Pricing Swap Credit Risk with Copulas

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Outline

- Motivation and tools
- Cherubini and Luciano (2002) model
- Vulnerable pricing kernels
- Vulnerable call and put options
- Swap credit risk: the copula approach
- Dependence of swap credit risk and interest rate risk
- Empirical evidence

Counterpart risk in derivatives

- Most of the derivative contracts, particularly options, forward and swaps, are traded on the OTC market, and so they are affected by credit risk
- Credit risk may have a relevant impact on the evaluation of these contracts, namely,
  - The price and hedge policy may change
  - Linear contracts can become non linear
  - Dependence between the price of the underlying and counterparty default should be accounted for

Copula functions: the basics

- A function $z = C(u,v)$ is said copula iff

  z, u and v are in [0,1]

  $C(0,v) = C(u,0) = 0$, $C(1,v) = v$, $C(u,1) = u$

  $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) - C(u_1, v_1) \geq 0$ for all $u_2 > u_1$ and $v_2 > v_1$

- Sklar theorem: each joint distribution can be written as a copula function taking the marginal distributions as arguments, and vice versa, every copula function taking univariate distributions as arguments yields a joint distribution

Copula functions: examples

- Two risks A and B with joint probability $H(A,B)$ and marginal probabilities $H_a(A)$ and $H_b(B)$
- $H(A,B) = C(H_a , H_b)$, and C is a copula function.
- Simplest cases:

1) $C_{ind}(H_a , H_b) = H_a H_{b,}$ independent risks
2) $C_{max}(H_a , H_b) = \min(H_a,H_b)$ perfect positive dependent risks
3) $C_{min}(H_a , H_b) = \max(H_a + H_b - 1,0)$ perfect negative dependence risks

- Imperfect dependence (Fréchet bounds)

$$\max(H_a + H_b - 1,0) \leq C(H_a , H_b) \leq \min(H_a,H_b)$$

Copula function and dependence structure

- Using copula functions enables to specify marginal distributions and the dependence structure separately
- Copula functions are linked to non-parametric dependence statistics, as in example Kendall's τ or Spearman's ρ
- Notice that differently from non-parametric estimators, the linear correlation may not cover the whole range from – 1 to + 1, making the assessment of the relative degree of dependence involved.

Copula and *tail dependence*

- Copula functions may be used to compute an index of *tail dependence* assessing the evidence of simultaneous booms and crashes on different markets
- For example, in normal copulas $N(u_1, u_2, \ldots, u_N; \rho)$ extreme events are indipendent, while under the Student t copula with *v* degrees of freedom $T(u_1, u_2, \ldots, u_N; \rho, v)$ extreme events are dependent.

The Fréchet family of copulas

- Take

  $C(x,y) = \beta C_{min} + (1 - \alpha - \beta) C_{ind} + \alpha C_{max}, \quad \alpha, \beta \in [0,1]$

  $C_{min} = \max(x + y - 1, 0)$, $C_{ind} = xy$, $C_{max} = \min(x,y)$
- The parameters α, β are linked to non-parametric dependence measures in analytical formulas
- Mixture copulas (Li, 2000) are a particular case in which copula is a linear combination of C_{max} and C_{ind} for positive dependent risks ($\alpha > 0$, $\beta = 0$), C_{min} and C_{ind} for the negative dependent ($\beta > 0$, $\alpha = 0$).

Dualities among copulas

- Consider a copula corresponding to the probability of the event A and B, $H(A,B) = C(H_a,H_b)$. Define the marginal probability of the complements A^c, B^c as $\underline{H}_a=1 - H_a$ and $\underline{H}_b=1 - H_b$.
- The following duality relationships hold among copulas

- $H(A,B) = C(H_a,H_b)$
- $H(A^c,B) = H_b - C(H_a,H_b)$
- $H(A,B^c) = H_a - C(H_a,H_b)$
- $H(A^c,B^c) = 1 - H_a - H_b + C(H_a,H_b)$

Vulnerable digital call option

- Consider a vulnerable digital call (VDC) option paying 1 euro if Nikkei > K (event A). In this case, if the counterparty defaults (event B), the option pays the recovery rate RR.
- The payoff of this option is

$$VDC = P(t,T)[H(A,B^c)+RR\ H(A,B)]$$

$$= P(t,T)\ [H_a - H(A,B)+RR\ H(A,B)]$$

$$= P(t,T)H_a - (1 - RR)H(A,B)$$

$$= DC - P(t,T)\ Lgd\ C(H_a, H_a)$$

Vulnerable digital put option

- Consider a vulnerable digital put (VDP) option paying 1 euro if Nikkei ≤ K (event A^c). In this case, if the counterparty defaults (event B), the option pays the recovery rate RR.
- The payoff of this option is

$$VDP = DP - P(t,T)(1 - RR)H(A^c,B)$$
$$= P(t,T)\underline{H}_a - P(t,T)(1 - RR)H(A^c,B)$$
$$= P(t,T)\underline{H}_a - P(t,T)(1 - RR)[H_b - C(H_a, H_b)]$$
$$= P(t,T)(1 - H_a) - P(t,T)\ Lgd\ [H_b - C(H_a, H_b)]$$
$$= P(t,T) - VDC - P(t,T)\ Lgd\ H_b$$

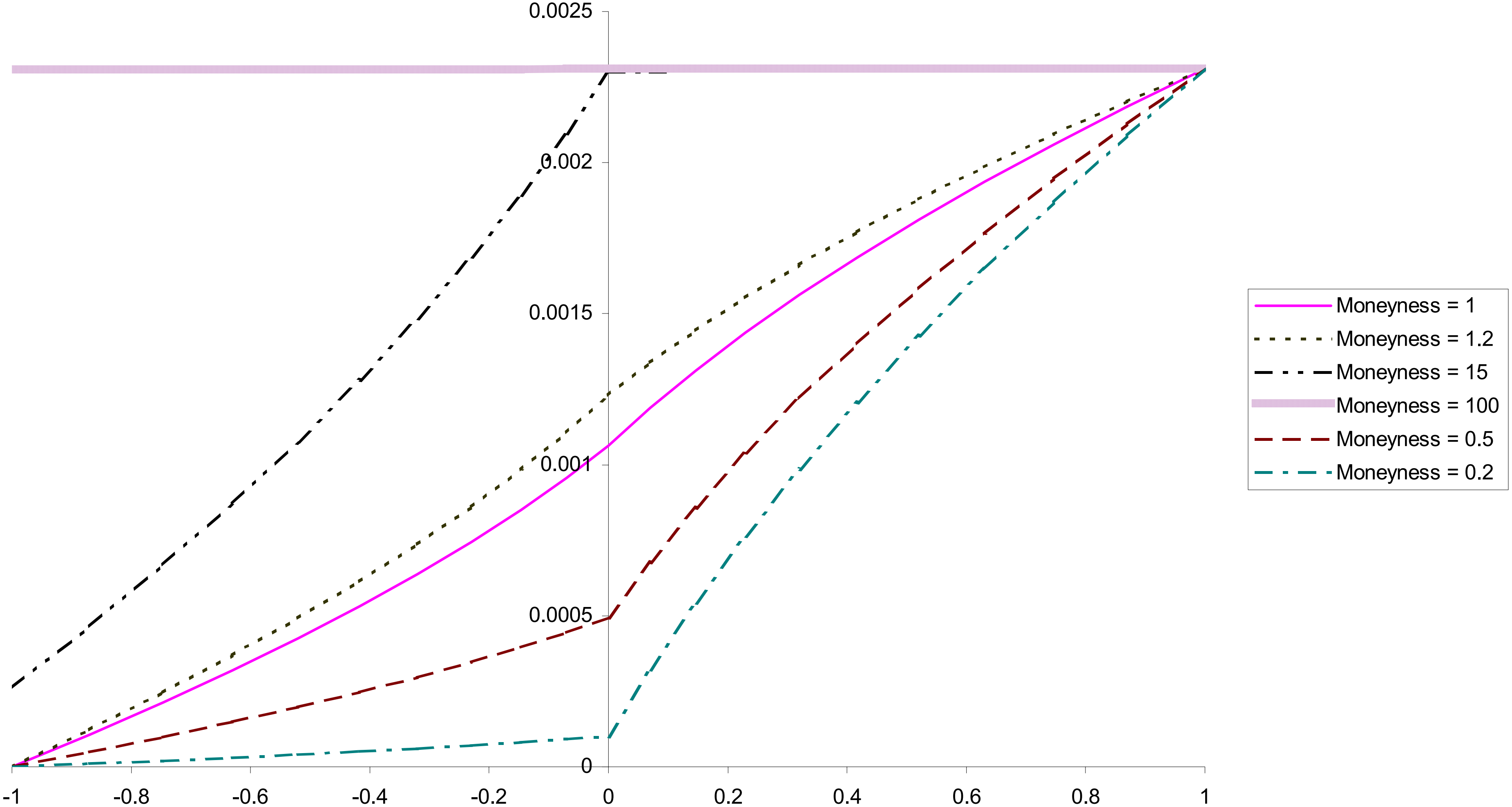
Vulnerable digital put call parity

- Define the expected loss $EL = Lgd\ H_b$.
- If $D(t,T)$ is a defaultable ZCB issued by the counterparty we have

  $D(t,T) = P(t,T)(1 – EL)$
- Notice that copula duality implies a clear no-arbitrage relationship

  $VDC + VDP = P(t,T) – P(t,T)\ EL = D(t,T)$
- Buying a vulnerable digital call and put option from the same counterparty is the same as buying a defaultable zero-coupon bond

Digital Vulnerable Options



From the pricing kernel to options

- The basic idea goes back to Breeden and Litzenberger (1978)
- Integrating the pricing kernel (i.e. the cumulative or decumulative risk neutral distribution) we may recover put and call prices
- From simple digital call and put options we can recover call and put prices: simply set Pr(S(T) ≤ u) = Q(u)

$$-\frac{1}{P(t,T)}\frac{\partial CALL}{\partial K} = \Pr(S(T) > K) \Rightarrow CALL = P(t,T)\int_K^{\infty}(1 - Q(\eta))d\eta$$

$$\frac{1}{P(t,T)}\frac{\partial PUT}{\partial K} = \Pr(S(T) \le K) \Rightarrow PUT = P(t,T)\int_0^{K} Q(\eta)d\eta$$

Copula functions and the pricing of structured financial products

- Digital products structured with digital bivariate options:

Coupon = 10% if Nasdaq > K_1 and Nikkey > K_2

Coupon = 10% C(Q(Nasdaq > K_1),Q(Nikkey > K_2))

- Call options on the minimum of a basket of assets (Everest)

$$Call(\min(S_1, S_2, \ldots S_N), K, T) =$$

$$P(t,T)\int_K^\infty C\left[Q(S_1(T) > \eta), Q(S_2(T) > \eta), \ldots Q(S_N(T) > \eta)\right] d\eta$$

Vulnerable call and put options

$$VC(S,t:K,T)=\int_K^{\infty} VDC(\eta)d\eta =$$

$$=\int_K^{\infty} DC(\eta)d\eta - P(t,T)Lgd\int_K^{\infty} C[1-Q(\eta),H_b]d\eta =$$

$$=C(S,t:K,T)-P(t,T)Lgd\int_K^{\infty} C[1-Q(\eta),H_b]d\eta$$

$$VP(S,t:K,T)=P(S,t:K,T)-P(t,T)Lgd\int_0^{K} \widehat{C}[Q(\eta),H_b]d\eta$$

$$\widehat{C}\equiv H_b - C[1-Q(\eta),H_b]$$

Vulnerable put-call parity

$$VP(S,t:K,T)+S(t)=P(S,t:K,T)+S(t)-P(t,T)Lgd\int_0^K \widehat{C}\left[Q(\eta),H_b\right]d\eta =$$

$$=C(S,t:K,T)+KP(t,T)-P(t,T)Lgd\int_0^K \widehat{C}\left[Q(\eta),H_b\right]d\eta$$

$$=C(S,t:K,T)+KP(t,T)(1-EL)+P(t,T)Lgd\int_0^K C\left[1-Q(\eta),H_b\right]d\eta$$

$$=VC(S,t:K,T)+KD(t,T)+P(t,T)Lgd\int_0^\infty C\left[1-Q(\eta),H_b\right]d\eta$$

Vulnerable call options: counterpart risk

- Independence

$C(.;K,T)*EL$

- Perfect positive dependence

$\max[K^* - K,0]P(t,T)EL - Lgd\ C(.;\max(K^*,K),T)$

- Perfect negative dependence

$$Lgd\ [C(.;\max(K^{**},K),T) - C(.;K,T)] + \max[K^{**} - K,0]\ P(t,T)Lgd(1 - H_b)$$

$Q(S(T)>K^*) = H_b$ $\qquad$ $Q(S(T)>K^{**}) = 1 - H_b$

Vulnerable put options: counterpart risk

- Independence

  $P(.;K,T)*EL$

- Perfect positive dependence

  $\max[K - K^{**},0]P(t,T)\ EL - Lgd\ \ P(.;\min(K^*,K),T)$

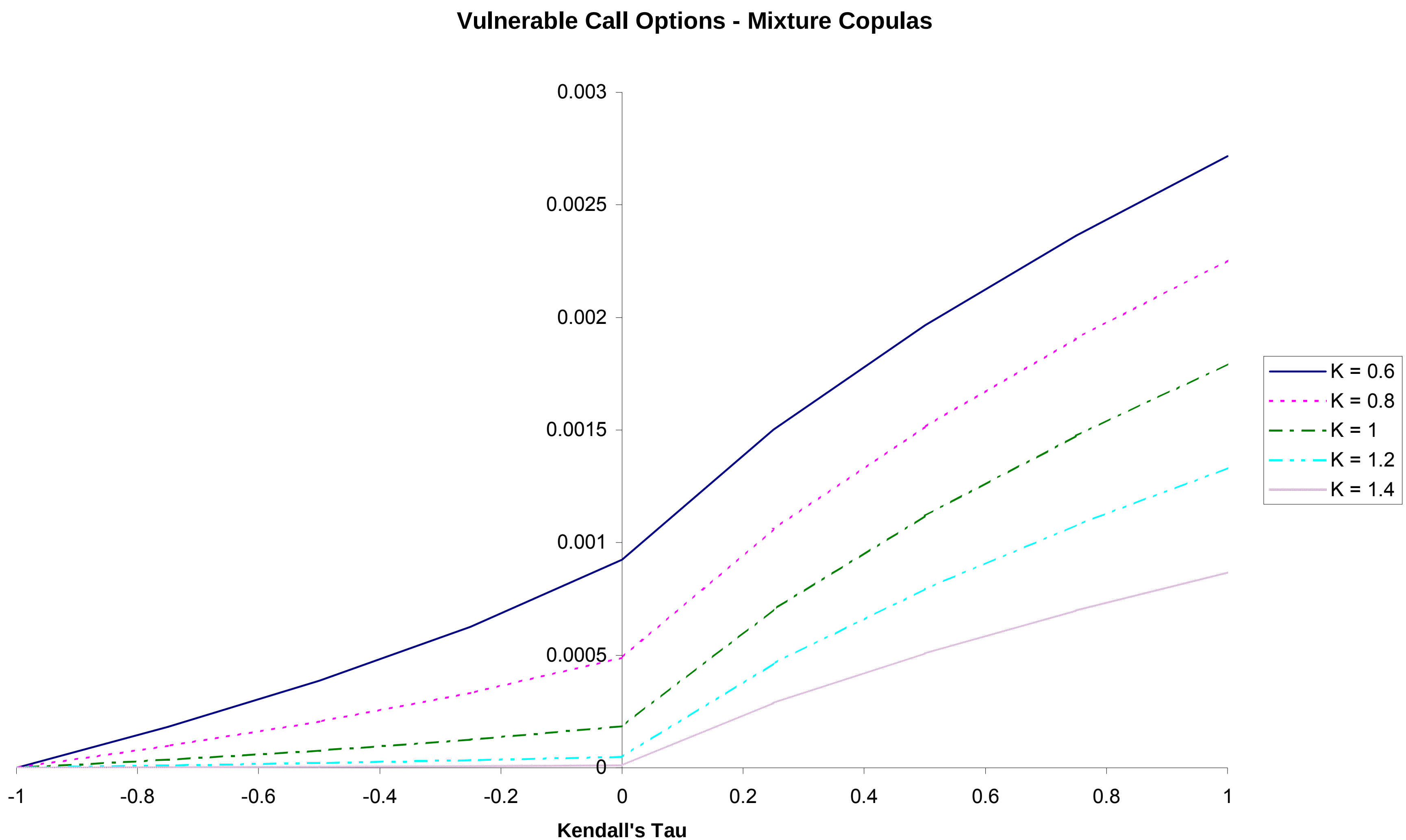
- Perfect negative dependence

  $Lgd\ [P(.;\min(K^*,K),T) - P(.;K,T)]$

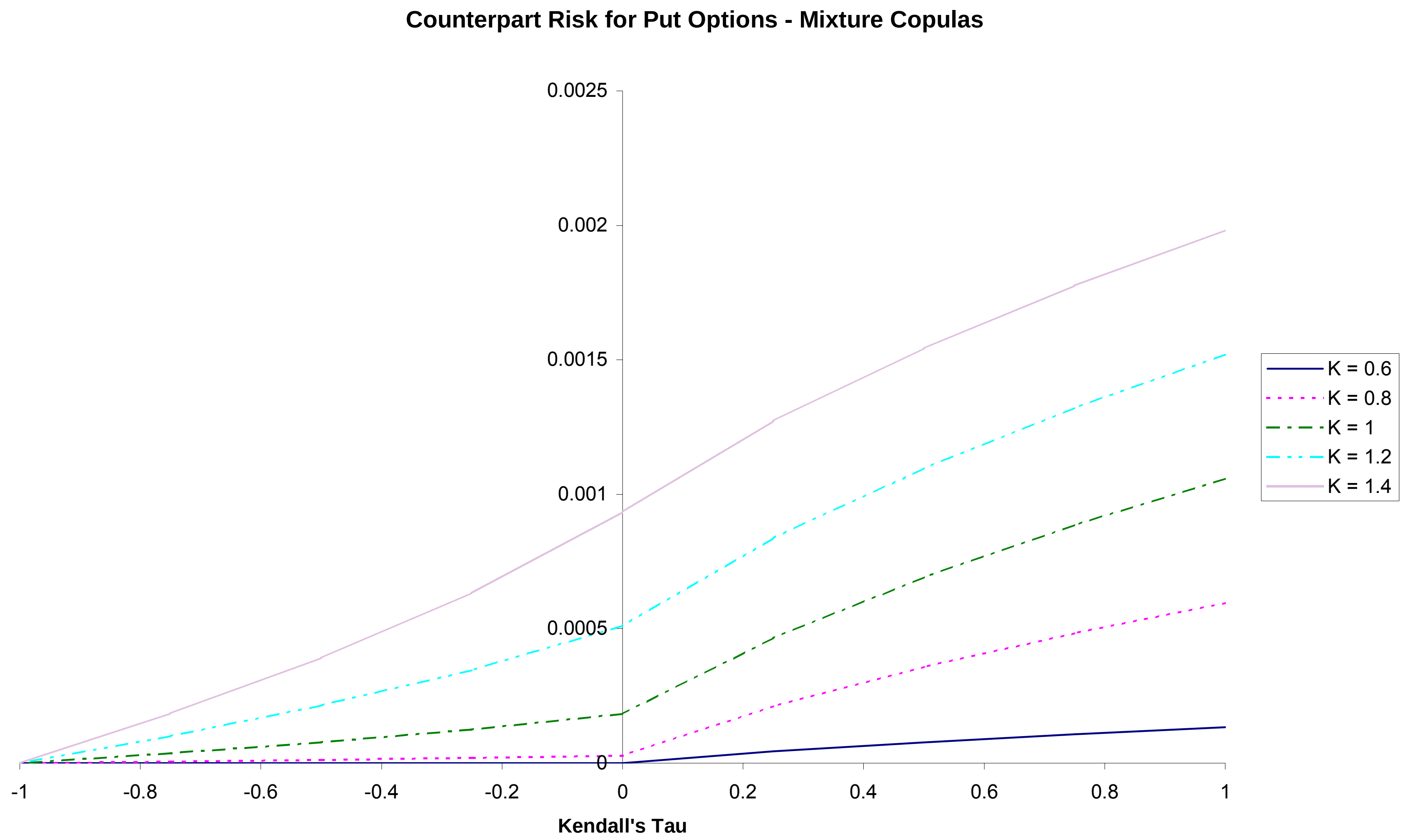
  $+ \max[K - K^*,0]P(t,T)Lgd(1 - H_b)$

$Q(S(T)>K^*) = H_b$ $\qquad$ $Q(S(T)>K^{**}) =1 - H_b$

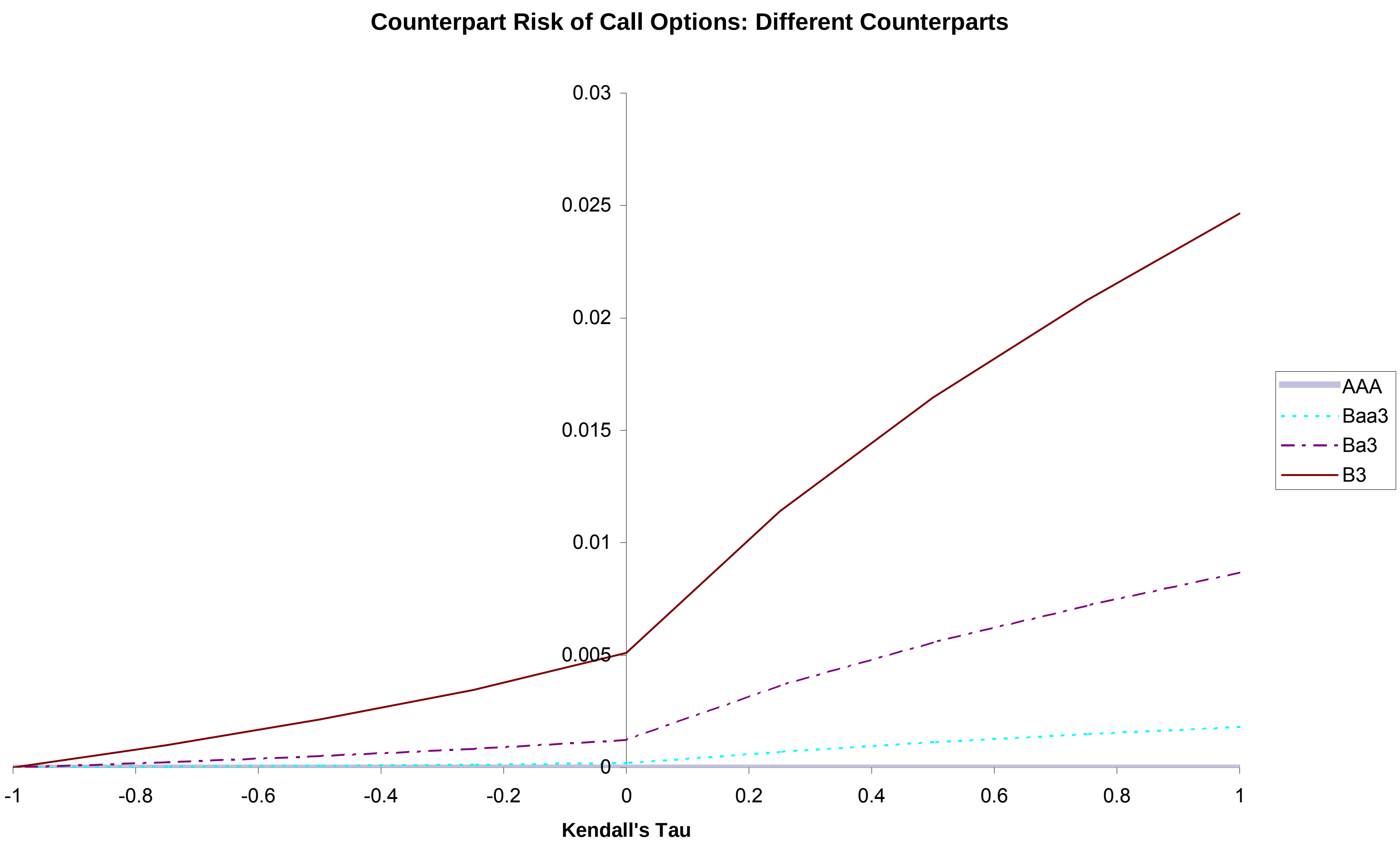
Counterpart Risk: Call Options



Counterpart risk: Put Options



Vulnerable call: different ratings



Swap credit risk

- In a swap contract, both the parties are exposed to counterparty risk. Credit risk should account for the joint event of the counterparty defaulting over the life on the contract, and the contract being "in the money" for the other
- If counterparty A pays fixed the risk for her is…
- while for B…

The approach due to Sorensen and Bollier, 1994 suggests to represent swap credit risk as a portfolio of payer or receiver swaptions. Their approach rests on the hypothesis of independence of the swap rate term structure and credit risk of the counterparties.

Strike rate *k* of the swaptions equals the swap rate at inception or depends on the collateral where applicable

$$\mathrm{Lgd}_{\mathrm{B}} \sum_{\mathrm{i=j}}^{\mathrm{n-1}} P(t, t_{i+1}) \max\left(sr(t_j, t_n) - k, 0\right)$$

$$\mathrm{Lgd}_{\mathrm{A}} \sum_{\mathrm{i=j}}^{\mathrm{n-1}} P(t, t_{i+1}) \max\left(k - sr(t_j, t_n), 0\right)$$

Pricing swap credit risk with copulas

- Denoting $G_B(t_j)$ the survival probability of counterparty B beyond time t_j. Then her default probability between time t_{j-1} and t_j is $G_B(t_{j-1}) – G_B(t_j)$. Furthermore, assume that, under the appropriate swap measure $Q(u) = Pr(sr(t_j) \leq u)$
- Swap credit risk for the fixed payer can be evaluated as

$$Lgd_B \sum_{i=j}^{n-1} P(t, t_{i+1}) \int_K^{\infty} C(1 - Q(\eta), G_B(t_{j-1}) - G_B(t_j)) d\eta$$

Credit risk for the fixed payer

- Independence (Sorensen and Bollier, 1994)

  $Lgd_B[G_B(t_{j-1}) - G_B(t_j)]PayerSwaption(t;t_j,t_n,k)$
- Perfect positive dependence

  $Lgd_B \max[k^*(t_j) - k,0]A(t, t_j, t_n) [G_B(t_{j-1}) - G_B(t_j)] -$

  $Lgd_B\ PayerSwaption(.;\max(k^*(t_j),k))$
- Perfect negative dependence

  $Lgd_B[PayerSwaption(.;\max(k^{**}(t_j),k),T) -$

  $PayerSwaption(.;K,T)]$

  $+ \max[k^{**}(t_j) - k,0]\ A(t, t_j, t_n)Lgd_B(1 - (G_B(t_{j-1}) - G_B(t_j)))$

$Q(sr(t_j)>k^*(t_j)) = G_B(t_{j-1}) - G_B(t_j)$ $Q(sr(t_j)>k^{**}(t_j)) =1 -(G_B(t_{j-1}) - G_B(t_j))$

$$A(t,t_j,t_n) = \sum_{i=j}^{n-1} P(t,t_{i+1})$$

Credit risk for the fixed receiver

- Independence (Sorensen and Bollier, 1994)

$Lgd_B[G_B(t_{j-1}) - G_B(t_j)]ReceiverSwaption(t;t_j,t_n,k)$

- Perfect positive dependence

$Lgd_B[ReceiverSwaption(.;\min(k^*(t_j),k)) - ReceiverSwaption(.;k)]$
$+ \max[k - k^*(t_j),0]\ A(t, t_j, t_n)\ Lgd(1 - (G_B(t_{j-1}) - G_B(t_j)))$

- Perfect negative dependence

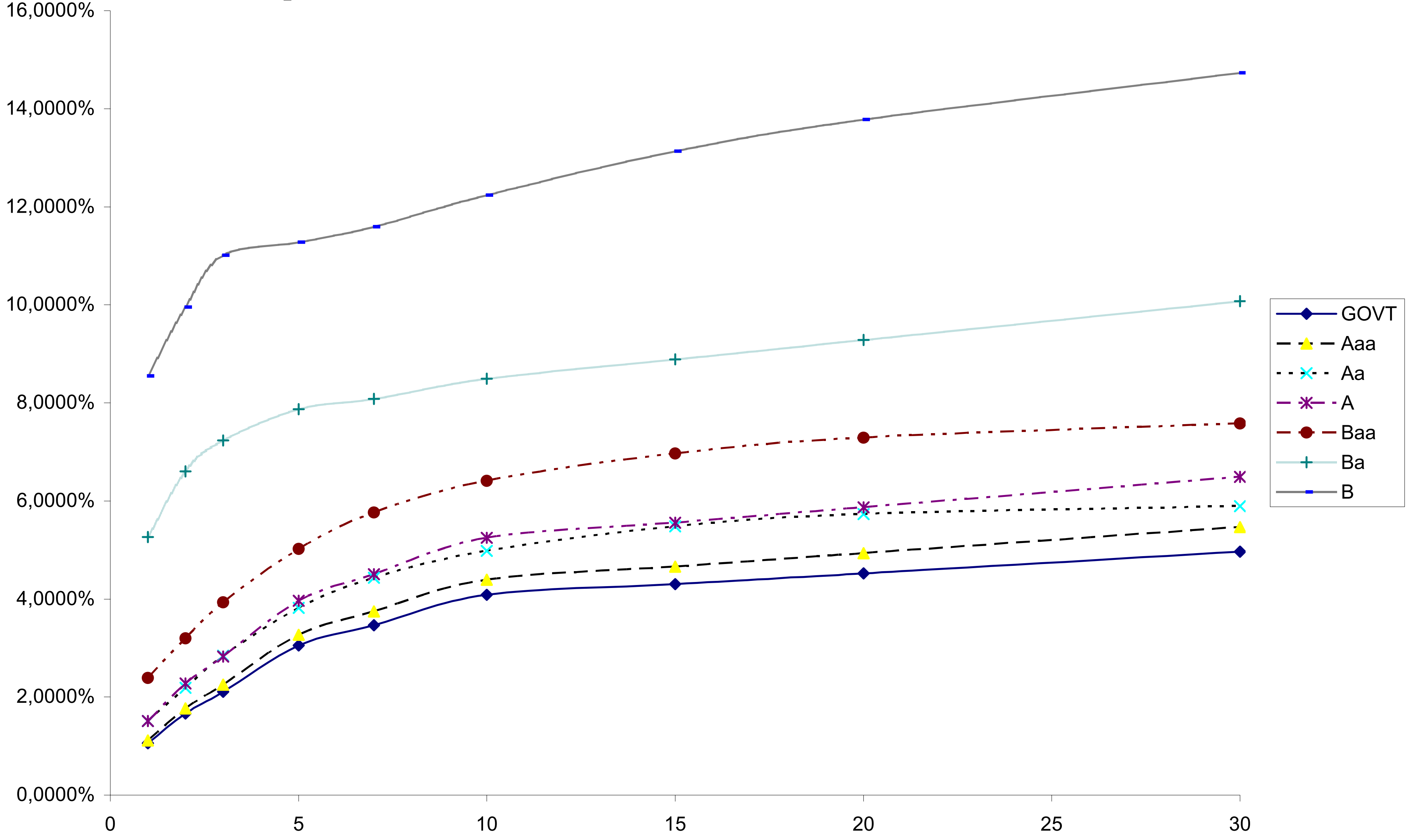
$Lgd_B \max[k - k^{**},0]\ A(t, t_j, t_n)\ [G_B(t_{j-1}) - G_B(t_j)]$
$- Lgd_B\ ReceiverSwaption\ (.;\min(k^{**},k))$

$Q(sr(t_j)>k^*(t_j)) = G_B(t_{j-1}) - G_B(t_j)$

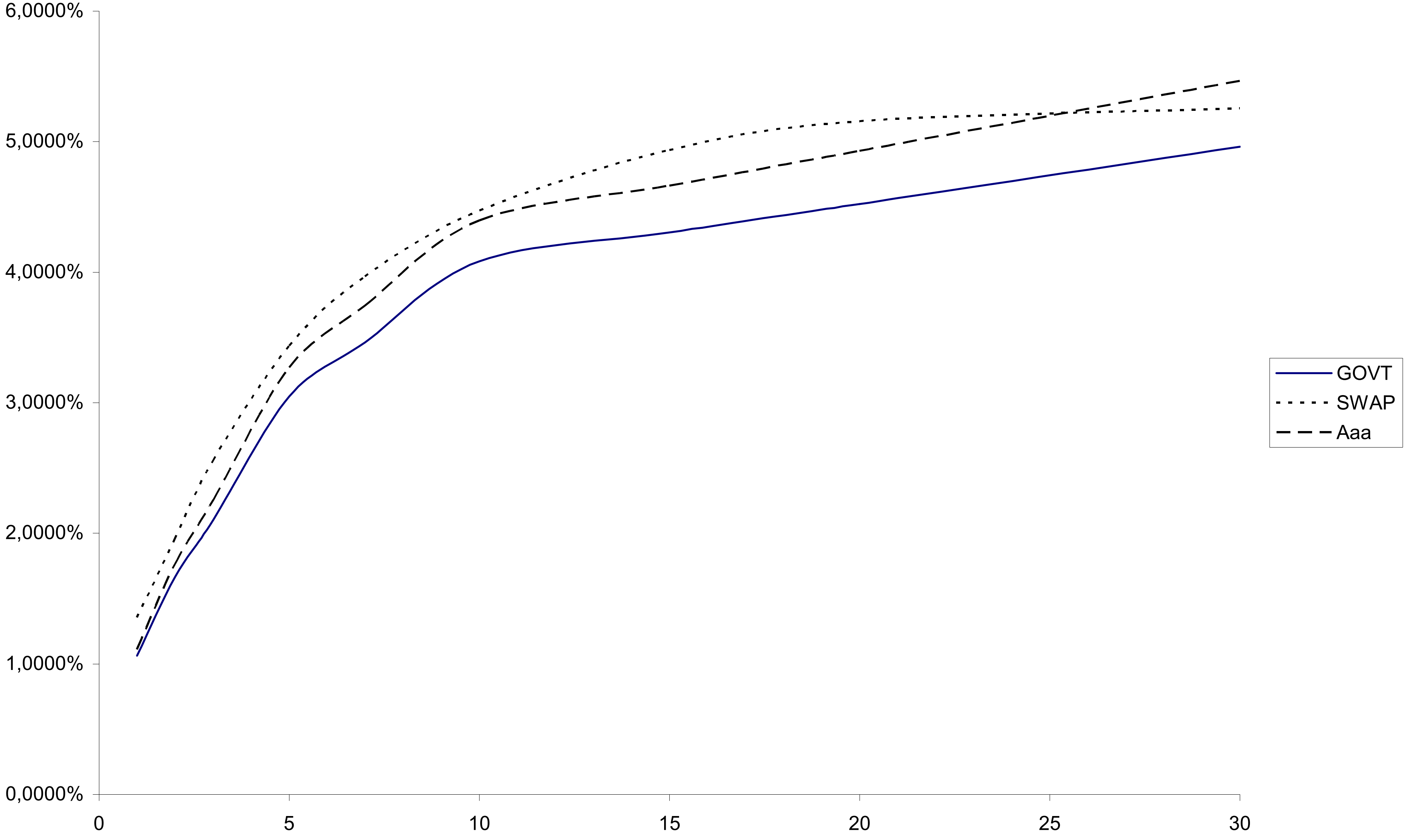
$Q(sr(t_j)>k^{**}(t_j)) = 1 - (G_B(t_{j-1}) - G_B(t_j))$

$$A(t,t_j,t_n) = \sum_{i=j}^{n-1} P(t,t_{i+1})$$

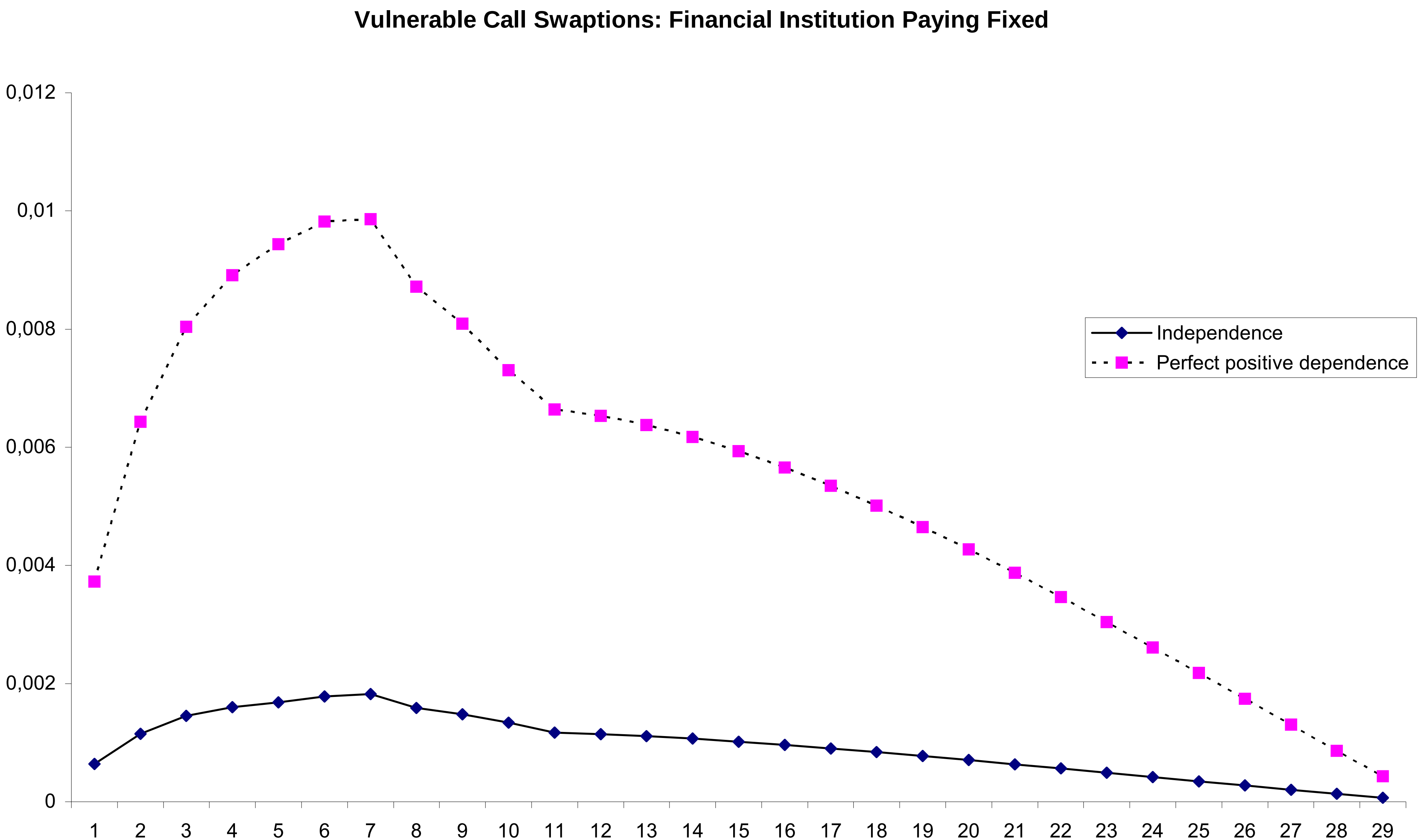
Empirical evidence: the data



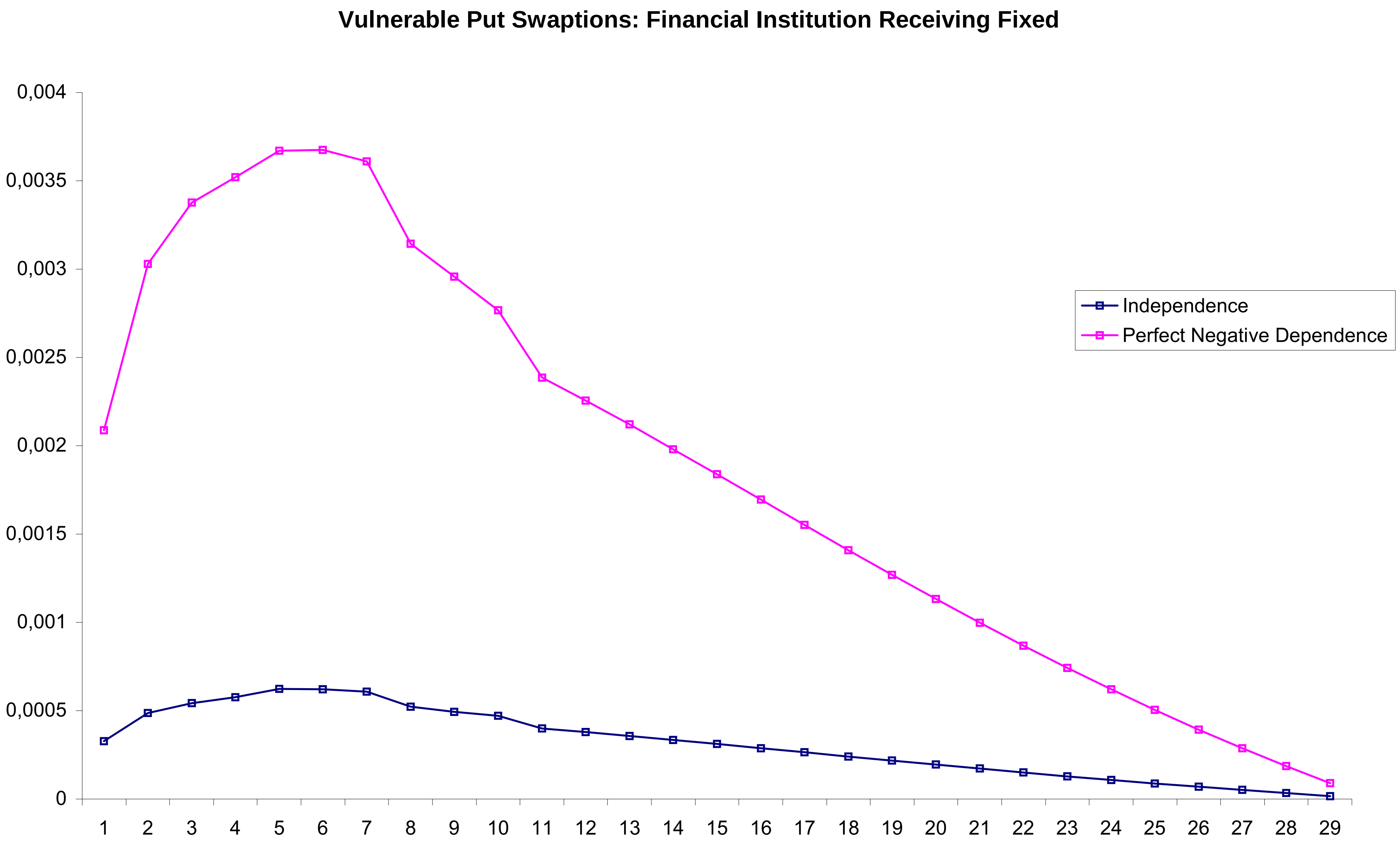
Swap spread and credit spread



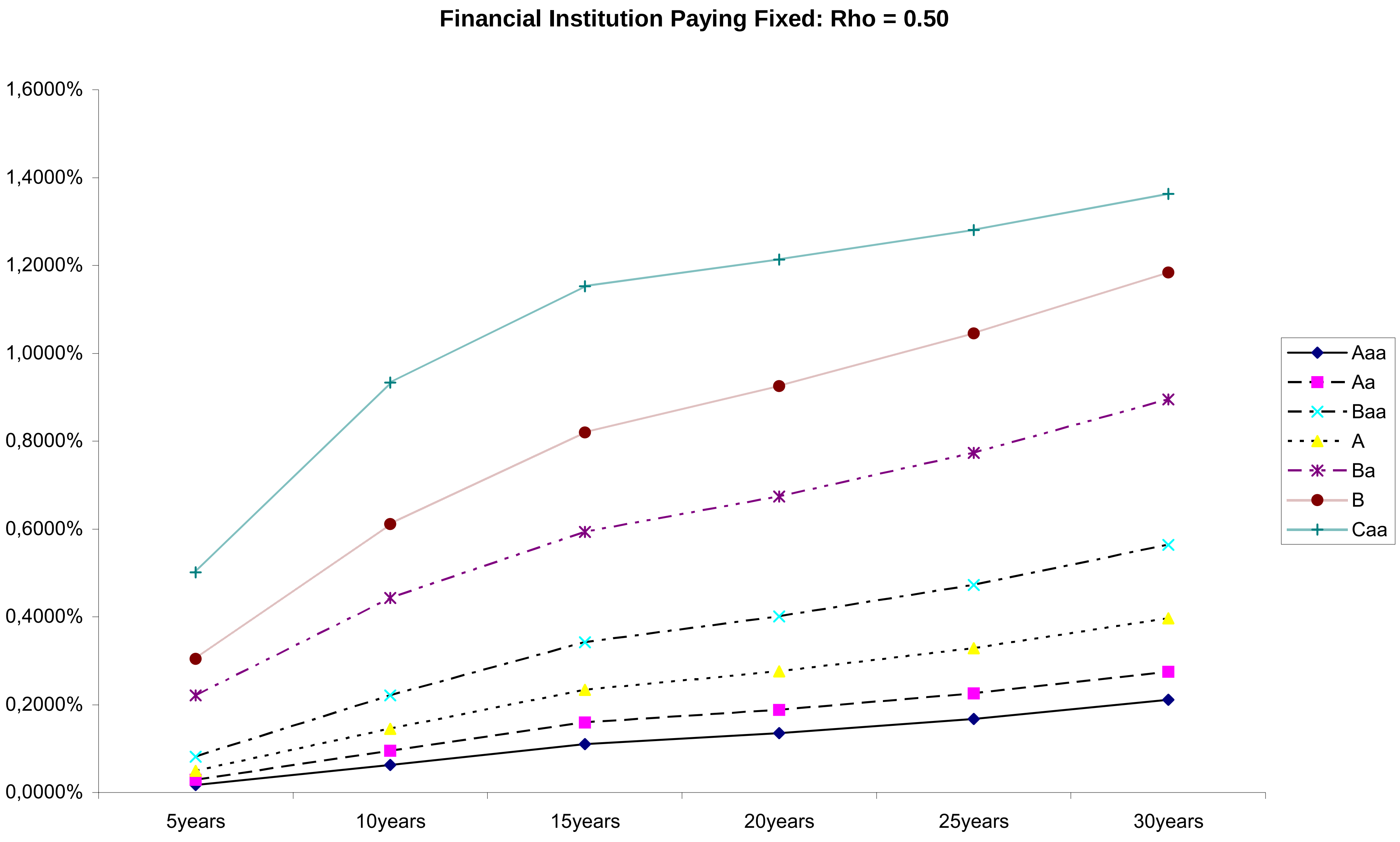
Credit risk for the fixed payer



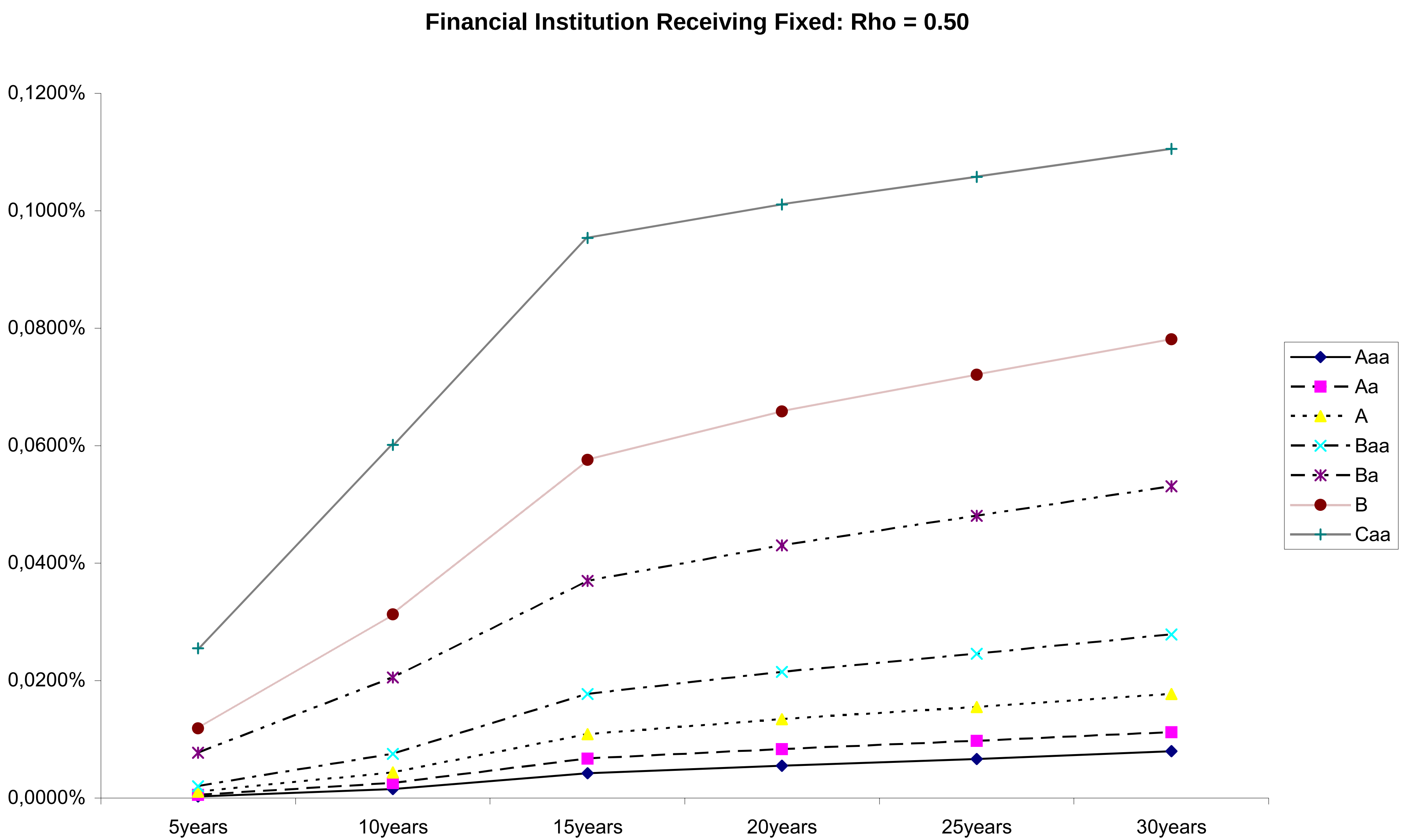
Credit risk for the fixed receiver



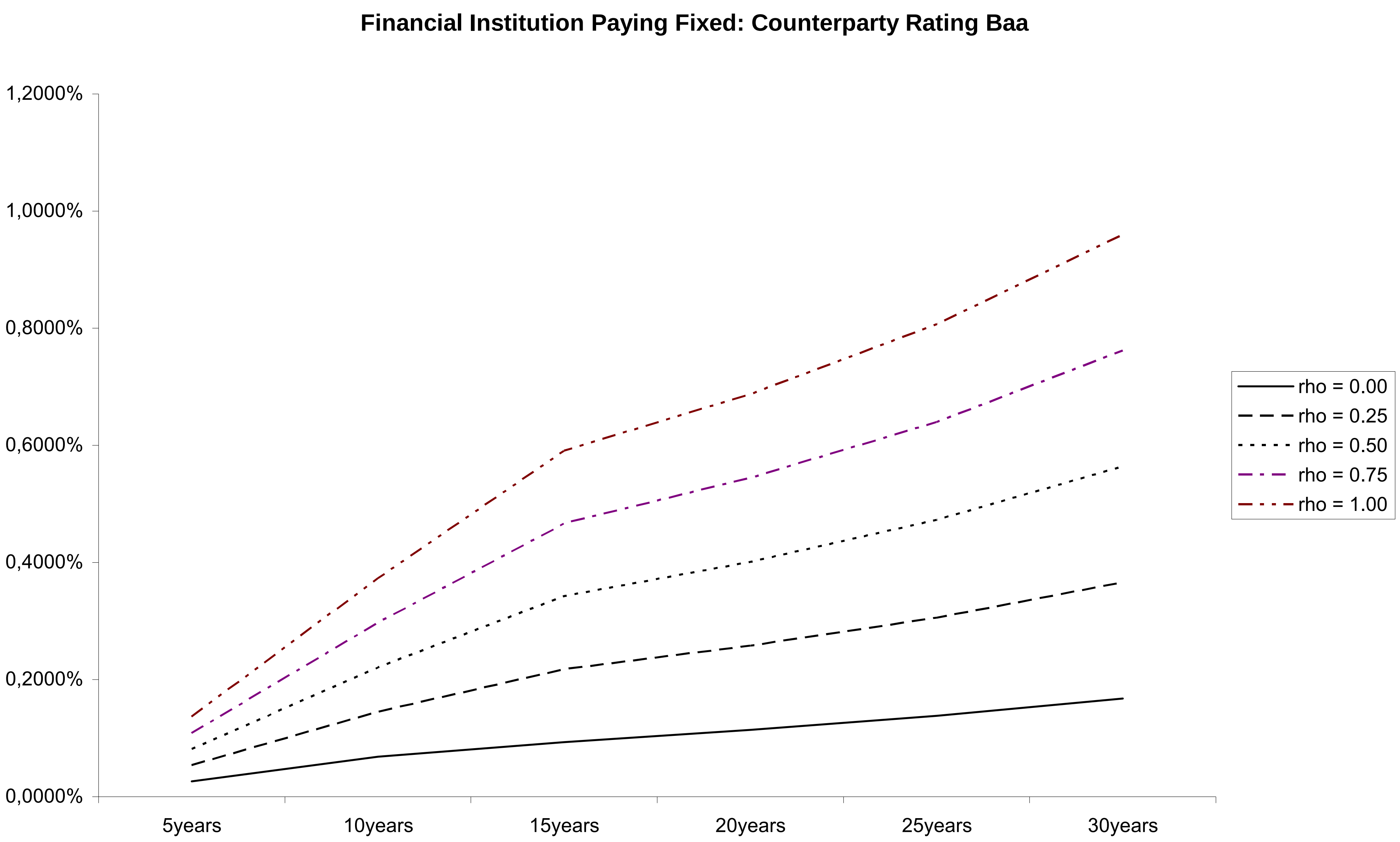
Counterparty risk spread: ratings



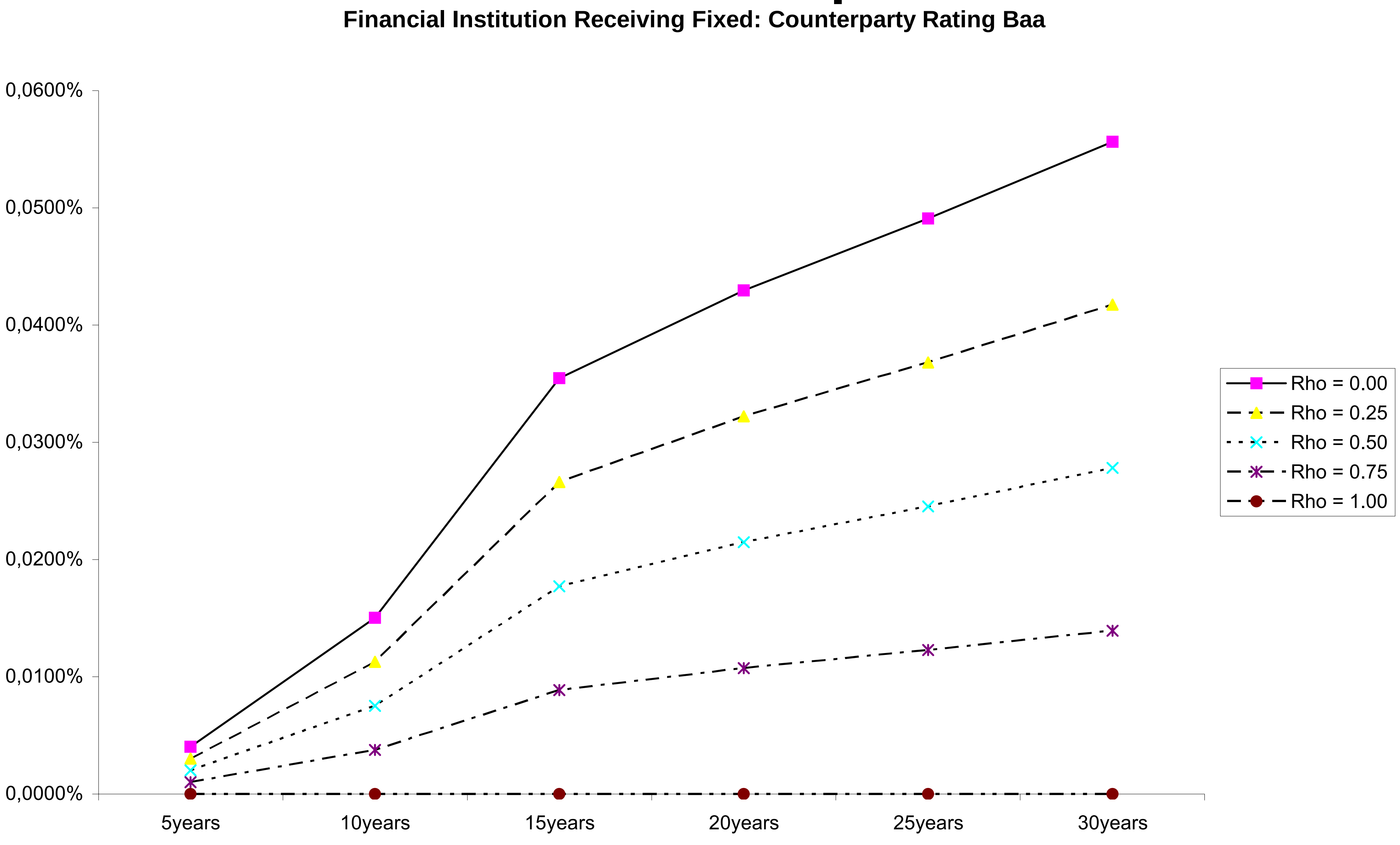
Counterparty risk spread: ratings



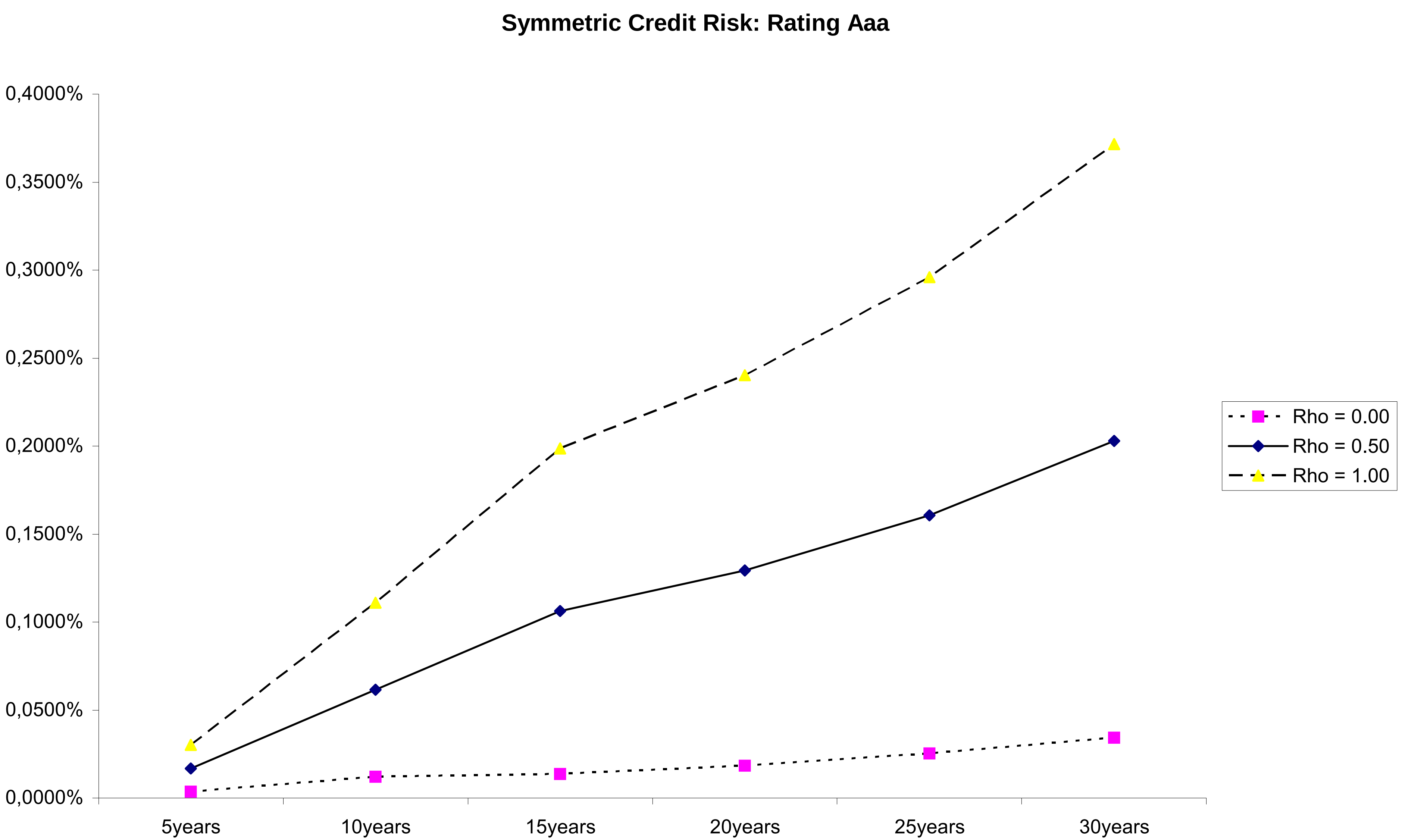
The effects of dependence



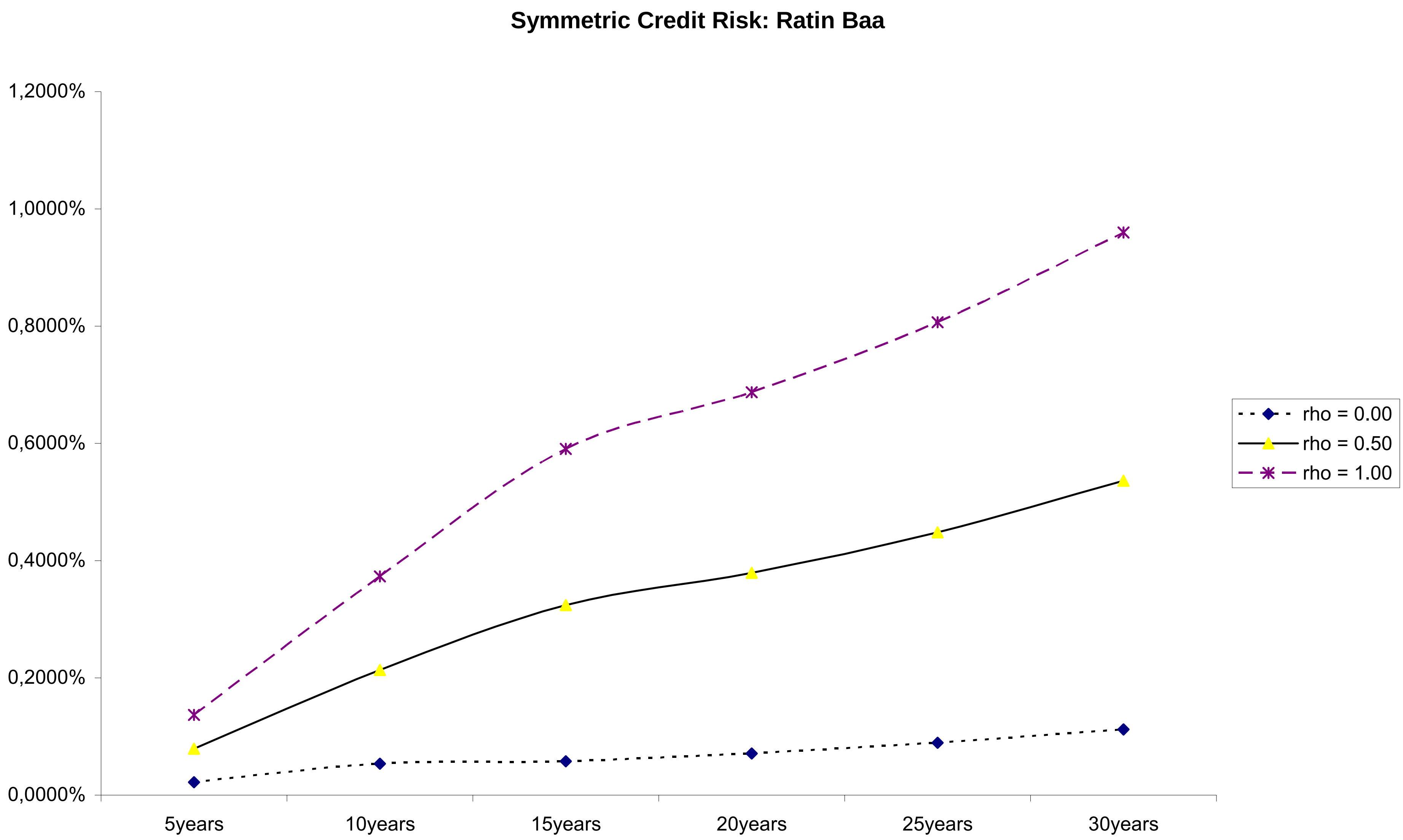
The effects of dependence



Symmetric risk: Aaa



Symmmetric risk: Baa



Swap credit risk: Aaa

	Fixed-Payer					
Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0041%	0,0150%	0,0223%	0,0294%	0,0385%	0,0502%
0,25	0,0106%	0,0390%	0,0664%	0,0821%	0,1029%	0,1306%
0,5	0,0171%	0,0630%	0,1104%	0,1348%	0,1672%	0,2109%
0,75	0,0237%	0,0870%	0,1545%	0,1875%	0,2316%	0,2913%
1	0,0302%	0,1110%	0,1986%	0,2402%	0,2959%	0,3716%

	Fixed-Receiver					
Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0005%	0,0030%	0,0085%	0,0111%	0,0134%	0,0160%
0,25	0,0004%	0,0023%	0,0064%	0,0083%	0,0100%	0,0120%
0,5	0,0003%	0,0015%	0,0043%	0,0055%	0,0067%	0,0080%
0,75	0,0001%	0,0008%	0,0021%	0,0028%	0,0033%	0,0040%
1	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%

Swap credit risk: Aa

Fixed-Payer

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0074%	0,0245%	0,0352%	0,0444%	0,0555%	0,0692%
0,25	0,0179%	0,0598%	0,0973%	0,1163%	0,1406%	0,1721%
0,5	0,0285%	0,0952%	0,1593%	0,1883%	0,2256%	0,2750%
0,75	0,0390%	0,1305%	0,2214%	0,2603%	0,3107%	0,3780%
1	0,0495%	0,1658%	0,2834%	0,3323%	0,3958%	0,4809%

Fixed-Receiver

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0010%	0,0051%	0,0134%	0,0166%	0,0194%	0,0225%
0,25	0,0008%	0,0038%	0,0101%	0,0125%	0,0146%	0,0168%
0,5	0,0005%	0,0025%	0,0067%	0,0083%	0,0097%	0,0112%
0,75	0,0003%	0,0013%	0,0034%	0,0042%	0,0049%	0,0056%
1	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%

Swap credit risk: A

Fixed-Payer

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0142%	0,0406%	0,0572%	0,0711%	0,0878%	0,1082%
0,25	0,0318%	0,0926%	0,1454%	0,1734%	0,2080%	0,2523%
0,5	0,0495%	0,1446%	0,2337%	0,2756%	0,3282%	0,3963%
0,75	0,0671%	0,1965%	0,3219%	0,3779%	0,4484%	0,5404%
1	0,0848%	0,2485%	0,4102%	0,4801%	0,5686%	0,6845%

Fixed-Receiver

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0021%	0,0087%	0,0218%	0,0268%	0,0309%	0,0355%
0,25	0,0016%	0,0065%	0,0163%	0,0201%	0,0232%	0,0266%
0,5	0,0011%	0,0044%	0,0109%	0,0134%	0,0155%	0,0177%
0,75	0,0005%	0,0022%	0,0054%	0,0067%	0,0077%	0,0089%
1	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%

Swap credit risk: Baa

Fixed-Payer

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0259%	0,0686%	0,0935%	0,1141%	0,1385%	0,1678%
0,25	0,0536%	0,1448%	0,2178%	0,2574%	0,3056%	0,3658%
0,5	0,0813%	0,2210%	0,3420%	0,4007%	0,4726%	0,5637%
0,75	0,1090%	0,2971%	0,4663%	0,5440%	0,6397%	0,7617%
1	0,1367%	0,3733%	0,5905%	0,6873%	0,8068%	0,9597%

Fixed-Receiver

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0040%	0,0150%	0,0355%	0,0429%	0,0491%	0,0556%
0,25	0,0030%	0,0113%	0,0266%	0,0322%	0,0368%	0,0417%
0,5	0,0020%	0,0075%	0,0177%	0,0215%	0,0245%	0,0278%
0,75	0,0010%	0,0038%	0,0089%	0,0107%	0,0123%	0,0139%
1	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%

Swap credit risk: Ba

			Fixed-Payer			
Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0918%	0,1675%	0,1987%	0,2279%	0,2637%	0,3058%
0,25	0,1564%	0,3053%	0,3960%	0,4509%	0,5185%	0,6004%
0,5	0,2209%	0,4432%	0,5933%	0,6739%	0,7733%	0,8949%
0,75	0,2855%	0,5810%	0,7906%	0,8968%	1,0282%	1,1894%
1	0,3501%	0,7188%	0,9878%	1,1198%	1,2830%	1,4839%

			Fixed-Receiver			
Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0154%	0,0410%	0,0740%	0,0861%	0,0961%	0,1062%
0,25	0,0115%	0,0308%	0,0555%	0,0646%	0,0721%	0,0797%
0,5	0,0077%	0,0205%	0,0370%	0,0430%	0,0481%	0,0531%
0,75	0,0038%	0,0103%	0,0185%	0,0215%	0,0240%	0,0266%
1	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%

Swap credit risk: B

Fixed-Payer						
Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,1402%	0,2564%	0,3081%	0,3482%	0,3935%	0,4444%
0,25	0,2221%	0,4337%	0,5641%	0,6369%	0,7193%	0,8141%
0,5	0,3040%	0,6110%	0,8201%	0,9256%	1,0451%	1,1838%
0,75	0,3859%	0,7882%	1,0761%	1,2143%	1,3709%	1,5536%
1	0,4678%	0,9655%	1,3322%	1,5030%	1,6966%	1,9233%

Fixed-Receiver						
Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0237%	0,0626%	0,1152%	0,1316%	0,1442%	0,1563%
0,25	0,0178%	0,0470%	0,0864%	0,0987%	0,1081%	0,1172%
0,5	0,0119%	0,0313%	0,0576%	0,0658%	0,0721%	0,0782%
0,75	0,0059%	0,0157%	0,0288%	0,0329%	0,0360%	0,0391%
1	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%

Swap credit risk: Caa

Fixed-Payer

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,2808%	0,4669%	0,5158%	0,5340%	0,5578%	0,5883%
0,25	0,3908%	0,7003%	0,8342%	0,8738%	0,9193%	0,9755%
0,5	0,5007%	0,9336%	1,1526%	1,2136%	1,2808%	1,3627%
0,75	0,6106%	1,1670%	1,4710%	1,5535%	1,6422%	1,7500%
1	0,7206%	1,4004%	1,7894%	1,8933%	2,0037%	2,1372%

Fixed-Receiver

Correl. Rho	5years	10years	15years	20years	25years	30years
0	0,0509%	0,1203%	0,1907%	0,2022%	0,2115%	0,2211%
0,25	0,0382%	0,0902%	0,1431%	0,1516%	0,1587%	0,1658%
0,5	0,0255%	0,0602%	0,0954%	0,1011%	0,1058%	0,1106%
0,75	0,0127%	0,0301%	0,0477%	0,0505%	0,0529%	0,0553%
1	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%	0,0000%

IF You Wanna Know More..