

# Risk-neutralizing statistical distributions.

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# Remarks on Risk Pricing

- For the purposes of this presentation a Risk is a random outcome, in general path dependent, but primarily for our focus here a single outcome, on which we wish to trade a contingent claim.
- The Risk is said to be not priced if the price of the contingent claim matches the discounted expected payoff under the statistical distribution.
- Alternatively, it is priced, if a change of measure or probability is embedded in the pricing.

# The Role of Correlation Refined

- In general we have a presumed change of measure for the economy, either unique or selected on a variety of principles that we denote by  $\Lambda$ .
- Let the risk in question be denoted by the random variable  $X$ .
- It is often argued that the risk  $X$  is not priced if it is uncorrelated with the measure change  $\Lambda$ .
- This suffices for positions linear in  $X$  but if for example

$$X = A\varepsilon$$

for a random outcome  $A$  and an independent zero mean unit variance disturbance  $\varepsilon$  and  $\Lambda$  changes the law of  $A$  then there is a change of probability on  $X$  while  $X$  and  $\Lambda$  remain uncorrelated.

- The self sufficient filtration containing the risk  $X$  is defined as a filtration containing  $X$  with the property “loosely speaking” of containing all variables useful in predicting functions of  $X$ . Here  $A$  may be useful in predicting  $X^2$ .
- In the refined view all variables in a self sufficient filtration containing  $X$  must be orthogonal to  $\Lambda$  for there to be no change of probability.
- Hence we could have a change of probability via change of law for a variable in the self sufficient filtration containing  $X$ . Of course we are interested in the smallest self sufficient filtration containing the risk in question from the perspective of parsimony in the implied correlation tests.
- For further details we refer to the working paper by Atlan, Geman, Madan and Yor (2004).

# Statistical Risks

- Many risks of interest are associated with loss events.
- Examples of relevance to this conference include the losses of bond holders on corporate defaults.
- The losses of corporate stake holders on corporate bankruptcy.
- The losses of corporate stake holders on credit downgrades.

- Though there is little or no statistical information on bankruptcy or downgrade by a particular entity just as we have no information on death by cancer by a particular individual, there is often considerable statistical information when we pool individuals by various statistical categories of interest.
- We therefore put ourselves in a context where we have some reasonable understanding of the statistical risk in question.
- The question we focus on is that of the pricing of a reasonably well understood statistical risk.

# Exponential Tilts

- By standard principles we know that prices,  $w$ , of state contingent cash flows  $c(\omega)$  are determined by

$$w = e^{-r} E^P [\Lambda c]$$

where  $P$  is the statistical measure and  $\Lambda$  is the economy wide change of measure density.

- Assuming  $c$  is adapted to level of loss  $L$  of a known statistical type we apply iterated expectations and write

$$\begin{aligned} w &= e^{-r} E^P [E^P [\Lambda|L] c(L)] \\ &= e^{-r} E^P [g(L)c(L)] \end{aligned}$$

- This expression shows that in general we wish to evaluate a tilted expectation of the final cash flow, tilted by the function  $g(L)$ .

- If the statistical density for  $L$  is  $p_L(L)$  then its risk neutral counterpart takes the form

$$q_L(L) = g(L)p_L(L)$$

- For a risk that is bounded in an interval we may consider a local approximation of the tilt function in the form of a renormalized exponential and write

$$q_L(L) = \kappa e^{\alpha_L L} p_L(L)$$

where  $\kappa$  is a renormalizing constant.

- This approach to risk neutralization has a long history in financial economics and we cite Duffie (1992) in connection with the Black-Merton-Scholes model, followed by Heston (1993), Heath, Jarrow and Morton (1992), Naik and Lee (1990), Sondermann (1991) and Gerber and Shiu (1996).

# Tilt Identification

- The question at issue in implementing such a strategy is that of identifying the local tilt coefficient to be used.
- This is especially critical when dealing with risks that have no liquid markets in the underlying derivatives.
- We define the projected pricing kernel as

$$y_L(L) = \frac{q_L(L)}{p_L(L)}$$

- Our interest is in determining

$$\alpha_L = \frac{\partial \ln(y_L(L))}{\partial L}.$$

# Tilt Coefficients for Non-Traded Risks

- We suppose that the self sufficient filtration containing the risk in question has some financial variable like a market index with a liquid underlying options market.
- We wish to relate  $\alpha_L$  to possibly observed tilts in the associated options market.
- For this purpose we let  $S$  denote the level of the financial index with an active options market.
- Our assumption of the membership of  $S$  in the self sufficient filtration associated with  $L$  translates to the assertion that the conditional density of  $L$  given  $S$

$$\psi(L|S)$$

is nontrivially dependent on  $S$ .

- We may then write the joint density of  $L$  and  $S$  as

$$h(L, S) = p_S(S)\psi(L|S).$$

# The Risk Neutral and Statistical Marginals

- We may then derive the statistical marginals for  $L$  by

$$p_L(L) = \int_0^\infty p_S(S)\psi(L|S)dS$$

- And the risk neutral marginal supposing only that the law of  $S$  undergoes a change as

$$q_L(L) = \int_0^\infty q_S(S)\psi(L|S)dS$$

- From which it follows that

$$\begin{aligned} y_L(L) &= \int_0^\infty y_S(S) \frac{p_S(S)\psi(L|S)}{\int_0^\infty p_S(S)\psi(L|S)dS} \\ &= \int_0^\infty y_S(S)\psi(S|L)dS \\ y_S(S) &= \frac{q_S(S)}{p_S(S)}. \end{aligned}$$

## Mean Value Theorem and a Direct Relation

- By the mean value theorem we may now write the existence of a function  $\phi(L)$  such that

$$y_L(L) = y_S(\phi(L))$$

- From which we get that

$$\alpha_L = \alpha_S \phi'(L)$$

where

$$\alpha_S = \left. \frac{\partial \ln(y_S(S))}{\partial S} \right|_{S=\phi(L)}$$

- We may then infer the nontraded exponential tilt  $\alpha_L$  by identifying the options tilt  $\alpha_S$  and estimating the relationship  $\phi(L)$ .
- We refer to  $\alpha_S$  as the options tilt.
- We refer to  $\phi'(L)$  as the options tilt adjustment appropriate for the risk at hand.

# Modeling the Options Tilt

- We have observed from other studies and we cite Jackwerth (2000) and Carr, Geman, Madan and Yor (2002) that the function  $y_S(S)$  is *U-shaped* allowing for risk aversion from both the long and short side of the market raising risk neutral tail probabilities in both tails
- This leads us to the tentative specification 
$$y_S(S) = \alpha_P e^{-\alpha_P S} \mathbf{1}_{S < S_0} + \alpha_C e^{\alpha_C S} \mathbf{1}_{S > S_0}$$
 where we expect put side  $\alpha_P$  and call side  $\alpha_C$  to be positive reflecting the conjectured *U-shape* with a minimum taken here at the current spot  $S_0$ .

- To determine these coefficients one estimates the statistical and risk neutral densities on the two sides separately by times series methods applied to large negative and positive returns followed by nonlinear least squares fits to out-of-the-money call and put options to estimate the risk neutral density.
- The coefficients are then obtained by regression of  $\ln(q_S(S)) - \ln(p_S(S))$  on  $S$  separately on the two sides.

# Estimating Option Tilt Adjustments

- For this purpose we adopt our two sided exponential model for  $y_S(S)$  and adopt the local linear model for the dependence of  $S$  on  $L$  given by

$$S = \alpha + \beta L + \varepsilon$$

- An exact computation in this simplified context yields directly that

$$\alpha_L = \begin{cases} \alpha_C \beta & \beta > 0 \\ \alpha_P |\beta| & \beta < 0 \end{cases}$$

## A Specific Local Loss Risk Contract

- We consider the pricing of a loss reinsurance contract with a deductible of  $K$  and a cap on the coverage of  $B$ .
- The payoff made by the reinsurer is
$$CF(L) = \text{Min}(\text{Max}(L - K, 0), B) - \text{Max}(L - (K + B), 0)$$
- This contract is a call spread on the loss level and is localized between  $K$  and  $K + B$ .

# Choice of Risk Neutral Density

- After an analysis of a variety of extreme value distributions we recommend the use of the Weibull density for the pricing of the embedded call options.
- The specific price is

$$w = e^{-r} \int_0^{\infty} CF(L)q(L)dL$$

- We adopt the Weibull form for  $q$  so that

$$q(L) = \exp\left(-\left(\frac{L}{c}\right)^a\right) \frac{aL^{a-1}}{c^a}$$

- The density has the mean  $\mu$  and standard deviation  $\sigma$  given by

$$\mu = c\Gamma\left(1 + \frac{1}{a}\right)$$

$$\sigma = c\sqrt{\Gamma\left(1 + \frac{2}{a}\right) - \Gamma\left(1 + \frac{1}{a}\right)^2}$$

- The parameter  $c$  is a scaling parameter and the parameter  $a$  is the shape parameter.
- $\Gamma(x)$  is the gamma function.

# Weibull Call Option Pricing Model

- Taking Call payoff discounted expectations under this density we obtain the Weibull Call option pricing model for the strike  $X$  in the form

$$C = e^{-r} [L^*W_1 - XW_2]$$

$$L^* = c\Gamma\left(1 + \frac{1}{a}\right) = \mu$$

$$W_1 = 1 - \text{gammainc}\left(\left(\frac{X}{c}\right)^a, 1 + \frac{1}{a}\right)$$

$$W_2 = \exp\left(-\left(\frac{X}{c}\right)^a\right)$$

- where *gammainc* is the incomplete gamma function.

# Details for Estimating Options Tilts

- The basic problem is that of obtaining the statistical and risk neutral densities at a common horizon.
- The risk neutral may be backed out from option prices for a horizon matching the options maturity.
- The statistical is best observed at the daily frequency that falls far short of the options maturity.
- We also focus attention on the two sides separately using Weibull forms for the two sides.

# The Statistical Density at the Options Maturity

- We define nonnegative upside and downside daily returns by

$$R_u = \frac{S}{S_0}, S > S_0$$

$$R_d = \frac{S_0}{S}, S < S_0$$

- For both random variables we extract the larger upside and downside returns and specify the Weibull density estimated by maximum likelihood by

$$f(R) = \exp\left(-\left(\frac{R-1}{c}\right)^a\right) \frac{a(R-1)^{a-1}}{c^a}$$

- To get to the options horizon we employ scaling, a strategy supported by some other ongoing studies.
- We employ the hypothesis that the  $N$  day return  $R_N$  has the distribution

$$(R_N - 1) \stackrel{law}{=} \sqrt{N} (R - 1)$$

- It follows that  $R_N$  is Weibull with scale parameter  $c\sqrt{N}$  and shape parameter  $a$ .
- These distributions are estimated for the statistical densities on the upside and downside for the parameters  $c_u^S, a_u^S, c_d^S, a_d^S$ .

# The Risk Neutral Density for Options

- We fit the call prices

$$cv(K, t) = e^{-rt} \int_{\frac{K}{S_0}}^{\infty} (S_0 R - K) f(R) dR$$

$$pv(K, t) = e^{-rt} \int_{\frac{S_0}{K}}^{\infty} \left( K - \frac{S_0}{R} \right) f(R) dR$$

to the prices of out-of-the-money call and put options to infer risk neutral parameters

$$c_u^{RN}, a_u^{RN}, c_d^{RN}, a_d^{RN}.$$

- Option tilts on the two sides follow on performing the appropriate regressions.

## Insuring FDIC Losses

- In 1991, the FDICIA authorized the Federal Deposit Insurance Corporation, “to obtain private reinsurance covering not more than 10 percent of any loss the Corporation incurs with respect to an insured depository institution.”
- The 2000 Options Paper produced by the FDIC view reinsurance as one way “to use market information to differentiate risks without imposing a particular funding structure on insured institutions.”
- We use PHLX / KBW Bank Index (BKX) prices and options written on this index as the sufficient statistic with a liquid options market to estimate the tilt for FDIC losses.
- This index is capitalization-weighted and composed of 24 geographically diverse stocks representing national money center banks and leading regional institutions.

# The MMC Quote

- MMC Enterprise Risk provided the FDIC with two quotes on reinsurance.
- Quote 1, for a 2 billion dollar coverage at the risk level of one basis point the quote was 4 million dollars.
- Quote 2, for a 0.5 billion dollar coverage at the risk level of one percentage point the quote was 10 million dollars.
- We compare the results of our methodology with these quotes.

# Estimating the statistical distribution of the BKX Index

- Time series data on the Bank Index (BKX) for 1500 days ending on September 28, 2001.
- Extract the top 25% of daily returns.
- The statistical Weibull parameters are estimated by maximum likelihood for the up and down side as follows.

	<b>BKX UP Side</b>	<b>BKX DOWN Side</b>
$c^S$	0.0345	0.0335
$a^S$	2.5324	2.7593

# Estimating the risk-neutral distribution of the BKKX Index

- Data on index options for every second-wednesday of each month over the year beginning in October 2000 and ending in September 2001
- The risk neutral parameters,  $c_u^{RN}$ ,  $a_u^{RN}$ ,  $c_d^{RN}$ ,  $a_d^{RN}$  are estimated by calibrating a Weibull call option model prices to the prices of the out-of-the-money call options.

**TABLE 1: Risk-neutral Weibull parameter estimates on extreme 2-month BXX call and put options.**

Date	maturity	cdrn	adrn	curn	aurn
Oct. 2000	.1804	0.0563	0.6465	0.0821	1.1453
Nov. 2000	.1968	0.0627	0.7004	0.0176	0.4245
Dec. 2000	.1779	0.0636	0.6932	0.0272	0.5128
Jan. 2001	.1781	0.0665	0.6701	0.0540	0.6997
Feb. 2001	.1779	0.0527	0.6749	0.0659	0.9728
Mar. 2001	.1779	0.0342	0.4549	0.0914	1.0725
Apr. 2001	.1753	0.0736	0.6703	0.0867	0.9985
May 2001	.1945	0.0632	0.7057	0.0680	0.9862
Jun. 2001	.1753	0.0365	0.6122	0.0524	1.0211
Jul. 2001	.1945	0.0538	0.6794	0.0734	1.1855
Aug. 2001	.1917	0.0406	0.6049	0.0637	1.1206
Sep. 2001	.1563	0.1303	1.0758	0.0913	1.1556
Mean Level	.1814	0.0612	0.6824	0.0645	0.9413

## Estimating the tilt coefficient

<b>BKX</b>	up	down
$c^S$	0.0345	0.0335
$a^S$	2.5324	2.7593
$c^{RN}$	0.0645	0.0612
$a^{RN}$	0.9413	0.6824

- Estimate the following regression equation on the price level in the range between 1% to 0.01% return levels in two months using scaling to adjust the statistical probability.

$$\log \left( \frac{q(S)}{p(S)} \right) = k + \alpha S + \epsilon.$$

- The estimated options tilt coefficients on the upside and down side are

$$\alpha_{up} = 0.1739$$

$$\alpha_{down} = -0.6387$$

- We observe the conjectured and expected *U* – *shape* with

$$\alpha_C = 0.1739$$

$$\alpha_P = 0.6387$$

# Estimating the Tilt Adjustment

- To estimate the slope coefficient for the conditional distribution of the index level given the *FDIC* loss level we regressed the *NASDAQ* bank index on the level of *FDIC* losses over the period 1981 – 2001 with the result

$$S = 1165.43 - .1698 L + \varepsilon$$

(191.31) (0.0675)

$$R^2 = 28.76\%$$

- We therefore employ the down side tilt with

$$\begin{aligned}\alpha_L &= \alpha_P |\beta| \\ &= 0.6387 * 0.1698 \\ &= 0.1085\end{aligned}$$

# FDIC Annual Loss Levels

Year	Loss (in \$Billions)	Number
1986	1.775	145
1987	2.023	203
1988	6.921	280
1989	6.199	207
1990	2.785	169
1991	6.148	127
1992	3.675	122
1993	0.646	41
1994	0.179	13
1995	0.085	6
1996	0.038	5
1997	0.005	1
1998	0.234	3
1999	0.841	7
2000	0.039	6

## Estimating the parameters of the statistical Weibull distribution of the FDIC's losses

$$\mu = 2.106 = c\Gamma\left(1 + \frac{1}{a}\right)$$

$$\sigma = 2.497 = c\sqrt{\Gamma\left(1 + \frac{2}{a}\right) - \Gamma\left(1 + \frac{1}{a}\right)^2}$$

$$1 + \frac{2.497^2}{2.106^2} = \frac{\Gamma\left(1 + \frac{2}{a}\right)}{\Gamma\left(1 + \frac{1}{a}\right)^2}$$

	<b>Statistical</b>
$c$	1.9317
$a$	0.8472
$\mu$	\$2.106 billion
$\sigma$	\$2.497 billion

## Risk-neutralizing the FDIC's statistical distribution

- We risk neutralize the statistical distribution in the context of a reinsurance quote estimate given to the FDIC.
- The MMC Enterprise Risk (MMC) provided two rough reinsurance price estimates in a report submitted to the FDIC:
  - Annual premium on a \$2 billion coverage at a .01 % risk level is \$4 million.
  - Annual premium on a \$0.5 billion coverage at 1% risk level is \$10 million.

$$w_1(B = \$2 \text{ billion}, K_1) = \$4 \text{ million}$$

$$w_2(B = \$0.5 \text{ billion}, K_2) = \$10 \text{ million}$$

# Implied Strikes

Note that the probability of a loss amount exceeding the strike is given by

$$P(L > K) = \theta = 1 - F(K)$$

For the Weibull cumulative distribution function,

$$F(K) = 1 - \exp \left[ - \left( \frac{L}{c} \right)^a \right]$$

Hence,

$$P(L > K) = -\log(\theta) = \left( \frac{K}{c} \right)^a$$

Thus, strike is expressed as

$$K = c (-\log(\theta))^{\frac{1}{a}}$$

Substituting the estimates of  $c$ ,  $a$ , and the risk level we obtain

$$K_1 = \$26.56\text{billion}$$

and

$$K_2 = \$11.72\text{billion}$$

## Parameters of the Risk-neutral Weibull distribution

- Using Proposition 1, the tilt coefficient is the difference between the derivative of the logarithm of the risk neutral and statistical densities,

$$\frac{d \log q(L)}{dL} - \frac{d \log p(L)}{dL} = 0.1085$$

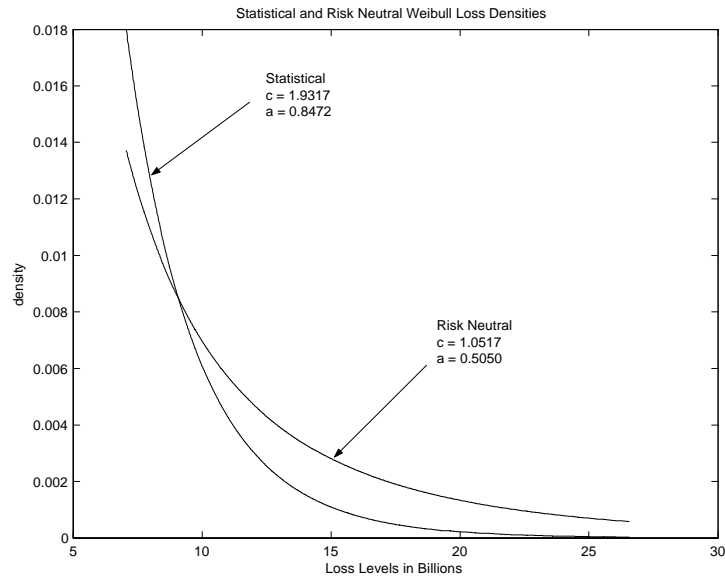
- We evaluate the derivative of the logarithm of  $p$  at the two strikes of 11.72 and 26.56 to be  $-.3638$  and  $-.2871$ .
- This provides us two equations

$$\left. \frac{d \log q(L)}{dL} \right|_{11.72} = -0.2533$$

$$\left. \frac{d \log q(L)}{dL} \right|_{26.56} = -0.1786$$

from which we can simultaneously solve for the parameters  $c$  and  $a$  of  $q(L)$

- The estimated risk neutral parameters are:  $c = 1.0442$  and  $a = 0.6054$ .



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# Statistical vs. Risk-Neutral Loss Distribution

## Pricing the reinsurance contract

When the two call-spreads are priced using the estimated risk-neutral distribution, we obtain

Contract	Model	MMC
w(.01, \$.5bil., \$11.72 bil.)	\$6.16 mil	\$10 mil
w(.0001,\$2bil., \$26.56 bil.)	\$1.39 mil	\$4 mil

Contract	Statistical
w(.01, \$.5bil., \$11.72bil.)	\$4.5 mil.
w(.0001,\$2bil., \$26.56bil.)	\$150,000

## Pricing the complete coverage

- The risk neutral density for complete coverage (\$26.56 billion) is

$$q(L) = \frac{e^{0.1085L}p(L : \mu = 2.106, \sigma = 2.497)}{\int_0^{26.56} e^{0.1085L}p(L : \mu = 2.106, \sigma = 2.497)dL}.$$

- The aggregate premium to be collected

$$\Pi = \int_0^{26.56} Lq(L)dL.$$

- This integral is computed to be \$3.096 billion.
- For the level of insured deposits around \$1909.9 billion this is a premium of around 16.21 cents per 100 dollars of deposit.
- Calculated statistically, the value of the integral is \$2.1032 billion representing deposit insurance premiums of 11 cents per \$100 deposits.

# Summary

- When a risk has to be priced and we have statistical data on the physical probability measure we propose in this paper a risk neutralization strategy.
- The strategy is based on estimating an exponential tilt coefficient inferred from options markets and adjusted for the application at hand.
- The options tilt is estimated by a time series analysis of daily returns scaled to the option maturity to construct the physical measure.
- The risk neutral options measure is estimated by standard methods of equity derivatives.
- A regression gives the options tilt.

- The tilt adjustment is based on an analysis of the conditional density of the index given the loss level and the slope coefficient provides the tilt adjustment.
- The tilted physical measure is then applied to develop prices for the contingent claims of interest on the risk at hand.
- The method is illustrated on pricing FDIC loss reinsurance contracts.