THE MARKET PRICE OF CREDIT RISK

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Abstract

The credit risk premium is empirically documented to be a significant component of credit spreads. However, its determinants are not fully understood. We offer a structural model of the credit risk premium in which investors have incomplete information about a firm's default barrier. The premium has two components. One is standard and accounts for investors' aversion towards price volatility that is due to the diffusive fluctuation of the firm value. The other is an event premium induced by investors' uncertainty about the firm's true distance to default, which causes jumps in security prices at default. The event premium is an explicit function of the running minimum firm value that is determined by investors' prior distribution of the unobserved default barrier.

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1 Introduction

Aggregate credit risk is one of the most pervasive threats in today's financial markets. It comes from the dependence across issuers of credit sensitive securities on the economic environment. It also arises from issuers interacting directly with their business partners, which can lead to a "contagious" propagation of distress. Taken together, these risks cannot be completely diversified away. It is a standard economic principle that undiversifiable risks commands a premium. In other words, risk-averse investors must be compensated for assuming undiversifiable credit risk.

The credit risk premium is empirically well-documented.[1] It is of central importance to financial practitioners and academicians since it connects the two main purposes of a credit model. First, a credit model is used to forecast the probability of default. As such, the model must reflect the historical default experience. However, a credit model is also a tool for pricing and hedging credit sensitive securities. In this context, it must fit market prices. In order to build a coherent model that serves both purposes, we need to understand the relationship between actual defaults and prices of credit sensitive securities. This is where the risk premium comes into play. It maps the actual likelihood of default to the *pricing* or *martingale* likelihood of default that is used to price securities.

In this article, we analyze the credit premium in the context of the I^2 incomplete information model, described in Giesecke & Goldberg (2004*a*). In I^2, a firm defaults if its value falls below a barrier that investors cannot observe. First passage models based on complete information have been considered by Black & Cox (1976), Leland (1994), Longstaff & Schwartz (1995) and many others. In these complete information models, credit risk is driven exclusively by uncertainty about the firm value and the risk premium takes a familiar form. The required excess return on any credit sensitive security issued by or referenced on the firm is equal to its risk times the market price of that risk. Here "risk" is measured in terms of diffusive price volatility. The market price of risk is given by the excess return on the firm per unit of firm risk.

However, this representation of the credit premium neglects the short-term uncertainty surrounding the default event. Indeed, in the complete information models cited above, the distance of the firm to default is always observable, so default is predictable. The existence of short-term uncertainty in the credit market is highlighted by the prevalence of positive short-term credit spreads and the precipitous drops in equity and bond prices that occur at default. Empirical observation shows that equity drops to near zero. Bondholders usually lose something but generally do not lose everything. Consequently, net firm value, which is equal to the sum of equity and debt values, also drops at default. In order to fit market prices, a credit model must take account of these discontinuities.

[1] Amato (2005) and Berndt, Douglas, Duffie, Ferguson & Schranz (2005) estimate the credit premium using data on credit swap spreads. Collin-Dufresne, Goldstein & Helwege (2002), Driessen (2005), Elton, Gruber, Agrawal & Mann (2001) and Fons (1987) analyze the premium using corporate bond price data. Liu, Longstaff & Mandell (2006) estimate the premium using interest rate swap spread data.

In a growing literature initiated by Duffie & Lando (2001), it is shown that the forecasts of a structural model become more realistic if the assumption that investors are completely informed is relaxed. This amounts to specifying an investor filtration that does not distinguish events that cannot be publicly observed. In I^2, the distance to default cannot be observed by investors so default is a totally inaccessible event; it comes unannounced. This leads to model forecasts of positive short spreads that are consonant with empirical observation. The jumps in security prices induced by this short-term uncertainty may command an event risk premium, over and above the premium due to diffusive price volatility. Berndt et al. (2005), Collin-Dufresne et al. (2002) and Driessen (2005) find that the event premium is a significant factor in credit swap and bond markets. In this article, we examine the determinants of the risk premium in the case where investors have incomplete information about a firm's default barrier.

The risk premium corresponds to a pricing measure, which can be represented by its density with respect to the physical measure. The space of densities for I^2 is parameterized by pairs of processes that are predictable in the investors' filtration. Thus, the risk premium can always be decomposed into two economically meaningful components. The *diffusive risk premium*, which is realized as a change to the drift term in the security's price process, is proportional to the security's diffusive price volatility. The proportionality factor can be interpreted as a market price of diffusion risk in the firm value. Since investors have incomplete information about the default barrier, there is also an *event risk premium*, which accounts for investors' risk aversion towards the downward jump in prices upon default. It prescribes the mapping between instantaneous default probabilities under the physical measure and the pricing measure. The event premium is a deterministic function of the running minimum firm value process that is determined by investors' prior distributions of the unobserved default barrier under the physical and pricing measures. Thus, the event premium is realized as a change to the default barrier distribution.

This economic picture is based on the explicit parametrization of the full space of martingales on our probability space. Our analysis uses the powerful martingale representation results of Jacod (1977). Every uniformly integrable martingale can be represented in terms of a Brownian motion that drives the diffusion-type uncertainty in the firm value and the compensated default jump martingale, which represents the jump-type uncertainty in the firm value. The risk premia are proportional to the martingale coefficients in this representation. Our analysis extends a similar representation result of Kusuoka (1999), who considers the square integrable case. An analogous martingale representation theorem for a filtration generated by a Lévy process is in Kunita (2004).

The paper is organized as follows. In Section 2, we outline the I^2 model assumptions and some immediate consequences. The underlying probabilistic structure is discussed in the Appendix. In Section 3 we analyze the space of pricing measures, and in Section 4 we study the risk premium decomposition. In Section 5 we derive generalized reduced form formulae for the prices of credit sensitive securities subject to fractional recovery.

2 The I^2 model

2.1 Assumptions

The uncertainty in the economy is modeled with a complete probability space $(\Omega, \mathcal{G}, \pi)$. We consider a fixed firm and make the following assumptions. The probabilistic structure underlying these assumptions is discussed in more detail in Appendix A.

A1. Capital structure of the firm:
The firm is financed by equity and debt. Debt is senior to equity.

A2. Gross firm value:
The gross firm value X_t is the present value at time t of all future cash flows generated by the firm. It follows a geometric Brownian motion under the measure π. This is described by the equation

$$\frac{dX_t}{X_t} = \mu^\pi dt + \sigma dW_t^\pi, \quad X_0 > 0, \tag{1}$$

where $\mu^\pi \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W^π is a standard Brownian motion with respect to the measure π and its augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ defined by $\mathcal{F}_t = \mathcal{F}_0 \vee \sigma(W_s^\pi : s \leq t)$ where $\mathcal{F}_0$ is the collection of null sets in $\mathcal{G}$. The equation (1) has the unique strong solution

$$X_t = X_0 e^{V_t} \tag{2}$$

where $V_t = mt + \sigma W_t$ is a Brownian motion with drift $m = \mu^\pi - \frac{1}{2}\sigma^2$.

A3. Default time:
We assume that the firm defaults if the gross firm value X falls to some barrier. This *default threshold* is modeled by a random variable $d \in (0, X_0)$ that is independent of $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. Defining the normalized default threshold $D = \log(d/X_0) \in (-\infty, 0]$, we can write the random default time τ as

$$\tau = \inf\{t > 0 : V_t \leq D\}. \tag{3}$$

Associated to the default time τ is the *indicator process* N defined as

$$N_t = 1_{\{\tau \leq t\}}. \tag{4}$$

A4. Information structure:
Investors observe the gross firm value and the default but not the level d. Therefore, their information is *incomplete*. This puts investors at a disadvantage relative to firm management. The value of d depends on the firm's liabilities and is assumed

to be firm inside information. In mathematical terms, the public information flow is modeled by the augmented, right continuous[2] filtration $\mathbb{G}$ generated by

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(N_s : s \leq t). \tag{5}$$

A5. Threshold prior:
Lacking definite knowledge of the default point, investors agree on a prior distribution function G^π on the normalized default threshold D with respect to π. We assume G^π has a strictly positive density function g^π.

A6. Credit sensitive claims:
The firm has issued credit sensitive claims including equity and debt. A general claim is characterized by its payoff $c_T \in L^1(\Omega, \mathcal{F}_T, \pi)$ at a horizon $T \leq \bar{T}$, where $\bar{T} > 0$ is a fixed finite horizon. The payoff is made if there was no default by T.[3] Mathematically, the time T payoff is

$$C_T = c_T 1_{\{\tau > T\}}. \tag{6}$$

If the firm defaults, a recovery payment is made. We follow the fractional recovery convention introduced by Duffie & Singleton (1999). Let R be an $\mathbb{F}$-predictable process with values in $[0, 1]$. If the firm defaults at time t, a fraction R_t of the market value of the claim just prior to default is recovered. The fraction $1 - R_t$ represents *bankruptcy costs*. Therefore, the payoff at the default time is

$$C_\tau = R_\tau C_{\tau-} 1_{\{\tau \leq T\}}. \tag{7}$$

A credit sensitive claim is characterized by the triple (T, c_T, R).

A7. Dynamics of credit sensitive claim prices:
The value dynamics of the credit sensitive claim (T, c_T, R) with respect to $(\pi, \mathbb{G})$ are described by the stochastic differential equation

$$\frac{dC_t}{C_{t-}} = d\mu_C^\pi(t) + \sigma_C(t) dW_t^\pi - (1 - R_t) dN_t, \quad C_0 > 0. \tag{8}$$

Here, $\mu_C^\pi = (\mu_C^\pi(t))_{t \geq 0}$ is a $\mathbb{G}$-adapted process that starts at zero and has continuous paths of finite variation. It describes the *cumulative* growth rate of C. Further, $\sigma_C = (\sigma_C(t))_{t \geq 0}$ is a strictly positive $\mathbb{G}$-predictable process that describes the diffusive volatility of C. The processes μ_C and σ_C are chosen such that (8) is well-defined.

A8. Riskless bonds:
On the financial market investors can trade in riskless bonds. Given some constant riskless rate r, these are valued at e^{rt} at time t.

[2] See Bélanger, Shreve & Wong (2004, Appendix A) for a proof of right continuity.

[3] The choice $c_T \in \mathcal{F}_T$ is without loss of generality: For any $\mathcal{G}_T$-measurable random variable c, we have $c 1_{\{\tau > T\}} = \bar{c} 1_{\{\tau > T\}}$, where $\bar{c}$ is $\mathcal{F}_T$-measurable.

2.2 Consequences

The model assumptions outlined above have several consequences which are important in the sequel. Once again, Appendix A provides more details with respect to the underlying probabilistic structure.

C1. Observability of defaults:
Assumption A4 implies that the random default time τ is a *stopping time* in the filtration $\mathbb{G}$. It means that at each point in time, the default status of the firm can be observed. Note that τ is not a stopping time in the firm value filtration $\mathbb{F}$.

C2. Unpredictability of defaults:
Assumptions A3–A5 imply that the default time τ is *totally inaccessible* in the filtration $\mathbb{G}$. In mathematical terms, $\pi[\tau = \sigma < \infty] = 0$ for all $\mathbb{G}$-predictable times σ. On an intuitive level, it means that default cannot be anticipated. This is economically reasonable. Since investors are not privileged to firm inside information, they do not know the true distance between gross firm value and the default barrier. The unpredictability of defaults is consistent with the sudden downward jumps in the market value of debt and equity.

C3. Default trend and compensator:
The default indicator N is a submartingale in the investor filtration $\mathbb{G}$. It admits a unique Doob-Meyer decomposition into the sum of a $(\pi, \mathbb{G})$-martingale H^π and a non-decreasing $\mathbb{G}$-predictable process called the compensator of N. Proposition 6.1 in Giesecke (2006) implies that the $(\pi, \mathbb{G})$-compensator is given by $A^\pi_{\cdot\wedge\tau}$, where

$$A^\pi_t = -\log G^\pi(M_t) \tag{9}$$

is the $\mathbb{F}$-adapted *trend* and where M is the historical low of log-firm values:

$$M_t = \min_{s\leq t} V_s.$$

Since M is of finite variation and G^π has density g^π by A5, we get

$$dA^\pi_t = -\frac{g^\pi(M_t)}{G^\pi(M_t)} dM_t. \tag{10}$$

This equation shows that A^π increases only when $-M$ does, which is when $M_t = V_t$ and assets reach their historical low. The set of times $\{t \geq 0 : M_t = V_t\}$ has Lebesgue measure zero, and therefore the I^2 compensator does not admit an intensity. This means a positive process λ^π such that $A^\pi_t = \int_0^t \lambda^\pi_s ds$ does not exist.

C4. Gross firm value and survival information:
The $\mathbb{F}$-Brownian motion W^π is also a Brownian motion in investors' filtration $\mathbb{G}$.

This is a consequence of the independence of W^π and the default barrier, see A3. A formal discussion is in Appendix A.

It follows that any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale: a process that is fair with respect to the information described by the augmented Brownian filtration $\mathbb{F}$ remains fair in the larger filtration $\mathbb{G}$, which contains survival information in addition.

C5. Net firm value:
The net firm value $\mathcal{X}$ is the sum of the values of equity S and debt. Assumption A4 implies that the dynamics of $\mathcal{X}$ with respect to $(\pi, \mathbb{G})$ are of the form (8). They are described by the equation

$$\frac{d\mathcal{X}_t}{\mathcal{X}_{t-}} = d\mu^\pi_{\mathcal{X}}(t) + \sigma_{\mathcal{X}}(t) dW^\pi_t - J_t dN_t, \quad \mathcal{X}_0 > 0. \tag{11}$$

If we let B denote the value of all debt securities issued by the firm, then the cumulative growth rate $\mu^\pi_{\mathcal{X}}$ of the net firm value satisfies

$$d\mu_{\mathcal{X}}(t) = \frac{1}{\mathcal{X}_{t-}} \big(S_{t-} d\mu^\pi_S(t) + B_{t-} d\mu^\pi_B(t) \big)$$

where μ^π_S and μ^π_B are the cumulative growth rates of equity and bonds, respectively. Similarly, the volatility of the net firm value $\sigma_{\mathcal{X}}$ can be expressed as

$$\sigma_{\mathcal{X}}(t) = \frac{1}{\mathcal{X}_{t-}} \big(S_{t-} \sigma_S(t) + B_{t-} \sigma_B(t) \big)$$

where σ_S and σ_B are the volatilities of equity and bonds, respectively. At default, $\mathcal{X}$ jumps downwards, mirroring the losses of equity and bonds. If default were to occur at time t, the combined losses relative to $\mathcal{X}$ are

$$J_t = \frac{1}{\mathcal{X}_{t-}} \big(S_{t-} + (1 - R_t) \cdot B_{t-} \big).$$

The value $J_t \mathcal{X}_{t-}$ represents the costs of bankruptcy, see A6. This value is lost to third parties at default. Thus, the net firm value $\mathcal{X}_t$ differs from the gross firm value X_t, which represents the present value of the future cash flows generated by the firm (A2). It follows that the I^2 model assumptions A1-A8 are not consistent with the Modigliani-Miller theorem. See Giesecke & Goldberg (2004*b*) for a discussion.

3 Pricing measures

In this section we analyze the set of pricing measures for I^2. Let $T \leq \bar{T}$, where $\bar{T} > 0$ is a finite horizon that we fix throughout. A pricing or equivalent martingale measure is characterized by two properties.

M1. Martingale property:
The discounted price process $(C_t e^{-rt})_{t \leq T}$ of any traded credit sensitive security (T, c_T, R) must be a $\mathbb{G}$-martingale with respect to the pricing measure.

M2. Equivalence:
The pricing measure and physical measure P belong to the same class. In other words, they agree on which sets in $\mathcal{G}_T$ have zero measure.

Let $\mathcal{P}$ denote the set of measures on $(\Omega, \mathcal{G}_T)$ satisfying M1 and M2. The mathematical conditions determining $\mathcal{P}$ arise from a fundamental economic result in Delbaen & Schachermayer (1997) that goes back to Harrison & Kreps (1979) and Harrison & Pliska (1981): Under broad assumptions, $\mathcal{P}$ is non-empty if and only if the security prices generated by the elements in $\mathcal{P}$ do not admit arbitrage opportunities. Further, $\mathcal{P}$ consists of a single measure if and only if markets are complete.

Throughout this section, the reference measure π is the physical measure P. This means assumption A1-A8 hold under P. We examine the set $\mathcal{P}$ of martingale measures equivalent to P in more detail. Our analysis sheds light on the structure of the risk adjustment, which provides an economic link between P and the measures in $\mathcal{P}$.

The space $\mathcal{P}$ sits inside the set $\mathcal{E}$ of measures that are equivalent to the physical measure.[4] Each $Q \in \mathcal{E}$ can be identified with a P-martingale $Z = Z(Q)$. In Theorem 3.1, we show that the space $\mathcal{E}$ is parameterized by a pair of $\mathbb{G}$-predictable processes α and β. The representation in Theorem 3.1 depends on the filtration $\mathbb{G}$ and the measure class of P but not on which securities are traded.

The next step is to express the Q-price processes of traded securities in terms of its associated P-martingale Z. Direct analysis of the Theorem 3.1 representation for $Z = Z(\alpha, \beta)$ gives rise to necessary and sufficient conditions on α and β given in Theorem 3.3 for which the price processes are Q-martingales.

3.1 Equivalent measures

Let $L^1 = L^1(P)$ denote the P-integrable functions on $(\Omega, \mathcal{G}_T)$. The relationship between the physical measure P and any $Q \in \mathcal{E}$ is expressed in terms of a positive random variable $Z_T = dQ/dP \in L^1$ for which $E^P[Z_T] = 1$. The variable Z_T is called the Radon-Nikodym derivative of Q with respect to P. Let Z be the right-continuous version of the P-martingale defined by

$$Z_t = E^P[Z_T \,|\, \mathcal{G}_t], \quad t \leq T. \tag{12}$$

Then for all bounded $Y \in \mathcal{G}_T$, the martingale Z satisfies the equation

$$E^Q[Y \,|\, \mathcal{G}_t] = \frac{1}{Z_t} E^P[Y Z_T \,|\, \mathcal{G}_t], \quad t \leq T. \tag{13}$$

[4]Note that the physical measure is in $\mathcal{E}$ but not usually in $\mathcal{P}$. In cases where the physical measure is in $\mathcal{P}$ the risk premium is zero.

In Theorem 3.1 below we represent the density process Z in terms of the two martingales that generate the uncertainty in I^2 under the physical measure. These are the Brownian motion W^P that underlies the gross firm value process and the compensated jump martingale $H^P = N - A^P_{\cdot\wedge\tau}$ associated with the default process N. Recall from C3 that the trend $A^P = -\log G^P(M)$, where G^P is investors' prior distribution on the default barrier and M is the historical low of the gross firm value. Note that H^P vanishes in case we follow traditional structural models in assuming that investors can observe the default barrier. This is because the default time is predictable with complete information.

Theorem 3.1. *The density process Z satisfies*

$$Z_t = \exp\left(-\int_0^t \alpha_s dW^P_s - \frac{1}{2}\int_0^t \alpha_s^2 ds + \int_0^t \log(1+\beta_s)dN_s - \int_0^{t\wedge\tau} \beta_s dA^P_s\right) \tag{14}$$

where α and $\beta > -1$ are $\mathbb{G}$-predictable processes. For a sequence of $\mathbb{G}$-stopping times T_n that increase to T, these processes satisfy

$$E^P\left[\int_0^{T_n} \alpha_s^2 ds\right] < \infty \quad \textit{and} \quad E^P\left[\int_0^{T_n} |\beta_s| dA^P_s\right] < \infty. \tag{15}$$

Conversely, $\mathbb{G}$-predictable processes α and β that satisfy (15) and for which $Z_T = Z_T(\alpha, \beta)$ defined by (14) satisfies $E^P[Z_T] = 1$, correspond to a measure $Q = Q(\alpha,\beta) \in \mathcal{E}$.

Kusuoka (1999) proves a result similar to Theorem 3.1 under the additional assumption that Z is square integrable.[5] Our result for I^2 is not subject to this restriction. Kunita (2004) proves an analogous result for a filtration generated by a Lévy process.

The proof of Theorem 3.1 is based on the following proposition.

Proposition 3.2. *The density process Z can be expressed as a sum*

$$Z_t = 1 + \int_0^t a_s dW^P_s + \int_0^t b_s dH^P_s, \tag{16}$$

where a and b are $\mathbb{G}$-predictable processes. For a sequence of $\mathbb{G}$-stopping times T_n that increase to T, these processes satisfy

$$E^P\left[\int_0^{T_n} a_s^2 ds\right] < \infty \quad \textit{and} \quad E^P\left[\int_0^{T_n} |b_s| dA^P_s\right] < \infty. \tag{17}$$

Proof. For $Z_T \in L^2(\Omega, \mathcal{G}_T, P)$, the representation (16) holds under more stringent growth conditions on a and b: $E^P[\int_0^T a_s^2 ds] < \infty$ and $E^P[\int_0^T b_s^2 dA^P_s] < \infty$. See for example Kusuoka (1999). Here, we follow a different line of reasoning to obtain a broader result under weaker restrictions on the coefficients.

[5] Kusuoka (1999) also assumes that the trend is of the form $A^P_t = \int_0^t \lambda^P_s ds$ for an intensity process λ^P. The proof of his representation Theorem 2.3 does however not require this assumption.

Let M_P be the collection of all martingales with respect to the stochastic basis $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T}, P)$ that take the form (16) and satisfy (17).

A special case of Jacod (1977, Theorem 2) is that if P is the unique measure on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T})$ for which every element of M_P is a martingale, then every martingale on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T}, P)$ is in M_P. We consider the subset $\mathcal{M}_P$ of M_P that consists of $(W_t, H_t, W_t^2 - t)_{0\le t\le \bar{T}}$. For readability we suppress the superscript P on W and H. It suffices to show that P is unique in the sense above with respect to the elements in $\mathcal{M}_P$.

Suppose that $P' \in M_P$. Since W and $(W_t^2 - t)$ are continuous martingales for both P and P' with respect to $\mathbb{G}$, Lévy's theorem implies that W is a Brownian motion under both measures. It follows that $P(W_{t_i} \in A_i; i = 1, 2, \dots, n) = P'(W_{t_i} \in A_i; i = 1, 2, \dots, n)$ for Borel sets A_i so that $P = P'$ on sets in $\mathcal{F}_T$.

Further, since H is a martingale for both P and P', the uniqueness of the Doob-Meyer decomposition implies that $A^P_{\cdot\wedge\tau}$ is the compensator of N for both P and P'. Let A^P and $A^{P'}$ be the $\mathbb{F}$-predictable trends of N under P and P' respectively. From the martingale property of $N - A^P_{\cdot\wedge\tau}$ and formula (9), $A^P_{t\wedge\tau} = A^{P'}_{t\wedge\tau}$ so that A^P and $A^{P'}$ agree for $t \le \tau$. We show that they agree on $[0, \infty)$ almost surely. Let Γ be the infimum of all times at which the trends A^P and $A^{P'}$ disagree. Then Γ is an $\mathbb{F}$-stopping time which is an upper bound for τ. This means the running minimum log-firm value M_Γ is less than the default barrier D. But D is not observable in the filtration $\mathbb{F}$. It follows that $\Gamma = \infty$ almost surely and the trends A^P and $A^{P'}$ are indistinguishable.

Let $U \in \mathcal{F}_T$. Then

$$E^{P'}[1_U(1 - N_t)] = E^{P'}[1_U e^{-A_t^{P'}}] = E^{P'}[1_U e^{-A_t^P}] = E^P[1_U e^{-A_t^P}]$$

where the first equation follows from Giesecke (2006, Theorem 4.5) and the third equation follows from the fact that the argument of the expectation is $\mathcal{F}_T$-measurable. Since every set in $\mathcal{G}_T$ can be arbitrarily well approximated by finite unions and complements of sets of the form $U \cap \{\tau \le t\}$, it follows that P and P' agree on $\mathcal{G}_T$. Thus, P is the unique measure for which the processes in $\mathcal{M}_P$ are martingales.

Now, Jacod (1977, Theorem 2) implies that every martingale on $(\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{0\le t\le T}, P)$ can be represented as in equation (16) with coefficients satisfying conditions (17). □

We now give the proof of Theorem 3.1.

Proof. According to Jacod & Shiryaev (1987, Proposition 3.5a), $P[\inf_t Z_t > 0] = 1$. It follows that P-almost surely, $Z_{t_-} > 0$ for all t. Further, $1/Z_-$ is locally bounded. For if not, then the $\mathbb{G}$-stopping times $\Gamma_n = \inf_t(Z_t < 1/n)$ increase to a stopping time Γ that is strictly less than T on a set $U \in \mathcal{G}_T$ of positive measure. But then $E^P[Z_{\Gamma-}1_U] = 0$, contradicting the fact that P-almost surely, $Z_{t_-} > 0$ for all t.

Let $\alpha = a/Z_-$ and $\beta = b/Z_-$ where a and b are the $\mathbb{G}$-predictable processes defined in Proposition 3.2. Since $1/Z_-$ is locally bounded, α and β satisfy (15) and thus, for positive $t \le T$, the integrals $\int_0^t \alpha_s^2\, ds$ and $\int_0^t |\beta_s|\, dA_s^P$ are finite almost surely. For if there is a positive measure set $U \in \mathcal{G}$ on which one of these integrals diverges, we can choose a large n so that $U \cap \{T^n > t\}$ has positive measure and one of the integrals $E^P[\int_0^{T_n} \alpha_s^2 ds]$ and $E^P[\int_0^{T_n} |\beta_s| dA_s^P]$ will diverge as well.

Define a semimartingale Y by $Y_t = -\int_0^t \alpha_s dW_s^P + \int_0^t \beta_s dH_s^P$, for $t \leq T$. Equation (16) can be rewritten as

$$Z_t = 1 + \int_0^t Z_{s-} dY_s, \quad t \leq T \tag{18}$$

so Z is the stochastic exponential of Y. By Theorem 37 in Chapter II of Protter (2004),

$$Z_t = \exp\left(Y_t - \frac{1}{2}[Y,Y]_t^c\right) \prod_{s \leq t} (1 + \Delta Y_s) \exp(-\Delta Y_s) \tag{19}$$

where $[Y,Y]^c$ is the (path-by-path) continuous part of $[Y,Y]$, and $\Delta Y_t = Y_t - Y_{t-}$ is the jump of Y at time t. Since W^P is a P-Brownian motion and $A_{\cdot\wedge\tau}^P$ is of finite variation, $[Y,Y]_t = \int_0^t \alpha_s^2 ds + \int_0^t \beta_s^2 dN_s$ so $[Y,Y]_t^c = \int_0^t \alpha_s^2 ds$. Since τ is totally inaccessible, the compensator of N is continuous. It follows that H^P is continuous except for a jump of size 1 at τ. Therefore, $\Delta Y_t = \beta_\tau 1_{\{t=\tau\}}$. With these observations, formula (14) is a consequence of formula (19).

For the converse, suppose that there are $\mathbb{G}$-predictable processes α and β satisfying the conditions (15). Let $Z_T = Z_T(\alpha, \beta)$ be defined by (14). Then $Z_T > 0$ and if $E^P[Z_T] = 1$, it is the density dQ/dP of an equivalent measure $Q = Q(\alpha, \beta) \in \mathcal{E}$ with respect to P. □

3.2 When are the price processes martingales?

The absence of arbitrage opportunities implies the existence of at least one martingale measure. Theorem 3.1 states that each measure $Q \in \mathcal{E}$ can be identified with a pair of processes (α, β). In Theorem 3.3 below we give necessary and sufficient conditions on (α, β) so that $Q(\alpha, \beta) \in \mathcal{P}$. These conditions are formulated in terms of the cumulative drift μ_C^P and the diffusive volatility σ_C of the value of a credit sensitive claim (T, c_T, R), see assumption A7. Recall that the trend $A^P = -\log G^P(M)$.

Theorem 3.3. *The discounted price process of the credit sensitive claim (T, c_T, R) is a martingale under $Q(\alpha, \beta)$ with α and β satisfying conditions (15) if and only if*

$$\mu_C^P(t) - \int_0^t (r + \sigma_C(s)\alpha_s) ds = \int_0^{t\wedge\tau} (\beta_s + 1)(1 - R_s) dA_s^P, \quad 0 \leq t \leq T. \tag{20}$$

Remark 3.4. *Equation (20) implies that the cumulative drift μ_C^P has continuous paths of finite variation that are not absolutely continuous with respect to the Lebesgue measure. This is due to the fact that the I^2 model does not admit an intensity, see C3.*

Remark 3.5. *For the $\mathbb{G}$-predictable process β there exists an $\mathbb{F}$-predictable process $\bar{\beta}$ such that $\beta_t = \bar{\beta}_t$ on the set $\{t \leq \tau\}$, cf. Jeulin & Yor (1978). Thus the integrand of the right hand side of equation (20) can be taken to be $\mathbb{F}$-adapted. In this situation, the $\mathcal{G}_t$-measurable random variable given by the left hand side of equation (20) takes the form $f(t \wedge \tau)$ where f is $\mathbb{F}$-adapted.*

Proof. From (13), it is equivalent to require that $(\bar{C}_t Z_t)_{t\le T}$, where $\bar{C}_t = C_t e^{-rt}$, is a $(P,\mathbb{G})$-martingale. From (18),

$$dZ_t = Z_{t-}(-\alpha_t dW_t^P + \beta_t dH_t^P). \tag{21}$$

Noting assumption A7, by integration by parts

$$d\bar{C}_t = d(C_t \cdot e^{-rt}) = \bar{C}_{t-}(d\mu_C^P(t) - rdt + \sigma_C(t)dW_t^P - (1-R_t)dN_t). \tag{22}$$

Since the cumulative growth rate μ_C^P has paths of finite variation (A7), the process defined by the Stieltjes integral $\int_0^t \bar{C}_{s-} d\mu_C^P(s)$ has paths of finite variation. Using this and Protter (2004, Chapter IV, Theorem 22), we get

$$[\bar{C}, Z]_t = \int_0^t \bar{C}_{s-} Z_{s-}(-\sigma_C(s)\alpha_s ds - \beta_s(1-R_s)dN_s). \tag{23}$$

Integrating by parts, substituting equations (21), (22) and (23), and applying the Doob-Meyer decomposition $A^P_{\cdot\wedge\tau} + H^P = N$,

$$\begin{aligned} d(\bar{C}_t Z_t) = &\bar{C}_{t-} Z_{t-}(d\mu_C^P(t) - (r + \sigma_C(t)\alpha_t)dt - (\beta_t + 1)(1-R_t)dA^P_{t\wedge\tau} \\ &+ (\sigma_C(t) - \alpha_t)dW_t^P + (\beta_t R_t + R_t - 1)dH_t). \end{aligned} \tag{24}$$

We see that $(\bar{C}_t Z_t)_{t\le T}$ is a $(P,\mathbb{G})$-martingale if and only if the drift in (24) vanishes. □

4 Risk premia

As in the previous section, the reference measure is the physical measure P. We fix $T \in (0, \bar{T}]$. Theorem 3.1 states that each pricing measure corresponds to a pair of predictable processes α and β satisfying conditions (15). We examine the relationship between these processes and the risk premia demanded by investors. We start with an observation that is a standard consequence of Girsanov's Theorem. We omit the proof.

Proposition 4.1. *Under the pricing measure $Q = Q(\alpha,\beta)$ with α and β satisfying conditions (15), the process $(W_t^Q)_{t\le T}$ given by $W_t^Q = W_t^P + \int_0^t \alpha_s ds$ is a $\mathbb{G}$-standard Brownian motion and the process $(H_t^Q)_{t\le T}$ given by $H_t^Q = N_t - \int_0^{t\wedge\tau}(1+\beta_s)dA_s^P$ is a $\mathbb{G}$-martingale.*

The interpretation of α as a risk premium can be seen in the value dynamics of a credit sensitive claim given in assumption A7. It follows from Proposition 4.1 that

$$p_C(t) = \mu_C^P(t) - \mu_C^Q(t) = \int_0^t \alpha_s \sigma_C(s) ds. \tag{25}$$

Equation (25) shows that the excess cumulative growth p_C on the claim demanded by risk-averse investors has paths that are absolutely continuous with respect to Lebesgue measure. The excess growth rate $dp_C(t)/dt$ is proportional to the diffusive price volatility

$\sigma_C(t)$. The proportionality factor α_t equals the excess return per unit of diffusive risk. It can thus be interpreted as the market price of Brownian motion driven diffusion-type risk in firm values. The *diffusive risk premium* $\alpha\sigma_S$ is a $\mathbb{G}$-predictable process that depends on the underlying Brownian motion and the default time.

To understand the role of β, note that Proposition 4.1 implies that the $(Q, \mathbb{G})$-compensator of N is given by

$$A^Q_{t\wedge\tau} = \int_0^{t\wedge\tau} (1 + \beta_s) dA^P_s. \tag{26}$$

From equation (26) and the heuristic relation $dA^\pi_{t\wedge\tau} = E^\pi[dN_t \mid \mathcal{G}_t]$ implied by the $(\pi, \mathbb{G})$-martingale property of the compensated process $N - A^\pi_{\cdot\wedge\tau}$, we get that

$$Q[t < \tau \le t + dt \mid \mathcal{G}_t] = (1 + \beta_t) P[t < \tau \le t + dt \mid \mathcal{G}_t] \tag{27}$$

on the no-default set $\{t < \tau\}$. Hence, β_t provides the mapping between the instantaneous Q-default probability and the instantaneous P-default probability. This suggests the interpretation of β as the *default event risk premium* that is demanded by investors as compensation for assuming the risk of a downward jump in security prices due to default. Note that this event premium is zero in traditional structural credit models in which it is assumed that investors can observe the default barrier. This is because there is no short-term default risk with complete information. Prices do not jump at default.

The infinitesimal Q-default probability (27) can be interpreted as the undiscounted pre-default price of an insurance contract that pays one dollar if default occurs over the next infinitesimal period of time, and zero otherwise. If investors are risk-neutral with respect to default event risk, they value default insurance with the P-default probability. Here $\beta = 0$ and default probabilities are equal under P and Q. If investors love jump risk, their value of the insurance contract is lower than what is suggested by the P-default probability. Here $\beta \in (-1, 0)$ and the Q-default probability is less than the P-probability. If investors are averse to jump risk, they demand a spread for the default insurance contract that exceeds the physical default spread. In this case $\beta > 0$ and the Q-default probability exceeds the P-probability.

The event risk premium β is linked to the distribution of the unobserved default barrier. First note that the relationship (26) extends to the trends under P and Q:[6]

$$A^Q_t = \int_0^t (1 + \beta_s) dA^P_s. \tag{28}$$

[6]The random measures on $\mathbb{R}_+$ associated with the $\mathbb{F}$-predictable trends $A^P(\omega)$ and $A^Q(\omega)$ are both concentrated on the set $\{t \ge 0 : V_t(\omega) = M_t(\omega)\}$. Hence, for almost all $\omega \in \Omega$, they are absolutely continuous with respect to each other. Theorem 68 in Dellacherie & Meyer (1982, Chapter VI) states that the corresponding density process γ is $\mathbb{F}$-predictable. Equation (26) implies that $\gamma_t = 1 + \beta_t$ on the set $\{t \le \tau\}$, where β is $\mathbb{F}$-predictable by Remark 3.5. Using an argument similar to that used in the proof of Proposition 3.2, we can then show that $\gamma_t = 1 + \beta_t$ on $[0, \infty)$.

Using formula (10), equation (28) can be rewritten in terms of the barrier distribution G^P and density g^P under P as

$$A_t^Q = -\int_0^t \frac{g^P(M_s)}{G^P(M_s)}(1+\beta_s)dM_s \tag{29}$$

where M is the running minimum log-firm value. Comparing formula (29) with the formula for A^Q obtained from equations (9) and (10) applied under the measure Q, we see that on the set $\{t \geq 0 : V_t = M_t\}$ the event premium is determined by the relation

$$1 + \beta_t = \frac{g^Q(M_t)/G^Q(M_t)}{g^P(M_t)/G^P(M_t)}. \tag{30}$$

Equation (30) shows that $\beta_t = f(M_t)$ for a deterministic function f that is given in terms of the distributions of the default barrier under the physical and pricing measures. The typical case of event risk aversion $\beta \geq 0$ corresponds to the case where the conditional probability that a small increment in minimum firm value causes default, given that default has not yet occurred, is greater for the pricing measure than for the physical measure.

5 Valuing credit sensitive claims

A credit sensitive claim is characterized by a triple (T, c_T, R). Here, $c_T \in \mathcal{F}_T$ is the payoff at the horizon $T \in (0, \bar{T}]$ if there was no default, and $1-R$ is the fractional loss in the pre-default market value of the claim if the firm defaults before T. For $Q \in \mathcal{P}$, a no-arbitrage, pre-default price $C_t = C_t(Q)$ of this claim at time $t \leq T$ is given by

$$C_t = E^Q\big[e^{-r(T-t)}c_T 1_{\{\tau>T\}} + e^{-r(\tau-t)}R_\tau C_{\tau-}1_{\{\tau\leq T\}} \,|\, \mathcal{G}_t\big]. \tag{31}$$

Formula (31) has the disadvantage of involving the default time τ explicitly. We provide an alternative reduced form formula that is based on the recovery-adjusted Q-trend $A^Q(R)$, which is defined in terms of the Q-trend A^Q by the formula

$$A_t^Q(R) = \int_0^t (1-R_s)dA_s^Q. \tag{32}$$

One approach to calculate A^Q is to reason as in the previous two sections. Suppose assumptions A1-A8 hold under the actual P. Then the P-trend $A^P = -\log G^P(M)$ by C3. Given the density $Z(\alpha,\beta)$ associated with the chosen pricing measure Q, we then calculate A^Q from A^P via formula (28). A second approach is to suppose A1-A8 hold under Q. Then the Q-trend $A^Q = -\log G^Q(M)$ by C3.

Proposition 5.1. *Suppose the trend A^Q is continuous. If the process Y given by*

$$Y_t = e^{-r(T-t)}E^Q\big[c_T e^{A_t^Q(R)-A_T^Q(R)} \,|\, \mathcal{G}_t\big], \quad t \leq T, \tag{33}$$

has paths that are continuous at the default time, then the credit sensitive claim (T, c_T, R) admits an arbitrage-free value $C_t = Y_t(Q)$ on the no-default set $\{\tau > t\}$ at time $t \leq T$.

The following observation is required to prove Proposition 5.1.

Lemma 5.2. *For each $Q \in \mathcal{P}$, a $(Q, \mathbb{G})$-martingale $H^Q(R)$ is defined by*

$$H_t^Q(R) = (1 - R_\tau)N_t - A_{t\wedge\tau}^Q(R), \quad t \leq T.$$

Proof. The process $(1-R_\tau)N$ is zero before τ at which time it jumps to $(1-R_\tau) \in \mathcal{G}_\tau$ and stays there. Since R is $\mathbb{G}$-predictable, $(1 - R_\tau)N$ is $\mathbb{G}$-adapted. Since $A_{\cdot\wedge\tau}^Q(R)$ is clearly $\mathbb{G}$-adapted, so is the process $H^Q(R)$. We show that $H^Q(R)Z$ is a P-martingale. We have

$$[H^Q(R), Z]_t = \int_0^t Z_{s-}\beta_s(1 - R_s)dN_s.$$

Integration by parts together with (26) and (32) yields that

$$\begin{aligned} d(Z_t H_t^Q(R)) &= H_{t-}^Q(R)dZ_t + Z_{t-}((1 - R_\tau)dN_t - dA_{t\wedge\tau}^Q(R)) + Z_{t-}\beta_t(1 - R_t)dN_t \\ &= H_{t-}^Q(R)dZ_t + Z_{t-}(1 + \beta_t)((1 - R_\tau)dN_t - (1 - R_t)dA_{t\wedge\tau}^P) \\ &= H_{t-}^Q(R)dZ_t + Z_{t-}(1 + \beta_t)(1 - R_t)dH_t^P, \end{aligned}$$

where H^P is the compensated jump P-martingale. Since Z is also a P-martingale, $ZH^Q(R)$ is a P-martingale as well. This is equivalent to $H^Q(R)$ being a Q-martingale. □

Proof of Proposition 5.1. First note that if A^Q is continuous, then $A^Q(R)$ is as well continuous. Let $K_t = e^{-rT}E^Q[c_T e^{-A_T^Q(R)} \,|\, \mathcal{G}_t]$ such that $Y_t = e^{A_t^Q(R)+rt}K_t$. Setting $\bar{Y}_t = Y_t e^{-rt} = e^{A_t^Q(R)}K_t$, by integration by parts we find that

$$d\bar{Y}_t = e^{A_t^Q(R)}dK_t + \bar{Y}_{t-}dA_t^Q(R).$$

We now define the process U by

$$U_t = (1 - N_t)\bar{Y}_t + \int_0^t R_s\bar{Y}_{s-}dN_s;$$

our goal is to show that $(U_t)_{t\leq T}$ is a Q-martingale. Since $\bar{Y}$ does not jump at τ by assumption, $\Delta\bar{Y}\Delta(1 - N) = 0$ and we have by integration by parts

$$\begin{aligned} dU_t &= (1 - N_{t-})e^{-A_t^Q(R)}dK_t - \bar{Y}_{t-}((1 - R_\tau)dN_t - dA_{t\wedge\tau}^Q(R)) \\ &= (1 - N_{t-})e^{-A_t^Q(R)}dK_t - \bar{Y}_{t-}dH_t^Q(R), \end{aligned}$$

for the Q-martingale $H^Q(R)$, cf. Lemma 5.2. Since K is also a Q-martingale, $(U_t)_{t\leq T}$ is as well a Q-martingale (all integrands are predictable and bounded for $T < \bar{T}$).

By the martingale property of U we get

$$(1 - N_t)Y_t e^{-rt} = E^Q\Big[(1 - N_T)Y_T e^{-rT} + \int_t^T e^{-rs}R_s Y_{s-}dN_s \,\Big|\, \mathcal{G}_t\Big].$$

But this implies the valuation formula since $Y_T = c_T$. □

Proposition 5.1 extends Corollary 5.5 in Giesecke (2006) to the case of non-trivial recovery. Proposition 5.1 is essentially Theorem 1 in Duffie & Singleton (1999) if the Q-trend is of the form $A_t^Q = \int_0^t \lambda_s^Q ds$ for an intensity process λ^Q. Recall from C3 that under assumptions A1-A8, the trend is continuous but does not admit an intensity. Bélanger et al. (2004) and Elliott, Jeanblanc & Yor (2000) derive price representations similar to that in Proposition 5.1 under different sets of assumptions.

The process C in Proposition 5.1 uniquely defines the price of the claim only if markets are complete. In the incomplete case, Proposition 5.1 leads to an interval of arbitrage-free prices for (T, c_T, R).

The continuity assumption on the process Y in Proposition 5.1 can be relaxed by introducing an absolutely continuous change of measure; see Collin-Dufresne, Goldstein & Hugonnier (2004) for details.

A Probabilistic model structure

We detail the probabilistic structure underlying our model described in Section 2.

We introduce two probability spaces. The first is the filtered space $(\Omega_1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t\geq 0}, \pi_1)$ supporting the standard Brownian motion $\tilde{W}$. The second is the space $(\Omega_2, \mathcal{F}^2, \pi_2)$ supporting the random variable $\tilde{d}$. Here we may set $\Omega_2 = (0, X_0)$ for some constant $X_0 > 0$ and $\tilde{d} = \omega_2$ for $\omega_2 \in \Omega_2$, and $\mathcal{F}^2 = \sigma(\tilde{d})$. Our reference probability space is

$$(\Omega, \mathcal{G}, \pi) = (\Omega_1 \times \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, \pi_1 \otimes \pi_2)$$

where the state of the world $\omega \in \Omega$ is the pair (ω_1, ω_2).

On this space, we define the standard Brownian motion $W^\pi(\omega) = \tilde{W}(\omega_1)$ and the random default barrier $d(\omega) = \tilde{d}(\omega_2)$ considered in assumptions A2 and A3, respectively. Notice that we do not observe ω_2, cf. assumption A5. Corresponding to assumption A3, we also introduce the random time τ by setting

$$\tau(\omega) = \inf\{t > 0 : V_t(\omega) \leq D(\omega)\},$$

where $V_t = mt + \sigma W_t$ for constants $m \in \mathbb{R}$ and $\sigma > 0$ and $D = \log(d/X_0)$ is the normalized default barrier. The measure π_2 induces a distribution function G^π of the normalized barrier D via $\pi_2(0, X_0 e^x) = G^\pi(x)$ for all $x \in (-\infty, 0)$. The distribution function G^π is often called "prior." Letting $M_t(\omega) = \min_{s\leq t} V_s(\omega)$, we can write

$$\{\tau(\omega) > t\} = \{M_t(\omega) > D(\omega)\}.$$

Consider the standard filtration $\mathbb{F} = \mathcal{F}_{t\geq 0}$ generated by W^π on $(\Omega, \mathcal{G}, \pi)$. All the sets in $\mathbb{F}$ are of the form $F \times \Omega_2$ and $F \times \emptyset$ for $F \in \mathcal{F}_t^1$. Let $(\mathcal{S}_t)$ be the standard filtration generated by the indicator process $(1_{\{\tau \leq t\}})$. Corresponding to assumption A5, we can now introduce the enlarged filtration $\mathbb{G}$ on the reference space by setting $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{S}_t$.

The $(\pi, \mathbb{F})$-Brownian motion W^π is also a Brownian motion in the enlarged filtration $\mathbb{G}$. Indeed, because W^π ignores ω_2,

$$E^\pi[W_t^\pi|\mathcal{G}_s] = E^\pi[W_t^\pi|\mathcal{F}_s] = W_s^\pi, \quad t \geq s,$$

proving the martingale property of W^π in $\mathbb{G}$. Since the quadratic variation $[W^\pi, W^\pi]_t = t$ does not depend on the filtration, the result follows by Lévy's theorem.

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