Pricing Swap Credit Risk with Copulas

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Abstract

We apply copula functions to evaluate counterpart risk in swap transactions. Using copulas allows to generalise the approach proposed by Sorensen and Bollier (1994), allowing for dependence between swap rates and counterparty default.. Counterpart risk is represented by a sequence of vulnerable swaptions, which are priced using Cherubini and Luciano (2002) approach. Using copulas grants maximum flexibility with the choice of term structure and default risk models, as well as with the specification of the dependence structure between interest rate and credit risk. Closed form hedging and pricing formulas are derived for extreme dependence cases and for copula functions of the Fréchet family. An empirical application based on actual market data has shown that dependence affects both the level and the slope credit spreads, particularly for the case in which a credit institution is paying fixed. The effect is reversed in the case in which the financial institution pays floating.

1 Introduction

Swap contracts feature among the most mature derivative products traded in the financial market. While exotic derivative contracts are designed and sold in limited quantity to niches of selected and qualified investors, plain vanilla swap contracts are extensively used across the world, serving widely different customers with respect to size, sector and business, as well as with their degree of financial sophistication. Not only swaps are used by large financial institutions, investors and corporations as a quick tool to "ride the yield curve", they are also proposed by commercial banks to small firms as a risk management service to hedge against adverse movements in term structures and exchange rates. Of course, as one of the most typical examples of *over-the-counter* (OTC) derivatives, the swap contract is exposed to counterpart risk, that is the risk that one of the two parties goes bankrupt before the maturity of the contract. If upon

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default a swap contract has negative value for the party that goes bankrupt, the other party will join the group of creditors to recover a percentage of the swap market value (substitution cost). If instead the value of the contract is positive for the bankrupt party, the other party will close out the swap by paying the market value, according to an arrangement called "full two-way payment", which has has been adopted under the ISDA master agreement since 1992.

Allowing for credit risk in a swap transaction is particularly relevant both for the pricing of the contract and for risk evaluation. Even the new IAS accounting standards require that counterpart risk be considered in the marking to market process of derivative OTC contracts books. When counterpart risk is taken into account, evaluation of linear products is significantly more involved than that of options. In fact, contrary to options, for which the only relevant default risk is that of the short end of the contract, in linear products who bear counterpart risk depends on the market value of the contract at each point in time: the relevant default risk will be that of the party for which the contract has negative market value. As an outcome, if the two parties to the contract have asymmetric credit risk, that is one is riskier than the other, the market sensitivity of the contract would change over its life as the value turns from negative to positive or vice versa. To put it in a different way, asymmetric credit risk turns a linear product, such as a swap, into a non-linear one. From this point of view, it is not surprising that in one of the first approaches to the problem, due to Sorensen and Bollier (1994), swap credit risk was evaluated as a sequence of swaptions, whose exercise was conditional on default of the party for which the contract was out-of-the-money, (i.e. with negative market value). This approach is particularly interesting because it allows to evaluate counterpart risk consistently with those term structure and credit risk evaluation models that are globally chosen by the bank to price its whole fixed income portfolio. A shortcoming of the approach is that it does not allow for dependence between term structure movements and default of one of the two parties in the contract. A possible solution to the problem was suggested by Duffie and Huang (1996), who applied an affine reduced form model to price the two legs of the swap, conditional on the sign of the market value at each point in time: in this case, interest and credit risk correlation was accounted for by using the whole defaultable term structures of the two parties. This solution comes at the cost of losing some flexibility in the specification of term structure and credit risk models, with respect to the option based model. Here we propose a different solution to the problem, based on copula functions. The key idea is to stick to the option based representation of counterpart risk proposed by Sorensen and Bollier (1994), using copula functions to generalise the approach beyond the independence assumption: in fact, Sorensen and Bollier approach can be looked at as a particular instance of a copula based approach, in which a very specific copula, the product one, is chosen. The copula approach to vulnerable options proposed by Cherubini and Luciano (2002) provides a natural extension of the model. This will enable to account for interest and credit risk correlation and will allow to preserve the flexibility of the option based approach.

The plan of the paper is as follows. In section 2 we will provide the basic lay-out of the model, applying the copula based approach to vulnerable options to the evaluation of counterpart risk in a plain vanilla swap. In section 3 we will focus on specific cases in which counterpart risk can be priced and hedged in closed form: Sorensen and Bollier (1994) is a specific instance, corresponding to the product copula, to which we add the cases of the Fréchet copula bounds, representing perfect positive and negative dependence between interest rates and counterparty default. Section 3 will specify the analysis to the case, investigated elsewhere in the literature, of counterparties with symmetric credit risk. Section 4 proposes an example of application of the model to the determination of credit spreads applied to a corporate counterparty entering a plain vanilla swap transaction with a credit institution. Section 5 concludes and discusses possible extensions of the model.

2 The Model

In this section we lay out the basic set up of the model. We proceed in two stages. First, we describe the loss the would be incurred if one of the two parties were to default during the life of a plain vanilla swap contract. Of course, the loss for each of the two parties will depend on the joint event of default of the counterparty and a positive market value of the swap contract. The second step would then be to use copula functions to specify this joint probability and to recover the pricing kernel for the evaluation of credit risk in the swap transaction. The specification of both the marginal distributions of default risk of the counterparties and dependence between default and swap rates for each one of them would enable us assess analytically the relevance of counterparties asymmetries in the pricing of the swap contract. Furthermore, using extreme dependence scenarios for the two parties in the contract would allow to check value bounds and super-replication strategies for credit risk pricing and hedging for each one of them.

2.1 Swap credit risk: the pay-off

Consider a simple plain vanilla swap, in which the long end, which is denoted A, pays a stream of fixed coupons and receives a stream of floating payments, indexed to some reference short term rate interest rate, such as the Libor or Euribor rates. The payments occur at a set of dates $\{t_1, t_2, ..., t_n\}$ and they are to be evaluated at current time t. We assume the transaction occurs over natural time lags, so that the accrual period of the coupon and the term of the reference rate coincide, and the new coupon is set equal to the reference rate observed at the beginning of the period and payed at the end of it. This rules out the need to specify convexity and timing adjustments that would make the analysis more involved: the same strategy and idea proposed for the plain vanilla case would however carry over to these more complex instances. The discount factor curve observed at time t is denoted $\{D(t,t_1), D(t,t_2), ..., D(t,t_n)\}$. The forward rates evaluated at time t are $\{f(t,t_1,t_2), ...f(t,t_i,t_{i+1}), ..., f(t,t_{n-1},t_n)\}$. A standard no-arbitrage argument implies that the net present value (NPV) of the swap at time t, viewed from the party paying fixed, is computed as

$$NPV(t) = \sum_{i=0}^{n-1} D(t, t_{i+1}) \left(f(t, t_i, t_{i+1}) - k \right)$$

where k is the fixed coupon.Using the definition of swap rate (a weighted average of forward rates) we can rewrite the equation as

$$NPV(t) = \sum_{i=0}^{n-1} D(t, t_{i+1}) \left(sr(t, t_n) - k \right)$$

where $sr(t, t_n)$ denotes the swap rate, i.e. the flow of fixed payments corresponding to a flow of indexed payments over the schedule $\{t_1, t_2, ..., t_n\}$. It is well known that at inception the swap contract is worth zero, so that if the swap contract is stipulated at time t we have $k = sr(t, t_n)$ by convention. If swap rates increase, the swap will add value to the party paying fixed. Now, assume that the counterparty paying floating, which we denote B, would default at some time $t_{j-1} \leq \tau < t_j$. Under this scenario, the loss to the party paying fixed, discounted back to time t, would then be

$$Lgd_B \sum_{i=j}^{n-1} D(t, t_{i+1}) \max \left(sr(t_j, t_n) - k, 0 \right)$$

where Lgd_B is the loss given default figure of counterparty B. As it is immediately evident, the loss of the party paying fixed corresponds to the pay-off of a payer swaption. By the same token, the party paying floating will be exposed to default of the counterparty A if the swap rate falls short of the rate stipulated at the origin of the contract, yielding a market value of the loss evaluated at time t

$$Lgd_A \sum_{i=j}^{n-1} D(t, t_{i+1}) \max \left(k - sr(t_j, t_n), 0 \right)$$

where Lgd_A is the loss given default of counterparty A. In this case, the pay-off representing the exposure corresponds to a receiver swaption.

2.2 Swap credit risk: the copula approach

The evaluation problem described above can be viewed as a particular instance of the loss incurred in a vulnerable option transaction. The option represents the opportunity to enter a payer or receiver swap at the strike rate k, which corresponds to the swap rate stipulated initially. The vulnerability feature of the option relyies on the fact that its exercise is not only contingent on the contract ending up in the money, but also on the event of default of the counterparty. Of course, if the balance sheets of one or both the two parties in the contract are affected by changes in the term structure, the two events can be correlated. Here we follow the copula-based approach described in Cherubini and Luciano (2002) to account for dependence between the underlying of an option contract and default of the writer of the contract. The main idea in that approach is to derive the price of vulnerable call and put options as the integrals of vulnerable pricing kernels. In our case it is convenient to work under the swap market measure, so that the value of the swaption can be factorized into an annuity numeraire and the expected value of a pay-off function depending on the swap rate. The price of the vulnerable payer swaption is written in this case as

$$Lgd_B \sum_{i=j}^{n-1} D(t, t_{i+1}) \int_k^{\infty} Q(sr(t_j, t_n) > u, t_{j-1} \leq \tau_B < t_j)\, du$$

where τ_B is the default time of counterpart B. Using copula functions the above relationship can be written as

$$Lgd_B \sum_{i=j}^{n-1} D(t, t_{i+1}) \int_k^{\infty} \widetilde{C}_B \left[\overline{Q}(u), \overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right] du$$

where $\overline{Q}(u)$ denotes the risk-neutral probability, under the time t_j swap market measure, of the event $sr(t_j, t_n) > u$, and $\overline{G}_B(t)$ is the survival probability, under the same measure, of counterparty B beyond time t. $\widetilde{C}_B[x, y]$ is a bivariate function fulfilling the requirements for a copula function. As it is well known, copula functions enable to represent whatever joint probability as a function of the marginal ones (see Nelsen 1999 for a mathematical introduction to copulas and Cherubini et al. 2004 for financial applications). In this case, function $\widetilde{C}[x, y]$ is such that

$$Q(sr(t_j, t_n) > u, t_{j-1} \leq \tau_B < t_j) = \widetilde{C}_B\left[\overline{Q}(u), \overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right]$$

Following the same argument, we may compute counterpart risk for the end of the contract paying floating:

$$Lgd_A \sum_{i=j}^{n-1} D(t, t_{i+1}) \int_0^{k} C_A\left[Q(u), \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right] du$$

where $Q(u) \equiv 1 - \overline{Q}(u)$ is the probability distribution of the swap rate under the time t_j swap market measure, and $\overline{G}_A(t)$ is the survival probability, under the same measure, of counterparty A beyond time t. Notice that the copula function $C_A[x, y]$ is conceptually different from the function $\widetilde{C}_A[x, y]$ used above, beyond the obvious difference that they are referred to different counterparts. In fact, we have

$$Q(sr(t_j, t_n) \leq u, t_{j-1} \leq \tau_A < t_j) = C_A\left[Q(u), \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right]$$

So, if we drop for a minute the subscripts A and B, the copula function $C[x,y]$ represents dependence between the event that the swap rate is **lower** than u and default of the counterparty, while copula function $\widetilde{C}[x,y]$ determines dependence between the event that the swap rate is **greater** than u and default of the counterparty. It may be easily proved that the two copula functions are linked by the relationship

$$C\left[Q(u),\overline{G}(t_{j-1})-\overline{G}(t_j)\right]=\overline{G}(t_{j-1})-\overline{G}(t_j)-\widetilde{C}\left[\overline{Q}(u),\overline{G}(t_{j-1})-\overline{G}(t_j)\right]$$

It may also be checked (Cherubini and Luciano, 2002) that the above equation rules out arbitrage opportunities between put and call options. Using this no-arbitrage relationship we may evaluate the counterpart risk for the party paying floating as

$$\begin{aligned}&Lgd_A\sum_{i=j}^{n-1}D(t,t_{i+1})\left[\overline{G}_A(t_{j-1})-\overline{G}_A(t_j)\right]\\&-Lgd_A\sum_{i=j}^{n-1}D(t,t_{i+1})\int_0^k\widetilde{C}_A\left[\overline{Q}(u),\overline{G}_A(t_{j-1})-\overline{G}_A(t_j)\right]du\end{aligned}$$

3 Hedging swap credit risk

Hedging swap credit risk in the model described above calls for replicating portfolios of vulnerable swaptions. Since the credit risk involved in these options is obviously the core feature one is willing to hedge, the determination of the proper hedging policy will rely heavily on the specification of copula function selected. As an alternative possibility, one can recover hedging bounds for the values of these vulnerable swaptions, corresponding to extreme dependence cases. Exploiting the results in Cherubini and Luciano (2002) it is possible to recover closed form solutions for these super-hedging policies. These results enable to determine closed form hedging policies and prices for a specific family of copula specifications, i.e. the mixture one.

3.1 Indepence

The first obvious case is that of independence between default of the counterparties and movements in the swap rate structure. This actually reproduces the results which were first derived in Sorensen and Bollier (1994). In this case, the credit risk of a long position in a swap transaction can be written as

$$
\begin{aligned}
& Lgd_B \left[\overline{G}_B (t_{j-1}) - \overline{G}_B (t_j)\right] \sum_{i=j}^{n-1} D (t, t_{i+1}) \int_k^{\infty} \overline{Q} (u) \, du \\
= \; & Lgd_B \left[\overline{G}_B (t_{j-1}) - \overline{G}_B (t_j)\right] SWAPTION_P (t; t_j, k, t_n)
\end{aligned}
$$

where $SWAPTION_P (t; t_j, k, t_n)$ denotes the value of a payer swaption giving the right to enter a swap at time t_j for a swap rate equal to k and maturity at time t_n.

Following the same argument, we may compute credit risk prices for the party paying floating as

$$
\begin{aligned}
& Lgd_A \left[\overline{G}_A (t_{j-1}) - \overline{G}_A (t_j)\right] \sum_{i=j}^{n-1} D (t, t_{i+1}) \int_0^{k} Q (u) \, du \\
= \; & Lgd_A \left[\overline{G}_A (t_{j-1}) - \overline{G}_A (t_j)\right] SWAPTION_R (t; t_j, k, t_n)
\end{aligned}
$$

where $SWAPTION_R (t; t_j, k, t_n)$ denotes the value of a receiver swaption.

3.2 Perfect positive dependence

Let us now assume that the credit risk of the counterparty is negatively affected by an increase in interest rates. To assume the worst possibile scenario, we may conceive the case in which default of the counterparty and increases of the swap rate structure are perfectly dependent. This is of course bad news for the the party paying fixed, because the contract will be valuable for her when the swap rate increases, which is exactly when the counterparty will go bust. In order to represent this perfect dependence case, we set the copula function $\widetilde{C} [x, y]$ equal to the upper Fréchet bound, so that

$$
\begin{aligned}
\widetilde{C}_B \left[\overline{Q} (u), \overline{G}_B (t_{j-1}) - \overline{G}_B (t_j)\right] &= \min \left[\overline{Q} (u), \overline{G}_B (t_{j-1}) - \overline{G}_B (t_j)\right] \\
&= C_{\max} \left[\overline{Q} (u), \overline{G}_B (t_{j-1}) - \overline{G}_B (t_j)\right]
\end{aligned}
$$

Notice that in this case the credit risk of the swap contract can be evaluated in closed form as

$$
\begin{aligned}
& Lgd_B \sum_{i=j}^{n-1} D(t,t_{i+1}) \int_k^\infty \min\left[\overline{Q}(u), \overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right] du \\
= \; & Lgd_B \sum_{i=j}^{n-1} D(t,t_{i+1}) \max(k^*(t_j) - k, 0) \left[\overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right] + \\
& Lgd_B \sum_{i=j}^{n-1} D(t,t_{i+1}) \int_{\max(k,k^*(t_j))}^\infty \overline{Q}(u)\, du \\
= \; & Lgd_B \sum_{i=j}^{n-1} D(t,t_{i+1}) \max(k^*(t_j) - k, 0) \left[\overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right] + \\
& + Lgd_B SWAPTION_P\left(t; t_j, \max\left(k, k^*(t_j)\right), t_n\right)
\end{aligned}
$$

with

$$
k^*(t_j) = \overline{Q}^{-1}\left(\overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right)
$$

On the contrary, perfect dependence between the swap rate increases and counterparty default is good news for the party payng floating: in this case, the counterparty will default exactly when the swap is almost worthless for her. To formalise this case, notice that

$$
\begin{aligned}
C\left[Q(u), \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right] &= \overline{G}(t_{j-1}) - \overline{G}(t_j) \\
& \quad - \min\left[\overline{Q}(u), \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right] \\
&= \max\left[\left(\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right) - \overline{Q}(u), 0\right] \\
&= \max\left[Q(u) + \left(\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right) - 1, 0\right] \\
&\equiv C_{\min}\left[Q(u), \overline{G}(t_{j-1}) - \overline{G}(t_j)\right]
\end{aligned}
$$

and we obtain the lower Fréchet bound copula. The value of credit risk for the party paying floating can be evaluated in closed form as

$$
\begin{aligned}
& Lgd_A \sum_{i=j}^{n-1} D(t,t_{i+1}) \int_0^k \max\left[Q(u) + \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j) - 1, 0\right] du \\
= \; & Lgd_A \sum_{i=j}^{n-1} D(t,t_{i+1}) \int_{\min(k,k^*(t_j))}^k \left[Q(u) + \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j) - 1\right] du \\
= \; & Lgd_A \left[SWAPTION_R(t; t_j, k, t_n) - SWAPTION_R\left(t; t_j, \min\left(k, k^*(t_j)\right), t_n\right)\right] \\
& + \max\left(k - k^*(t_j), 0\right) Lgd_A \sum_{i=j}^{n-1} D(t,t_{i+1}) \left[\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j) - 1\right]
\end{aligned}
$$

Notice that the credit risk is zero unless $k > k^*(t_j)$, so that $Q(k^*(t_j)) > 1 - \left(\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right)$. So, for the credit risk of the transaction to have positive value the swap must be very deep in the money for the party paying floating. In general, we then expect the credit risk value to be equal to zero for the party paying floating, if default of the counterparty paying fixed is perfectly dependent on increases of the swap rates.

3.3 Perfect negative dependence

It may be the case that the counterparty is negatively affected by a decrease in interest rates. In this case. we set

$$\begin{aligned}\widetilde{C}_B\left[\overline{Q}(u), \overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right] &= \max\left[\overline{Q}(u) + \overline{G}_B(t_{j-1}) - \overline{G}_B(t_j) - 1, 0\right] \\ &= C_{\min}\left[\overline{Q}(u), \overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right]\end{aligned}$$

The credit risk for the party paying fixed is computed as

$$\begin{aligned}&Lgd_B\left[SWAPTION_P(t; t_j, k, t_n) - SWAPTION_P(t; t_j, \max(k, k^{**}(t_j)), t_n)\right] \\ &+ \max(k^{**}(t_j) - k, 0)\, Lgd_B \sum_{i=j}^{n-1} D(t, t_{i+1})\left[\overline{G}_B(t_{j-1}) - \overline{G}_B(t_j) - 1\right]\end{aligned}$$

with

$$k^{**}(t_j) = \overline{Q}^{-1}\left(1 - \overline{G}_B(t_{j-1}) - \overline{G}_B(t_j)\right)$$

Notice that credit risk is zero unless $k < k^{**}(t_j)$, so that $\overline{Q}(k^{**}(t_j)) > 1 - \left(\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right)$. So, the party paying fixed will have credit risk only if the swap contract is very deep in the money for her. In general, the credit risk adjustment can be safely set to zero under this extreme scenario.

Negative dependence between default of the counterparty and the swap rate is instead very relevant for the party paying floating. In the case of perfect negative dependence we get in fact

$$\begin{aligned}C_A\left[Q(u), \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right] &= \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j) \\ &\quad - \max\left[\overline{Q}(u) + \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j) - 1, 0\right] \\ &= \overline{G}_A(t_{j-1}) - \overline{G}_A(t_j) \\ &\quad - \max\left[\left(\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right) - Q(u), 0\right] \\ &= \min\left[Q(u), \left(\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right)\right] \\ &\equiv C_{\max}\left[Q(u), \overline{G}(t_{j-1}) - \overline{G}(t_j)\right]\end{aligned}$$

Credit risk of the counterparty paying floating can be computed in this case as

$$
\begin{aligned}
& Lgd_A \sum_{i=j}^{n-1} D(t,t_{i+1}) \int_0^k \min\left[Q(u), +\overline{G}_A(t_{j-1}) - \overline{G}_A(t_j)\right] du \\
= \quad & Lgd_A \sum_{i=j}^{n-1} D(t,t_{i+1}) \max(k - k(t_j)) \\
& + SWAPTION_R(t; t_j, \min(k, k^{**}(t_j)), t_n)
\end{aligned}
$$

4 The case of symmetric counterparties

We tackle the case in which the two counterparties have the same credit risk. In our approach, this means not only that loss given default and survival probability figures are equal across all the reset dates ($Lgd_A = Lgd_B$ and $\overline{G}_A(t_j) = \overline{G}_B(t_j)$ for all $j = 1, 2, ..n-1$). but also that dependence between default and the swap rate is the same for the two parties ($C_A[x,y] = C_B[x,y]$). In this case we may compute the upfront payment to make up for the credit risk in a standard interest rate swap. Assuming the fixed coupon payment is set equal to the swap rate, the credit risk adjustment for the swap turns out to be

$$
NPV = Lgd \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} D(t,t_{i+1}) \left[\begin{array}{c} \overline{G}(t_{j-1}) - \overline{G}(t_j) \\ -\int_0^\infty \widetilde{C}\left[\overline{Q}(u), \overline{G}(t_{j-1}) - \overline{G}(t_j)\right] du \end{array} \right]
$$

Notice that the adjustment does not depend on the spot swap rate. In the independence case the equation simplifies to

$$
NPV_\perp = Lgd \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} D(t,t_{i+1}) \left(\overline{G}(t_{j-1}) - \overline{G}(t_j)\right) [1 - fsr(t,t_j,t_n)]
$$

where $fsr(t,t_j,t_n)$ is the forward swap rate, that is the fixed payment stipulated at time t for a forward start swap beginning at time t_j and ending at time t_n.The result is obtained by setting $\widetilde{C}[x,y] = xy$ and by exploiting the martingale property of forward swap rates under the swap market measure. In the case of perfect positive dependence between default and swap rates we get

$$
NPV_{\max} = Lgd \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} D(t,t_{i+1}) \left[\begin{array}{c} \left(\overline{G}(t_{j-1}) - \overline{G}(t_j)\right)(1 - k^*(t_j)) \\ -\int_{k^*}^\infty \overline{Q}(u)\, du \end{array} \right]
$$

with

$$k^*\left(t_j\right) = \overline{Q}^{-1}\left(\overline{G}\left(t_{j-1}\right) - \overline{G}\left(t_j\right)\right)$$

The case of perfect negative dependence yields finally

$$NPV_{\min} = Lgd \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} D\left(t, t_{i+1}\right) \left[\begin{array}{c} \left(\overline{G}\left(t_{j-1}\right) - \overline{G}\left(t_j\right)\right)\left(1 - k^{**}\left(t_j\right)\right) \\ + \int_0^{k^{**}} Q\left(u\right) du \end{array} \right]$$

with

$$k^{**}\left(t_j\right) = Q^{-1}\left(\overline{G}\left(t_{j-1}\right) - \overline{G}\left(t_j\right)\right)$$

5 An example of application

We now give a practical example of implementation of the model. Consider a financial institution which is entering a plain vanilla swap contract with a corporate counterparty. As for the credit standing of the counterparty, we assume to observe the term structure of default intensities and take the worst case scenario concerning the loss given default figure ($Lgd = 1$). To make the example realistic, we use the default intensity structure bootstrapped from credit default swap prices referred to a famous Italian industrial corporate name: the default intensity was then simply extrapolated for maturities beyond 10 years, for which credit default swap quotes were not available. The term structure of the risk-free rate, along with that of the defaultable counterparty, assuming again zero recovery rate, are depicted in figure 1. The credit spread ranges from 3% to 5%.

ftbpF4.8715in3.0364in0ptfigure1.bmp.bmpIn order to price the swaptions, the standard Black model is used, and for the sake of simplicity we use a flat 15% volatility surface. The boundary values $k^*\left(t_j\right)$ and $k^{**}\left(t_j\right)$ are recovered immediately as

$$\begin{aligned} k^*\left(t_j\right) &= fsr\left(t, t_j, t_n\right) \exp\left[-\frac{\sigma_{j,n}^2}{2}\left(t_j - t\right) - N^{-1}\left(\overline{G}\left(t_{j-1}\right) - \overline{G}\left(t_j\right)\right)\sigma_{j,n}\sqrt{t_j - t}\right] \\ k^{**}\left(t_j\right) &= fsr\left(t, t_j, t_n\right) \exp\left[-\frac{\sigma_{j,n}^2}{2}\left(t_j - t\right) + N^{-1}\left(\overline{G}\left(t_{j-1}\right) - \overline{G}\left(t_j\right)\right)\sigma_{j,n}\sqrt{t_j - t}\right] \end{aligned}$$

where $\sigma_{j,n}$ denotes swap rate volatility and $N^{-1}(.)$ is the inverse of the standard normal distribution. Accounting for dependence between interest rates and counterparty default may have a dramatic impact on the evaluation of credit risk in swap transactions. A visual representation of the effect is represented in figure 2, where credit risk, viewed from the counterparty paying fixed, is evaluated for every payment of a 30-year swap contract. From the figure, it may be glanced that the extreme scenario of perfect positive dependence yields a much higher profile of risk. The typical hump feature of the risk profile is also accentuated.

ftbpF4.7046in2.9326in0ptfigure2.bmp.bmp In order to give a more precise idea of the magnitude of credit risk adjustment in the swap transactions, we report the impact for different maturities and several degrees of dependence between increases in the swap rates and default of the corporate counterparty. Dependence is modelled using a mixture copula, namely we set

$$\begin{aligned}\widetilde{C}_B\left[\overline{Q}\left(u\right),\overline{G}_B\left(t_{j-1}\right)-\overline{G}_B\left(t_j\right)\right] &= \rho\min\left[\overline{Q}\left(u\right),\overline{G}_B\left(t_{j-1}\right)-\overline{G}_B\left(t_j\right)\right] \\ &\quad +\left(1-\rho\right)\overline{Q}\left(u\right)\left[\overline{G}_B\left(t_{j-1}\right)-\overline{G}_B\left(t_j\right)\right]\end{aligned}$$

The use of this copula function enables to exploit the closed form solutions recovered for the extreme dependence cases, and provides an immediate interpretation in terms of non-parametric measures of association. For example, it may be easily proved that ρ directly corresponds to *Spearman's rho* rank correlation statistics.

Table 1 describes credit credit risk for the credit institution in the case in which it has to pay fixed while receiving floating payments. In order to ease the comparison among the different maturities, the credit risk impact is measured in terms of spreads over the risk-free swap rate. As expected, the credit risk impact increases with the maturity of the contract: for example, assuming a 25% rank correlation between the swap rate and default of the corporate counterparty, the credit institution will charge a credit spread ranging from about 14 basis points for a 5 year contract up to over 80 basis points for a 30 year contract. The impact of different degrees of dependence is also substantial. For a 10 year swap, for example, the credit risk impact changes from about 17 basis points in the case of independence, to 38.5 for 50% rank correlation, and up to over 60 basis points in the case of perfect dependence. The slope of the credit spreads is also substantially affected by an increase in dependence: for example, while in the independence case the 30 year credit spread (40 basis points) is about 4 times that of the 5 year one, with 75% rank correlation the 30 year figure gets about 8 times bigger than the 5 year one.

Table 1. Swap credit risk: credit institution paying fixed

	5 years	10 years	15 years	20 years	25 years	30 years
$\rho = 0.00$	0.1033%	0.1683%	0.2341%	0.2948%	0.3465%	0.3965%
$\rho = 0.25$	0.1399%	0.2764%	0.4264%	0.5719%	0.7001%	0.8242%
$\rho = 0.50$	0.1764%	0.3846%	0.6187%	0.8491%	1.0537%	1.2520%
$\rho = 0.75$	0.2130%	0.4928%	0.8110%	1.1262%	1.4073%	1.6797%
$\rho = 1.00$	0.2496%	0.6010%	1.0034%	1.4033%	1.7609%	2.1074%

In table 2 we report the credit spread charged by the credit institution in the case in which the counterparty pays fixed and receives floating payments from the credit institution. The first feature emerging from the data, also noticed in Sorensen and Bollier (1994), is that the credit adjustments are lower, even in the independence case, with respect to the case in which the financial institution was paying fixed: this is due to the upward sloping shape of the term structure, so that payer swaptions are more in the money than receiver swaptions. The credit risk impact of rank correlation is reversed, as it is expected from the description of the model. Remember in fact that in this case the value of the swap is negatively affected by an increase in interest rates, so that the associated increase in default probability of the counterparty corresponds to a lower amount of credit exposure. So, a 50% rank correlation roughly cuts by half the credit spread required: figures fall from 2.45 basis points to 1.23 for the 10 year maturity and from 21 basis points to 10.5 for the 30 year one.

Table 2. Swap credit risk: credit institution paying floating

	5 years	10 years	15 years	20 years	25 years	30 years
$\rho = 0.00$	0.0012%	0.0245%	0.0723%	0.1276%	0.1728%	0.2103%
$\rho = 0.25$	0.0009%	0.0184%	0.0542%	0.0957%	0.1296%	0.1577%
$\rho = 0.50$	0.0006%	0.0123%	0.0362%	0.0638%	0.0864%	0.1052%
$\rho = 0.75$	0.0003%	0.0061%	0.0181%	0.0319%	0.0432%	0.0526%
$\rho = 1.00$	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%

6 Conclusions and further extensions

In this paper we have extended the approach first proposed by Sorensen and Bollier (1994) to allow for dependence between interest rate movements and default of the counterparty. The credit risk adjustment is represented by a stream of vulnerable swaptions. These swaptions are prices using copula functions as proposed in Cherubini and Luciano (2002). Particularly, using the Fréchet bound copulas corresponding to perfect positive and negative dependence between interest rate movements and counterparty default enables to recover closed form hedging and pricing solutions. An empirical application based on actual market data has shown that dependence affects both the level and the slope credit spreads, particularly for the case in which the credit institution pays fixed. The rank correlation effect is reversed in the case in which the financial institution pays floating.

As for possible extensions of this model, we cannot avoid mentioning that today's market practice is heading toward new techniques to mitigate the credit risk in swap transactions. Among these practices, the most well-known involve the use of collateral and marking to market schemes (see Johannes and Sundaresan, 2003), as well as netting agreements. Of course, these practices are particularly oriented to a market of sophisticated users, particularly involving financial institutions on both sides of the deal, so that our approach would remain useful as a practical guide to determine the credit spreads in most of the transactions with corporate customers that still represent the rule in the commercial banking business. However, the model can be modified to accomodate some of the risk mitigating practices quoted above, and some of these modifications can be particularly easy. For example, accounting for collateral would only call for adjustment of the strike price of the swaptions involved in the evaluation. On the other side, accounting for a marking to market scheme and a periodic reset of the swap rate would suggest to use vulnerable ratchet swaptions. These and other extensions of the model (think of the presence of caps and floors, or quanto adjustments) are left as suggestions for future research.

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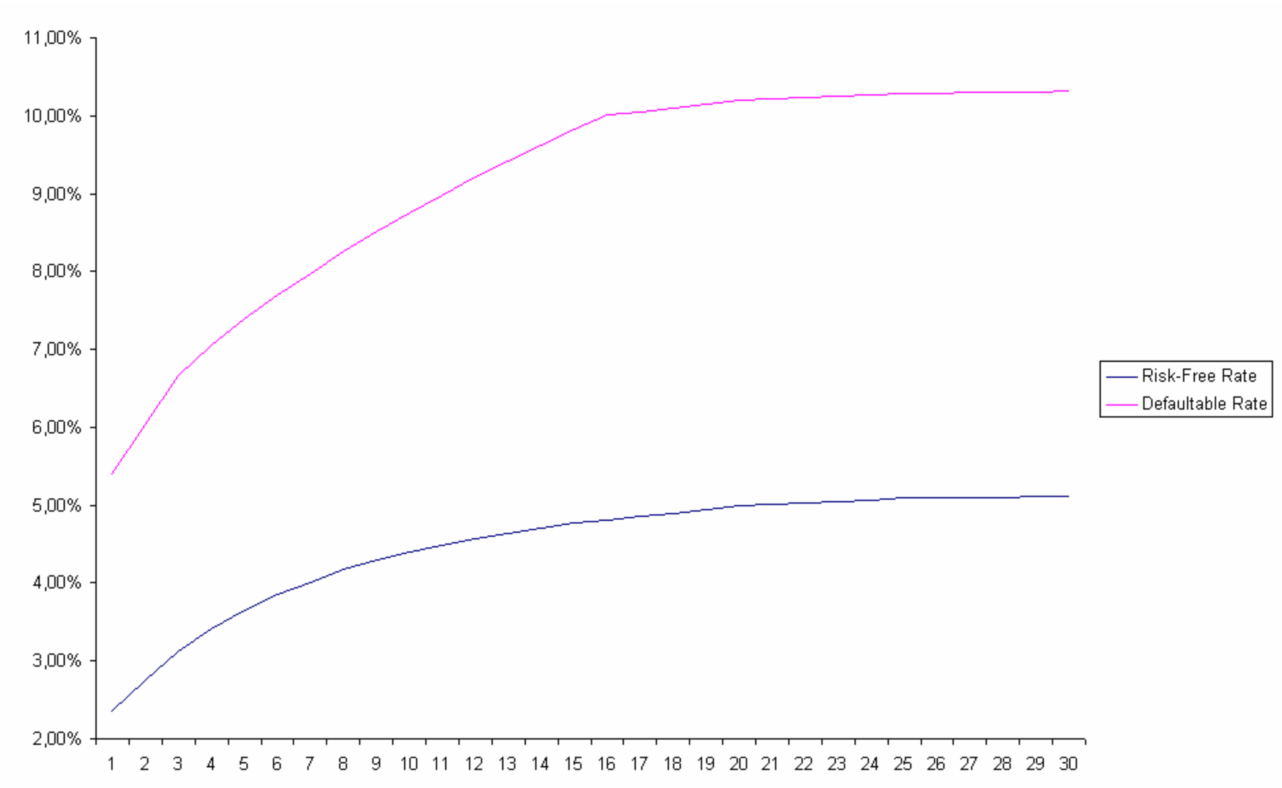

Figure 1 - Default-free and defaultable zero coupon yield curve

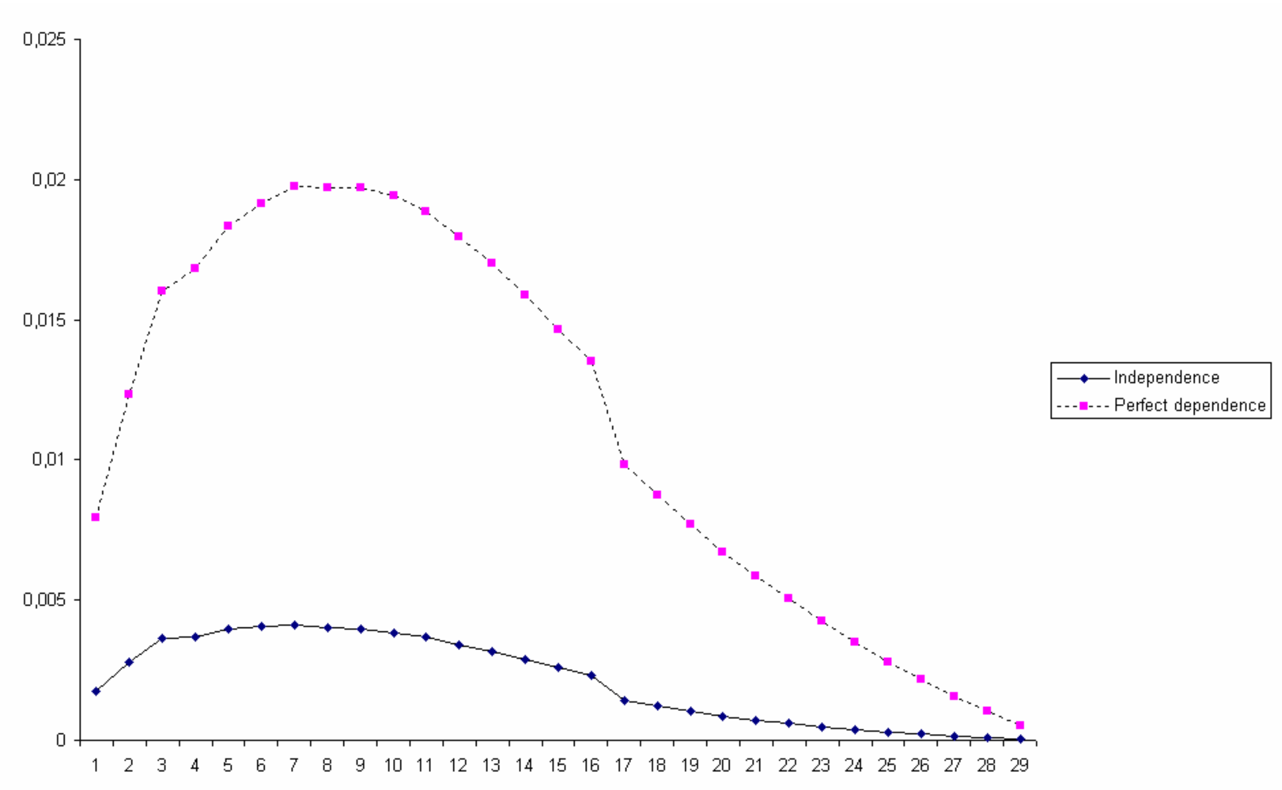

Figure 2 – Risk profile of a 30 year payer swap – Different dependence structures