Correlated Defaults and the Valuation of Defaultable Securities

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Motivation

- Modeling correlated defaults is a top priority in credit risk research.
  - The valuation of certain credit derivatives, such as basket credit default swaps, is sensitive to default correlation in the underlying portfolio.
  - Banks need to set aside capital against defaults in their loan portfolio by computing the credit VaR.
  - The pricing of complex structures may require consideration of counterparty risk.

Current methods

- Copula-based approaches:
  - Marginal distribution can be from any model, and default dependency summarized by copula functions.
  - Li (2000), Schonbucher and Schubert (2001).
- Structural approaches:
  - Correlation of asset levels and default boundaries.
  - Hull and White (2001), Giesecke (2001).
- Intensity-based approaches:
  - Common factors in default intensities, and firm-specific counterparty risk.
  - Partial solution offered by Jarrow and Yu (2001) for a simplified default dependency structure.

Our contribution

- We fully develop the intensity-based approach.
  - A recursive procedure can be used to simulate default times with intensities dependent on common factors and past defaults.
- We compare this approach to existing methods.
  - It can be considered as an extension of Schonbucher and Schubert (2001) where the copula function itself depends on the sample path of the common factors.
- We apply this approach to the valuation of
  - defaultable bonds,
  - credit default swaps,
  - and basket credit default swaps.

Preliminaries

- A typical reduced-form model assumes that the default time τ is defined as

$$\tau = \inf \left\{ t : \int_0^t \lambda_s ds \geq E \right\},$$

  where λ is the default intensity dependent on exogenous common factors X, and E is a unit exponential random variable independent of X.

- $N_t = 1_{\{t \geq \tau\}}$ defines a Cox process representing default, with intensity λ.

The "conditionally independent construction"

- With multiple default times $\left(\tau^1, \ldots, \tau^I\right)$, this definition can be extended with a collection of iid unit exponential random variables $\left(E^1, \ldots, E^I\right)$.
- This implies that the default times are independent conditional on the information contained in X, and default correlation arises because of correlation at the intensity level.
- Hull and White (2001) and Schonbucher and Schubert (2001) have expressed doubt on the ability of this type of models to match the level of empirical default correlations.

Basic setup

- We consider $\tau = \left(\tau^1, \ldots, \tau^I\right)$ defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P\right)$ where $\mathcal{F}_t$ is the minimal sigma-field needed to support both X and τ.
- We assume that τ^i has $(P, \mathcal{F}_t)$-intensity λ^i.
- In other words, λ^i is the conditional hazard rate of τ^i:

$$\begin{aligned} \lambda_t^i 1_{\{t<\tau^i\}} &= \lim_{\Delta t \to 0+} \frac{E\left(N_{t+\Delta t}^i - N_t^i | \mathcal{F}_t\right)}{\Delta t} \\ &= \lim_{\Delta t \to 0+} \frac{P\left(t < \tau^i \leq t + \Delta t | \mathcal{F}_t\right)}{\Delta t}. \end{aligned}$$

Some technical issues

- Unlike the "conditionally independent construction," $N_t^i = 1_{\{t \geq \tau^i\}}$ is no longer a Cox process.
- Generally (Kusuoka 1999), the survival probability

$$P\left(\tau^i > T|\mathcal{F}_t\right) \neq 1_{\{\tau^i > t\}} E\left(\exp\left(-\int_t^T \lambda_s^i ds\right)|\mathcal{F}_t\right).$$

- The conditional expectation above jumps at τ^i, violating the "no-jump condition" of Duffie, Schroder and Skiadas (1996).
- Although a "trick" (CGH 2003) can be used to restore this formula under a change of measure, anything that requires the joint distribution of default times can pose a problem.

How to construct the default times?

- We use the "total hazard construction" of Norros (1986) and Shaked and Shanthikumar (1987).
- Given their conditional hazard rates, this method generates stopping times from iid unit exponentials.
- It results from the fact that the total hazards accumulated by the stopping times by the time they occur are iid unit exponential random variables.
- We extend this construction so that the conditional hazard rates (intensities) can also depend on exogenous common factors, a case of broader appeal.

Case 1: "Internal histories"

- Let $\lambda_i\left(t|n\right)$ denote the intensity for i given n past defaults, $t_{k_1},\ldots,t_{k_n}$, where $0=t_{k_0}<t_{k_1}<\cdots<t_{k_n}<t<\tau^i$.
- Define the total hazard accumulated by i by time t given n observed defaults as

$$\psi_i\left(t|n\right)=\sum_{m=1}^{n}\Lambda_i\left(t_{k_m}-t_{k_{m-1}}|m-1\right)+\Lambda_i\left(t-t_{k_n}|n\right),$$

  where $\Lambda_i\left(s|m\right)=\int_{t_{k_m}}^{t_{k_m}+s}\lambda_i\left(u|m\right)du$ is the total hazard accumulated by i for a period of length s following the mth default.
- Also define an inverse function:

$$\Lambda_i^{-1}\left(x|n\right)=\inf\left\{s:\Lambda_i\left(s|n\right)\geq x\right\},\ x\geq 0.$$

The following recursive procedure constructs a collection of random variables $\widehat{\tau} = \left(\widehat{\tau}^1, \ldots, \widehat{\tau}^I\right)$ with the same joint distribution as the original default times.

1. Draw a collection of iid unit exponentials $E = \left(E^1, \ldots, E^I\right)$.

2. Let

$$
\begin{aligned}
k_1 &= \arg\min_{1 \le i \le I} \left\{ \Lambda_i^{-1} \left(E^i | 0 \right) \right\}, \\
\widehat{\tau}^{k_1} &= \Lambda_{k_1}^{-1} \left(E^{k_1} | 0 \right).
\end{aligned}
$$

3. Assume that the values of $\left(\widehat{\tau}^{k_1}, \ldots, \widehat{\tau}^{k_{m-1}}\right)$ are already given, where $m \ge 2$. Define the set $I_{m-1} = \{k_1, \ldots, k_{m-1}\}$ and $\overline{I}_{m-1}$ as the set of firms excluding I_{m-1}. Recall that $\psi_i\left(t | m-1\right)$ is the total hazard accumulated by firm i given the first $m-1$ defaults, let

$$
\begin{aligned}
k_m &= \arg\min_{i \in \overline{I}_{m-1}} \left\{ \Lambda_i^{-1} \left(E^i - \psi_i \left(\widehat{\tau}^{k_{m-1}} | m-1 \right) | m-1 \right) \right\}, \\
\widehat{\tau}^{k_m} &= \widehat{\tau}^{k_{m-1}} + \Lambda_{k_m}^{-1} \left(E^{k_m} - \psi_{k_m} \left(\widehat{\tau}^{k_{m-1}} | m-1 \right) | m-1 \right).
\end{aligned}
$$

4. If $m = I$ then stop. Otherwise, increase m by 1 and go to Step 3.

Case 2: External common factors

- Denote by $\omega \in \mathcal{F}^X_\infty$ a complete sample path of exogenous common factors X.
- Given ω, the intensities collapse into functions with explicit dependence on the default times.
- The previous procedure can be used to construct random variables $\widehat{\tau}^i$ with $\mathcal{G}_t$-conditional hazard rate λ^i, where $\mathcal{G}_t = \mathcal{F}^X_\infty \vee \mathcal{F}^1_t \vee \cdots \vee \mathcal{F}^I_t$.
- Since we assume λ^i to be $\mathcal{F}_t$-predictable, $\widehat{\tau}^i$ has $\mathcal{F}_t$-conditional hazard rate λ^i as well.
- Replace Step 1 with "Draw a complete sample path of X. Draw iid unit exponentials $E = \left(E^1, \ldots, E^I\right)$ independent of X."

A specific copula-based approach

- Schonbucher and Schubert (2001) define τ^i as

$$\tau^i = \inf\left\{t : \int_0^t h_s^i ds \geq E^i\right\},$$

where h^i is an $\mathcal{F}_t^X$-adapted process.

- They assume that the unit exponentials $\left(E^1, \ldots, E^I\right)$ are governed by a joint distribution function $C\left(U^1, \cdots, U^I\right)$, where $U^i = \exp\left(-E^i\right)$ and C is a copula.

- The "conditionally independent construction" corresponds to choosing the product copula:

$$C\left(U^1, \cdots, U^I\right) = U^1 U^2 \cdots U^I.$$

Relationship between the two approaches

- If only the internal history of default is observed, the two approaches are equivalent in the following sense:
  - Given the joint distribution F, we can always find the marginal distributions and a copula C that stitches the marginals into F.
  - Given the joint distribution F, we can always find the conditional hazard rates and apply the total hazard construction to recover F.
- If we also observe the process X, then the joint distribution and the associated copula are conditional on $\omega \in \mathcal{F}_{\infty}^{X}$.
  - In this case, our approach is an extension of Schonbucher and Schubert (2001).

Example 1: Defaultable bond pricing, $I = 2$

- Two default times τ^A and τ^B, with the following intensities:

$$\begin{aligned} \lambda_t^A &= a_1 + a_2 1_{\{t \geq \tau^B\}}, \\ \lambda_t^B &= b_1 + b_2 1_{\{t \geq \tau^A\}}. \end{aligned}$$

- Jarrow and Yu (2001) solve a simplified case:

$$\begin{aligned} \lambda_t^A &= a, \\ \lambda_t^B &= b_1 + b_2 1_{\{t \geq \tau^A\}}. \end{aligned}$$

- Complete solution can be derived by total hazard construction (alternative solutions given by Kusuoka 1999 and CGH 2003).
- Pricing of B coincides with the simplified case of JY (Bielecki and Rutkowski 2002).
  - We can ignore the default dependency of A when pricing B.

Example 2: Defaultable bond pricing, $I = 3$

- Three default times:

$$\begin{aligned}
\lambda_t^A &= a_{10} + a_{12} 1_{\{t \geq \tau^B\}} + a_{13} 1_{\{t \geq \tau^C\}}, \\
\lambda_t^B &= a_{20} + a_{21} 1_{\{t \geq \tau^A\}} + a_{23} 1_{\{t \geq \tau^C\}}, \\
\lambda_t^C &= a_{30} + a_{31} 1_{\{t \geq \tau^A\}} + a_{32} 1_{\{t \geq \tau^B\}}.
\end{aligned}$$

- Can be applied to the pricing of CDS, where the buyer, the seller, and the underlying all have credit quality issues.
- When pricing A, we can ignore the impact of A on B and C, but we should not ignore the default dependency between B and C.

- An alternative solution is to use the CGH formula:

$$P\left(\tau^A > t\right) = E'\left(\exp\left(-\int_0^t \lambda_s^A ds\right)\right)$$

under a change of measure to P' that puts zero probability on $\{\tau^A \leq t\}$.

- The expectation above should be evaluated by assuming a joint distribution for τ^B and τ^C determined by the following conditional hazard rates:

$$\begin{aligned}\lambda_t^B &= a_{20} + a_{23} 1_{\{t \geq \tau^C\}}, \\ \lambda_t^C &= a_{30} + a_{32} 1_{\{t \geq \tau^B\}}.\end{aligned}$$

Note that τ^A has been dropped from this system.

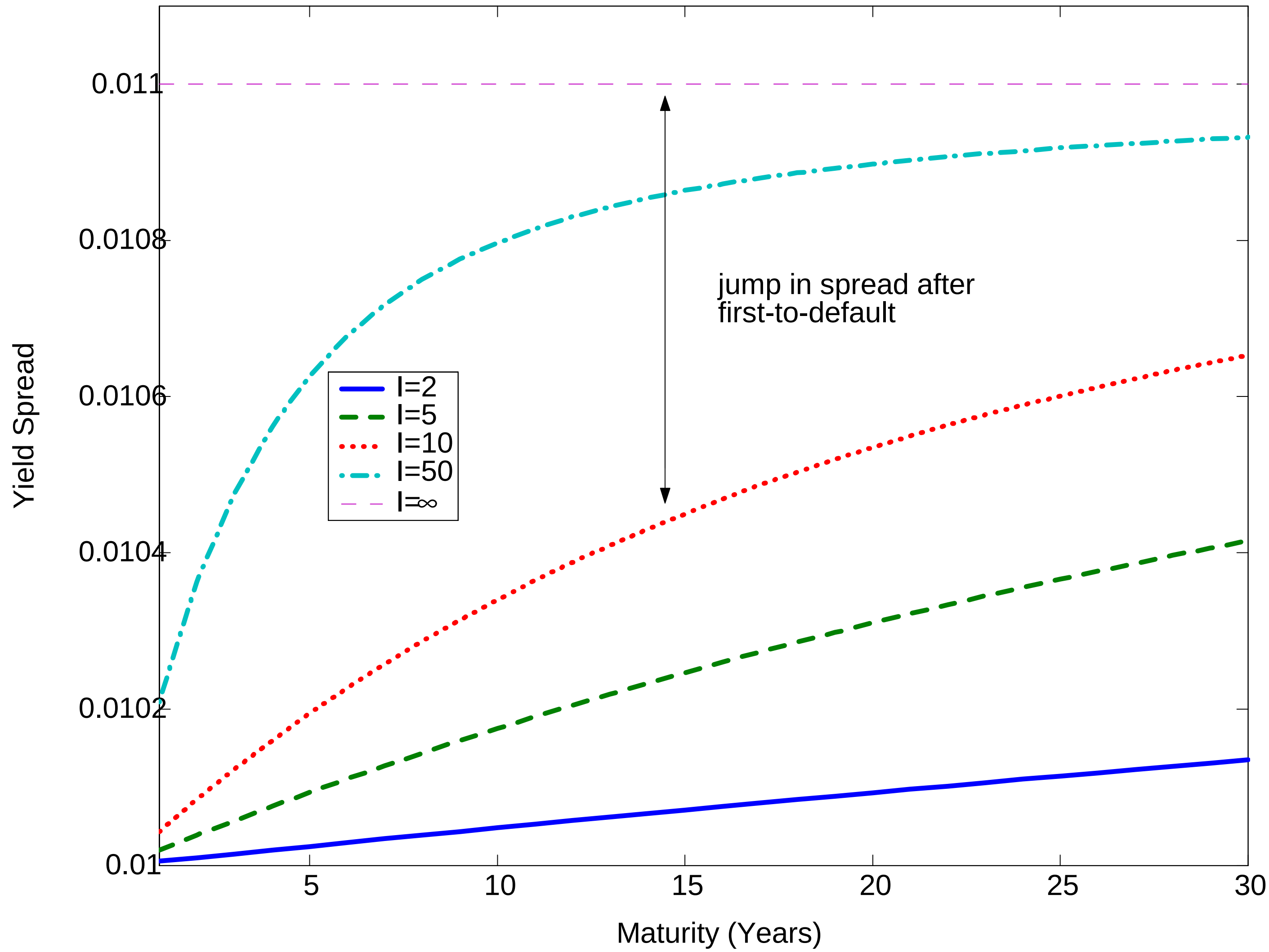
Example 3: Defaultable bond pricing, FTD

- Empirical evidence shows that there are market-wide response to major credit events.
  - Lang and Stulz (1992): stock returns to bankruptcy filing.
  - Newman and Rierson (2003): credit spreads to large debt offering.
  - Collin-Dufresne, Goldstein and Helwege (2003): credit spreads to large change in credit spreads.
- We consider the following intensities:

$$\lambda_t^i = a_1 + a_2 1_{\{t \geq \tau_F\}},$$

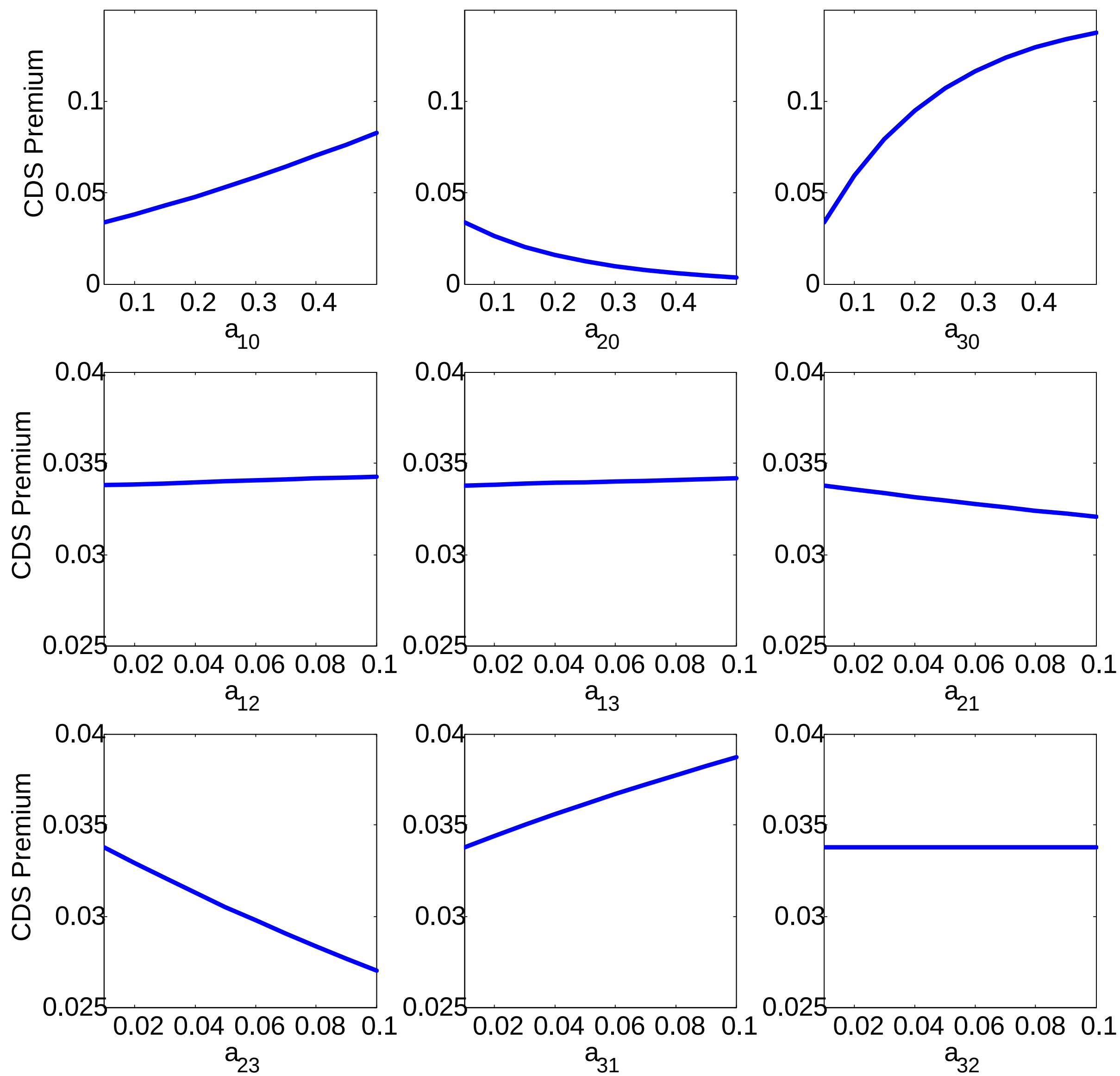
where $\tau_F = \min\left(\tau^1, \ldots, \tau^I\right)$ is the first-to-default time.

- This model can be calibrated to bond price changes around credit events.


Yield Spread
0.011
0.0108
0.0106
0.0104
0.0102
0.01
5
10
15
20
25
30
Maturity (Years)
I=2
I=5
I=10
I=50
I=∞
jump in spread after
first-to-default

Example 4: CDS pricing with counterparty risk

- In contrast to the typical arrangement, we assume that the buyer will pay until its own default or the expiration of the CDS, whichever is earlier.
- We also assume that the default protection is paid at CDS expiration.
- Generally, anything that elevates the credit risk of the buyer or reference asset, or reduces the credit risk of the seller, will increase the CDS premium.
  - This can be a direct effect or an indirect effect.

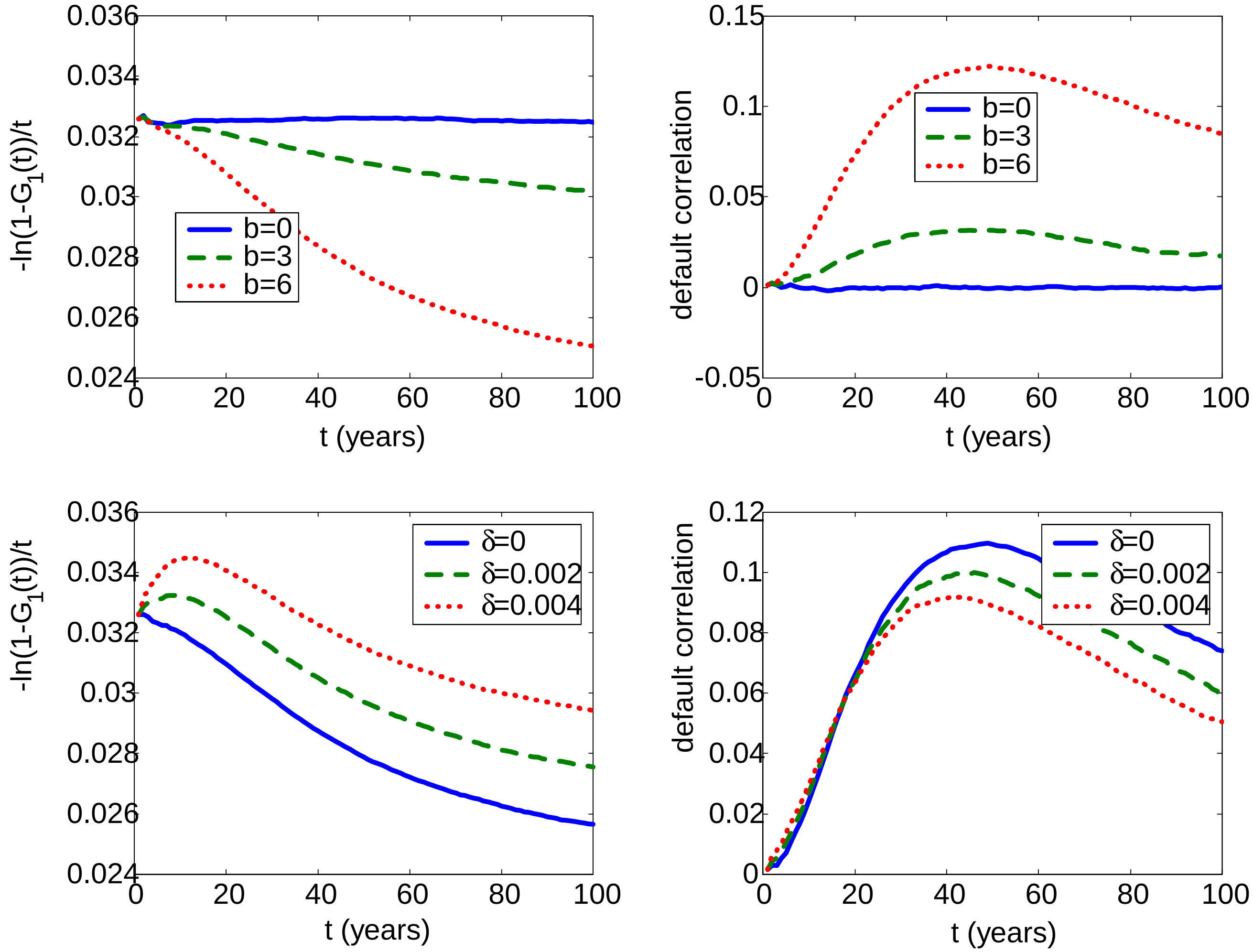

CDS Premium
0.1
0.05
0
0.1 0.2 0.3 0.4
a10
a20
a30
0.04
0.035
0.03
0.025
0.02 0.04 0.06 0.08 0.1
a12
a13
a21
a23
a31
a32

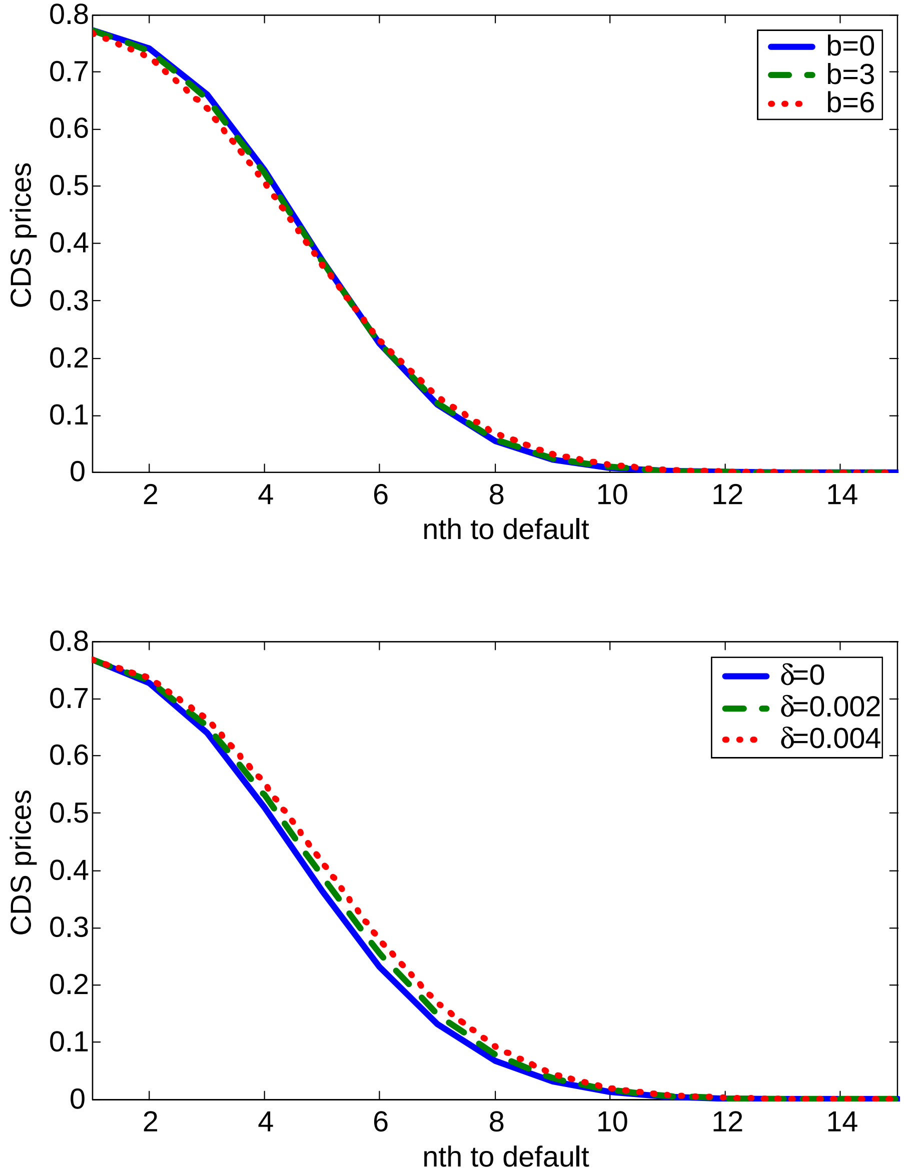
Example 5: Pricing basket CDS

- Basket CDS pays \$1 at expiration if the nth-to-default in a portfolio occurs prior to expiration.
- The premium is proportional to $P\left(\tau^{(n)} \leq T\right)$.
- We assume the following state-dependent intensity for I reference credits:

$$\lambda_t^i = a + bF_t + \delta 1_{\left\{t \geq \tau^{(1)}\right\}}.$$

- Both the common factor and the default contagion can contribute to default correlation in this portfolio.


0.036
0.034
0.032
0.03
0.028
0.026
0.024
-ln(1-G1(t))/t
b=0
b=3
b=6
0
20
40
60
80
100
t (years)
0.15
0.1
0.05
0
-0.05
default correlation
b=0
b=3
b=6
t (years)
-ln(1-G1(t))/t
δ=0
δ=0.002
δ=0.004
t (years)
0.12
0.1
0.08
0.06
0.04
0.02
0
default correlation
δ=0
δ=0.002
δ=0.004
t (years)


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
CDS prices
2
4
6
8
10
12
14
nth to default
b=0
b=3
b=6
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
CDS prices
2
4
6
8
10
12
14
nth to default
δ=0
δ=0.002
δ=0.004

Conclusion

- We introduce an intensity-based model of correlated defaults that offers more flexibility than models based on Cox processes.
- A simulation-based approach can be used to generate default times with intensities that depend on external common factors and past defaults.
- This approach is equivalent to a copula-based approach where the copula function is "dynamic," changing with the histories of exogenous state variables.
- We present illustrative examples, but further research is needed to make use of the full generality of our approach.