{G}_t = \mathcal{F}_\infty^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^I$. Since λ^i is in fact assumed to be $\mathcal{F}_t$-predictable, $\widehat{\tau}^i$ has $\mathcal{F}_t$-conditional hazard rate λ^i as well. Therefore, we replace Step 1 of the previous procedure with:

1'. Draw a complete sample path of X. Draw i.i.d. unit exponentials $E = \left(E^1, \ldots, E^I\right)$ independent of X.

We then have the following result:

Proposition 1 *Construct $\widehat{\tau}$ according to Steps 1' and 2-4. Let $\mathcal{F}_t$ be the minimal sigma-field needed to support X and $\widehat{\tau}$, then $\widehat{\tau}^i$ has $\mathcal{F}_t$-intensity λ^i, for $1 \leq i \leq I$.*

Remark 5 When the $\mathcal{F}_t$-intensities are in fact $\mathcal{F}_t^X$-adapted and there is no explicit dependence on other default times, the sequence of past defaults is irrelevant and one recovers the definition of τ^i as

$$\tau^i = \Lambda_i^{-1}\left(E^i\right) = \inf\left\{t : \Lambda_i\left(t\right) \geq E^i\right\}, \tag{17}$$

where

$$\Lambda_i\left(t\right) = \int_0^t \lambda_s^i ds. \tag{18}$$

This yields the standard construction where the default times are conditionally independent given X.

2.3 Comparison with the Copula-Based Approach

Due to the ability of copula functions to isolate the dependence structure in multivariate distributions [see Nelsen (1999) for an introduction to copulas], they have become increasingly popular in the modeling of correlated defaults. In this subsection we take a comparative look at a recent copula-based approach outlined by Schonbucher and Schubert (2001, SS), which contains elements of a reduced-form model.

Simply put, this approach assumes that τ^i is defined as

$$\tau^i = \inf \left\{ t : \int_0^t h_s^i ds \geq E^i \right\}, \tag{19}$$

where h^i is an $\mathcal{F}_t^X$-adapted process. However, instead of maintaining that the unit exponentials $\left(E^1, \ldots, E^I\right)$ are independently distributed, it assumes that they are governed by a joint distribution function $C\left(U^1, \cdots, U^I\right)$, where $U^i = \exp\left(-E^i\right)$ and C is a copula function. Note, in particular, that SS show that h^i is the intensity of τ^i with respect to the filtration $\mathcal{F}_t^X \vee \mathcal{F}_t^i$, and not the intensity in the most general sense (with respect to the full filtration $\mathcal{F}_t$).

To understand the relationship between the SS copula approach and the total hazard construction, we again start with a simple case with no state variables, where a collection of random variables $\tau = \left(\tau^1, \ldots, \tau^I\right)$ have a sufficiently smooth joint distribution $F\left(t_1, \ldots, t_I\right) = F\left(\tau^1 \leq t_1, \ldots, \tau^I \leq t_I\right)$ and marginal distribution $G_i\left(t_i\right) = G_i\left(\tau^i \leq t_i\right)$. By Sklar's Theorem [see Nelsen (1999)], there exists a copula function C such that

$$F\left(t_1, \ldots, t_I\right) = C\left(G_1\left(t_1\right), \ldots, G_I\left(t_I\right)\right). \tag{20}$$

Then, let

$$h_t^i = \frac{1}{1 - G_i(t)} \frac{dG_i(t)}{dt}. \tag{21}$$

This is the probability per unit time of τ^i occurring in the next instant, conditional on it has not occurred prior to t, i.e. the hazard rate of τ^i when other defaults are ignored.

When we apply the SS construction using the above hazard rate h^i and copula function C, it is clear that the resulting default times will have a joint distribution given by F. On the other hand, the total hazard construction starts from the conditional hazard rates given all available information, including observed defaults. As shown by Shaked and Shanthikumar (1987), this procedure can also yield default times with the joint distribution F. It is in this sense that the two approaches are "equivalent" ways for modeling correlated default times with respect to their internal histories.

Turning now to the more general case with the addition of state variables X, the total hazard construction as described in Steps 1' and 2-4 gives default times with joint distribution F_ω conditional on a particular sample path $\omega \in \mathcal{F}_\infty^X$. If the dependence structure of this distribution is given by a copula function, then this copula will generally depend on ω. On the other hand, we see that in the SS construction, the copula is invariant to the sample path of X. As a result, only the marginal distributions of the default times (ignoring other defaults) change with the state variables, while the dependence structure does not.[6] It is in this sense that the copula-based approach is "static," whereas the intensity-based approach truly incorporates the dynamics of the state variables.

3 Applications

3.1 Pricing Defaultable Bonds

For simplicity, we price zero-coupon bonds with zero recovery and under a constant default-free short rate r. The defaultable bond price is then proportional to the conditional survival probability, $P(\tau^i > T|\mathcal{F}_t)$. In this subsection we discuss the behavior of this entity under various assumptions of the default correlation structure.

3.1.1 The Case of Two Firms

Jarrow and Yu (2001) present a "looping default" case with two firms, where the default of one firm affects the default intensity of the other firm. In other words, they assume two default times, τ^A and τ^B, with the following intensities with respect to the internal history of defaults:

$$\lambda_t^A = a_1 + a_2 1_{\{t \geq \tau^B\}}, \tag{22}$$
$$\lambda_t^B = b_1 + b_2 1_{\{t \geq \tau^A\}}. \tag{23}$$

[6] Of course, as discussed in Schonbucher and Schubert (2001), the copula function changes at the time of an observed default.

Unable to provide an explicit solution for the distribution of the default times, they turn to a simpler case with

$$\begin{aligned}\lambda_t^A &= a, &(24)\\ \lambda_t^B &= b_1 + b_2 1_{\{t\geq\tau^A\}}. &(25)\end{aligned}$$

Namely, they assume that firm A is a "primary" firm not affected by counterparty default and firm B is a "secondary" firm whose default intensity jumps at the default of firm A. Below, we present a solution to the more general case using the total hazard construction.

First, note that with no default up to t, the total hazards accumulated from 0 to t are $\Lambda_A(t) = a_1 t$ and $\Lambda_B(t) = b_1 t$. If firm A has defaulted at t_1, the total hazard accumulated by firm B from t_1 to t_1+t is $\Lambda_B(t|t_1) = (b_1+b_2)\,t$. Similarly, if firm B has defaulted at t_2, one can set $\Lambda_A(t|t_2) = (a_1+a_2)\,t$. Using the inverse of these functions, we can construct two default times $\widehat{\tau}^A$ and $\widehat{\tau}^B$ from two independent unit exponential random variables E^A and E^B as:

$$\widehat{\tau}^A = \begin{cases} \frac{E^A}{a_1}, & \frac{E^A}{a_1} \leq \frac{E^B}{b_1}, \\ \frac{E^B}{b_1} + \frac{1}{a_1+a_2}\left(E^A - \frac{a_1}{b_1}E^B\right), & \frac{E^A}{a_1} > \frac{E^B}{b_1}, \end{cases} \tag{26}$$

$$\widehat{\tau}^B = \begin{cases} \frac{E^A}{a_1} + \frac{1}{b_1+b_2}\left(E^B - \frac{b_1}{a_1}E^A\right), & \frac{E^A}{a_1} \leq \frac{E^B}{b_1}, \\ \frac{E^B}{b_1}, & \frac{E^A}{a_1} > \frac{E^B}{b_1}. \end{cases} \tag{27}$$

From this, we derive the joint density of $\widehat{\tau}^A$ and $\widehat{\tau}^B$, which is also shared by the original default times τ^A and τ^B, as

$$f(t_1,t_2) = \begin{cases} a_1(b_1+b_2)\,e^{-(a_1-b_2)t_1-(b_1+b_2)t_2}, & t_1 \leq t_2, \\ b_1(a_1+a_2)\,e^{-(b_1-a_2)t_2-(a_1+a_2)t_1}, & t_1 > t_2. \end{cases} \tag{28}$$

It follows that the marginal densities of τ^A and τ^B are

$$\begin{aligned} g_A(t_1) &= \frac{(a_1+a_2)\,b_1}{b_1-a_2}\left(e^{-(a_1+a_2)t_1} - e^{-(a_1+b_1)t_1}\right) + a_1 e^{-(a_1+b_1)t_1}, &(29)\\ g_B(t_2) &= \frac{(b_1+b_2)\,a_1}{a_1-b_2}\left(e^{-(b_1+b_2)t_2} - e^{-(a_1+b_1)t_2}\right) + b_1 e^{-(a_1+b_1)t_2}, &(30)\end{aligned}$$

and their marginal distributions are

$$\begin{aligned} G_A(t_1) &= 1 - \frac{b_1 e^{-(a_1+a_2)t_1} - a_2 e^{-(a_1+b_1)t_1}}{b_1-a_2}, &(31)\\ G_B(t_2) &= 1 - \frac{a_1 e^{-(b_1+b_2)t_1} - b_2 e^{-(a_1+b_1)t_1}}{a_1-b_2}. &(32)\end{aligned}$$

From these expressions, we see that neither G_A depends on b_2, nor G_B on a_2. In fact, the expression above for G_B coincides with Jarrow and Yu's solution for default times with intensities specified by (24) and (25). Intuitively, what occurs to firm B after the default of firm A should have no impact on the distribution of firm A's default time. This is why in computing G_A, we can effectively ignore the default dependency for firm B.[7]

For another interpretation of this result we invoke the Collin-Dufresne, Goldstein and Hugonnier (2003a) formula, where the survival probability can be written as

$$P\left(\tau^A > t_1\right) = E'\left(\exp\left(-\int_0^{t_1} \lambda_s^A ds\right)\right) \tag{33}$$

under a change of measure to P' that puts zero probability on $\left\{\tau^A \leq t_1\right\}$. Hence the intensity λ^B is almost surely equal to b_1 under P'. Equation (33) then suggests that one can correctly obtain the survival probability of firm A by treating firm B as having a constant intensity b_1.

3.1.2 The Case of Three Firms

The case of $I = 3$ is not merely a mechanical repetition of the case of $I = 2$. Instead, our interest in this case is motivated by several considerations. First and foremost and as shall be revisited below, this case is relevant to the modeling of credit default swaps, where the credit quality of the seller, the buyer, and the reference asset, as well as their interaction may all play a role in valuation. Second, this is the simplest case where default dependency matters in a non-trivial sense for the pricing of defaultable bonds. In other words, the previous intuition that one can ignore the default dependency of other firms when pricing a defaultable bond does not hold in this case. Lastly, we use this case to demonstrate a general procedure for using the CGH formula to derive the marginal distribution of any number of correlated default times.

We note the definition of the default intensities:

$$\begin{aligned}
\lambda_t^A &= a_{10} + a_{12} 1_{\{t \geq \tau^B\}} + a_{13} 1_{\{t \geq \tau^C\}}, &(34)\\
\lambda_t^B &= a_{20} + a_{21} 1_{\{t \geq \tau^A\}} + a_{23} 1_{\{t \geq \tau^C\}}, &(35)\\
\lambda_t^C &= a_{30} + a_{31} 1_{\{t \geq \tau^A\}} + a_{32} 1_{\{t \geq \tau^B\}}. &(36)
\end{aligned}$$

[7] This interesting observation and its interpretation are first given in Bielecki and Rutkowski (2002, Chapter 9).

The most general specification would have interactions between the default times, which we leave out in the interest of brevity.

Solution using the Total Hazard Construction First, we use the total hazard construction to identify the joint density of the default times. To derive $f(t_1, t_2, t_3)$ where $t_1 < t_2 < t_3$, we note that

$$\begin{aligned} \widehat{\tau}^A &= \frac{E^A}{a_{10}}, & (37)\\ \widehat{\tau}^B &= \widehat{\tau}^A + \frac{E^B - a_{20}\widehat{\tau}^A}{a_{20} + a_{21}}, & (38)\\ \widehat{\tau}^C &= \widehat{\tau}^B + \frac{E^C - a_{30}\widehat{\tau}^A - (a_{30} + a_{31})\left(\widehat{\tau}^B - \widehat{\tau}^A\right)}{a_{30} + a_{31} + a_{32}}, & (39) \end{aligned}$$

where we assume that

$$\begin{aligned} \frac{E^A}{a_{10}} &< \frac{E^B}{a_{20}}, & (40)\\ \frac{E^A}{a_{10}} &< \frac{E^C}{a_{30}}, & (41)\\ \frac{E^B - \frac{a_{20}}{a_{10}}E^A}{a_{20} + a_{21}} &< \frac{E^C - \frac{a_{30}}{a_{10}}E^A}{a_{30} + a_{31}}. & (42) \end{aligned}$$

Following a similar procedure for the other five permutations of (t_1, t_2, t_3), we derive the joint density as:

$$f(t_1, t_2, t_3) = \begin{cases} a_{10}(a_{20} + a_{21})(a_{30} + a_{31} + a_{32}) \times \\ e^{-(a_{10} - a_{21} - a_{31})t_1 - (a_{20} + a_{21} - a_{32})t_2 - (a_{30} + a_{31} + a_{32})t_3}, \ t_1 < t_2 < t_3, \\ a_{10}(a_{30} + a_{31})(a_{20} + a_{21} + a_{23}) \times \\ e^{-(a_{10} - a_{31} - a_{21})t_1 - (a_{30} + a_{31} - a_{23})t_3 - (a_{20} + a_{21} + a_{23})t_2}, \ t_1 < t_3 < t_2, \\ a_{20}(a_{10} + a_{12})(a_{30} + a_{31} + a_{32}) \times \\ e^{-(a_{20} - a_{12} - a_{32})t_2 - (a_{10} + a_{12} - a_{31})t_1 - (a_{30} + a_{31} + a_{32})t_3}, \ t_2 < t_1 < t_3, \\ a_{30}(a_{10} + a_{13})(a_{20} + a_{21} + a_{23}) \times \\ e^{-(a_{30} - a_{13} - a_{23})t_3 - (a_{10} + a_{13} - a_{21})t_1 - (a_{20} + a_{21} + a_{23})t_2}, \ t_3 < t_1 < t_2, \\ a_{20}(a_{30} + a_{32})(a_{10} + a_{12} + a_{13}) \times \\ e^{-(a_{20} - a_{32} - a_{13})t_2 - (a_{30} + a_{32} - a_{13})t_3 - (a_{10} + a_{12} + a_{13})t_1}, \ t_2 < t_3 < t_1, \\ a_{30}(a_{20} + a_{23})(a_{10} + a_{12} + a_{13}) \times \\ e^{-(a_{30} - a_{23} - a_{12})t_3 - (a_{20} + a_{23} - a_{12})t_2 - (a_{10} + a_{12} + a_{13})t_1}, \ t_3 < t_2 < t_1. \end{cases} \tag{43}$$

This joint density comes from differentiating the product density of three independent unit exponential random variables, treating them as total hazards accumulated up to the respective default times.

We then integrate out t_2 and t_3 to obtain the marginal density $g_A(t_1)$ for τ^A as:

$$\begin{aligned}
g_A(t_1) &= a_{10} e^{-(a_{10}+a_{20}+a_{30})t_1} + \\
&\frac{a_{20}(a_{10}+a_{12})}{a_{20}-a_{12}-a_{32}} \left(e^{-(a_{10}+a_{12}+a_{30}+a_{32})t_1} - e^{-(a_{10}+a_{20}+a_{30})t_1}\right) + \\
&\frac{a_{30}(a_{10}+a_{13})}{a_{30}-a_{13}-a_{23}} \left(e^{-(a_{10}+a_{13}+a_{20}+a_{23})t_1} - e^{-(a_{10}+a_{20}+a_{30})t_1}\right) + \\
&\frac{a_{20}(a_{30}+a_{32})(a_{10}+a_{12}+a_{13})}{a_{20}-a_{32}-a_{12}} \left(\frac{1}{a_{30}+a_{32}-a_{13}}\left(1-e^{-(a_{30}+a_{32}-a_{13})t_1}\right) - \right. \\
&\left. \frac{1}{a_{20}+a_{30}-a_{12}-a_{13}}\left(1-e^{-(a_{20}+a_{30}-a_{12}-a_{13})t_1}\right)\right) e^{-(a_{10}+a_{12}+a_{13})t_1} + \\
&\frac{a_{30}(a_{20}+a_{23})(a_{10}+a_{12}+a_{13})}{a_{30}-a_{23}-a_{13}} \left(\frac{1}{a_{20}+a_{23}-a_{12}}\left(1-e^{-(a_{20}+a_{23}-a_{12})t_1}\right) - \right. \\
&\left. \frac{1}{a_{20}+a_{30}-a_{12}-a_{13}}\left(1-e^{-(a_{20}+a_{30}-a_{12}-a_{13})t_1}\right)\right) e^{-(a_{10}+a_{12}+a_{13})t_1}. \qquad (44)
\end{aligned}$$

This result shows that $g_A(t_1)$ is independent of a_{21} or a_{31}, which is consistent with the prior intuition. However, $g_A(t_1)$ is not independent of a_{23} and a_{32}. In other words, one should not ignore the default dependency between τ^B and τ^C when computing the marginal distribution of τ^A.

Solution using the CGH Formula One could also use the CGH formula (33) to find the marginal distribution of τ^A. First, note that by Lemma 2 of CGH, since the compensated martingale $M_t^j = N_t^j - \int_0^{t\wedge\tau^j} \lambda_s^j ds$, $j = B, C$ does not jump at τ^A, the intensities for the default times τ^B and τ^C remain unchanged under P'. On the other hand, the intensity for τ^A is zero under P'. Since, by the total hazard construction, the joint distribution of default times is completely determined by their conditional hazard rates, we can explicitly compute the expectation on the right hand side of (33), letting $\lambda_t^A = a_{10} + a_{12}1_{\{t\geq\tau^B\}} + a_{13}1_{\{t\geq\tau^C\}}$ and treating τ^B and τ^C as having joint density determined by

$$\lambda_t^B = a_{20} + a_{23}1_{\{t\geq\tau^C\}}, \qquad (45)$$
$$\lambda_t^C = a_{30} + a_{32}1_{\{t\geq\tau^B\}}. \qquad (46)$$

This joint density of two default times can be determined by suitably modifying (28). An explicit calculation using this method produces results identical to (44).

The above illustration suggests that in a system of I interacting default times, we can obtain the marginal distribution of any given default time using the CGH formula. The expectation therein should be computed with respect to the joint density of the other $I-1$ default times, ignoring the impact of the given default time on this remaining collection. Notice that, although this procedure reduces the dimensionality of the problem by 1, in a large system it still requires the joint distribution of a large number of default times, leaving numerical simulation based on the total hazard construction as the only choice.

3.1.3 A Simple Specification with Multiple Firms

The type of default dependency modeled in this paper is in fact based on solid empirical evidence. For example, Lang and Stulz (1992) find that there are significant industry-wide abnormal stock returns in response to a bankruptcy filing. Newman and Rierson (2003) show that European telecom credit spreads widened by 10 basis points in June 2000 in response to a large issue by Deutsche Telekom. Evidence that a single "credit event," defined as a large monthly change in an individual corporate bond yield spread, can affect the entire corporate bond market index is presented by Collin-Dufresne, Goldstein and Helwege (2003b).

To model such contagion effects in a large portfolio (say with I different names), one may wish to start from something quite general, say

$$\lambda_t^i = a_{i0} + \sum_{j=1}^{I} a_{ij} 1_{\{t \geq \tau^j\}}, \tag{47}$$

where $a_{ii} = 0$. Here, a_{ij} represents the increase in the ith intensity given a credit event for the jth name. We can then price credit-risky securities whose payoffs are linked to this portfolio by Monte Carlo simulation based on the total hazard construction.

Clearly, the above specification would be vacuous in the absence of a methodology to calibrate the individual coefficients. A sensible approach following Lang and Stulz (1992), for example, is to estimate the coefficients from bond price changes in reaction to credit events in the same industry.

Consistent with this approach, we consider the following simple specification:

$$\lambda_t^i = a_1 + a_2 1_{\{t \geq \tau_F\}}, \tag{48}$$

where $\tau_F = \min\left(\tau^1, \ldots, \tau^I\right)$ is the first-to-default time. This case differs from the more general specification (47) in two aspects. First, all firms are assumed to have the same default intensity.[8] Second, only the first-to-default is relevant for generating default dependency. The first assumption is consistent with estimating a_2 from the average price reaction of a large cross-section of bonds to "typical" credit events. The second can be interpreted as a reasonable approximation in a population of high quality credits where defaults are relatively infrequent.

As in the case of $I = 3$, the marginal distribution of the default times as described in (48) can be solved in two ways. First, using the total hazard construction we note that the first-to-default time $\widehat{\tau}_F$ can be written as

$$\widehat{\tau}_F = \min\left(\frac{E^1}{a}, \frac{E^2}{a}, \ldots, \frac{E^I}{a}\right), \tag{49}$$

and the individual default times are given through their respective unit exponential random variables as

$$\widehat{\tau}^i = \frac{E^i - a_1\widehat{\tau}_F}{a_1 + a_2} + \widehat{\tau}_F. \tag{50}$$

Using the fact that $\widehat{\tau}_F = \min\left(E^1/a, \widehat{\tau}'_F\right)$ where $\widehat{\tau}'_F = \min\left(E^2, \ldots, E^I\right)/a$, and $\widehat{\tau}'_F$ is an exponential random variables with rate $(I-1)\,a_1$ independent of E^1, we can derive the marginal density of τ^1 as:

$$g_1(t_1) = \frac{(I-1)\,a_1\,(a_1+a_2)\,e^{-(a_1+a_2)t_1} - I a_1 a_2 e^{-I a_1 t_1}}{(I-1)\,a_1 - a_2}. \tag{51}$$

Second, using the fact that under the probability measure P', defined in (10) where $i = 1$, τ_F is an exponential random variable with rate $(I-1)\,a_1$, we can use the CGH formula to compute the survival probability of τ^1 as

$$P\left(\tau^1 > t_1\right) = \frac{(I-1)\,a_1 e^{-(a_1+a_2)t_1} - a_2 e^{-I a_1 t_1}}{(I-1)\,a_1 - a_2}. \tag{52}$$

Further differentiation yields results identical to (51).

[8] This assumption is similar to the use of symmetric copulas in default dependence modeling, which is also motivated by tractability as well as ease of calibration.

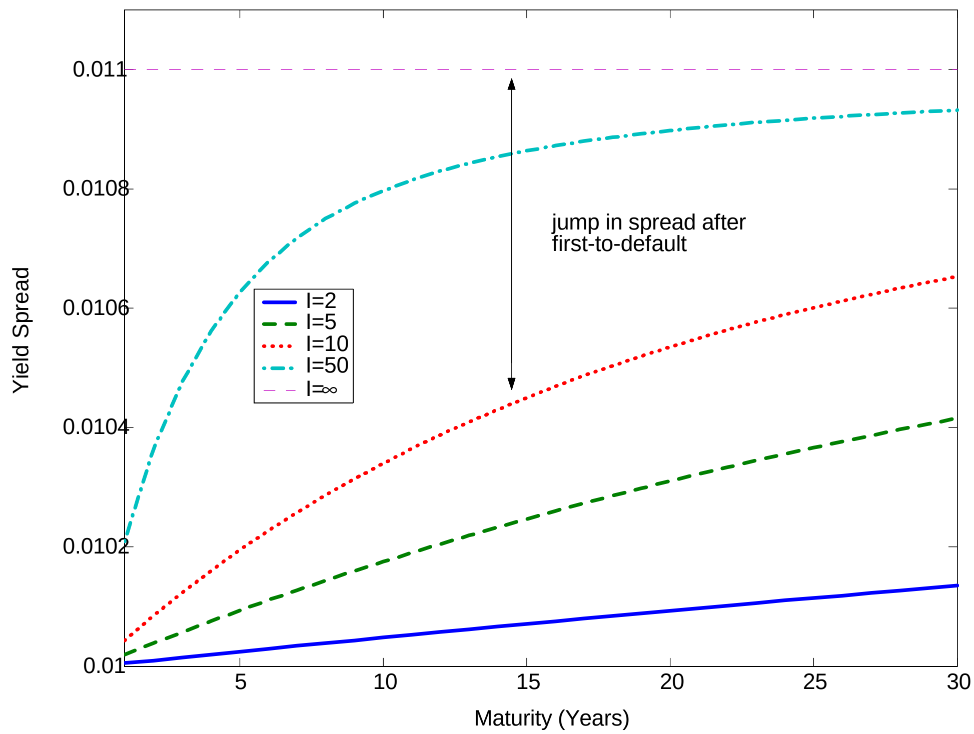

Figure 1: The term structure of yield spreads on defaultable zero-coupon bonds. We assume that $a_1 = 0.01$, $a_2 = 0.001$, and $I = 2, 5, 10, 50$, or ∞.

Remark 6 Taking the limit of $I \to \infty$ in (52), we see that τ^1 converges to a default time with intensity $a_1 + a_2$ in the sense that the survival probability converges to $\exp(-(a_1 + a_2) t_1)$. Intuitively, as I increases, the first-to-default occurs sooner. In the limit of $I \to \infty$, τ_F is equal to 0 and the intensity of τ^1 is equal to $a_1 + a_2$ almost surely.

Remark 7 Ignoring the default dependency in this example amounts to computing the survival probability as

$$P^{\text{incorrect}}\left(\tau^1 > t_1\right) = E\left(\exp\left(-\int_0^{t_1}\left(a_1 + a_2 1_{\{s \geq \tau_F\}}\right) ds\right)\right), \tag{53}$$

treating τ_F as exponentially distributed with rate Ia_1. This is the same as replacing $I-1$ in (52) with I. The two are clearly different and the difference is larger for smaller I, disappearing as $I \to \infty$.

In Figure 1, we present a numerical illustration of this case. Specifically, we assume a constant interest rate, $a_1 = 0.01$, $a_2 = 0.001$, and price a defaultable zero-coupon bond. The figure plots the yield spread on such a bond as a function of maturity for various values of I, assuming that no default has occurred in the system. As a reference, the thin dotted line represents the case where the first-to-default has just occurred (or, as mentioned in Remark 7, the case of $I \to \infty$). The gap between other lines and the thin dotted line represents the jump in yield spread at the first-to-default, which can be used to identify the coefficient a_2.

For example, for an industry that consists of 10 firms, if one estimates that the spreads on 5-year zero-coupon corporate bonds are equal to 150 bps, and that a default in the industry adds another 10 bps to these spreads, we can imply out the default intensity parameters as:

$$a_1 = 0.1464,\ a_2 = 0.00136. \tag{54}$$

3.2 Pricing Credit Default Swaps

In this subsection we revisit the three-firm example in Section 3.1.2. The focus here is on the pricing of credit default swaps (CDS), an insurance-like security which helps to transfer credit risk. Specifically, we assume that the buyer of the CDS agrees to make continuous premium payments at a fixed rate until the expiration of the contract. The seller agrees to pay \$1 at the expiration of the contract if the reference credit defaults prior to expiration.[9]

Using the notation from Section 3.1.2, we assume that the CDS buyer has default time τ^A with intensity λ^A, the seller, τ^B with λ^B, and the reference credit, τ^C with λ^C. We also assume a short rate r to be used for discounting cash flows. The present value of the premium payment from the buyer is then

$$E\left(\int_0^T \exp\left(-\int_0^s r_u du\right) y 1_{\{s<\tau^A\}} ds\right), \tag{55}$$

where y denotes the CDS premium rate and T the time to expiration of the contract. The indicator function reflects the fact that the premium payment stops given a default by the CDS buyer.

The present value of the seller's payment at the time of default by the reference credit is

$$E\left(\exp\left(-\int_0^T r_u du\right) 1_{\{\tau^C \leq T, \tau^B > T\}}\right). \tag{56}$$

The indicator here shows that the default protection is effective if the reference credit event occurs prior to the CDS expiration, and then only if the seller is still alive to honor its obligations. Setting the two sides of the contract equal in value, we obtain the CDS premium as

$$y = \frac{E\left(\exp\left(-\int_0^T r_u du\right) 1_{\{\tau^C \leq T, \tau^B > T\}}\right)}{E\left(\int_0^T \exp\left(-\int_0^s r_u du\right) 1_{\{s<\tau^A\}} ds\right)}. \tag{57}$$

Remark 8 When λ^A, λ^B, λ^C, and the short rate r are all $\mathcal{F}_t^X$-adapted processes, equation (57) can be reduced to [following Lando (1998)]:

$$y = \frac{E\left(\exp\left(-\int_0^T \left(r_u + \lambda_u^B\right) du\right) - \exp\left(-\int_0^T \left(r_u + \lambda_u^B + \lambda_u^C\right) du\right)\right)}{E\left(\int_0^T \exp\left(-\int_0^s \left(r_u + \lambda_u^A\right) du\right) ds\right)}. \tag{58}$$

[9] In a typical CDS contract, the premium payment stops when the reference credit defaults, upon which the default protection is paid. With our assumptions, CDS pricing is more sensitive to the assumed default dependency structure. We also abstract away from realistic features such as recovery rate, physical or cash settlement, cheapest-to-deliver options, accrued interest, and considerations for debt restructuring. For an excellent introduction to the conventions of the CDS market, see Berndt et al. (2004).

This is similar to an expression for the CDS premium in Longstaff, Mithal and Neis (2003). The difference is due to our assumptions about the timing of the cash flows and consideration for buyer and seller credit risk.

Instead of attempting analytical solutions as in Section 3.1.2, we use Monte-Carlo simulation to numerically evaluate the expectations in (57). Specifically, we assume a base case with the following parameters: $r = 0.05$, $T = 5$, $a_{10} = a_{20} = a_{30} = 0.05$, and $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.01$, where the coefficients a_{ij} are defined in reference to (34)-(36). We then vary individual coefficients, keeping others fixed at their base case values, and examine the corresponding change in the CDS premium y.

The results of this exercise are easy to understand. Generally, anything that elevates the credit risk of the buyer or reference asset, or reduces the credit risk of the seller, will increase the CDS premium. This can be a direct effect, as illustrated in the top three panels in Figure 2, or an indirect effect, as shown in the other six panels. Notably, an increase in a_{23} causes the default correlation between the seller and the reference credit to go up, reducing the CDS premium. This is the essence of an example on CDS pricing in Jarrow and Yu (2001). Here we see that default contagion can also propagate between the buyer and the seller (a_{12} and a_{21}), and between the reference and the buyer (a_{31} and a_{13}).

The absence of an effect due to changes in a_{32} can be understood as follows. Since a protection payment requires that the seller does not default until after the reference credit event, this coefficient has no effect on the numerator of (57). Instead, it enters CDS valuation by altering the distribution of the buyer's default time. If changing a_{10} constitutes a direct effect on the distribution of τ^A, changing a_{31} a first order effect, then this would be a second order effect, too small to be picked up by numerical simulations presented here.

3.3 Pricing Basket Credit Default Swaps

In this subsection we apply the simulation-based approach to the valuation of basket credit default swaps. An example of a basket CDS is a contract that pays \$1 if the first-to-default out of a portfolio of reference credits occurs prior to expiration. For simplicity, we assume that the payment (if any) occurs at expiration, and that the buyer pays a premium at the initiation of the swap. With a constant interest rate r, this allows us to write the premium on the

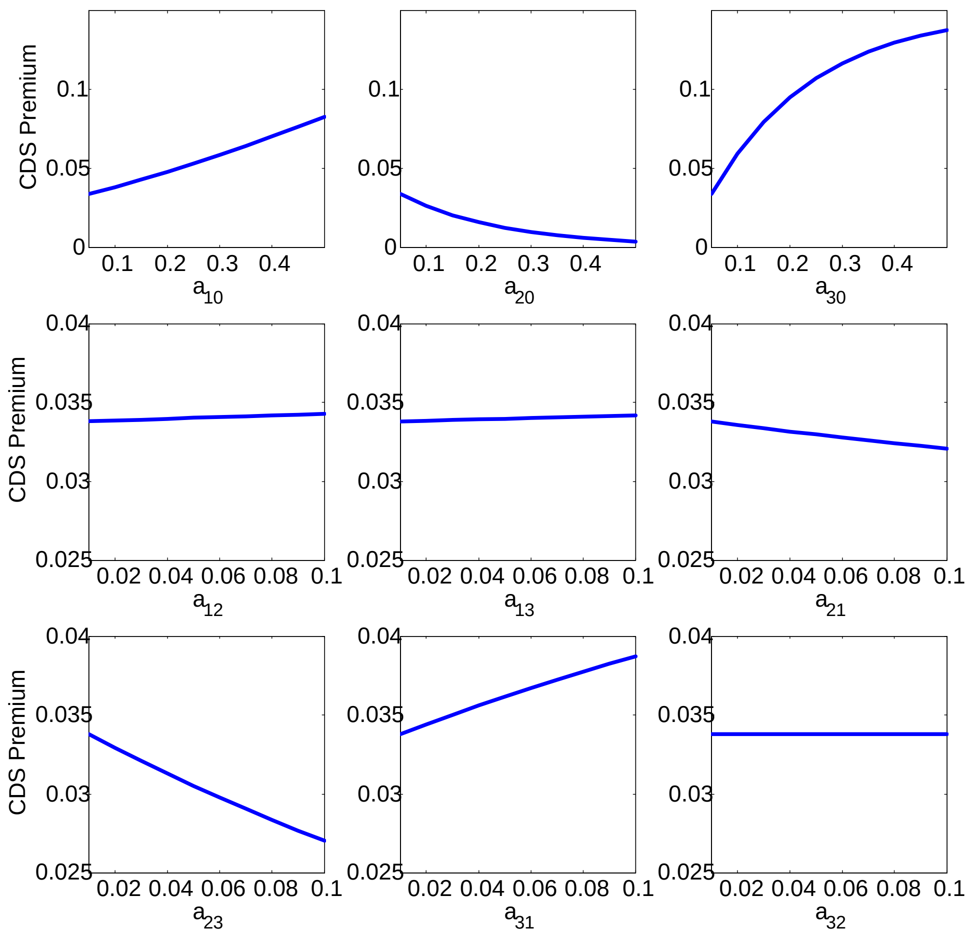

Figure 2: Changes in the CDS premium with respect to the default intensity parameters. All parameters are at their base case values except for the one being plotted.

nth-to-default CDS as

$$y_n = \exp(-rT)\, P\left(\tau^{(n)} \leq T\right), \tag{59}$$

where T is the maturity of the swap and $\tau^{(n)}$ denotes the nth-to-default time.

Evidently, the pricing of basket CDS is sensitive to default correlation among the reference credits. The case where all defaults tend to happen at the same time is clearly different from a case where all defaults are independent. To examine the effect of the default correlation structure on basket CDS pricing, we assume the following default intensity for I reference credits:

$$\lambda_t^i = a + bF_t + \delta 1_{\left\{t \geq \tau^{(1)}\right\}}, \tag{60}$$

which generalizes (48) by including a common factor F. This specification recognizes that default correlation can arise by two mechanisms: 1) correlation of the intensities through exposure to the same factor F; 2) default contagion through a jump in the intensity of size δ at the first-to-default. With empirically motivated choices for the coefficients, we can then ask what fraction of the total default correlation is attributed to each source.

To simulate the default times, we set an overall horizon of, say, $N = 100$ years, and choose an interval of one year for discretization. Following the total hazard construction, we first generate a discrete-time sample path of length N for the process F_t, summarized by $\{F_j\}_{0 \leq j \leq 100}$. Then, for each independent unit exponential E^i, we find the integer n_i such that

$$\sum_{j=0}^{n_i - 2} (a + bF_j) < E^i \leq \sum_{j=0}^{n_i - 1} (a + bF_j). \tag{61}$$

The smallest such integer $n = \min(n_1, \ldots, n_I)$ is defined as the first-to-default time. In this procedure, a simple predictable process is used to approximate F_t, thus turning the integral in the definition of the total hazard into a summation, resulting in equation (61).

Having identified the first-to-default, one can show that the default time $\widehat{\tau}^i$ can be determined by choosing an integer m_i such that

$$\sum_{j=n}^{m_i - 2} (a + bF_j + \delta) < E^i - \sum_{j=0}^{n-1} (a + bF_j) \leq \sum_{j=n}^{m_i - 1} (a + bF_j + \delta), \tag{62}$$

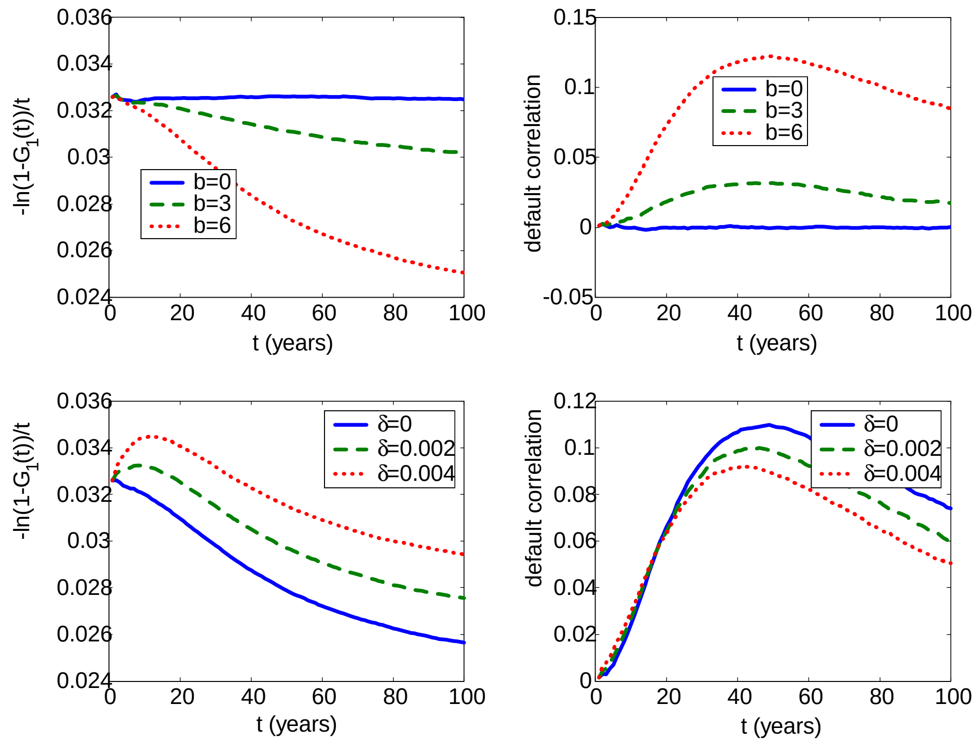

Figure 3: The effect of the common factor and default contagion on the marginal distribution and default correlation functions.

and then setting $\widehat{\tau}^i$ to m_i. If nothing less than 100 can be found to satisfy this inequality, then we record the respective default time as being greater than 100 years. We repeat the above procedure many times in order to generate the marginal distribution and default correlation functions over the domain $[0, 100]$.

As an illustration of this procedure, assume that there are 10 firms in the industry. We take F_t to be the first common factor in the Driessen (2003) model. This is a square-root diffusion with parameters $\kappa = 0.03$, $\theta = 0.005$, and $\sigma = 0.016$. In the base case we take $a = 0.004$ and $b = 5.707$, which correspond to Driessen's Baa intensity parameters. In addition, we assume that $\delta = 0.002$. This represents roughly a five percent jump over the long-run mean of the default intensity.

To study the effect of the common factor, we set $\delta = 0$ and change b while maintaining the long-run mean of the default intensity at the base case value (which means adjusting the value of a simultaneously). The results are presented in the upper panels of Figure 3. In the case where $b = 0$, we let $a = 0.032535$. One can see that there is zero default correlation between the firms and that the marginal distribution reflects a constant intensity equal to a. As b is increased, the survival (default) probability increases (decreases) slightly due to the effect of Jensen's inequality. The elevation of default correlation, however, is much more noticeable.

In the lower panels of Figure 3, we also verify the impact of default contagion on the outputs of the model. Specifically, we move away from the base case by increasing δ from 0 to 0.004. As indicated by the plot, this increases default probability for all maturities. Default correlation, on the other hand, initially increases with δ but decreases with δ at longer maturities. This is because at longer maturities the contagion effect has already resulted in increased intensities, while the factor sensitivity of the intensities has not changed.

We then take this example to the pricing of basket CDS. Specifically, we assume a portfolio with 30 reference credits with the above intensity, a constant interest rate of 5%, and a time to expiration of 5 years.

Figure 4 demonstrates how the correlation structure can affect the valuation of basket CDS. First, with the overall mean of the intensity fixed, an increase in the dependence on the common factor increases (decreases) the probability that a large (small) number of defaults are observed prior to the expiration of the swap. This is due to the default clustering effect introduced by the common factor exposure of the default intensities.

A second way to inject more default correlation into the system is to increase the parameter δ, which describes the extent of default contagion. As shown by the lower panel, an increase in δ raises the basket CDS premium for all n. The first-to-default premium is unchanged, however, as δ has no effect on the distribution of the first-to-default time.

4 Conclusion

We introduce an intensity-based model of correlated defaults that offers more flexibility than models based on the conditionally independent construction (i.e. Cox processes). Specifically, we assume that the default intensities are

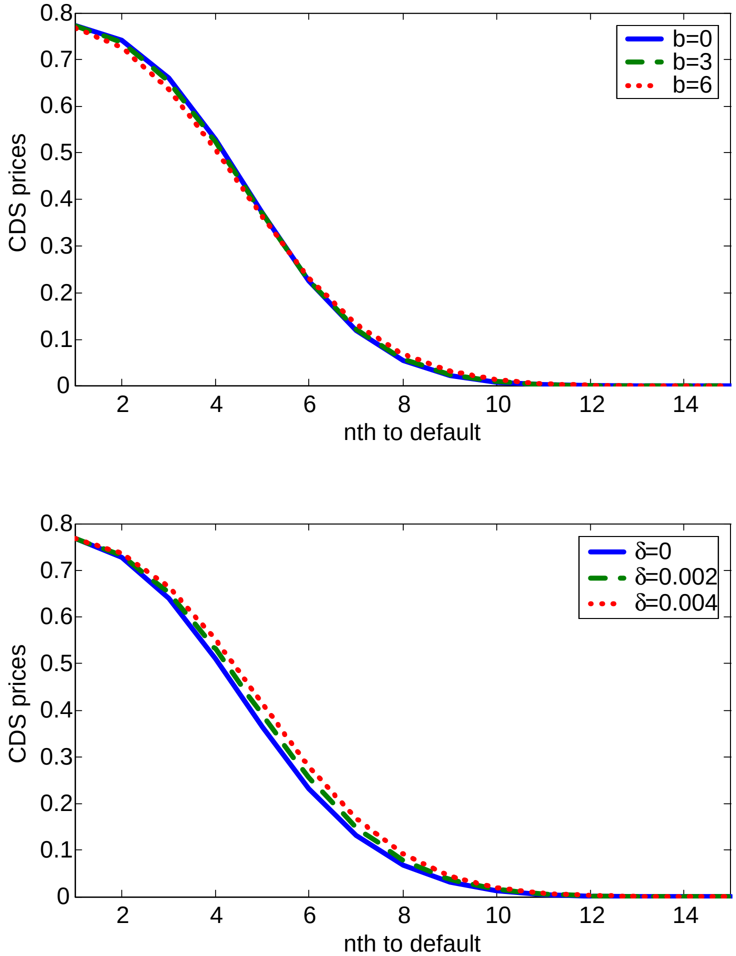

Figure 4: The effect of the common factor and default contagion on basket CDS premiums.

driven by external common factors as well as defaults in the system. We show that the "total hazard construction" of Norros (1987) and Shaked and Shanthikumar (1987) can be extended to the simulation of default times with such generalized intensities.

This approach can accommodate a wide range of default dependency structures. In fact, we show that the Schonbucher and Schubert (2001) hybrid approach that combines a single-obligor intensity-based model with a copula function at the portfolio level can be considered as a special case of our framework. In particular, the state-dependent joint distribution of default times is completely determined by the default intensities.

Our approach is amenable to a wide range of applications to the valuation of defaultable securities. We present simple examples of defaultable bonds, and credit default swaps of the regular and basket type, where pricing is sensitive to default dependency. These examples, however, are only illustrative in nature, and further research is needed to make use of the full generality of our approach.

References

[1] Berndt, A., R. Douglas, D. Duffie, M. Ferguson, and D. Schranz, 2004, Measuring default risk premia from default swap rates and EDFs, Working paper, Stanford University.

[2] Bielecki, T., and M. Rutkowski, 2002, *Credit risk: Modeling, valuation and hedging* (Springer-Verlag, Berlin).

[3] Brémaud, P., 1981, *Point process and queues: Martingale dynamics* (Springer-Verlag, New York).

[4] Collin-Dufresne, P., R. S. Goldstein, and J. Hugonnier, 2003a, A general formula for valuing defaultable securities, Working paper, CMU.

[5] Collin-Dufresne, P., R. S. Goldstein, and J. Helwege, 2003b, Is credit event risk priced? Modeling contagion via the updating of beliefs, Working paper, CMU.

[6] Davis, M., and V. Lo, 2001, Infectious defaults, *Quantitative Finance* 1, 382-387.

[7] Driessen, J., 2003, Is default event risk priced in corporate bonds? Working paper, University of Amsterdam.

[8] Duffie, J. D., M. Schroder, and C. Skiadas, 1996, Recursive valuation of defaultable securities and the timing of resolution of uncertainty, *Annals of Applied Probability* 6, 1075-1090.

[9] Duffie, J. D., and K. J. Singleton, 1999a, Modeling term structures of defaultable bonds, *Review of Financial Studies* 12, 687-720.

[10] Duffie, J. D., and K. J. Singleton, 1999b, Simulating correlated defaults, Working paper, Stanford University.

[11] Hull, J., and A. White, 2001, Valuing credit default swaps II: Modeling default correlations, *Journal of Derivatives* 8(3), 12-22.

[12] Jarrow, R. A., D. Lando, and S. M. Turnbull, 1997, A Markov model for the term structure of credit risk spread, *Review of Financial Studies* 10, 481-523.

[13] Jarrow, R. A., and S. M. Turnbull, 1995, Pricing derivatives on financial securities subject to credit risk, *Journal of Finance* 50, 53-85.

[14] Jarrow, R., and F. Yu, 2001, Counterparty risk and the pricing of defaultable securities, *Journal of Finance* 56, 1765-1800.

[15] Kusuoka, S., 1999, A remark on default risk models, *Advances in Mathematical Economics* 1, 69-82.

[16] Lando, D., 1998, On Cox processes and credit risky securities, *Review of Derivatives Research* 2, 99-120.

[17] Lang, L., and R. Stulz, 1992, Contagion and competitive intra-industry effects of bankruptcy announcements, *Journal of Financial Economics* 32, 45-60.

[18] Longstaff, F. A., S. Mithal, and E. Neis, 2003, The credit default swap market: Is credit protection priced correctly? Working paper, UCLA.

[19] Nelsen, R. B., 1999, *An introduction to copulas* (Springer-Verlag, Berlin).

[20] Newman, Y., and M. Rierson, 2003, Illiquidity spillovers: Theory and evidence from European telecom bond issuance, Working paper, Stanford University.

[21] Norros, I., 1986, A compensator representation of multivariate life length distributions, with applications, *Scandinavian Journal of Statistics* 13, 99-112.

[22] Schonbucher, P., 2000, A LIBOR market model with default risk, Working paper, University of Bonn.

[23] Schonbucher, P., and D. Schubert, 2001, Copula-dependent default risk in intensity models, Working paper, University of Bonn.

[24] Shaked, M., and G. Shanthikumar, 1987, The multivariate hazard construction, *Stochastic Processes and Their Applications* 24, 241-258.

[25] Yu, F., 2003, Default correlation in reduced-form models, Working paper, UC-Irvine.