Performance of Dynamic Hedging Strategies: The Tale of Two Trading Desks *

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Abstract. Suppose an investment bank consists of two desks trading in equity and equity options, and it operates in a market where equity returns are leptokurtic. It is well known (Schweizer 1994) that the optimal mean-variance trading strategy for the bank as a whole is path-dependent. This paper examines quasi-optimal strategies that preserve the path-independent nature of Black–Scholes option hedging coefficients without excessively compromising bank's overall efficiency. More generally, I investigate the issue of risk-adjusted performance measurement and attribution between two desks trading in derivative and the underlying asset, respectively.

It is shown that both the optimal and quasi-optimal hedging strategies require close coordination between the equity and option desks, insofar as the optimal volume of option sales depends crucially on the relative performance of the two desks. Closed-form expressions for the Sharpe ratio and Certainty Equivalent Growth Rate as well as numerical results for a model calibrated to historical FTSE 100 equity index returns are given.

Keywords: Incremental Sharpe ratio, risk-adjusted returns, dynamic performance measurement, investment–hedging separation, option pricing, mean-variance hedging, Lévy process

JEL classification code: G11, C61
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1. Introduction

Consider an investor who trades in equities and equity options, and who operates in a market where equity returns are leptokurtic. This model has one important feature typical of real financial markets – the trading of options is a risky activity. No matter how frequently one chooses to hedge there is no way of replicating a given option perfectly. Furthermore, the market incompleteness is fundamental in that the model remains incomplete after adding a finite number of derivative assets (in contrast to e.g. Liu and Pan, 2003).

In this paper I think of the investor as an investment bank with two trading desks: equity and options. But if one is predominantly interested in dynamic equity trading, with relatively few option trades on the side, my investor looks very much like a hedge fund, which is the preferred interpretation in the existing literature on performance measurement of multi-period investments (see references below).

It is well known (Schweizer 1994) that in an incomplete market the optimal dynamic mean-variance trading strategy for the bank as a whole introduces path dependency into the hedging strategy of the option desk. This paper examines quasi-optimal strategies that preserve the path-independent nature of Black–Scholes option hedging coefficients without excessively compromising bank's overall efficiency.

To paraphrase Dybvig (1988), my aim is to identify simple hedging strategies that do not "throw away million dollars". It is shown that both the optimal and quasi-optimal strategies require close coordination between the equity and option desks, insofar as the optimal volume of option sales depends crucially on the relative performance of the two desks. The distribution of wealth between the equity and option desks has some impact on the overall performance of quasi-optimal strategies, but no bearing on the performance of the dynamically optimal strategy.

I present two ways of measuring the ex-ante performance of dynamic investment strategies — i) the unconditional Sharpe ratio of the hedged portfolio at maturity; and ii) the certainty equivalent growth rate of terminal wealth as measured by quadratic utility[1]. I provide numerical results for a model calibrated to historical FTSE 100 equity index returns for different combinations of equity trading and option hedging strategies.

Much of the literature on Sharpe ratio evaluation in multiperiod setting is in the spirit of complete markets, exploiting the duality between the maximal Sharpe ratio and the standard deviation of the pricing kernel (cf. Hansen and Jagannathan (1991)). In a complete market the maximal unconditional Sharpe ratio can be attained by dynamic

[1] Dybvig (1988) presents a different metric based on the insightful observation that agents, regardles of their specific preferences, should hold more wealth in states with lower state price. Dybvig's approach is preference-free but it relies entirely on market completeness and implies that buy-and-hold strategies are efficient. In contrast, my methodology requires specific preferences but it works for incomplete markets and, in line with other researchers, identifies buy-and-hold strategies in a dynamic setting as inefficient. However, like Dybvig (1988), I report efficiency loss in money terms.

trading in the underlying asset. Since any contingent claim is by default redundant, adding a fairly priced option to the dynamically optimal portfolio will not increase its unconditional Sharpe ratio. However, if one considers only *buy-and-hold* equity strategies as a point of departure then the Sharpe ratio will naturally improve if one also uses static option positions, see Leland (1999) and Goetzmann et al. (2002), since each option trade acts as an imperfect substitute for dynamic trading in the stock. In aforementioned papers this phenomenon is interpreted as a failure of Sharpe ratio to capture the true nature of investment opportunities[2].

Cvitanić et al. (2004) concentrate on optimal dynamic equity trading with different target horizons. They identify so-called horizon problem in which an investor with, say, 1 year horizon will obtain inferior performance if he or she delegates to a fund manager who maximizes quarterly performance.

Evaluation of risk-adjusted ex-ante investment performance in an incomplete market has received little attention to date. It is best tackled by solving the primal utility maximization[3] which leads to the solution of a modified mean-variance hedging problem[4]. In fact, the primal approach brings some advantages to a complete market because it explicitly identifies the optimal dynamic trading in equities.

This paper extends the existing literature (cf. Leland 1999, Goetzmann et al. 2002, Nielsen and Vassalou 2004, Cvitanić et al. 2004) i) by allowing for leptokurtic stock returns; and ii) by identifying the dynamically optimal equity trading strategy. However, the scope of the paper

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[2] In reality, this phenomenon merely reflects investor's failure to understand that in a *multi-period* setting one is allowed to use *dynamic strategies* and that optimal strategies as a rule require *dynamic rebalancing*, which makes buy-and-hold strategies inefficient almost by default. Cochrane (2001) warns:

> ...keep in mind that the unconditional mean-variance frontier *includes* returns on managed portfolios. This definition is eminently reasonable. If you are trying to minimize variance for given mean, why tie your hands to fixed-weight portfolios?

This observation is not specific to Sharpe ratio and quadratic utility but holds universally for any utility function and any performance measure derived from it, cf. Černý (2003). To put it differently, there is no reason why Sharpe ratio should only be attached to static positions. Sharpe ratio of dynamic investment strategies is computed exactly the same way as that of static multi-period investments, namely as the unconditional mean excess return divided by its standard deviation. In this context it is misleading to talk of different Sharpe ratios for the same *asset* (as in Nielsen and Vassalou, 2004), when one really has in mind the unconditional Sharpe ratio of different dynamic *portfolios* based on the same asset.

[3] In an incomplete market the Hansen and Jagannathan (1991) bound turns into a duality between the maximal unconditional Sharpe ratio attainable by trading in the basis assets and the variance of so-called variance-optimal measure (cf. Schweizer (1996)).

[4] In a multi-period setting the link between risk minimization and quadratic utility is exploited, for example, in Duffie and Richardson (1991) and Li and Ng (2000). For a systematic analysis of the relationship between certainty equivalent growth rate from quadratic utility and Sharpe ratio see Černý (2004b, Chapter 3) and Theorems 2.1 and 2.3 in this paper. Černý and Kallsen (2005) give an extensive survey of the mean–variance hedging literature and evaluate the maximal unconditional Sharpe ratio and the optimal equity trading strategy in a general semimartingale model.

is broader. My main motivation is to say something about the performance of alternative option hedging strategies. As mentioned above the optimal dynamic mean-variance hedging strategy introduces a path dependency which is absent in the standard Black–Scholes model. Since in practice no one uses path-dependent hedging coefficients, it is important to ask how much a Black–Scholes-like hedge would take away from the optimal performance. As it turns out, one can only answer this question meaningfully if option hedging is viewed as an incremental activity to an already existing equity trading strategy. I evaluate the investment performance in situations where neither the equity portfolio nor the option hedging strategy are necessarily optimal and attribute performance between equity and option trades as a function of option price.

The paper is organized as follows. Section 2 introduces notation and provides main theoretical results on multi-period ex-ante performance measurement. In Section 3 I compare the performance of dynamically optimal equity trading with static buy-and-hold strategy. Section 4 discusses the first best solution — optimal equity trading combined with optimal option hedging. Section 5 combines optimal equity trades with Black–Scholes-like hedging and gauges the magnitude of efficiency loss. Section 6 looks at buy-and-hold equity positions combined with suboptimal hedging and Section 7 concludes. Longer proofs are relegated to the appendix.

2. Setup

Consider an arbitrage-free market with two assets, stock and a risk-free bank account. The stock bears no dividends and its price at time t is denoted by S_t. The one-period total risk-free return is denoted by $R > 0$, the corresponding rate of return being $R - 1$. By X I denote the excess return on the stock

$$X_t := S_t / S_{t-1} - R.$$

$G_t^{x,\theta}$ denotes the value of financial portfolio generated by dynamic self-financing strategy investing θ_t pounds in the stock at time t and starting with initial wealth $G_0 = x$

$$G_t^{x,\theta} := R^t \left(x + \sum_{j=0}^{t-1} \theta_j \frac{X_{j+1}}{R^{j+1}} \right).$$

For simplicity I assume that the state space in the model is finite[5]. Admissible trading strategies are those adapted to stock price filtration.

Suppose an investment bank consists of two trading desks, equity and options. Assume that the equity desk trades purely in the stock, while the option desk issues a European option (or a book of options) maturing at time T. Once the option(s) are issued at time 0 there are

[5] This represents no loss of generality as the results shown here extend naturally to exponential Lévy models, see Černý (2005)

no additional option trades until maturity, that is the option position is completely static. Suppose the investment bank starts with initial financial capital x and wishes to maximize the expected quadratic utility of its terminal capital B_T at time T, given its coefficient of relative local risk aversion at risk-free wealth equals $\tilde{\gamma}$. In other words, as shown in Černý (2004b, Chapter 3), the objective of the bank is to minimize

$$\min_{\theta,\eta} \mathrm{E}\left[\left((1+\tilde{\gamma}^{-1})R^T x - B_T(x,\theta,\eta)\right)^2\right] \tag{1}$$

$$B_T(x,\theta,\eta) := \left(G_T^{x+\eta C_0,\theta} - \eta H\right) \tag{2}$$

where η is the volume of contingent claim with pay-off H, sold at time 0 at price C_0 and held to maturity. The quantity $(1+\tilde{\gamma}^{-1})R^T x$ represents the (unattainable) bliss point on the bank's utility function.

It may seem at this stage that the results of my analysis depend crucially on the initial level of capital x and the attitude to risk $\tilde{\gamma}$. I will show shortly, however, that one can disentangle the influence of both x and $\tilde{\gamma}$ and it is enough to analyze a simpler problem that boils down to the maximization of multiperiod Sharpe ratio[6].

To measure the performance of a particular, not necessarily optimal, investment strategy I compute the *certainty equivalent growth rate*

$$\mathrm{CEG}_{\tilde{\gamma}}(\eta,\theta) = \frac{W_{\tilde{\gamma}}(\eta,\theta)}{R^T x} - 1, \tag{3}$$

where $W_{\tilde{\gamma}}(\eta,\theta) < (1+\tilde{\gamma}^{-1})R^T x$ is the certainty equivalent wealth determined implicitly from

$$\mathrm{E}\left[\left((1+\tilde{\gamma}^{-1})R^T x - B_T(x,\theta,\eta)\right)^2\right] =: \left((1+\tilde{\gamma}^{-1})R^T x - W_{\tilde{\gamma}}(\eta,\theta)\right)^2. \tag{4}$$

THEOREM 2.1 (Performance measures). *The Certainty Equivalent Growth Rate (CEG) is inversely proportional to local risk aversion*

$$\mathrm{CEG}_{\tilde{\gamma}}(\eta,\theta) = \frac{1}{\tilde{\gamma}}\mathrm{CEG}_1(\tilde{\gamma}\eta,\tilde{\gamma}\theta). \tag{5}$$

The initial wealth can be factored out from CEG as follows

$$\mathrm{CEG}_{\tilde{\gamma}}(\eta,\theta) = \frac{1}{\tilde{\gamma}}\left(1 - \sqrt{\mathrm{E}\left[\left(1 - B_T(0,\tilde{\theta},\tilde{\eta})\right)^2\right]}\right), \tag{6}$$

$$\tilde{\theta} := \tilde{\gamma}\frac{\theta}{R^T x},\quad \tilde{\eta} := \tilde{\gamma}\frac{\eta}{R^T x} \tag{7}$$

Furthermore, if $\mathrm{SR}(\eta,\theta)$ *denotes the unconditional Sharpe ratio of the terminal wealth distribution* $B_T(x,\theta,\eta)$,

$$\mathrm{SR}(\eta,\theta) := \frac{\mathrm{E}\left[B_T(x,\theta,\eta) - R^T x\right]}{\sqrt{\mathrm{Var}\left(B_T(x,\theta,\eta)\right)}} = \frac{\mathrm{E}\left[B_T(0,\theta,\eta)\right]}{\sqrt{\mathrm{Var}\left(B_T(0,\theta,\eta)\right)}}, \tag{8}$$

[6] Meaningful factorization of x and $\tilde{\gamma}$ is not specific to quadratic utility but can be performed for any HARA utility, cf. Cerny (2004, Chapter 3).

then one obtains

$$\max_{\alpha} \mathrm{CEG}_1(\alpha\eta, \alpha\theta) = 1 - \sqrt{\left(1 + \mathrm{SR}^2(\eta, \theta)\right)^{-1}}. \tag{9}$$

Theorem 2.1 shows that one can fully characterize the quadratic utility performance of any investment strategy if one can evaluate the quadratic expectation in (6). Equation (5) shows that investor's risk aversion only affects the extent to which the investor benefits from the risky investment. When the investment is scaled optimally to reflect investor's risk aversion (equation 9) the CEG has a one-to-one relationship with the Sharpe ratio of the investment strategy. Conversely, one can gauge the Sharpe ratio of a given option hedging strategy from the corresponding utility maximization. The latter is very convenient for computation since it can be reduced to the well understood task of mean-variance hedging.

Let me now relate the standard one-period CAPM results to quadratic utility optimization.

THEOREM 2.2. *Consider n investments with excess returns $Y_i, i = 1, \dots, n$. Define $\mu_i := E[X_i], \Omega_{ij} := E[X_i X_j], \Sigma_{ij} := Cov(X_i, X_j)$ and assume Σ is non-singular. Then Σ^{-1} and Ω^{-1} exist and the maximal squared Sharpe ratio satisfies*

$$\mathrm{SR}^2 := \sup_{\alpha \in \mathbb{R}^n} \frac{(\alpha'\mu)^2}{(\alpha'\Sigma\alpha)^2} = \mu^\intercal \Sigma^{-1} \mu = \frac{\mu^\intercal \Omega^{-1} \mu}{1 - \mu^\intercal \Omega^{-1} \mu}.$$

Furthermore the optimal money amount $\hat{\alpha}_i$ to be invested into asset i by an investor with quadratic utility, safe wealth G_{safe} and local relative risk aversion $\tilde{\gamma}$ equals

$$\begin{aligned} \alpha &:= \arg \inf_{\alpha \in \mathbb{R}^n} E\left[\left(G_{safe}\left(1 + \tilde{\gamma}^{-1}\right) - (G_{safe} + \alpha' X)\right)^2\right] \\ &= \frac{G_{safe}}{\tilde{\gamma}} \Omega^{-1} \mu = \frac{G_{safe}}{\tilde{\gamma}} \Sigma^{-1} \mu \left(1 + \mathrm{SR}^2\right)^{-1} \end{aligned}$$

Proof. See Appendix A. ■

In the sequel I will use this result with $n = 2$, Y_1 representing the terminal wealth of equity investment and Y_2 representing the option hedging error, for a specific choice of dynamic investment and hedging strategies.

I conclude this section with an explicit recipe for the maximization of Sharpe ratio of dynamic trading strategies via the solution of a modified mean–variance hedging problem.

THEOREM 2.3 (Maximization of multiperiod Sharpe ratio). *Let Θ be a linear (sub-)space of stock trading strategies. In the notation of Theorem 2.1 define*

$$\begin{aligned} \{\hat{\eta}_{\tilde{\gamma}}, \hat{\theta}_{\tilde{\gamma},\Theta}\} &:= \arg \max_{\eta, \theta \in \Theta} \mathrm{CEG}_{\tilde{\gamma}}(\eta, \theta), \\ \mathrm{CEG}_{\tilde{\gamma},\Theta} &:= \max_{\eta, \theta \in \Theta} \mathrm{CEG}_{\tilde{\gamma}}(\eta, \theta). \end{aligned}$$

Furthermore define

$$\{\hat{\eta}, \hat{\theta}_\Theta\} := \arg\min_{\eta, \theta \in \Theta} \mathrm{E}\left[(1 - B_T(0, \theta, \eta))^2\right].$$

Then one has

$$\begin{aligned} \hat{\eta}_{\tilde{\gamma}} &= \hat{\eta}\frac{R^T x}{\tilde{\gamma}}, \hat{\theta}_{\tilde{\gamma},\Theta} = \hat{\theta}_\Theta \frac{R^T x}{\tilde{\gamma}} \\ \mathrm{SR}^2_\Theta &:= \mathrm{SR}^2(\hat{\eta}, \hat{\theta}_\Theta) = \frac{1}{\min_{\eta \in \mathbb{R}} \min_{\theta \in \Theta} \mathrm{E}\left[\left(1 + \eta H - G_T^{\eta C_0, \theta}\right)^2\right]} - 1, \end{aligned}$$

and

$$\begin{aligned} \mathrm{CEG}_{\tilde{\gamma},\Theta} &= \frac{1}{\tilde{\gamma}} \mathrm{CEG}_{1,\Theta}, \\ \mathrm{CEG}_{1,\Theta} &= 1 - \sqrt{\left(1 + \mathrm{SR}^2_\Theta\right)^{-1}}, \end{aligned}$$

that is one can rephrase the task of maximizing the Sharpe ratio as a two-stage procedure where the inner optimization evaluates the expected squared hedging error of the optimal mean–variance hedge for a contingent claim

$$Y = 1 + \eta H,$$

with initial wealth ηC_0 and the outer optimization chooses the optimal volume of option trade η as a function of the option price C_0.

Proof. Appendix A. ■

From now on I will restrict my attention to a model with IID stock returns, denoting

$$\begin{aligned} \mu &:= \mathrm{E}\left[S_{t+1}/S_t\right], \\ \sigma^2 &:= \mathrm{Var}(S_{t+1}/S_t). \end{aligned}$$

3. Optimal equity trading vs. buy-and-hold strategy

THEOREM 3.1. *Suppose the option position is zero. The optimal amount of wealth to be invested in the stock is given by*

$$\theta_t^e := aR(\bar{V}_t - G_t),$$

where

$$\begin{aligned} \bar{V}_t &:= (1 + \tilde{\gamma}^{-1}) R^t x, \\ a &:= \frac{\mu - R}{\sigma^2 + (\mu - R)^2}, \\ b &:= 1 - \frac{(\mu - R)^2}{\sigma^2 + (\mu - R)^2}. \end{aligned}$$

The performance in terms of CEG and the maximal Sharpe ratio is given by

$$\begin{aligned}\mathrm{CEG}_{\tilde{\gamma},e} &:= 1-\sqrt{\left(1+\mathrm{SR}_e^2\right)^{-1}},\\ \mathrm{SR}_e^2 &:= \mathrm{SR}^2\left(0,\theta^e\right)=b^{-T}-1.\end{aligned}$$

Proof. Use Theorems 4.1 and 4.2 with $Y=\bar{V}_T$, $\theta^e:=\varphi(x,\bar{V}_T)$, noting that $\varepsilon^2_{0\varphi}(\bar{V}_T)=0$ and $V_t(\bar{V}_T)=\bar{V}_t$. The certainty equivalent growth and the Sharpe ratio of the optimal strategy are obtained from Theorem 2.3. ■

Let us examine the nature of the optimal equity investment in more detail. Recall from Černý (2004b, Chapter 3) that

$$\tilde{\gamma}_t := \left(\frac{\bar{V}_T}{G_t R^{T-t}}-1\right)^{-1} = \left(\frac{\bar{V}_t}{G_t}-1\right)^{-1}, \tag{10}$$

can be interpreted as the investor's coefficient of relative risk aversion evaluated at risk-free wealth *as perceived at time* t. Specifically, at $t=0$ one has $\tilde{\gamma}_0=\tilde{\gamma}$. The wealth G_t may be risky as of time 0 but once at time t the investor could deposit G_t in the risk-free bank account and finish with risk-free wealth $G_t R^{T-t}$. With (10) in hand one can write the optimal strategy more naturally as

$$\theta_t^e = a\frac{RG_t}{\tilde{\gamma}_t}.$$

The parameter a represents the optimal proportion of risk-free wealth invested in the stock for a myopic investor with unit local risk aversion. Thus θ_t^e is a *constant proportion* strategy with stochastically changing risk aversion. It is a variation on the classical doubling strategy in which the investor leverages up when the investment performs poorly, only to switch back to safe assets when his or her wealth has risen sufficiently. As the value of bank's portfolio approaches the target value $\bar{V}_t$ the level of local risk aversion $\tilde{\gamma}_t$ steadily increases and the bank gradually switches from equities to safe assets.

The quantity b is related to one-period Sharpe ratio of the stock return $b=(1+\mathrm{SR}^2)^{-1}$. Thus the instantaneous Sharpe ratio is a sufficient statistic for the ranking of dynamically optimal strategies in a model with IID stock returns, and, by extension, for log stock price driven by an arbitrary Lévy process[7]. In contrast, the instantaneous Sharpe ratio does not provide enough information to rank buy-and-hold strategies as shown in the following theorem.

THEOREM 3.2. *Suppose the equity desk pursues a buy-and-hold strategy with initial capital x and a constant number of α_{bh} shares, $\theta_t^{bh}:=\alpha_{bh}S_t$. The quasi-optimal number of shares is given by*

$$\alpha_{bh}S_0 = \frac{xR^T}{\tilde{\gamma}}\frac{\mu^T-R^T}{(\sigma^2+\mu^2)^T-\mu^{2T}}\left(1+\mathrm{SR}_{bh}^2\right)^{-1}.$$

[7] This observation is made in a special case by Nielsen and Vassalou (2004) in a complete market model where log returns follow a Brownian motion with drift.

and the mean-variance performance of such a strategy equals

$$\begin{aligned}\mathrm{SR}_{bh}^2 &= \frac{\left(\mu^T - R^T\right)^2}{(\sigma^2+\mu^2)^T - \mu^{2T}},\\ \mathrm{CEG}_{\tilde{\gamma},bh} &= \frac{1}{\tilde{\gamma}}\left(1-\sqrt{\left(1+\mathrm{SR}_{bh}^2\right)^{-1}}\right).\end{aligned}$$

Proof. Since returns are IID one obtains

$$\mathrm{E}\left[S_T/S_0\right] = \mu^T, \qquad \mathrm{E}\left[(S_T/S_0)^2\right] = \left(\mu^2+\sigma^2\right)^T.$$

Apply Theorem 2.2 with $n = 1$ and $Y_1 = \frac{S_T}{S_0} - R^T$. $\mathrm{CEG}_{\tilde{\gamma},bh}$ is then given by Theorem 2.3 with Θ representing buy-and-hold strategies. ■

3.1. Model calibration and numerical results for equity trading strategies

I fix the performance of buy-and-hold strategy at the levels of Sharpe ratio equal to $0.25, 0.5, 0.75, 1.00$ and let the volatility of annual stock return take values $0.2, 0.3, 0.4$ while the annualized interest rate equals 0%[8]. Thus I have

$$\begin{gathered} T = 1 \text{ year}\\ \frac{\mu^T - R^T}{\sqrt{(\sigma^2+\mu^2)^T - \mu^{2T}}} \in \{0.25, 0.5, 0.75, 1.00\},\\ \sigma_{bh}{:=}\sqrt{(\sigma^2+\mu^2)^T - \mu^{2T}} \in \{0.2, 0.3, 0.4\},\\ R^T = 1.0.\end{gathered}$$

Numerical results are reported in Table I. For example, if the annual buy-and-hold Sharpe ratio is 0.25 and the volatility of buy-and-hold share return is 20% then the annual SR of the optimal strategy is 0.26. In terms of certainty equivalent wealth this represents $3.3\% - 3.0\%$ difference for an agent with unit risk aversion. The difference is applied to the safe wealth $R^T x$. For an agent with risk aversion $\tilde{\gamma} = 5$, who according to Grinold and Kahn (1999) would be an aggressive investor, the difference is five times smaller. When the buy and hold volatility doubles to 40% so does the gain from following the dynamic strategy. For an agent with $\tilde{\gamma} = 5$ the gain now stands at $(3.6\% - 3.0\%)/5$ equal to 12 basis points out of risk-free wealth.

To compare the performance of alternative strategies in monetary terms I assume as in Dybvig (1988) that the bank's initial capital is $x = 2$ billion. The risk-free rate is assumed to be zero and the risk aversion is set to 5. The magnitude of losses is broadly comparable to those computed by Dybvig (1988) who however uses a different metric (see footnote 1).

[8] It makes very little difference when the interest rate is increased to 2.5 or even 5.0%.

Table I. Performance of dynamically optimal and buy-and-hold equity trading strategies.

SR_{bh}	σ_{bh}	SR_e	$\mathrm{CEG}_{1,e}$	$\mathrm{CEG}_{1,bh}$	$\frac{\tilde{\gamma}\theta_0^e}{R^T x}$	$\frac{\tilde{\gamma}\theta_0^{bh}}{R^T x}$
0.25	20%	0.26	3.3%	3.0%	1.37	1.18
0.25	40%	0.28	3.6%	3.0%	0.77	0.59
0.5	20%	0.57	13.0%	10.6%	2.93	2.00
0.5	40%	0.61	14.6%	10.6%	1.73	1.00
0.75	20%	0.96	27.9%	20.0%	4.67	2.40
0.75	40%	1.07	31.6%	20.0%	2.89	1.20
1.00	20%	1.53	45.4%	29.3%	6.61	2.50
1.00	40%	1.79	51.2%	29.3%	4.25	1.25

Table II. The inefficiency of buy-and-hold equity trading strategy in money terms for an investor with x=2 billion and risk aversion $\tilde{\gamma} = 5$.

SR_{bh}	σ_{bh}	$x\mathrm{CEG}_{5,e}$ \$ million	$x\mathrm{CEG}_{5,bh}$ \$ million	$x\,(\mathrm{CEG}_{5,e} - \mathrm{CEG}_{5,bh})$ \$ million
0.25	20%	13.1	12.0	1.2
0.25	40%	14.4	12.0	2.4
0.5	20%	52.1	42.2	9.8
0.5	40%	58.3	42.2	16.0
0.75	20%	111.6	80.0	31.6
0.75	40%	126.3	80.0	46.3
1.00	20%	181.5	117.2	64.4
1.00	40%	205.0	117.2	87.8

4. Optimal trading in options and equities

To establish the necessary notation I start with the analysis of option hedging in isolation. Denote by Q the variance-optimal measure in my model, cf. Schweizer (1994), and define

$$Z_t := \prod_{k=t+1}^{T} (1 - aX_k). \tag{11}$$

To visualize the economic significance of the non-adapted process Z_t it is shown in Corollary 4.3 that $1 - Z_0$ is the *realized* excess return of the dynamically optimal equity portfolio. Following Černý (2005) I define conditional expectation under the variance-optimal measure Q as follows,

$$\mathrm{E}_t^Q[Y] := \mathrm{E}_t\left[Y Z_t / b^{T-t}\right],$$

for any $\mathcal{F}_T$-measurable random variable Y.

By construction Q is a martingale measure. Consider a contingent claim Y and define the *mean value process* $V(Y)$ by setting

$$V_t(Y) := \mathrm{E}_t^Q[Y/R^{T-t}].$$

When the market is complete $V_t(Y)$ gives the unique no-arbitrage price of the contingent claim Y. In an incomplete market it is natural to consider hedging strategies that in some way minimize the hedging error. I will call a hedging strategy θ dynamically optimal if it minimizes the expression

$$\mathrm{E}\left[\left(G_T^{x,\theta} - Y\right)^2\right].$$

THEOREM 4.1 (Dynamically optimal hedging). *The optimal hedging strategy for contingent claim Y with initial endowment x,*

$$\varphi(x,Y) := \arg\inf_{\theta} \mathrm{E}\left[\left(G_T^{x,\theta} - Y\right)^2\right],$$

is of the form

$$\varphi_t(x,Y) = \xi_t(Y) + aR(V_t(Y) - G_t^{x,\varphi(x,Y)}), \tag{12}$$

$$\xi_t(Y) := \frac{\mathrm{Cov}_t(V_{t+1}(Y), X_{t+1})}{\mathrm{Var}_t(X_{t+1})}, \tag{13}$$

$$V_t(Y) = \mathrm{E}_t\left[\frac{1 - aX_{t+1}}{b} V_{t+1}(Y)/R\right]. \tag{14}$$

Proof. See Černý (2005), Theorem 3.3. ■

In this paper I consider a single derivative asset with pay-off H. Occasionally, as in Theorem 4.5 below, it is convenient to allow for assets with payoffs different from H, but in all situations the problem at hand eventually boils down to $Y = H$. When there is no ambiguity as to the contingent claim being hedged I write simply φ, ξ and V instead of $\varphi(x,H), \xi(H)$ and $V(H)$. I refer to ξ as the *locally optimal hedge*[9]. The strategy φ is optimal but path-dependent, whereas ξ only depends on S but it is dynamically suboptimal. In a complete market there is no tracking error ($V_t - G_t^{V_0,\varphi} = 0$) and both strategies coincide.

In an incomplete market the hedging error of the optimal strategy is given as follows.

THEOREM 4.2 (Minimal hedging error). *Define the one-period realized hedging error of a perfectly balanced position*

$$e_t(Y) := RV_{t-1}(Y) + \xi_{t-1}(Y)X_t - V_t(Y) \textit{ for } t = 1, \ldots, T, \tag{15}$$

and set

$$\psi_t(Y) := \mathrm{E}[e_t^2(Y)] = \mathrm{E}\left[\mathrm{Var}_{t-1}\left(V_t(Y)\right) - \frac{(\mathrm{Cov}_{t-1}(V_t(Y), X_t))^2}{\mathrm{Var}_{t-1}(X_t)}\right].$$

[9] Thinking of φ_t as an explicit function of $V_t^{x,\varphi}$ I define ξ_t as the value of φ_t at $V_t^{x,\varphi} = H_t$. In other words ξ_t represents the optimal policy conditional on zero tracking error. In an incomplete market the tracking error takes non-zero values and a fortiori ξ is suboptimal.

Then

$$\mathrm{E}\left[\left(G_T^{x,\varphi(x,Y)} - Y\right)^2\right] = \left(bR^2\right)^T (x - V_0(Y)) + \varepsilon_{0\varphi}^2(Y), \tag{16}$$

$$\varepsilon_{0\varphi}^2(Y) := \sum_{t=1}^{T} \left(bR^2\right)^{T-t} \psi_t(Y), \tag{17}$$

$$G_T^{x,\varphi(x,Y)} - Y = R^T Z_0(x - V_0(Y)) + \sum_{t=1}^{T} R^{T-t} Z_t e_t(Y). \tag{18}$$

Proof. See Černý (2005), Theorem 3.3. Statement (18) follows from

$$G_t - V_t = R(G_{t-1} - V_{t-1})(1 - aX_t) + e_t,$$

multiplied on both sides by $R^{T-t}Z_t$ and summed over t. ■

COROLLARY 4.3. *The optimal wealth of the equity desk equals*

$$B_T(x, \theta^e, 0) = G_T^{x,\varphi(x,\bar{V}_T)} = R^T x + \frac{R^T x}{\tilde{\gamma}} (1 - Z_0). \tag{19}$$

THEOREM 4.4. *By* $\mathrm{SR}_{o\varphi}$ *denote the Sharpe ratio of the dynamically optimal option hedging strategy which invests the risk premium* $C_0 - V_0(H)$ *in the risk-free bank account and hedges the option optimally to maturity with initial capital* $V_0 = V_0(H)$. *Then*

$$\mathrm{SR}_{o\varphi} := \frac{R^T(C_0 - V_0)}{\varepsilon_{0\varphi}(H)}.$$

Proof. By Theorem 4.2 the terminal wealth of the above strategy reads

$$G = R^T(C_0 - V_0(H)) + \sum_{t=1}^{T} R^{T-t} Z_t e_t(H).$$

From Lemma 8.1 one obtains

$$\begin{aligned} E[G] &= R^T(C_0 - V_0), \\ \mathrm{Var}(G) &= \mathrm{E}\left[\left(\sum_{t=1}^{T} R^{T-t} Z_t e_t\right)^2\right] = \varepsilon_{0\varphi}^2. \end{aligned}$$

■

It is tempting to claim that $\mathrm{SR}_{o\varphi}$ represents the Sharpe ratio of the option hedging strategy. But such statement is, at this stage, meaningless because one can come up with alternative “hedging” strategies whose Sharpe ratio exceeds $\mathrm{SR}_{o\varphi}$. This is essentially possible because buying stock is inherently valuable to the investor even when there is no contingent claim to be hedged.

To deal with this situation one has to consider the joint performance of stock trading and option hedging, which is the subject of the next theorem.

THEOREM 4.5. *The optimal strategy for the bank as a whole can be implemented from a centralized trading desk that issues $\eta_{e\varphi}$ options and pursues the following investment-cum-hedging strategy*

$$\begin{aligned} \theta_t^{e\varphi} &= \varphi(x+\eta_{e\varphi}C_0, \bar{V}_T - \eta_{e\varphi}H) = \eta\xi_t + aR\left(\bar{V}_t + \eta V_t - G_t^{x,\theta^{e\varphi}}\right), \\ \eta_{e\varphi} &:= \frac{xR^T(C_0 - V_0)}{\tilde{\gamma}\varepsilon_{0\varphi}^2(H)}\left(1+\mathrm{SR}_{e\varphi}^2\right)^{-1}. \end{aligned}$$

The unconditional Sharpe ratio of the bank's optimal portfolio is given by

$$\mathrm{SR}_{e\varphi}^2 = \mathrm{SR}_e^2 + \mathrm{SR}_{o\varphi}^2,$$

in other words the certainty equivalent growth rate of the bank's wealth equals

$$\mathrm{CEG}_{\tilde{\gamma},e\varphi} = \frac{1}{\tilde{\gamma}}\left(1 - \sqrt{\left(1+\mathrm{SR}_{e\varphi}^2\right)^{-1}}\right).$$

Proof. Use Theorems 4.1 and 4.2 with $Y = \bar{V}_T + \eta H$, noting that $\varepsilon_{0\varphi}^2(\bar{V}_T + \eta H) = \eta^2\varepsilon_{0\varphi}^2(H), \xi(\bar{V}_T + \eta H) = \eta\xi(H)$ and $V_t(\bar{V}_T + \eta H) = \bar{V}_t + \eta V_t(H)$. The CEG and the Sharpe ratio of the optimal strategy are obtained from Theorem 2.3. ■

Theorem 4.5 identifies $\mathrm{SR}_{o\varphi}$ as the *incremental Sharpe ratio* of the option hedging strategy, in the sense that the Sharpe ratio of a pure equity investment SR_e will increase to $\sqrt{\mathrm{SR}_e^2 + \mathrm{SR}_{o\varphi}^2}$ when an optimal amount of an optimally hedged option trade is added to the initial equity position[10]. I will now examine the disaggregation of the optimal strategy.

THEOREM 4.6. *The optimal strategy of the bank can also be implemented by means of two autonomous trading desks:*

1. *an equity desk with initial capital x^e maximizing the multi-period Sharpe ratio of its wealth,*

$$\theta_t^e = aR\left(\bar{V}_t - G_t^{x^e,\theta^e}\right);$$

2. *an option trading desk with initial capital x^o issuing $\eta_{e\varphi}$ options and hedging them to maturity to minimize the expected squared hedging error*

$$\theta_t^o = \eta_{e\varphi}\xi_t + aR\left(\eta_{e\varphi}V_t - G_t^{x^o,\theta^o}\right).$$

The performance of the decentralized trading strategy is independent of the initial distribution of wealth between the two trading desks (as long as $x^e + x^o = x + \eta_{e\varphi}C_0$).

[10] The incremental Sharpe ratio is related to Dowd's (1999) investment rule based on the Sharpe ratio of a cumulative investment position. Dowd's rule rephrased in my terminology says "invest when the incremental Sharpe ratio of the new investment opportunity is positive".

Proof. One has $\theta^e + \theta^o = \theta^{e\varphi}$ whereby the statement follows from Theorem 4.5. ■

While the initial distribution of wealth has no effect on the overall bank performance it impacts crucially on the correlation of returns between the two trading desks. The next theorem examines the separation of hedging and investment in more detail. Previously, separation between optimal investment and hedging has been studied in Dybvig (1992). Dybvig considers projections under the objective measure P and as a result obtains very restrictive conditions for separation. In this paper the projection is computed under the variance-optimal measure Q and the separation obtains generally.

THEOREM 4.7. *The optimal strategy of the bank can be implemented in (at least) three different ways.*

1. *Equity desk with initial capital x inherits the aggregate level of risk aversion $\tilde{\gamma}$. Option desk runs a dynamically optimal hedge of $\eta_{e\varphi}$ options, treating $\eta_{e\varphi}C_0$ as its initial wealth. In this case the excess returns of equity and option desks are* negatively *correlated. The Sharpe ratio of the equity desk equals* SR_e *but the SR of the option desk is* lower than $\mathrm{SR}_{o\varphi}$.

2. *Equity desk with initial capital x inherits aggregate level of risk aversion $\tilde{\gamma}$. It also receives the risk premium $\eta_{e\varphi}(C_0 - V_0)$ from the option desk, treating this sum as a gain from trading. The option desk runs a dynamically optimal hedging strategy which is initially perfectly balanced. In this case the excess returns are uncorrelated. The Sharpe ratio of option desk is zero, but the Sharpe ratio of equity desk is* not *necessarily equal to the aggregate Sharpe ratio* $\sqrt{\mathrm{SR}_e^2 + \mathrm{SR}_{o\varphi}^2}$.

3. *Equity desk with initial capital x is assigned risk aversion*

$$\tilde{\gamma}_e := \tilde{\gamma}\left(1 + \frac{SR_{o\varphi}^2}{1 + SR_e^2}\right)$$

which is higher than the aggregate level of risk aversion $\tilde{\gamma}$. Option desk deposits the risk premium $\eta_{e\varphi}(C_0 - V_0)$ into the bank account and with the remaining amount $\eta_{e\varphi}V_0$ it runs a dynamically optimal hedging strategy which is initially perfectly balanced. The excess returns of the two desks are uncorrelated and the individual Sharpe ratios equal SR_e^2 *and* $\mathrm{SR}_{o\varphi}^2$*, respectively.*

Proof. See Appendix A. ■

5. Optimal equity trading, locally optimal option hedging

Suppose now the equity desk still trades optimally, but the option desk ignores the path-dependent portion of the dynamically optimal hedging strategy and simply uses the locally optimal hedging coefficient, which in my model only depends on the stock price and time. When considering locally optimal hedging in isolation from stock investment one obtains the following result[11],

$$
\begin{aligned}
G_T^{x,\xi(Y)} - Y &= R^T(x - V_0(Y)) + \sum_{t=1}^{T} R^{T-t} e_t(Y) && (20)\\
\mathrm{E}\left[\left(G_T^{x,\xi(Y)} - Y\right)^2\right] &= R^{2T}(x - V_0(Y))^2 + \varepsilon_{0\xi}^2(Y), && (21)\\
\varepsilon_{0\xi}^2(Y) &:= \sum_{t=1}^{T} R^{2(T-t)} \psi_t(Y), && (22)
\end{aligned}
$$

where the processes ξ, V and ψ are given in equations (13), (14) and (17).

By $\mathrm{SR}_{o\xi}$ denote the Sharpe ratio of the locally optimal option hedging strategy which invests the risk premium $C_0 - V_0$ in the risk-free bank account and hedges the option locally optimally to maturity with initial capital V_0:

$$\mathrm{SR}_{o\xi} := \frac{R^T(C_0 - V_0)}{\varepsilon_{0\xi}(H)}.$$

THEOREM 5.1. *Suppose the equity desk starts with initial capital x and inherits the aggregate risk aversion $\tilde{\gamma}$. If the option desk restricts itself to locally optimal hedging of η options, with initial capital ηC_0 then the quasi-optimal number of options to issue equals*

$$\eta_{\widetilde{e\xi}} := \frac{xR^T(C_0 - V_0)}{\tilde{\gamma}\varepsilon_{0\xi}^2(H)} \left(1 + \mathrm{SR}_e^2 + \mathrm{SR}_{o\xi}^2 + \mathrm{SR}_e^2\mathrm{SR}_{o\xi}^2\right)^{-1}$$

The Certainty Equivalent Growth Rate performance of the bank as a whole equals

$$\mathrm{CEG}_{\tilde{\gamma},\widetilde{e\xi}} = \frac{1}{\tilde{\gamma}}\left(1 - \sqrt{\left(1 + \mathrm{SR}_e^2 + \mathrm{SR}_{o\xi}^2\left(1 + \frac{\mathrm{SR}_e^2\mathrm{SR}_{o\xi}^2}{1 + \mathrm{SR}_e^2}\right)^{-1}\right)^{-1}}\right).$$

However, unlike in previous theorems, the quantity

$$\mathrm{SR}_e^2 + \mathrm{SR}_{o\xi}^2\left(1 + \frac{\mathrm{SR}_e^2\mathrm{SR}_{o\xi}^2}{1 + \mathrm{SR}_e^2}\right)^{-1}$$

cannot *be interpreted as the unconditional Sharpe ratio of the bank's aggregate position.*

Proof. See Appendix A. ■

THEOREM 5.2. *Suppose the equity desk starts with initial capital x and inherits the aggregate risk aversion $\tilde{\gamma}$. If the option desk pursues a locally optimal hedging of η options, then its quasi-optimal initial*

[11] See equation (12.76) in Černý (2004).

capital equals ηV_0*, that is the option desk should hand the risk premium* $\eta(C_0 - V_0)$ *over to the equity desk. In such a case the quasi-optimal number of options to issue equals*

$$\eta_{e\xi} := \frac{R^T x(C_0 - V_0)}{\tilde{\gamma}\varepsilon^2_{0\xi}(H)} \left(1 + \mathrm{SR}_e^2 + \mathrm{SR}_{o\xi}^2\right)^{-1}.$$

The Certainty Equivalent Growth Rate performance of the bank as a whole equals

$$\mathrm{CEG}_{\tilde{\gamma},e\xi} = \frac{1}{\tilde{\gamma}}\left(1 - \sqrt{\left(1 + \mathrm{SR}_{e\xi}^2\right)^{-1}}\right),$$

where $\mathrm{SR}_{e\xi}$ *is the unconditional Sharpe ratio of the bank's quasi-optimal portfolio,*

$$\mathrm{SR}_{e\xi}^2 = \mathrm{SR}_e^2 + \mathrm{SR}_{o\xi}^2.$$

Proof. See Appendix A. ■

Theorems 5.1 and 5.2 reveal that the distribution of initial capital between the two trading desks has an impact on the overall performance of the bank. If the option desk keeps the option premium then the certainty equivalent growth rate is observationally equivalent to incremental Sharpe ratio of $\mathrm{SR}_{o\xi}\left(1 + \frac{\mathrm{SR}_e^2 \mathrm{SR}_{o\xi}^2}{1+\mathrm{SR}_e^2}\right)^{-1/2}$ but if the option premium is handed over to the equity desk then the incremental Sharpe ratio is higher at $\mathrm{SR}_{o\xi}$. The story here mirrors that of Theorem 4.6. In Theorem 5.1 the bank has taken on too much equity investment while the volume of option trades is too low. If the equity position were reduced (by inflating the risk aversion of the equity desk to $\tilde{\gamma}\left(1 + \mathrm{SR}_{o\xi}^2 / \left(1 + \mathrm{SR}_e^2\right)\right)$) and simultaneously if the option position were increased from $\eta_{\widetilde{e\xi}}$ to $\eta_{e\xi}$ then the aggregate Sharpe ratio would increase to the quasi-optimal level $\sqrt{\mathrm{SR}_e^2 + \mathrm{SR}_{o\xi}^2}$ (in analogy to Theorem 4.6, strategy 3). The redistribution of initial wealth between the trading desks in Theorem 5.2 is an alternative way of achieving the quasi-optimal investment mix (corresponding to Theorem 4.6, strategies 1 and 2, where "dynamically optimal" hedging is to be replaced with "locally optimal" hedging).

Tables III and IV show numerical results for a 1 year horizon. The stock return distribution is calibrated to historical FT100 daily returns[12].

[12] I take the empirical distribution of nominal daily log returns from FT100 equity index using 100 equally-sized bins covering the entire range of observed values (leading to a centi-nomial lattice of stock prices). I then re-scale and re-center the distribution of log returns to match the theoretical annual values of mean and variance of returns used in the paper. This way the skewness and kurtosis of log returns are not altered at all. The kurtosis of historical daily log-return distribution is 5.47 as compared to 5.46 for its centinomial approximation. The kurtosis of historical daily return distribution is 5.40 compared to 5.37 for the centinomial approximation.

Table III. Performance of dynamically and locally optimal hedging. Incremental Sharpe ratio of optimal option hedging strategy $SR_{o\varphi} = 0.25$

SR_{bh}	σ_{bh}	$\mathrm{CEG}_{1,e\varphi}$	$\mathrm{CEG}_{1,e\xi}$	$\mathrm{CEG}_{1,e\tilde{\xi}}$	$x\,(\mathrm{CEG}_{5,e\varphi} - \mathrm{CEG}_{5,e\xi})$ \$ million	$x\left(\mathrm{CEG}_{5,e\xi} - \mathrm{CEG}_{5,e\tilde{\xi}}\right)$ \$ million
0.25	20%	6.00%	5.94%	5.93%	0.24	0.04
0.25	40%	6.29%	6.22%	6.20%	0.28	0.04
0.5	20%	15.05%	14.86%	14.84%	0.76	0.09
0.5	40%	16.51%	16.30%	16.27%	0.84	0.10
0.75	20%	29.13%	28.88%	28.86%	1.03	0.08
0.75	40%	32.65%	32.40%	32.38%	1.02	0.07
1.00	20%	46.01%	45.80%	45.80%	0.81	0.03
1.00	40%	51.73%	51.57%	51.57%	0.65	0.02

Table IV. Performance of dynamically and locally optimal hedging. Incremental Sharpe ratio of optimal option hedging strategy $SR_{o\varphi} = 0.50$

SR_{bh}	σ_{bh}	$\mathrm{CEG}_{1,e\varphi}$	$\mathrm{CEG}_{1,e\xi}$	$\mathrm{CEG}_{1,e\tilde{\xi}}$	$x\,(\mathrm{CEG}_{5,e\varphi} - \mathrm{CEG}_{5,e\xi})$ \$ million	$x\left(\mathrm{CEG}_{5,e\xi} - \mathrm{CEG}_{5,e\tilde{\xi}}\right)$ \$ million
0.25	20%	12.94%	12.75%	12.62%	0.77	0.51
0.25	40%	13.16%	12.94%	12.80%	0.89	0.54
0.5	20%	20.28%	19.64%	19.33%	2.54	1.23
0.5	40%	21.48%	20.77%	20.46%	2.84	1.25
0.75	20%	32.25%	31.34%	31.07%	3.65	1.06
0.75	40%	35.35%	34.43%	34.20%	3.67	0.93
1.00	20%	47.42%	46.67%	46.56%	3.02	0.43
1.00	40%	52.75%	52.13%	52.06%	2.48	0.28

6. Buy-and-hold equity position, quasi-optimal option hedging

The previous two sections assumed that the equity desk pursues optimal investment strategy. Here I assume instead that the equity investment in the form of a buy-and-hold strategy combined with locally optimal hedging.

THEOREM 6.1. *Suppose the equity desk follows a buy and hold strategy with the number of shares given in Theorem 3.2,*

$$\alpha_{bh} S_0 = \frac{xR^T}{\tilde{\gamma}} \frac{\mu^T - R^T}{(\sigma^2 + \mu^2)^T - \mu^{2T}} \left(1 + \mathrm{SR}_{\mathrm{bh}}^2\right)^{-1}.$$

If the option desk restricts itself to locally optimal hedging of η options, then the quasi-optimal number of options to issue equals

$$\eta = \frac{xR^T (C_0 - V_0)}{\tilde{\gamma}\varepsilon_{0\xi}^2} \left(1 + \mathrm{SR}_{\mathrm{bh}}^2 + \mathrm{SR}_{o\xi}^2 + \mathrm{SR}_{\mathrm{bh}}^2 \mathrm{SR}_{o\xi}^2\right)^{-1}.$$

The CEG of the bank's quasi-optimal portfolio equals

$$\mathrm{CEG}_{\tilde{\gamma},\widetilde{\mathrm{bh}\xi}} = \frac{1}{\tilde{\gamma}}\left(1-\sqrt{\left(1+\mathrm{SR}^2_{\widetilde{\mathrm{bh}\xi}}\right)^{-1}}\right),$$

$$\mathrm{SR}^2_{\widetilde{\mathrm{bh}\xi}} := \mathrm{SR}^2_{\mathrm{bh}} + \mathrm{SR}^2_{o\xi}\left(1+\frac{\mathrm{SR}^2_{\mathrm{bh}}\mathrm{SR}^2_{o\xi}}{1+\mathrm{SR}^2_{\mathrm{bh}}}\right)^{-1},$$

however, $\mathrm{SR}_{\widetilde{\mathrm{bh}\xi}}$ *is* not *the Sharpe ratio of the bank's quasi-optimal wealth.*

THEOREM 6.2. *If the option desk pursues a locally optimal hedging of* η *options, and the equity desk buys and holds* α *shares then the quasi-optimal values of* η *and* α *are given by*

$$\alpha_{\mathrm{bh}\xi} = \frac{xR^T}{\tilde{\gamma}}\frac{\mu^T - R^T}{(\sigma^2+\mu^2)^T - \mu^{2T}}\left(1+\mathrm{SR}^2_{\mathrm{bh}}+\mathrm{SR}^2_{o\xi}\right)^{-1}$$

$$\eta_{\mathrm{bh}\xi} = \frac{xR^T}{\tilde{\gamma}}\frac{(C_0 - V_0)}{\varepsilon^2_{0\xi}}\left(1+\mathrm{SR}^2_{\mathrm{bh}}+\mathrm{SR}^2_{o\xi}\right)^{-1}.$$

The terminal wealth of the two trading desks is uncorrelated and the unconditional Sharpe ratio of the bank's quasi-optimal portfolio equals

$$\mathrm{SR}^2_{\mathrm{bh}\xi} = \mathrm{SR}^2_{\mathrm{bh}} + \mathrm{SR}^2_{o\xi},$$

$$\mathrm{CEG}_{\tilde{\gamma},\mathrm{bh}\xi} = \frac{1}{\tilde{\gamma}}\left(1-\sqrt{\left(1+\mathrm{SR}^2_{\mathrm{bh}\xi}\right)^{-1}}\right).$$

Theorems 6.1 and 6.2 once again underline the degree of coordination between the equity and options trading desks. Moreover, Theorems 5.2 and 6.2 assert that $\mathrm{SR}_{o\xi}$ can be interpreted as the *incremental* Sharpe ratio of the locally optimal hedging strategy regardless of the investment strategy adopted by the equity desk.

7. Conclusions

I have examined the performance of optimal and suboptimal equity investment and option hedging strategies in a model with IID leptokurtic returns and frictionless trading. My findings show that the inefficiency of static equity positions compared to the optimal equity trading is more significant than the inefficiency of Black–Scholes-like hedging compared to dynamically optimal hedging. The inefficiency generally increases with the stock volatility and with the size of option risk premium.

I have considered two desks within an investment bank, trading in derivatives and the underlying asset, respectively. I have shown that one can separate the task of derivatives hedging from optimal investment in the underlying. At the same time, my analysis suggests that for a bank-wide optimal performance the two desks must coordinate trading volumes in line with their relative risk-adjusted performance.

There remains a significant value in future research into the impact of transaction costs and stochastic volatility on the present conclusions.

8. Proofs

LEMMA 8.1. *For process Z defined in (11) and for $e(Y)$ defined in (15) one has*

$$\mathrm{E}_t\left[Z_t\right] = \mathrm{E}_t\left[Z_t^2\right] = b^{T-t}, \tag{23}$$

$$\mathrm{E}\left[\left(\sum_{t=1}^{T} R^{T-t} e_t\right)^2\right] = \varepsilon_{0\xi}^2(Y) \tag{24}$$

$$\mathrm{E}\left[\left(\sum_{t=1}^{T} R^{T-t} Z_t e_t(Y)\right)^2\right] = \varepsilon_{0\varphi}^2(Y), \tag{25}$$

$$\mathrm{E}\left[Z_0 \sum_{t=1}^{T} R^{T-t} Z_t e_t(Y)\right] = 0, \tag{26}$$

$$\mathrm{E}\left[Z_0 \sum_{t=1}^{T} R^{T-t} e_t(Y)\right] = 0, \tag{27}$$

$$\mathrm{E}\left[\left(S_T/S_0 - R^T\right) \sum_{t=1}^{T} R^{T-t} Z_t e_t(Y)\right] = 0, \tag{28}$$

$$\mathrm{E}\left[\left(S_T/S_0 - R^T\right) \sum_{t=1}^{T} R^{T-t} e_t(Y)\right] = 0. \tag{29}$$

Proof. I will illustrate the proof of equation (26), the remaining statements are proved similarly. Using the law of iterated expectation for $t = 1, \ldots, T$ together with the definition of Z (11) one obtains

$$\begin{aligned}
\mathrm{E}\left[Z_0 Z_t e_t\right] &= \mathrm{E}\left[\prod_{k=1}^{t}(1 - aX_k) e_t \mathrm{E}_t\left[Z_t^2\right]\right] \\
&= b^{T-t}\mathrm{E}\left[\prod_{k=1}^{t-1}(1 - aX_k)\mathrm{E}_{t-1}\left[e_t(1 - aX_t)\right]\right].
\end{aligned}$$

Now $\mathrm{E}_{t-1}\left[e_t(1 - aX_t)\right] = 0$ since e_t, being an error from a least squares regression of V_t onto 1 and X_t, is by construction orthogonal to 1 as well as X_t. ■

Proof. (Theorem 2.1) One can rewrite the argument of the utility function as follows

$$\begin{aligned}
(1 + \tilde{\gamma}^{-1})R^T x \;-\; & B_T(x, \eta, \theta) = \tilde{\gamma}^{-1} R^T x - B_T(0, \eta, \theta) \\
&= \tilde{\gamma}^{-1}\left(R^T x - B_T(0, \tilde{\gamma}\eta, \tilde{\gamma}\theta)\right).
\end{aligned}$$

This implies

$$\begin{aligned}
R^T x - W_{\tilde{\gamma}}(\eta, \theta) &= \tilde{\gamma}^{-1}\left(R^T x - W_1(\tilde{\gamma}\eta, \tilde{\gamma}\theta)\right), \\
\mathrm{CEG}_{\tilde{\gamma}}(\eta, \theta) &= \frac{1}{\tilde{\gamma}}\mathrm{CEG}_1(\tilde{\gamma}\eta, \tilde{\gamma}\theta).
\end{aligned}$$

To show the correspondence between SR and CEG I define the "excess return"

$$X{:=}B_T(0,\eta,\theta)$$

and write

$$\begin{aligned}
\mathrm{E}\left[\left(R^T x-\alpha X\right)^2\right] &= \left(2R^T x-W_1(\alpha\eta,\alpha\theta)\right)^2\\
\sqrt{\mathrm{E}\left[(R^T x-\alpha X)^2\right]} &= R^T x-\left(W_1(\alpha\eta,\alpha\theta)-R^T x\right)\\
\sqrt{\mathrm{E}\left[\left(1-\frac{\alpha}{R^T x}X\right)^2\right]} &= 1-\mathrm{CEG}_1\left(\alpha\eta,\alpha\theta\right)\\
\max_{\alpha}\mathrm{CEG}_1\left(\alpha\eta,\alpha\theta\right) &= 1-\min_{\tilde\alpha}\sqrt{\mathrm{E}\left[(1-\tilde\alpha X)^2\right]}\\
&= 1-\sqrt{\min_{\tilde\alpha}\mathrm{E}\left[(1-\tilde\alpha X)^2\right]}\\
&= 1-\sqrt{1-\left(\mathrm{E}\left[X\right]\right)^2/\mathrm{E}\left[X^2\right]}\\
&= 1-\sqrt{\left(1+\mathrm{SR}^2(X)\right)^{-1}}
\end{aligned}$$

Finally, note that $\mathrm{SR}(X)=\frac{\mathrm{E}[B_T(0,\eta,\theta)]}{\sqrt{\mathrm{Var}(B_T(0,\eta,\theta))}}=\frac{\mathrm{E}\left[B_T(x,\eta,\theta)-R^T x\right]}{\sqrt{\mathrm{Var}(B_T(x,\eta,\theta))}}$, which completes the proof. ■

Proof. (**Theorem 2.2**)From Theorem (2.1) I have

$$\frac{1}{1+\mathrm{SR}^2(\alpha' X)}=\min_{\lambda\in\mathbb{R}}\mathrm{E}\left[(1-\lambda\alpha' X)^2\right],$$

and therefore

$$\begin{aligned}
\min_{\alpha\in\mathbb{R}^n}\frac{1}{1+\mathrm{SR}^2(\alpha' X)} &= \min_{\lambda\in\mathbb{R},\alpha\in\mathbb{R}^n}\mathrm{E}\left[\left(1-\lambda\alpha' X\right)^2\right]\\
&= \min_{\alpha\in\mathbb{R}^n}\mathrm{E}\left[(1-\alpha' X)^2\right].
\end{aligned}$$

The first order conditions for the right hand side read

$$\Omega\alpha=\mu,$$

whereby one obtains $\hat\alpha=\Omega^{-1}\mu$ and

$$\min_{\alpha\in\mathbb{R}^n}\frac{1}{1+\mathrm{SR}^2(\alpha' X)}=1-\mu'\Omega^{-1}\mu,$$

implying

$$\max_{\alpha\in\mathbb{R}^n}\mathrm{SR}^2(\alpha' X)=\frac{1}{1-\mu'\Omega^{-1}\mu}-1=\frac{\mu'\Omega^{-1}\mu}{1-\mu'\Omega^{-1}\mu}. \qquad (30)$$

Finally, one has $\Omega=\Sigma+\mu\mu'$, therefore $\Omega b=\Sigma b+\mu\mu' b$ which for $b:=\Omega^{-1}\mu$ yields

$$\mu=\Sigma b+\mu\mu' b,$$

and after rearrangement

$$\mu(1-\mu' b) = \Sigma b. \tag{31}$$

On left-multiplying both sides by $\mu'\Sigma^{-1}$ one has

$$\mu'\Sigma^{-1}\mu(1-\mu'\Omega^{-1}\mu) = \mu'\Omega^{-1}\mu,$$

which by virtue of (30) yields

$$\max_{\alpha\in\mathbb{R}^n} \mathrm{SR}^2(\alpha' X) = \mu'\Sigma^{-1}\mu.$$

Similarly, left-multiplication of (31) by Σ^{-1} yields

$$\hat{\alpha} = \frac{\Sigma^{-1}\mu}{1+\mu'\Sigma^{-1}\mu}.$$

■

Proof. (Theorem 2.3) From definition one has

$$\mathrm{SR}^2_\Theta = \max_{\eta\in\mathbb{R},\theta\in\Theta} \mathrm{SR}^2(\eta,\theta). \tag{32}$$

Use Theorem 2.1 to write

$$\begin{aligned}\mathrm{SR}^2(\eta,\theta) &= \frac{1}{\min_{\alpha\in\mathbb{R}} \mathrm{E}\left[(1-\alpha B_T(0,\eta,\theta))^2\right]} - 1 \\ &= \frac{1}{\min_{\alpha\in\mathbb{R}} \mathrm{E}\left[(1-B_T(0,\alpha\eta,\alpha\theta))^2\right]} - 1. \end{aligned} \tag{33}$$

(32) and (33) yield

$$\begin{aligned}\mathrm{SR}^2_\Theta &= \frac{1}{\min_{\eta\in\mathbb{R},\theta\in\Theta} \min_{\alpha\in\mathbb{R}} \mathrm{E}\left[(1-B_T(0,\alpha\eta,\alpha\theta))^2\right]} - 1 \\ &= \frac{1}{\min_{\eta\in\mathbb{R},\theta\in\Theta} \mathrm{E}\left[(1-B_T(0,\eta,\theta))^2\right]} - 1,\end{aligned}$$

where the last equality follows from the fact that Θ is a linear (sub)space. The rest follows from scaling properties proved in Theorem 2.1. ■

LEMMA 8.2. *Consider function $f:\mathbb{R}\longrightarrow\mathbb{R}$*

$$f(x) = A(1-Bx)^2 + Cx^2$$

for $A, C > 0$. Then

$$\begin{aligned}\hat{x} &:= \arg\min_{x\in\mathbb{R}} f(x) = \frac{AB}{AB^2+C}, \\ f(\hat{x}) &= \min_{x\in\mathbb{R}} f(x) = \frac{AC}{AB^2+C}.\end{aligned}$$

Proof. Straightforward. ■

Proof. (Theorem 4.7)

1. Apply (18) with $Y = \bar{V}_T + \eta_{e\varphi} H$

$$\begin{aligned}
B_T(x, \theta^e, \eta_{e\varphi}) &= G_T^{x+\eta_{e\varphi}C_0, \varphi(x+\eta_{e\varphi}C_0, \bar{V}_T+\eta_{e\varphi}H)} - \eta_{e\varphi}H - \bar{V}_T + \bar{V}_T \\
&= \bar{V}_T + R^T Z_0(x + \eta_{e\varphi}C_0 - V_0(\bar{V}_T + \eta_{e\varphi}H)) \\
&\quad + \sum_{t=1}^{T} R^{T-t} Z_t e_t(\bar{V}_T + \eta H) \\
&= R^T x \left(1 + \tilde{\gamma}^{-1}\right) + R^T Z_0(-\tilde{\gamma}^{-1}x + \eta C_0 - \eta_{e\varphi} V_0(H)) \\
&\quad + \eta_{e\varphi} \sum_{t=1}^{T} R^{T-t} Z_t e_t(H) \\
&= R^T x + \tilde{\gamma}^{-1} R^T x\,(1 - Z_0) + \eta_{e\varphi} R^T Z_0(C_0 - V_0(H)) \\
&\quad + \eta_{e\varphi} \sum_{t=1}^{T} R^{T-t} Z_t e_t(H). \qquad (34)
\end{aligned}$$

By virtue of Corollary 4.3 and (18) with $Y = H$ the last expression corresponds to an autonomous equity desk with initial wealth x and risk aversion $\tilde{\gamma}$ and an autonomous option desk issuing η options with initial wealth ηC_0 and hedging them optimally to maturity. The covariance between the two desks equals

$$\begin{aligned}
&\mathrm{Cov}\left(R^T x + \tilde{\gamma}^{-1} R^T x\,(1 - Z_0), \eta_{e\varphi} R^T Z_0(C_0 - V_0) + \eta_{e\varphi} \sum_{t=1}^{T} R^{T-t} Z_t e_t\right) \\
&= \eta_{e\varphi} \tilde{\gamma}^{-1} R^{2T} x (C_0 - V_0) \mathrm{Cov}(1 - Z_0, Z_0) \\
&\quad + \eta_{e\varphi} \tilde{\gamma}^{-1} R^T x \mathrm{Cov}\left(1 - Z_0, \sum_{t=1}^{T} R^{T-t} Z_t e_t\right) \\
&= \eta_{e\varphi} \tilde{\gamma}^{-1} R^{2T} x (C_0 - V_0) \left(\mathrm{E}\left[Z_0 - Z_0^2\right] - \mathrm{E}\left[1 - Z_0\right] \mathrm{E}\left[Z_0\right]\right) \\
&= -\eta_{e\varphi} \tilde{\gamma}^{-1} R^{2T} x (C_0 - V_0)(1 - b^T) b^T \leq 0
\end{aligned}$$

where the equality arises only in the trivial cases $C_0 = V_0$ or $b = 1$. The Sharpe ratio of the option desk equals

$$\frac{R^T b^T (C_0 - V_0)}{\sqrt{R^T b^T (C_0 - V_0)\,(1 - b^T) + \varepsilon_{0\varphi}^2}} \leq \mathrm{SR}_{o\varphi}$$

and the equality once again arises only in the trivial cases $C_0 = V_0$ or $b = 1$.

2. On rearranging (34) one has

$$R^T x (1 + \tilde{\gamma}^{-1}) + R^T Z_0 \left(\eta_{e\varphi}(C_0 - V_0) - \tilde{\gamma}^{-1} x\right) + \eta_{e\varphi} \sum_{t=1}^{T} R^{T-t} Z_t e_t(H).$$

On setting $\tilde{x} := x + \eta_{e\varphi}(C_0 - V_0)$ equation (18) implies

$$G^{\tilde{x}, \varphi(\tilde{x}, \bar{V}_T)} = R^T x (1 + \tilde{\gamma}^{-1}) + R^T Z_0 \left(\eta_{e\varphi}(C_0 - V_0) - \tilde{\gamma}^{-1} x\right),$$

which yields the claim.

3. On rearranging (34) one has

$$\begin{aligned} B_T(x, \theta^e, \eta_{e\varphi}) &= R^T x + R^T \left(\frac{x}{\tilde{\gamma}} - \eta_{e\varphi} \left(C_0 - V_0 \right) \right) (1 - Z_0) \\ &\quad + \eta_{e\varphi} \left(R^T \left(C_0 - V_0 \right) + \sum_{t=1}^{T} R^{T-t} Z_t e_t \right), \end{aligned}$$

which yields the claim if one sets $\tilde{\gamma}_e^{-1} := \tilde{\gamma}^{-1} - \eta_{e\varphi} \left(C_0 - V_0 \right) / x$.

■

Proof. (**Theorem 5.1**) In this proof all quantities V, ξ, e, ε relate to payoff H. By virtue of Theorem 2.1 one has to evaluate

$$\begin{aligned} f(\tilde{\eta}) &:= \mathrm{E}\left[\left(1 - B_T(0, \tilde{\theta}^e + \tilde{\eta}\xi, \tilde{\eta}) \right)^2 \right], \\ \tilde{\theta}^e &= \tilde{\gamma}\theta^e / (R^T x), \tilde{\eta} = \tilde{\gamma}\eta / (R^T x) \end{aligned}$$

Using (19) and (20) one obtains

$$f(\tilde{\eta}) = \mathrm{E}\left[\left(1 - (1 - Z_0) - \tilde{\eta} \left(R^T \left(C_0 - V_0 \right) + \sum_{t=1}^{T} R^{T-t} e_t \right) \right)^2 \right].$$

By virtue of (23), (26) and (9)

$$f(\tilde{\eta}) = \left(b^T - \tilde{\eta} R^T \left(C_0 - V_0 \right) \right)^2 + b^T (1 - b^T) + \tilde{\eta}^2 \varepsilon_{0\xi}^2.$$

Using Lemma 8.2 with $A = b^{2T}, B = b^{-T} R^T \left(C_0 - V_0 \right), C = \varepsilon_{0\xi}^2$ the optimal number of options to issue equals

$$\begin{aligned} \hat{\eta} &:= \arg\max_{\tilde{\eta}} f(\tilde{\eta}) = \frac{AB}{C + AB^2} = \frac{b^T R^T \left(C_0 - V_0 \right)}{\varepsilon_{0\xi}^2 + R^{2T} \left(C_0 - V_0 \right)^2}, \\ \max_{\tilde{\eta}} f(\tilde{\eta}) &= f(\hat{\eta}) = \frac{AC}{C + AB^2} + b^T (1 - b^T) \\ &= \frac{b^{2T} \varepsilon_{0\xi}^2}{\varepsilon_{0\xi}^2 + R^{2T} \left(C_0 - V_0 \right)^2} + b^T (1 - b^T) \\ &= b^T \frac{\varepsilon_{0\xi}^2 + R^{2T} \left(C_0 - V_0 \right)^2 (1 - b^T)}{\varepsilon_{0\xi}^2 + R^{2T} \left(C_0 - V_0 \right)^2} \end{aligned}$$

By Theorem 2.1 the CEG of the quasi-optimal strategy with $R^T x \hat{\eta} / \tilde{\gamma}$ options satisfies

$$\tilde{\gamma} \mathrm{CEG}_{\tilde{\gamma}, \widetilde{e\xi}} = 1 - \sqrt{f(\hat{\eta})}. \tag{35}$$

On defining

$$\mathrm{SR}_{\widetilde{e\xi}}^2 := \frac{1}{f(\hat{\eta})} - 1 = \mathrm{SR}_e^2 + \mathrm{SR}_{o\xi}^2 \left(1 + \frac{\mathrm{SR}_{o\xi}^2 \mathrm{SR}_e^2}{1 + \mathrm{SR}_e^2} \right)^{-1}$$

I conclude that a complete market with Sharpe ratio $\mathrm{SR}_{\widetilde{e\xi}}^2$ would give the same level of expected utility as the quasi-optimal strategy described above. It is *not* true, however, that $SR_{\widetilde{e\xi}}$ equals the Sharpe ratio of the bank's aggregate excess return for this quasi-optimal strategy because the proportion of equity investment is kept fixed and only the number of options is optimized. ■

Proof. (Theorem 5.2) Equity desk with initial wealth x and risk aversion $\tilde{\gamma}_e$ will earn excess return

$$G_T^{0,\theta^e} = G_T^{x,\theta^e} - R^T x = \frac{R^T x}{\tilde{\gamma}_e}(1 - Z_0), \tag{36}$$

by virtue of (19). On the other hand the option desk selling η options at price C_0 and hedging them locally optimally to maturity earns excess return of

$$\eta\left(G_T^{C_0,\xi} - H\right) = \eta\left(R^T(C_0 - V_0) + \sum_{t=1}^{T} R^{T-t} e_t\right).$$

By virtue of Lemma 8.1 the excess returns $(1 - Z_0)$ and $G_T^{C_0,\xi} - H$ are uncorrelated, with mean and covariance matrix

$$\begin{aligned}
\mu &= \begin{pmatrix} R^T(1 - b^T) \\ R^T(C_0 - V_0) \end{pmatrix}, \\
\Sigma &= \begin{pmatrix} R^{2T} b^T (1 - b^T) & 0 \\ 0 & \varepsilon_{0\xi}^2 \end{pmatrix}.
\end{aligned}$$

By virtue of Lemma 2.2 the maximal Sharpe ratio equals

$$\mathrm{SR}_{e\xi}^2 = \frac{\mu_1^2}{\Sigma_{11}} + \frac{\mu_2^2}{\Sigma_{22}} = \mathrm{SR}_e^2 + \mathrm{SR}_{0\xi}^2,$$

and the optimal amount invested in each asset is given by

$$\begin{aligned}
\alpha_1 &= \frac{R^T x}{\tilde{\gamma}} \frac{\mu_1}{\Sigma_{11}} \frac{1}{1 + \mathrm{SR}_{e\xi}^2} = \frac{x}{\tilde{\gamma}} \frac{1 + \mathrm{SR}_e^2}{1 + \mathrm{SR}_{e\xi}^2}, \\
\alpha_2 &= \frac{R^T x}{\tilde{\gamma}} \frac{\mu_2}{\Sigma_{22}} \frac{1}{1 + \mathrm{SR}_{e\xi}^2} =: \eta_{e\xi}.
\end{aligned}$$

In view of (36) the risk aversion of the equity desk should equal $\tilde{\gamma}_e = \tilde{\gamma}\frac{1+\mathrm{SR}_{e\xi}^2}{1+\mathrm{SR}_e^2}$ to achieve quasi-optimal aggregate performance. The quasi-optimal aggregate excess return can be rephrased as

$$\begin{aligned}
&\frac{R^T x}{\tilde{\gamma}_e}(1 - Z_0) + \eta_{e\xi}\left(R^T(C_0 - V_0) + \sum_{t=1}^{T} R^{T-t} e_t\right) \\
&= \frac{R^T x}{\tilde{\gamma}}\left(1 - Z_0 + \eta_{e\xi}(C_0 - V_0) Z_0\right) + \eta_{e\xi} \sum_{t=1}^{T} R^{T-t} e_t
\end{aligned}$$

where the first expression corresponds to the excess return of an equity desk with risk aversion $\tilde{\gamma}$, initial wealth x and initial gains from trading equal to $(C_0 - V_0)$ and the second expression is the hedging error of a locally optimal strategy which is initially perfectly balanced. ■

Proof. (**Theorem 6.1**)By virtue of Theorem 2.1 one has to evaluate

$$\begin{aligned} f(\tilde{\eta}) &:= \mathrm{E}\left[\left(1 - B_T(0, \tilde{\theta}^e + \tilde{\eta}\xi, \tilde{\eta})\right)^2\right], \\ \tilde{\theta}^e_t &= \tilde{\gamma}\alpha_{bh} S_t/(R^T x), \tilde{\eta} = \tilde{\gamma}\eta/(R^T x) \end{aligned}$$

Using (19) and (20) I obtain

$$f(\tilde{\eta}) = \frac{1}{1+\mathrm{SR}^2_{bh}} + \tilde{\eta}^2\left(R^{2T}\left(C_0 - V_0\right)^2 + \varepsilon^2_{0\xi}\right) - \frac{2\tilde{\eta}R^T\left(C_0 - V_0\right)}{1+\mathrm{SR}^2_{bh}}.$$

By virtue of (23), (26) and (9). First order conditions yield

$$\begin{aligned} \hat{\eta} &:= \arg\max_{\tilde{\eta}} f(\tilde{\eta}) = \frac{R^T\left(C_0 - V_0\right)}{R^{2T}\left(C_0 - V_0\right)^2 + \varepsilon^2_{0\xi}}\left(1+\mathrm{SR}^2_{bh}\right)^{-1}, \\ \max_{\tilde{\eta}} f(\tilde{\eta}) &= f(\hat{\eta}) = \frac{1}{1+\mathrm{SR}^2_{bh}} - \frac{R^{2T}\left(C_0 - V_0\right)^2}{\left(1+\mathrm{SR}^2_{bh}\right)^2\left(R^{2T}\left(C_0 - V_0\right)^2 + \varepsilon^2_{0\xi}\right)} \\ &= \frac{\left(1+\mathrm{SR}^2_{bh}\right)\left(1+\mathrm{SR}^2_{o\xi}\right) - \mathrm{SR}^2_{o\xi}}{\left(1+\mathrm{SR}^2_{bh}\right)^2\left(1+\mathrm{SR}^2_{o\xi}\right)}. \end{aligned}$$

By Theorem 2.1 the CEG of the quasi-optimal strategy with $R^T x\hat{\eta}/\tilde{\gamma}$ options satisfies

$$\tilde{\gamma}\mathrm{CEG}_{\tilde{\gamma},\widetilde{bh\xi}} = 1 - \sqrt{f(\hat{\eta})}. \tag{37}$$

On defining

$$\mathrm{SR}^2_{\widetilde{bh\xi}} := \frac{1}{f(\hat{\eta})} - 1 = \mathrm{SR}^2_{bh} + \mathrm{SR}^2_{o\xi}\left(1 + \frac{\mathrm{SR}^2_{o\xi}\mathrm{SR}^2_{bh}}{1+\mathrm{SR}^2_{bh}}\right)^{-1},$$

one concludes, similarly as in the proof of Theorem 5.1, that a complete market with Sharpe ratio $\mathrm{SR}^2_{\widetilde{bh\xi}}$ would give the same level of expected utility as the quasi-optimal strategy described above. ■

Proof. (**Theorem 6.2**) The excess return on α shares using buy and hold strategy equals

$$\alpha S_0(S_T/S_0 - R^T)$$

while the excess return on selling η options at price C_0 and hedging them locally optimally to maturity earns excess return of

$$\eta\left(G^{C_0,\xi}_T - H\right) = \eta\left(R^T\left(C_0 - V_0\right) + \sum_{t=1}^{T} R^{T-t} e_t\right).$$

By virtue of Lemma 8.1 the two returns are uncorrelated with mean and variance

$$\begin{aligned} \mu &= \begin{pmatrix} \mu^T - R^T \\ R^T\left(C_0 - V_0\right) \end{pmatrix}, \\ \Sigma &= \begin{pmatrix} (\mu^2+\sigma^2)^T - \mu^{2T} & 0 \\ 0 & \varepsilon^2_{0\xi} \end{pmatrix}. \end{aligned}$$

Theorem 2.2 then yields

$$\mathrm{SR}^2_{bh\xi} = \frac{\mu_1^2}{\sigma_{11}} + \frac{\mu_2^2}{\sigma_{22}} = \mathrm{SR}^2_{bh} + \mathrm{SR}^2_{o\xi},$$

and

$$\begin{aligned} \alpha_{bh\xi} S_0 &= \frac{\mu_1}{\sigma_{11}} \frac{1}{1 + \mathrm{SR}^2_{bh\xi}}, \\ \eta_{bh\xi} &= \frac{\mu_2}{\sigma_{22}} \frac{1}{1 + \mathrm{SR}^2_{bh\xi}}. \end{aligned}$$

■

References

Černý, A. (2003). Generalized Sharpe Ratios and asset pricing in incomplete markets. *European Finance Review* *7*(2), 191–233. Preprint ssrn.com//abstract=244731.

Černý, A. (2004). *Mathematical Techniques in Finance: Tools for Incomplete Markets*. Princeton University Press.

Černý, A. (2005). Optimal continuous-time hedging with leptokurtic returns. SSRN working paper, http://ssrn.com/abstract=713361. Accepted in *Mathematical Finance*.

Černý, A. and J. Kallsen (2005). On the structure of general mean–variance hedging strategies. SSRN working paper, http://ssrn.com/abstract=712743.

Cochrane, J. H. (2001). *Asset Pricing*. Princeton University Press.

Cvitanić, J., A. Lazrak, and T. Wang (2004). Sharpe ratio as a performance measure in a multi-period setting. Preprint.

Dowd, K. (1999). A value at risk approach to risk-return analysis. *Journal of Portfolio Management* *25*(4), 60–67.

Duffie, D. and H. Richardson (1991). Mean–variance hedging in continuous time. *Annals of Applied Probability* *1*, 1–15.

Dybvig, P. H. (1988). Inefficient dynamic portfolio strategies or how to throw away million dollars. *Review of Financial Studies* *1*(1), 67–88.

Dybvig, P. H. (1992). Hedging nontraded wealth: When is there separation of hedging and investment? In S. D. Hodges (Ed.), *Options: Recent Advances in Theory and Practice*, Volume 2, pp. 13–24. Manchester University Press.

Goetzmann, W. N., J. E. Ingersoll, Jr., M. I. Spiegel, and I. Welch (2002). Sharpening the Sharpe ratios. ICF Working Paper 02-08, Yale.

Grinold, R. C. and R. N. Kahn (1999). *Active Portfolio Management* (2nd ed.). McGraw-Hill.

Hansen, L. P. and R. Jagannathan (1991). Implications of security market data for models of dynamic economies. *Journal of Political Economy 99*(2), 225–262.

Leland, H. E. (1999). Beyond mean–variance: Performance measurement in a nonsymmetrical world. *Financial Analysts Journal 55*(1), 27–36.

Li, D. and W.-L. Ng (2000). Optimal dynamic portfolio selection: Dynamic multi-period formulation. *Mathematical Finance 10*(3), 387–406.

Liu, J. and J. Pan (2003). Dynamic derivative strategies. *Journal of Financial Economics 69*(3), 401–430.

Nielsen, L. T. and M. Vassalou (2004). Sharpe ratios and alphas in continuous time. *Journal of Financial and Quantitative Analysis 39*(1), 103–114.

Schweizer, M. (1994). Approximating random variables by stochastic integrals. *Annals of Probability 22*(3), 1536–1575.

Schweizer, M. (1996). Approximation pricing and the variance-optimal martingale measure. *Annals of Probability 24*(1), 206–236.