The Market Price of Credit Risk

Kay Giesecke

Cornell University

giesecke@orie.cornell.edu

www.orie.cornell.edu/∼giesecke

Lisa R. Goldberg

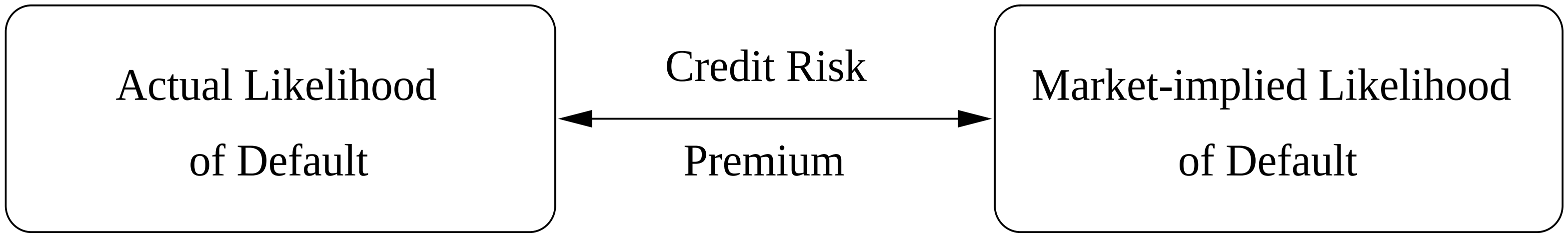
lrg@barra.com

We live in a dangerous world

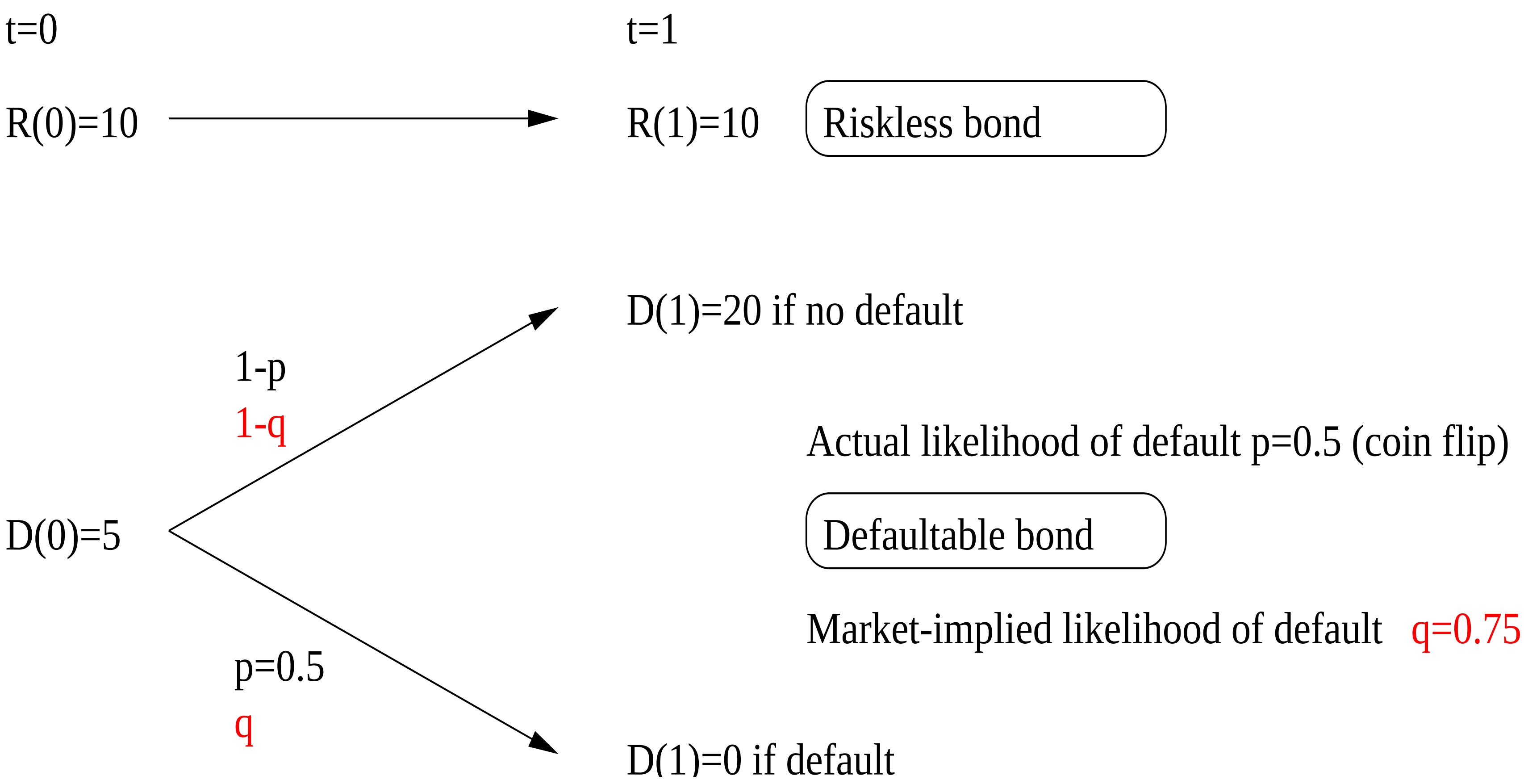
- Issuers of defaultable securities share a common dependence on the economic environment
- Aggregate credit risk cannot be diversified away
- Undiversifiable or *systematic risk* commands a premium
  - Risk-averse investors must be compensated for assuming systematic credit risk
- The credit premium is empirically well-documented, and theoretically complex
  - Why is it important to understand it?

How is a credit model used?

- To forecast the actual probability of default
  - Model must reflect historical default experience
  - *Actual* likelihood of default
- To estimate the value of default sensitive securities
  - Model must fit market prices
  - *Market-implied* likelihood of default



What's the difference?



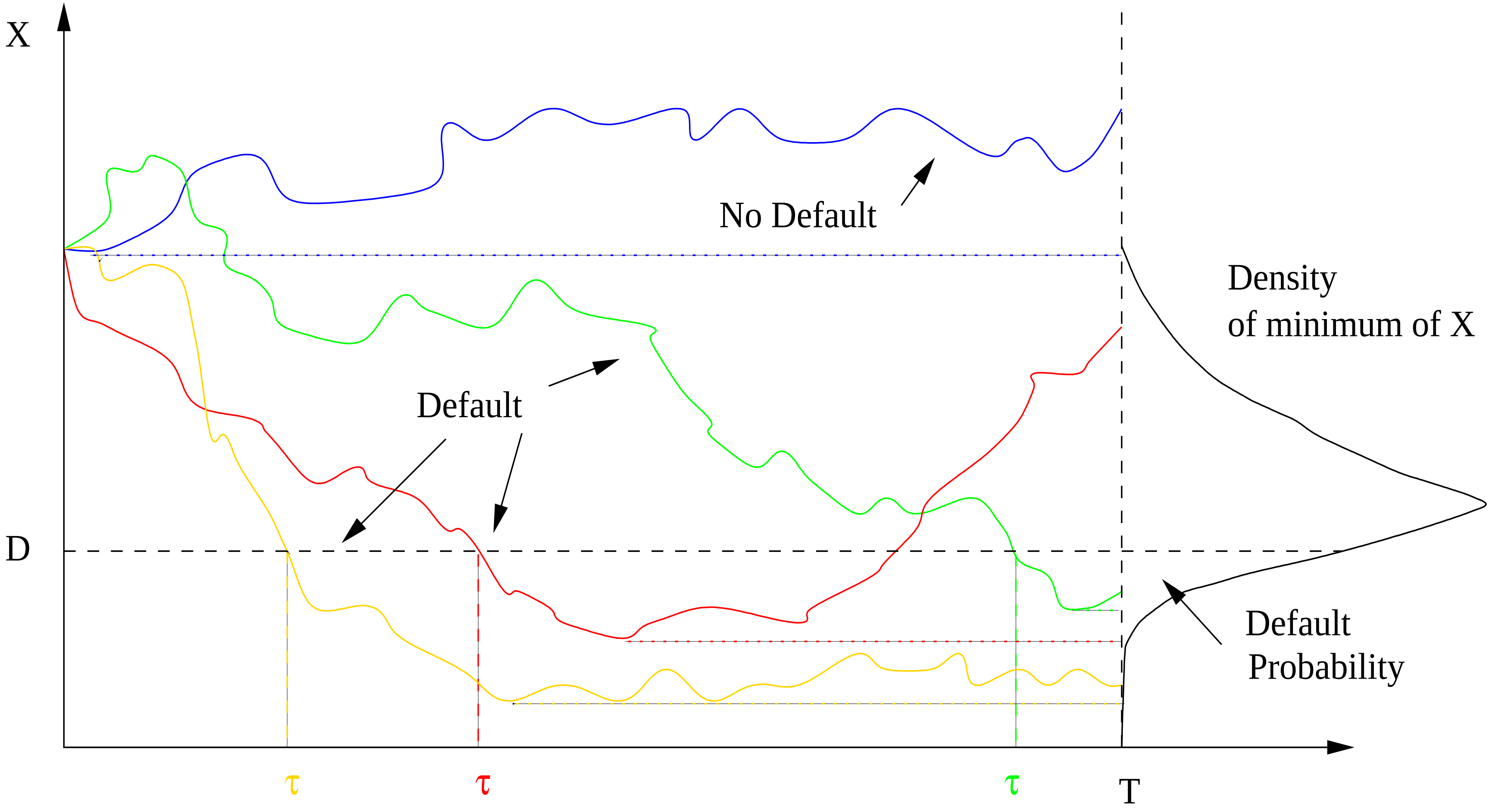
First passage structural credit model

- Default occurs when firm value X falls below a barrier D:

$$\tau = \inf\{t > 0 : X_t \leq D\}$$

- Requires
  - A model of firm value process X
  - A model of default barrier D
- Identifies equity as a down-and-out call option on the firm
- Default probability given by the distribution of $\min_{s \leq T} X_s$

First passage structural credit model



First passage credit risk premium

- Credit risk is exclusively driven by uncertainty about firm value
- Risk premium takes a familiar form (X a (μ, σ)-GBM)
  - Excess return on credit sensitive security is equal to its "risk" times the market price of that risk
  - Risk is measured in terms of diffusive price volatility
  - The market price of risk is given by

$$\frac{\mu - r}{\sigma},$$

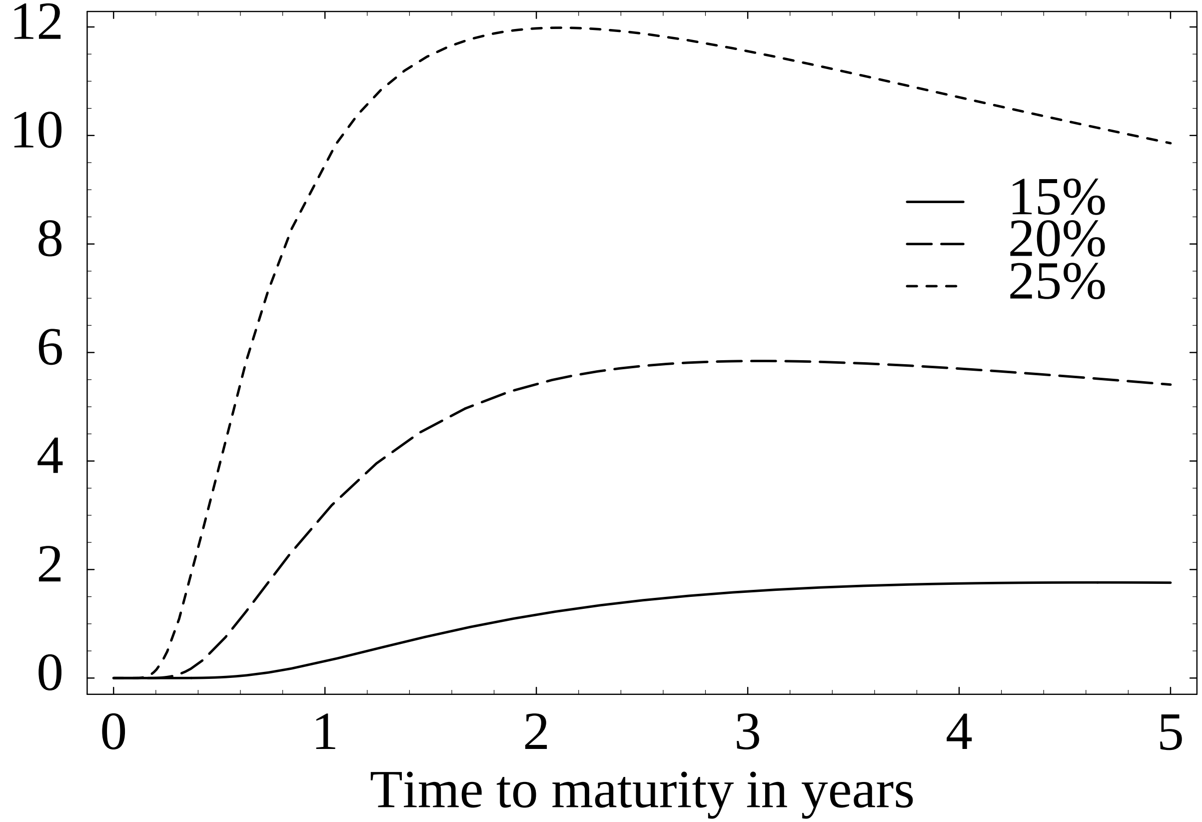
the excess return $\mu - r$ on the firm per unit of firm risk σ

Caveats

This simple representation of the risk premium neglects the short-term uncertainty surrounding the default

- Distance of firm to default is always certain
- Default is *predictable*, it has an announcing sequence $(\tau_n) \uparrow \tau$
  - Think of τ_n as the first time at which assets X fall dangerously close to the default barrier D
- Results in difficulties in fitting model to security prices
  - No jumps in defaultable security prices upon default
  - Zero short-term spreads

First passage credit spreads



What do investors really know?

- It is implicitly assumed that the information used to calibrate and run the model is publicly available
- Enron, WorldCom, and Tyco: incorrect corporate statements
  - Asset value, volatility, and growth rate?
  - Default trigger level?
- Relax the assumption that investors are completely informed
  - Distance of firm to default is uncertain
  - Default is *unpredictable* as in the classical reduced-form models
    - It has no announcing sequence

Incomplete information credit models

- Structure plus short-term uncertainty
- Economic reasonability and flexibility
  - Intuitive model parameters that can be calibrated to the quality of information available to investors
- Tractable pricing formulae and empirical plausibility
  - Non-zero short-term spreads
  - Jumps in defaultable security prices upon default
- Unified perspective on credit models
  - Traditional structural models and intensity based models are special cases

I^2 credit model

1. *Uncertainty*: probability space $(\Omega, \mathcal{G}, \pi)$ with filtration $(\mathcal{G}_t)$
2. *Pre-default firm value*: $X_t = X_0 e^{V_t}$, for V (m, σ)-Brownian motion
   - Corresponding historical log-asset low: $M_t = \min_{s \le t} V_s$
3. *Default time* τ: first passage of $X(\omega)$ to $D(\omega)$
4. *Information*: default barrier D unobservable
   - $\log(D/X_0)$ has continuous distribution function G
   - The filtration is defined by ($\mathcal{G}_0$ is the set of π-null sets)

$$\mathcal{G}_t = \mathcal{G}_0 \vee \sigma(V_s : s \le t) \vee \sigma(N_s : s \le t)$$

5. *Fractional recovery* process R: at τ, a defaultable security recovers a fraction R_τ of its pre-default value $C_{\tau-}$

Important consequences

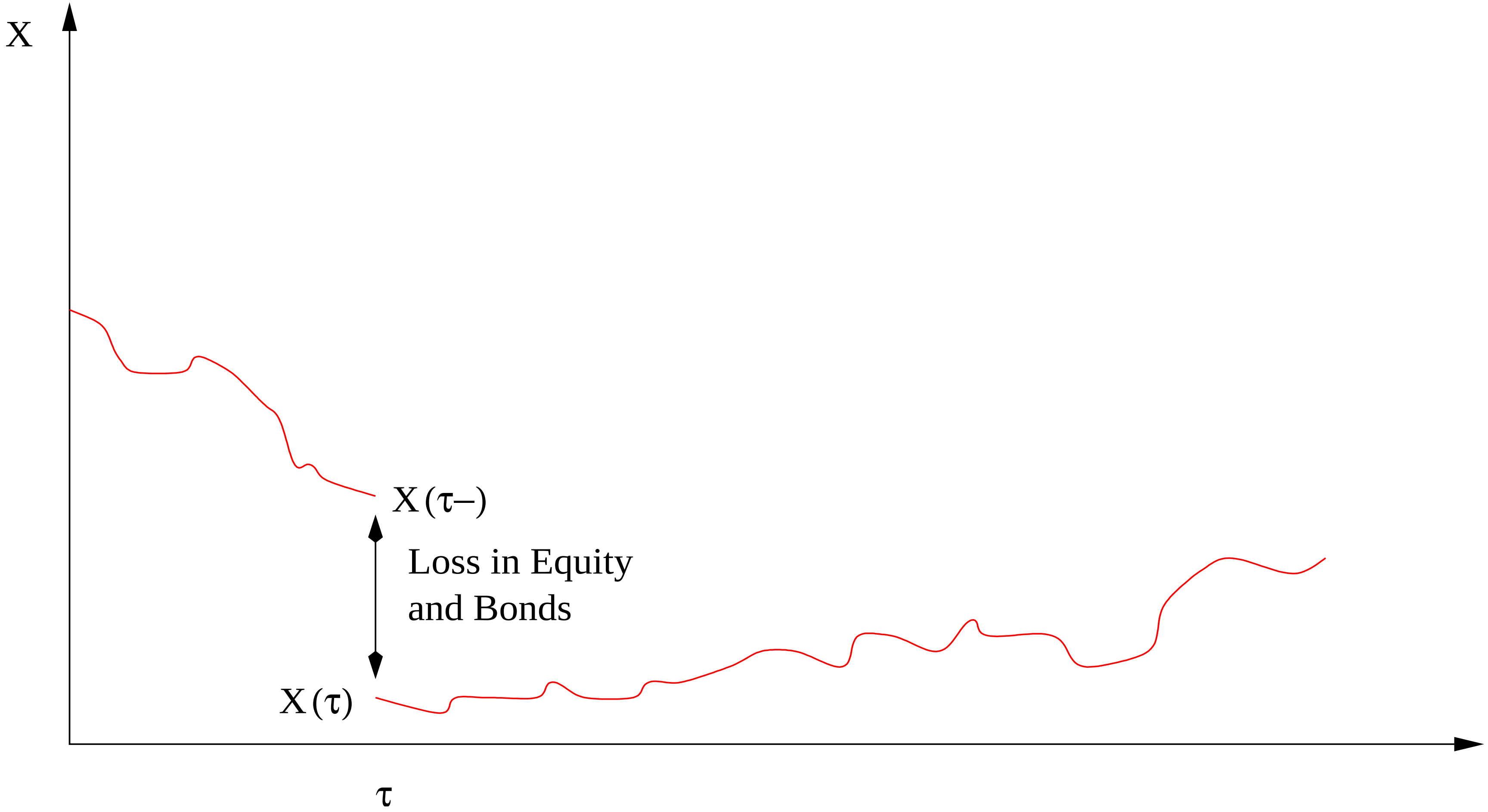
- The default stopping time is unpredictable
- Firm value $\mathcal{X}$ follows the *jump*-diffusion

$$\frac{d\mathcal{X}_t}{\mathcal{X}_{t-}} = \mu dt + \sigma dW_t - J_t dN_t$$

where $J_t \mathcal{X}_{t-}$ is the combined default loss in equity value S_t and debt value B_t if default were to occur at t, i.e.

$$J_t \mathcal{X}_{t-} = S_{t-} + (1 - R_t) \cdot B_{t-}$$

Jump in firm value



Probability of default

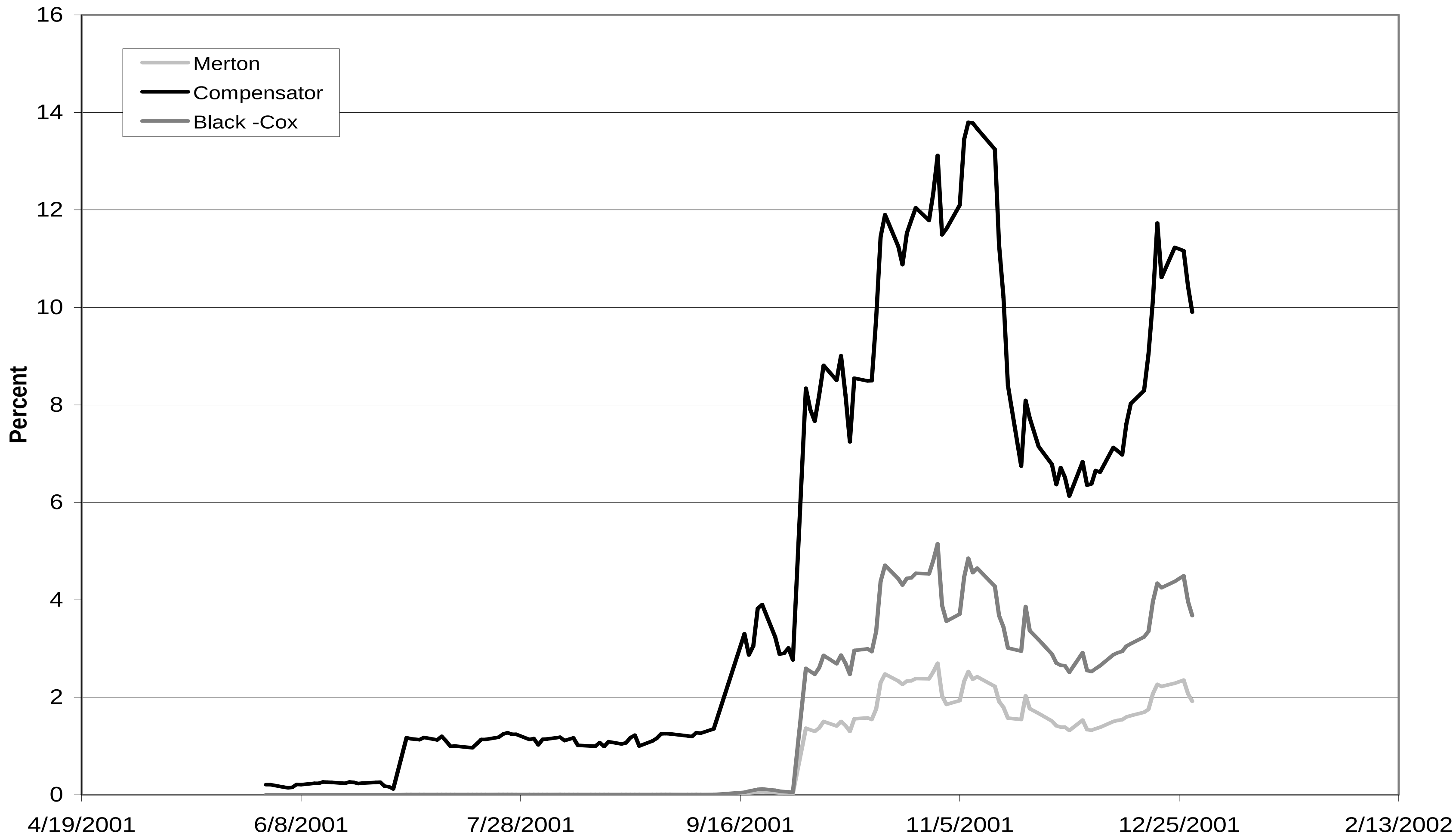
- Default trend $A^\pi_t = -\log G(M_t)$
  - Constructed with techniques from enlargement of filtrations
  - The stopped trend $A^\pi_{\cdot\wedge\tau}$ is the compensator in the Doob-Meyer decomposition of N, i.e. $N - A^\pi_{\cdot\wedge\tau}$ is a martingale
  - Fair compensation for short-term credit risk:

$$A^\pi_t = \lim_{h\downarrow 0} \frac{1}{h} \int_0^t \pi[s < \tau \le s + h \,|\, \mathcal{G}_s] ds$$

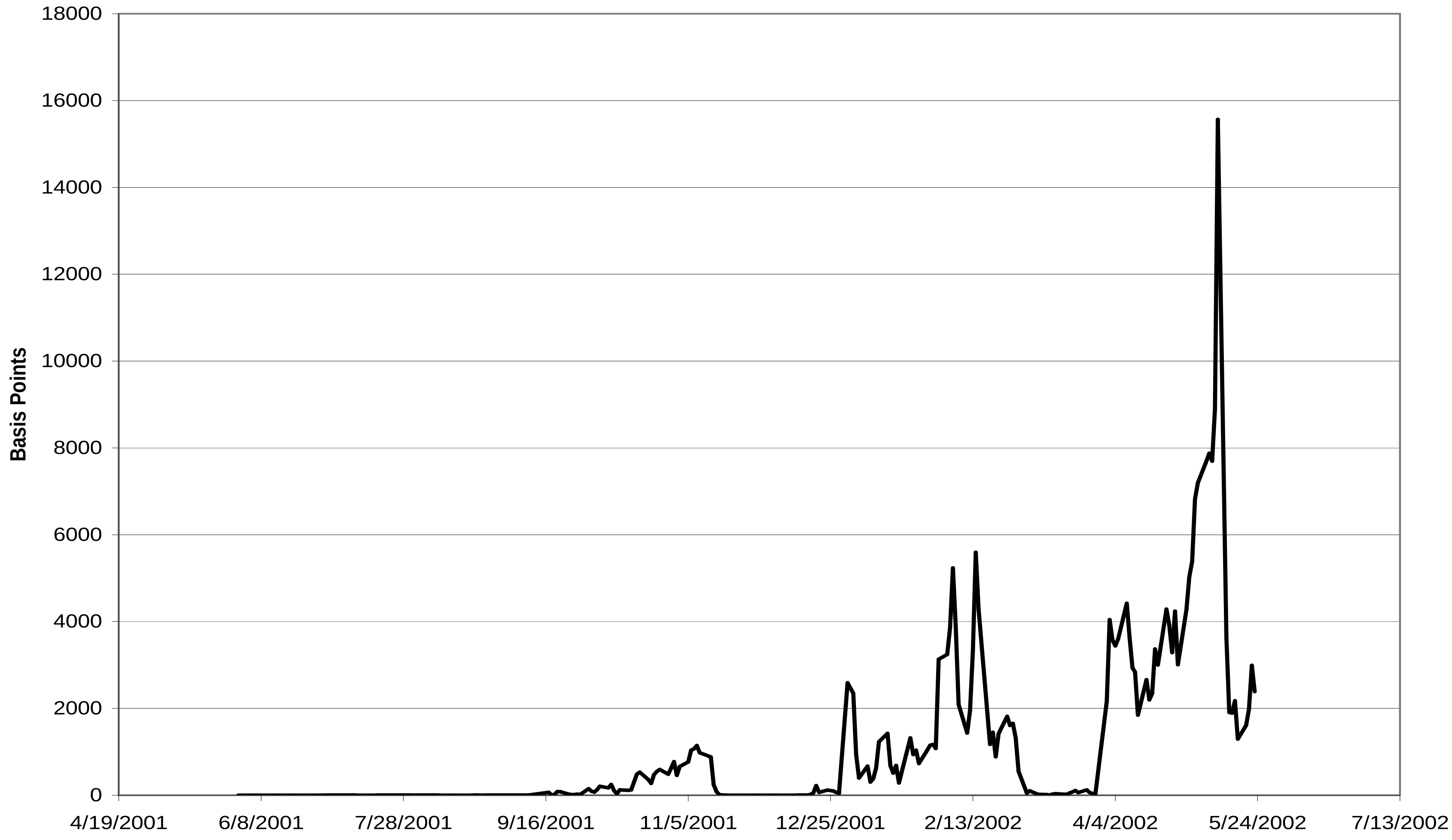
- Generalized reduced-form representation of the survival probability

$$\pi[\tau > T \,|\, \mathcal{G}_t] = E^\pi[e^{A^\pi_t - A^\pi_T} \,|\, \mathcal{G}_t]$$

Early reaction: United default probabilities



Non-zero short-term spreads of United



Actuarial pricing principle

- Consider a defaultable zero-coupon bond with zero recovery
  - Payoff $c_T = 1_{\{\tau > T\}}$ at maturity T
- According to the *actuarial principle*, the fair price is

$$C_t = e^{-r(T-t)} E^P[c_T \,|\, \mathcal{G}_t] = e^{-r(T-t)} P[\tau > T \,|\, \mathcal{G}_t]$$

  where P is the *physical* measure

- Implies that price spread covers only expected default loss
  - But investors are risk-averse and demand extra compensation, or a risk premium, for assuming the risk of losses
- Does not rule out arbitrage

No-arbitrage pricing principle

- Need to account for risk aversion: generate lower security prices
- Standard approach: retain the form of the principle

$$C_t = e^{-r(T-t)} E^{\pi}[c_T \mid \mathcal{G}_t]$$

  but substitute a *pricing measure* for π

- Events such as $\{\omega : \tau(\omega) \leq T\}$ are assigned new probabilities
  - They do not reflect the actual P-likelihood of default
  - They are consistent with market prices
- If π is a pricing measure, C defined as above accounts for both expected loss and risk aversion

Pricing measures

- The set of pricing measures $\mathcal{P}$ contains the measures Q such that
  1. Discounted price processes are Q-martingales
  2. Q and P agree on which sets in $\mathcal{G}_T$ have measure zero
- $\mathcal{P}$ is non-empty if and only if prices do not admit arbitrage
  - All pricing measures account for credit risk and each one corresponds to a particular risk premium
- Markets are complete and default can be hedged perfectly if and only if $|\mathcal{P}| = 1$

Characterizing the measures

- Each measure Q equivalent to P can be identified with its density $Z = Z(Q) \in L^1(P)$ with respect to P

- The P-martingale Z can be represented as

$$Z_t = \exp\left(-\int_0^t \alpha_s dW_s - \frac{1}{2}\int_0^t \alpha_s^2 ds + \int_0^t \log(1+\beta_s) dN_s - \int_0^{t\wedge\tau} \beta_s dA_s^P \right)$$

  where α and $\beta > -1$ are predictable and satisfy growth conditions

- This extends the familiar result which requires $Z \in L^2(P)$
  - Kusuoka (1999), El-Karoui & Martellini (2001)

- Necessary and sufficient conditions on (α, β) such that $Q(\alpha, \beta) \in \mathcal{P}$
  - Express prices under Q in terms of $Z(\alpha, \beta)$

Martingales and risk premia

- The space of $L^1(P)$-martingales is spanned by
  - The Brownian motion W:
    represents diffusion-type uncertainty in the firm value
  - The compensated jump martingale $N - A^P_{\cdot\wedge\tau}$:
    represents jump-type uncertainty in the firm value
- We use Jacod's (1977) martingale representation results
- The credit risk premia are proportional to the martingale coefficients
  - α is the *diffusive* risk premium per unit of diffusive risk
  - β is the *default event* risk premium

Diffusive risk premium

- Excess return $\mu_C^P - \mu_C^Q = \alpha \cdot \sigma_C$
- Realized as a change to the drift in the security's price process
  - As in Black-Scholes, adjust for aversion towards diffusive risk σ_C by lowering the likelihood of favorable firm value states
- Proportional to the security's diffusive price volatility
  - Proportionality factor α can be interpreted as the market price of diffusive firm value risk

Default event risk premium

- For the default trend

$$A_t^Q = \int_0^t (1 + \beta_s) dA_s^P$$

for $\beta > -1$ a predictable process. Therefore, heuristically,

$$Q[t < \tau \leq t + dt \,|\, \mathcal{G}_t] = (1 + \beta_t) P[t < \tau \leq t + dt \,|\, \mathcal{G}_t]$$

- Realized as a change to the default probability
  - Adjust for aversion towards default event risk by increasing the likelihood of bad states where default happens earlier
  - Not present in Black-Scholes and other structural credit models
- Analogous to the event risk premium in intensity-based models
  - Jarrow, Lando & Yu (2000), El-Karoui & Martellini (2001)

Default insurance

- The "value" of default insurance paying 1 at default is given by

$$\begin{aligned} dA_t^Q = E^Q[dN_t \,|\, \mathcal{G}_t] &= Q[t < \tau \leq t + dt \,|\, \mathcal{G}_t] \\ &= (1 + \beta_t) P[t < \tau \leq t + dt \,|\, \mathcal{G}_t] \end{aligned}$$

- $\beta = 0$: investors are indifferent towards default event risk
  - Value given by the P-default probability
- $\beta \in (-1, 0)$: investors love default event risk
  - Value lower than what is suggested by the P-default probability
- $\beta > 0$: investors are averse towards event risk
  - Value higher than what is suggested by the P-default probability
  - Empirical evidence: Driessen (2002)

Default-contingent claims

- Default-contingent claim (T, c_T, R): pay $c_T \in \mathcal{G}_T$ at T if there is no default by T and $R_\tau \cdot C_{\tau-}$ at default otherwise
  - $c_T = 1$: defaultable zero coupon bond with fractional recovery
- Fixing some martingale measure Q, a no-arbitrage price is given by

$$C_t = E^Q[e^{-r(T-t)} c_T 1_{\{\tau > T\}} + e^{-r(\tau-t)} R_\tau \cdot C_{\tau-} \cdot 1_{\{\tau \le T\}} \,|\, \mathcal{G}_t]$$

- More tractable representation?

Generalized reduced-form pricing

- If the process defined by

$$Y_t = e^{-r(T-t)} E^Q \left[c_T \exp\left(A_t^Q(R) - A_T^Q(R) \right) \middle| \mathcal{G}_t \right]$$

  is continuous at default, then the claim admits a no-arbitrage price of $C_t = Y_t(Q)$ on the no-default set $\{\tau > t\}$ at time $t \leq T$. Here

$$A_t^Q(R) = \int_0^t (1 - R_s) dA_s^Q$$

  is the pricing trend under fractional recovery

- Result holds in *all* credit models with unpredictable defaults
  - As in our context, a default intensity does not need to exist
  - If the intensity exists, results simplify to the formulae in Duffie & Singleton (1999)

Simultaneous calibration

- Equity, debt and CDS markets plus historical default rates
  - Huang and Huang (2003), Jarrow (2001)
  - Generalized reduced-form formulae
- Calibrate $(X, \mu, \sigma, \alpha, \beta, R)$
  - Generalized least squares, bounded influence estimators
  - Minimization: Levenberg-Marquardt method
  - Subject to arbitrage conditions on α and β
  - Firm-specific (X, μ, σ), market (α, β, R)
  - Equilibrium pricing measure

Specification of equity

- Classical setting: European call with known strike and maturity
- I^2: European down and out call with stochastic strike and barrier
  - Values?
  - Maturity?
- Stay tuned for some numbers...