Les Cahiers du CREF ISSN: 1707-410X

The Optimality of Uniform Pricing in IPOs: An Optimal Auction Approach

Moez Bennouri
Sonia Falconieri

CREF 07-16

August 2007

The Optimality of Uniform Pricing in IPOs: An Optimal Auction Approach

Moez Bennouri

HEC Montréal and CREF

3000, chemin de la Côte-Sainte-Catherine
Montréal, Québec H3T 2A7

E-Mail : moez.bennouri@hec.ca

Sonia Falconieri

Brunel University and CREF

Brunel Business School
Brunel University, Uxbridge
Mididlesex, UB8 3PH

Corresponding Author : sonia.falconieri@brunel.ac.uk

August 2007

Les Cahiers du CREF
CREF 07-16

The Optimality of Uniform Pricing in IPOs: An Optimal Auction Approach

Abstract

This paper uses an optimal auction approach to investigate the conditions under which uniform pricing in IPOs is optimal. We show that the optimality of a uniform price in IPOs depends crucially on whether the (optimal) allocation rule is restricted. These restrictions may stem from the retail investors' budget constraint as well as from the institutional investors' preferences. We show that the main determinant of the optimality of a uniform pricing rule is the existence and the *shape* of the retail investors' budget constraint. Institutional investors' preferences are instead shown to mainly affect the optimal allocation rule.

JEL Classification: D8, G2

Keywords: Mechanism design; IPO; Uniform price; Allocation rule

Résumé

Ce papier utilise les enchères optimales pour analyser les conditions sous lesquelles une procédure d'introductions en Bourse optimale est implémentée avec une règle de prix uniforme. Nous montrons que l'optimalité du prix uniforme dans les introductions en Bourse dépend d'une façon cruciale des restrictions imposées sur l'allocation des nouveaux titres. Ces restrictions peuvent provenir de contraintes budgétaires sur les investisseurs de détails aussi bien que des préférences non linéaires des investisseurs institutionnels. Nous trouvons que l'existence et la forme de contraintes budgétaires pour les investisseurs de détails affectent d'une façon déterminante l'optimalité d'une règle de prix uniforme. Par contre, les préférences des investisseurs institutionnels n'affectent que la règle optimale d'allocation.

Mots clés : Design de mécanismes, Introductions en Bourse; prix uniforme; règle d'allocation

Acknowledgements: *A previous version of this paper was circulated under the title "Price vs Quantity Discrimination in Optimal IPOs" (Tinbergen Institute DP 83/1).We thank Ralph Bachmann, Bruno Biais, François Derrien, Mathias Dewatripont, Paolo Fulghieri, Eugene Kandel, Alessandro Lizzeri, David Martimort, Enrico Perotti, Jean-Charles Rochet, Jean Tirole, Dan Weaver, seminar and conference participants at Tilburg University, Ecole Polytechnique de Tunis, HEC Montréal, Zicklin School of Business (Baruch College, NY), the 1999 EEA Conference, the CEPR Conference on Corporate Finance in Courmayer, March 2001, the V SAET Conference in Ischia, July 2001 and the NFA meeting (Quebec, September 2003), for helpful comments. At the beginning of this project we greatly benefited from discussions with GianLuigi Albano. All remaining errors are ours.*

1 Introduction

The literature on Initial Public Offerings (IPOs) has grown remarkably over the past few decades. Most of this literature, both theoretical (e.g. Rock, 1986, Allen and Faulhaber, 1989, and Benveniste and Spindt, 1989) and empirical (e.g. Beatty and Ritter, 1986, Ritter 1987, and Cornelli and Goldreich, 2001), has focused on the explanation of a number of apparent market anomalies surrounding IPOs, such as underpricing, long-term underperformance and hot issues markets.[1] Much less attention has been devoted to understanding whether the current IPO format, which imposes uniform pricing, is actually efficient. Firms which go public are currently forbidden from using price discrimination; they can however quantity discriminate by choosing an allocation rule and potentially rationing some investors. Virtually all papers in the IPO literature, to be consistent with current practice, simply assume uniform pricing. The only paper addressing this issue (Benveniste and Wilhelm, 1990) challenges the efficiency of the current regulatory constraints, and suggests that firms could improve on their IPO performance (raise the IPO proceeds) if they were allowed to price discriminate across investors.

The objective of this paper is to investigate the conditions under which uniform pricing is optimal, where the optimality is defined from the point of view of an issuer who wants to maximize the sale's expected proceeds. Our results show that the optimality of uniform pricing depends on whether the (optimal) allocation rule is subject to some restrictions. If this is the case, then discriminatory pricing is required to elicit information from the informed investor. If this is not the case, i.e. if the issuer has enough control over the allocation rule, quantity discrimination alone suffices to achieve optimality. We consider two types of restrictions on the allocation rule. The first is exogenous and refers to allocation constraints on retail investors. The second endogenously results from the non-linear preferences of institutional investors. Our results indicate that only the retail investors' allocation constraint critically matters for the optimality of uniform pricing. We furthermore show that the optimality of uniform pricing depends on both the existence of a budget constraint on retail investors, and the *shape* of this

[1] Ritter and Welch (2002) provide a useful concise review of the IPO literature, whereas for a fully comprehensive review of the theory and evidence on IPO activity see Jenkinson and Ljungqvist (2001).

constraint, as this will define the extent to which the allocation rule is restricted. Although institutional investors' preferences do not critically impact on optimal pricing, they do affect the optimal allocation rule. We show that with risk-neutral preferences, the optimal allocation rule gives priority to retail investors, with institutional investors being residual claimants. The opposite holds when preferences are non-linear.

In our IPO a firm wants to place a fixed number of unseasoned shares. There are two potential groups of investors: n institutional investors and a continuum of retail investors. Each institutional investor receives a private signal about the value of the asset on sale, whereas both the issuer and the retail investors are uninformed. The seller designs the IPO mechanism in order to elicit the private information from institutional investors and maximize the expected IPO proceeds. We employ an optimal auction approach to derive the optimal IPO, which consists in identifying the optimal allocation and pricing rule. We solve two different models. In the first, institutional investors are assumed to be risk neutral in line with most of the IPO literature. In the second, institutional investors have preferences that are concave in quantity but may place a higher valuation on the shares than do retail investors. The non-linear preferences case represents the main novelty of the paper. Each of these models is then solved under three different assumptions about retail investors' buying capacities: (i) no constraint on retail investors; (ii) retail investors are quantity constrained, i.e. they can only buy up to a maximum number of shares; and (iii) retail investors have budget constraints, i.e. they only have a maximum amount of cash that they can spend.

We show that the optimal IPO cannot be implemented with a uniform pricing rule under the second assumption, i.e. when retail investors are quantity constrained. In (most) of the other cases, we can show that uniform pricing is optimal in equilibrium.[2] Another contribution of the paper is the full characterization of the optimal allocation rule in all of the different environments considered, with interesting implications for the use of rationing in equilibrium. The explanation behind these results is that quantity constraints impose tighter restrictions on

[2] The case of concave preferences and retail investors' budget constraints cannot be explicitly solved due to the complexity of the optimization program. However, we are able to obtain some interesting qualitative results.

the seller's ability to use quantity discrimination as a tool to elicit information from institutional investors, which renders the complementary use of price discrimination necessary to achieve optimality.

The only other paper in the IPO literature that has dealt with the issue of optimal pricing is Benveniste and Wilhelm (1990). Our paper however differs from theirs in many respects, the most important of which is the methodology used. We apply optimal auction theory, which enables us to characterize the optimal IPO mechanism in a very general environment with respect to both the informational structure and investors' characteristics. Our model with risk-neutral institutional investors is also related to Maksimovic and Pichler (2006), hereafter MP. More specifically, the results we obtain in the case of risk-neutral institutional investors and no constraints on retail investors are effectively a generalization of MP to a continuous state space. It should be noted, however, that the two papers do differ in a number of respects. First, MP do not address the issue of the optimal pricing rule in their paper. The optimality of uniform pricing in their paper is simply a consequence of an equilibrium with zero underpricing. Secondly, whereas allocation constraints in MP are all exogenously imposed, in our model they may effectively arise endogenously when institutional investors' preferences are non-linear.

Our work yields a number of empirical predictions.

- The most direct implication is that uniform pricing and discriminatory pricing will lead to the same IPO proceeds if the constraints on the allocation rule are not too tight. As discriminatory pricing is forbidden in most countries, a direct test of this prediction is only possible in those few countries where discriminatory pricing has been used in IPOs. For example, Jagannathan and Sherman (2006) document the use of discriminatory IPO auctions in Taiwan and Japan.[3]

- Our results further predict that we should observe greater underpricing in IPOs when there are tighter restrictions on the allocation rule, e.g. the requirement that a minimum

[3] Discriminatory auctions are pay-your-bid auctions. There is, however, no example of discriminatory pricing in bookbuilding.

quantity of shares be allocated to institutional investors, which effectively translates into a maximum quantity constraint on retail investors.[4]

- Understanding which type of constraint - cash or quantity constraint - is more relevant in practice is ultimately an empirical question, the answer to which can have different implications for the overall efficiency of the offering mechanism.

- The results on the optimal allocation rule predict that institutional investors who disclose better information about the value of the stock will receive more shares. Cornelli and Goldreich (2001) indeed find that informed investors who reveal more information are typically allocated a larger fraction of the shares.

The paper is organized as follows. In the next section we set up the model and derive the sellers' maximization problem for generic institutional investor preferences with no constraints on the retail investors. The following sections then solve the model under different assumptions regarding institutional investors' preferences and retail investors' constraints. To guide the reader through the different sections, these are summarized in the table below:

*RI Constraints\II Preferences**	*Linear*	*Non-Linear*
No constraints	Section 3	Section 4.1.
Quantity Constraint	Section 3	Section 4.2
Cash Constraint	Section 5.1	Section 5.2

* RI = retail investors; II = institutional investors.

The last section concludes with a discussion of the results. All proofs are in the Appendix.

2 The Model

A firm would like to sell Q shares in an IPO, with Q fixed and, without loss of generality, normalized to 1. An intermediary is in charge of marketing the new shares. He is assumed to act in the firm's best interest. We will simply hereafter refer to the *seller* to denote the

[4] This practice is supported by theoretical arguments (Stoughton and Zechner, 1998) in that the issuer wants to have a minimum institutional ownership because institutional investors tend to monitor the firm more closely. The empirical relevance is more difficult to prove, although it has been for instance observed in some OpenIPOs run by WR Hambrecht. (We thank the referee for providing us with this useful piece of information).

firm-intermediary coalition. This is a standard way of modelling the role of an intermediary in the IPO literature. The seller wishes to maximize the proceeds from the sale. He faces both $n(>2)$ large institutional investors who hold private information about the firm's market value, and a fringe of retail and uninformed investors.

Institutional investors have private information in that each agent i receives a signal s_i about the market value of the new shares. Signals are i.i.d. according to a uniform distribution defined on $\Omega_i = [\underline{s}; \overline{s}]$ with $\underline{s} > 0$, so the cumulative distribution function is $F_i(s_i) = \dfrac{s_i - \underline{s}}{\Delta s}$, with $\Delta s = \overline{s} - \underline{s}$, and the density function $f_i(s_i) = \dfrac{1}{\Delta s}$. Let us also denote by $f(s)$ the joint density function so that $f(s) = f(s_1, ..., s_n) = \prod_i f_i(s_i)$, with $s = (s_1, ..., s_n) \in \Omega = \bigotimes_i \Omega_i = [\underline{s}, \overline{s}]^n$.[5] Each signal received by an institutional investor represents a piece of information about the market value of the new shares. We therefore assume that the value of the new shares, v, is a function of the vector of signals received by institutional investors. More precisely, we assume that $v(\cdot)$ is a function defined over Ω and is given by the average of all the signals, that is

$$v(s) = \frac{1}{n} \sum_i s_i. \tag{1}$$

The above function has two main properties: *a)* $\frac{\partial v(s)}{\partial s_i} > 0$, i.e. the asset value is increasing in each signal, and *b)* $\frac{\partial v(s)}{\partial s_i} = \frac{\partial v(s)}{\partial s_j}$ for any $i \neq j$, i.e. signals are equally weighted in the valuation function. This kind of informational structure is very common in auction theory (e.g. Bulow and Klemperer, 1996 and 2002) and is a straightforward generalization of the simple binomial informational structure adopted in other IPO papers (e.g. Benveniste and Wilhelm, 1990; Biais and Faugeron-Crouzet, 2002) to a continuous signal space.[6] It can furthermore be proved that our results hold for any generic valuation function $v(s) = \Psi(s)$ which satisfies properties *(a)* and *(b)* above (i.e. $\frac{\partial \Psi(s)}{\partial s_i} > 0$ and $\frac{\partial \Psi(s)}{\partial s_i} = \frac{\partial \Psi(s)}{\partial s_j}$ for any $i \neq j$). For the sake of tractability, we prefer a simple specification.

[5] Note that the uniform distribution of private signals satisfies the *increasing hazard rate assumption.* Indeed, we have that for the uniform distribution $\dfrac{\partial}{\partial s_i}\left[\dfrac{f_i(s_i)}{1 - F_i(s_i)}\right] = \dfrac{\partial}{\partial s_i}\left[\dfrac{1}{(\overline{s} - s_i)}\right] \geq 0$ for all i and all s_i. We show later in the paper that several of our results (in particular those in section 3) do not change qualitatively if we consider a more general distribution of signals satisfying the increasing hazard rate assumption.

[6] These papers assume that the signal investors receive can be either good or bad and the stock market value is monotonic in the number of good signals; we have the same but we use a continuum of signals.

Given a vector of signals s, each institutional investor i's preferences are given by the following utility function:

$$u_i(p_i, q_i, v(s)) = z(q_i, v(s)) - q_i p_i \qquad \text{for all } i \in \{1, 2, ...n\} \qquad (2)$$

where q_i is the quantity assigned to investor i and p_i is the price per share (s)he has to pay. We denote by $T_i = p_i q_i$ the total payment from investor i to the seller. The utility function in Equation (2) is linear in the transfer T_i. We make the following assumptions about the institutional investors' valuation function z (the subscripts denote derivatives with respect to variables):

A1 $z_1 > 0$ and $z_2 > 0$;

A2 $z_{11} \leq 0$;

A3 $z(0, v) = 0$ for all v;

A4 $z_{12} > 0$ (*single-crossing condition*);

A5 $z_{122} \leq 0$ and $z_{112} \geq 0$;

A6 $z_{12}(0, v) \geq 1$.[7]

We assume a continuum of competitive, uninformed and risk-neutral *retail* bidders. The total mass of these retail bidders is normalized to one. In the following sections, we will consider the possibility that these investors face an allocation constraint, with the constraint taking a number of different forms. Neither retail investors nor the seller hold any private information about the market value of the asset and only observe the density $f(\cdot)$ of signals.

In order to extract the information from the institutional investors, the seller designs a mechanism specifying the allocation and pricing rules for both institutional and retail investors. By using the *revelation principle,* we can focus on *direct mechanisms* in which the seller asks each institutional investor to announce his signal and then fixes quantities and prices as function

[7] Most of these assumptions are standard in the mechanism design literature. See Fudenberg and Tirole (1991) for a discussion.

of their announcements in such a way as to induce them to reveal their information truthfully (see Fudenberg and Tirole, 1991).

A mechanism is described by a pair of outcome functions (p, q) of the form $p : \Omega \rightarrow \mathbb{R}^{n+1}$ and $q : \Omega \rightarrow [0, 1]^{n+1}$ where $p = (p_1, ..., p_n, p_R)$ is a vector of prices and $q = (q_1, ..., q_n, q_R)$ is a vector of allocations. Thus if s is the vector of signals announced by institutional investors, each investor $i \in \{1, ..., n\}$ receives $q_i(s)$ shares and pays a price per share of $p_i(s)$, while retail investors receive $q_R(s)$ and pay a price per share of $p_R(s)$. We assume that all of the shares issued must be allocated to either institutional or retail investors. The seller's choice of the vector $\{q_i\}_{i=1,..n}$ implicitly determines the number of shares allocated to retail investors, q_R, which is given by:

$$q_R = 1 - \sum_i q_i \tag{3}$$

The set of possible strategies for the institutional investor i with signal s_i is Ω_i. Faced with a mechanism (p, q), his expected utility if he misrepresents his signal by announcing $\widehat{s_i}$ to the seller rather than his true signal s_i is

$$\begin{aligned} U_i(\widehat{s_i}, s_i) &= E_{s_{-i}}\{u_i(p_i(\widehat{s_i}, s_{-i}), q_i(\widehat{s_i}, s_{-i}), v(s_i, s_{-i}))\} \\ &= \int_{\Omega_{-i}} [z(q_i(\widehat{s_i}, s_{-i}), v(s_i, s_{-i})) - q_i(\widehat{s_i}, s_{-i})p_i(\widehat{s_i}, s_{-i})]\, f_{-i}(s_{-i})ds_{-i} \end{aligned} \tag{4}$$

where s_{-i} is the vector of all of the other institutional investors' signals, i.e. $s_{-i} = (s_1, .., s_{i-1}, s_{i+1}, ..., s_n)$ and $f_{-i}(s_{-i}) = \prod_{j \neq i} f_j(s_j)$.[8]

The optimal IPO mechanism for the seller is the solution to the following optimization program:

$$\max_{(p,q)} U_F = E_s\left[\sum_i T_i(s) + T_R(s)\right] = \int_\Omega \left(\sum_i T_i(s) + T_R(s)\right) f(s)ds \tag{5}$$

subject to the following standard constraints:

- *Retail Investors' Participation Constraint* (RPC). This requires the expected payoff for

[8]Note that the expected utility of agent i depends on the mechanism offered by the seller (p, q), which is omitted in our notation for the sake of simplicity.

retail investors to be greater than their reservation utility, which equals zero.

$$E_s\left[q_R(s)(v(s) - p_R(s))\right] = \int_{\Omega} q_R(s)\left[v(s) - p_R(s)\right] f(s)ds \geq 0 \qquad \text{(RPC)}$$

Thus the constraint is conditional on the initial distribution of the signals which implies that retail investors committ to buying the share without ever learning the reported signals. This in turn implies that they do not play strategically in the IPO game.

- *Institutional Investors' Participation Constraint* (IPC). The IPC ensures that each institutional investor is willing to participate in the offering conditional on his own signal. The expected utility of each institutional investor conditional on his signal should be greater than his expected utility when he does not participate in the IPO. The IPC is then written as follows:

$$\begin{aligned} U_i(s_i, s_i) &= E_{s_{-i}}\left\{z(q_i(s), v(s)) - q_i(s)p_i(s)\right\} \\ &= \int_{\Omega_{-i}} \left[z(q_i(s), v(s)) - p_i(s)q_i(s)\right] f_{-i}(s_{-i})ds_{-i} \geq 0 \qquad \text{(IPC)} \end{aligned}$$

This must be satisfied for all i and all s_i.

- *Institutional Investors' Incentive Compatibility Constraint* (IIC). This constraint ensures that each institutional investor has no incentive to misrepresent his type - the signal he receives - to the firm. The IIC then requires that each agent i be better off by truthfully announcing his signal. Using Equation (4) this may written as follows:

$$U_i(\widehat{s}_i, s_i) \leq U_i(s_i, s_i) \qquad \text{for all } s_i, \widehat{s}_i \text{ and } i, \qquad \text{(IIC)}$$

or, equivalently,

$$s_i \in \arg\max_{\widehat{s}_i} U_i(\widehat{s}_i, s_i) \qquad \text{for all } s_i, \widehat{s}_i \text{ and } i. \qquad \text{(IICa)}$$

- *Full Allocation Constraint* (FAC).

$$\sum_i q_i(s) + q_R(s) = 1 \text{ for all } s \qquad \text{(FAC)}$$

and the quantity non-negativity constraints:

$$q_i(s) \geq 0 \qquad \text{for all } i \text{ and } s. \qquad (6)$$

Finally, we will also introduce when appropriate the retail investors' allocation constraint. The above optimization program can be simplified by re-arranging the constraints. First, from the RPC and the maximand we can see that the seller's profit is increasing in the retail investors' payments, therefore at the optimum the RPC binds. We can then rewrite the RPC as follows:

$$\int_\Omega p_R(s)q_R(s)f(s)ds = \int_\Omega v(s)q_R(s)f(s)ds = \int_\Omega v(s)\left(1-\sum_i q_i(s)\right)f(s)ds \tag{7}$$

where we have replaced $q_R(s)$ from the FAC. We now turn to the IIC. By applying the envelope theorem to the maximization problem in Equation (IICa), we have:[9]

$$U_i'(s_i,s_i) = \frac{1}{n}\int_{\Omega_{-i}} z_2(q_i(s),v(s))f_{-i}(s_{-i})ds_{-i}. \tag{8}$$

Taking expectations over s_i yields

$$U_i(s_i,s_i) = U_i(\underline{s},\underline{s}) + \int_{\underline{s}}^{s_i}\left\{\frac{1}{n}\int_{\Omega_{-i}} z_2(q_i(\widetilde{s}_i,s_{-i}),v(s))f_{-i}(s_{-i})ds_{-i}\right\}d\widetilde{s}_i. \tag{9}$$

Since the seller's payoff is decreasing in the information rents paid to the informed investors, at the optimum he will set $U_i(\underline{s},\underline{s})=0$, so that the informed investors with the lowest evaluations receive zero rents at the optimum. Inverting integrals, taking the expectation over s_i and applying Fubini's theorem[10] to Equation (9) yields

$$\int_{\Omega_i} U_i(s_i,s_i)f_i(s_i)ds_i = \frac{1}{n}\int_{\Omega_{-i}}\left\{\int_{\Omega_i}\left[\int_{\underline{s}}^{s_i} z_2(q_i(\widetilde{s}_i,s_{-i}),v(s))d\widetilde{s}_i\right]f_i(s_i)ds_i\right\}f_{-i}(s_{-i})ds_{-i}. \tag{10}$$

Integration by parts of the integral $\int_{\Omega_i}\left[\int_{\underline{s}}^{s_i} z_2(q_i(\widetilde{s}_i,s_{-i}),v(s))d\widetilde{s}_i\right]f_i(s_i)ds_i$ yields the following

$$\begin{aligned}\int_{\Omega_i} U_i(s_i,s_i)f_i(s_i)ds_i &= \frac{1}{n}\int_\Omega\left[\frac{1-F_i(s_i)}{f_i(s_i)}\right]z_2(q_i(s),v(s))f(s)ds \\ &= \frac{1}{n}\int_\Omega(\overline{s}-s_i)z_2(q_i(s),v(s))f(s)ds.\end{aligned} \tag{11}$$

Finally, consider the IPC, which can be rewritten as follows for all i and all s_i

$$\int_{\Omega_{-i}} p_i(s)q_i(s)f_{-i}(s_{-i})ds_{-i} = \int_{\Omega_{-i}} z(q_i(s),v(s))f_{-i}(s_{-i})ds_{-i} - U_i(s_i,s_i) \geq 0 \tag{12}$$

[9] Notice that the equation below holds at the optimum, i.e. for $\widehat{s}_i = s_i$.

[10] Fubini's theorem states that we can invert integrals whenever the integrand is finite.

Taking expectations over s_i and using Equations (11) and (7) we can rewrite the seller's objective function as

$$\max_{\{q_i\}_1^n} \int_\Omega \left\{ v(s) + \sum_i \left[z(q_i(s), v(s)) - \frac{1}{n}(\overline{s} - s_i) z_2(q_i(s), v(s)) - v(s) q_i(s) \right] \right\} f(s) ds. \tag{13}$$

To be sure that truth-telling is indeed optimal, we need to check that the second-order conditions for the (IICa) equation are met. To do so, it is useful to rewrite the problem as follows

$$s_i \in \arg\min_{\widehat{s}_i} \left[U_i(\widehat{s}_i, \widehat{s}_i) - U_i(s_i, \widehat{s}_i) \right] \qquad \text{for all } s_i \text{ and } i. \tag{14}$$

Using assumptions 4 and 5, it is easy to check that a sufficient condition for truth-telling to be optimal is that the following monotonicity condition holds[11]

$$\frac{1}{n} E_{s_{-i}} \left[z_{12}(q_i(s), v(s)) \frac{\partial q_i(s)}{\partial s_i} \right] \geq 0 \tag{MC}$$

Summing up, optimal allocations are those resulting from the following optimization program:

$$\begin{cases} \max_{\{q_i\}_1^n} \int_\Omega \left\{ v(s) + \sum_i \left[z(q_i(s), v(s)) - \frac{1}{n}(\overline{s} - s_i) z_2(q_i(s), v(s)) - v(s) q_i(s) \right] \right\} f(s) ds \\ s.t: \\ \text{(i)} \quad U_i(\underline{s}, \underline{s}) = 0 \qquad \text{for all } i \\ \text{(ii)} \quad \frac{1}{n} E_{s_{-i}} \left[z_{12}(q_i(s), v(s)) \frac{\partial q_i(s)}{\partial s_i} \right] \geq 0 \qquad \text{for all } i \text{ and all } s_i \\ \text{(iii)} \quad q_i(s) \geq 0 \qquad \text{for all } i \text{ and all } s \\ \text{(iv)} \quad \sum_i q_i(s) \leq 1 \qquad \text{for all } s. \end{cases} \tag{$\mathcal{P}$}$$

Note that this program depends only on quantities. Once optimal quantities have been determined, optimal prices can be obtained from the participation constraints of both the institutional and the retail investors. Specifically, for institutional investors, plugging Equation (9) into Equation (12) yields the following price equation for all s_i and all i

$$\begin{aligned} \int_{\Omega_{-i}} p_i(s) q_i(s) f_{-i}(s_{-i}) ds_{-i} &= \int_{\Omega_{-i}} z(q_i(s), v(s)) f_{-i}(s_{-i}) ds_{-i} - \\ &\quad \int_{\Omega_{-i}} \left\{ \frac{1}{n} \int_{\underline{s}}^{s_i} z_2(q_i(\widetilde{s}_i, s_{-i}), v(s)) d\widetilde{s}_i \right\} f_{-i}(s_{-i}) ds_{-i} \end{aligned} \tag{15}$$

[11] In the mechanism design literature, the monotonicity condition is often referred to as an *implementability condition.*

Likewise, for retail investors, the optimal pricing rule must satisfy the (binding) participation constraint

$$\int_{\Omega} p_R(s)q_R(s)f(s)ds = \int_{\Omega} v(s)q_R(s)f(s)ds. \tag{16}$$

The seller's program in this set-up is quite different from that in a standard auction design problem where an uninformed seller faces usually only informed bidders. The participation of a class of uninformed bidders in the auction makes the problem rather different and interesting in the sense that it mitigates the adverse-selection problem and, thus, lowers the seller's cost of extracting the informed investors' private information.[12] What really matters however is not that retail investors are uninformed but rather that their information, if any, differs from that of the institutional investors and, more importantly, that they do not play strategically, i.e. it is impossible to elicit their information. This point has previously been made by Maksimovic and Pichler (2006).

Finally, note that once the optimal allocation(s) solving the program ($\mathcal{P}$) is specified, the seller has enough discretion in setting optimal pricing rule(s) to satisfy equations (15) and (16). For each defined allocation rule, prices must satisfy a set of integral equations for which there may exist a large set of potential solutions. In the following sections, we will consider different cases for which we can specify the allocation rule. Our main concern will then be to see whether a pricing rule satisfying equations (15) and (16) is uniform in the sense that all institutional investors and retail investors pay the same unit price.

3 Risk-neutral institutional investors

We start by analyzing the standard case in the IPO literature of risk-neutral institutional investors. We thus assume that the informed investors' valuation function is

$$z(q, v(s)) = v(s)q \tag{17}$$

[12] In a very similar framework, Malakhov (2006) investigates the impact of retail investors' participation on the issuer's revenues. It is shown that the seller's revenues are increasing in the the number of uninformed investors participating in the offering, as more uninformed investors lowers the outside option of informed investors and, consequently, as in our model, reduces the cost of gathering information.

and the utility function of investor i is

$$u_i(v(s), q_i, p_i) = [v(s) - p_i]\, q_i \tag{18}$$

We also assume that retail investors are quantity constrained, i.e. they cannot buy the whole quantity. Let $K < 1$ be the maximum quantity of shares they can buy. The quantity constraint is then

$$q_R(s) \leq K. \tag{19}$$

Notice that this constraint is exactly equivalent to requiring that a minimum quantity of shares be allocated to institutional investors. In other words we could just as well write it as $\sum_i q_i(s) \geq 1 - K$, which represents a quite common, though implicit, practice in IPOs. There may be several reasons why this is the case: due to the monitoring role played by institutional shareholders which potentially enhances the firm value (Mello and Parsons (1998), Stoughton and Zechner (1998)), or because of the tight links with the underwriter who tends in turn to favor his institutional clientele over retail demand in IPOs (Aggarwal, 2003, and Aggarwal et al., 2002).

In this case the seller's problem is the following:

$$\begin{aligned}
&\max_{\{q_i\}_1^n} \int_\Omega \left\{ v(s) - \textstyle\sum_i \left[\frac{1}{n}(\overline{s} - s_i) q_i(s) \right] \right\} f(s) ds \\
&s.t: \\
&(i) \quad U_i(\underline{s}, \underline{s}) = 0 \qquad \text{for all } i \\
&(ii) \quad E_{s_{-i}} \left[\frac{\partial q_i(s)}{\partial s_i} \right] \geq 0 \qquad \text{for all } i \text{ and all } s_i \\
&(iii) \quad q_i(s) \geq 0 \qquad \text{for all } i \text{ and all } s \\
&(iv) \quad \sum_i q_i(s) \leq 1 \qquad \text{for all } s. \\
&(v) \quad q_R(s) \leq K \qquad \text{for all } s
\end{aligned} \tag{$\mathcal{P}1$}$$

In the next proposition we characterize the optimal IPO mechanism in terms of allocation and pricing rules:

Proposition 1 *If institutional investors are risk neutral and retail investors can buy at most K shares, the optimal IPO is characterized as follows:*

1. ***(Allocation rule)*** *For all* $s \in \Omega$, *let* $s_m = \max\{s_1, ..., s_n\}$. *In equilibrium, the issuer allocates as many shares as possible to retail investors and the remaining shares to the institutional investor reporting* s_m, *that is* $q_R(s) = K$ *and* $q_m(s) = 1 - K$.[13]

2. ***(Pricing rule)*** *The optimal IPO requires discriminatory pricing.*

The above result regarding allocation can be shown to hold for any distribution function which satisfies the increasing hazard rate condition. To see why, note that from Equation (11) for a generic distribution of institutional investors' signals, the seller's objective function becomes

$$\int_{\Omega} \left\{ v(s) - \sum_{i} \left[\frac{1}{n} \left[\frac{1 - F_i(s_i)}{f_i(s_i)} \right] q_i(s) \right] \right\} f(s) ds. \tag{20}$$

As in the uniform distribution case, this function is decreasing in $q_i(s)$ and the information rents paid to institutional investors, now measured by the term $\int_{\Omega} \left\{ \sum_i \left[\frac{1}{n} \left[\frac{1 - F_i(s_i)}{f_i(s_i)} \right] q_i(s) \right] \right\} f(s) ds$, can be shown to be decreasing in s_i for any distribution satisfying the increasing hazard rate condition. These properties together ensure that the seller optimally allocates as much as possible to retail investors, i.e. up to their quantity constraint, with any residual quantity going to the institutional investor(s) with the highest reported signal.[14]

It is of interest to note that, even with institutional investors having linear preferences, a (maximum) quantity constraint on retail investors will prevent uniform pricing from being optimal in equilibrium. The institutional investor(s) receiving a positive quantity in this mechanism will pay on average a lower price than retail investors in order to incite him to report his true signal. To see this, consider the allocation rule in Proposition 1 and apply it to the participation constraint of retail investors and the IICs of institutional investors. After some manipulation, this yields

$$\int_{\Omega} \left[p_R(s) - p(s) \right] f(s) ds = \frac{1}{1-K} \int_{\Omega} \left\{ \sum_{i} \frac{1}{n} \left[\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i}) d\widetilde{s}_i \right] \right\} f(s) ds \tag{21}$$

[13]Note that as a result of the continuous distributions we consider in our model, the probability of having more than one agent announcing s_m is zero.

[14]It should be clear that the assumption of an increasing hazard rate is crucial for this result because it ensures that information rents increase in signals. Different assumptions regarding the behaviour of the hazard rate would of course affect the optimal allocation rule.

where $p(s)$ is the price paid by the institutional investor.[15] Since the right-hand side term of this equation is strictly positive, the average price paid by retail investors is higher than the price paid by the institutional investor.

In the case of no quantity constraints on retail investors, the result below follows directly from Proposition 1:

Corollary 1 *When informed investors are risk neutral and retail investors are not constrained, then the optimal offering is such that all the shares are sold to retail investors irrespective of the signal reported by the informed investors, at a marginal price of* $p_R = E(v(s))$.

This result is a generalization of MP to a continuous state space and implies that risk-neutral institutional investors receive no information rents because, in equilibrium, they are actually excluded from the offering. These results can be shown to hold for all distributions satisfying the increasing hazard rate condition.

With respect to this section's results, it is important to note that retail investors have priority in the allocation. This highlights the important role that uninformed investors play in the IPO process. Their presence mitigates the adverse-selection problem vis-a-vis the institutional investors, and consequently allows the seller to lower the cost of eliciting their private information by exploiting the competition between the two groups of investors. If retail investors have unlimited buying capacity, then the cost of information gathering is zero (Corollary 1).[16]

We could reinterpret the above two results in terms of a minimum quantity constraint on institutional investors. Proposition 1 then tells us that, in equilibrium, the issuer will make the constraint just binding. This, together with Corollary 1, leads us to conclude that such constraints are in fact detrimental to the efficiency of the selling mechanism.

In reality, however, we never observe an IPO in which all of the shares are placed with retail investors. In the light of this empirical regularity, our results then suggest that: a) the most relevant empirical situation is that of maximum (minimum) quantity constraints on retail

[15] Note that since there is only one institutional investor who receives a positive quantity we can apply the same price for all institutional investors without affecting the IICs.

[16] The role of retail investors has been already stressed by Benveniste et al. (1996), Biais and Faugeron-Crouzet (2002), Maksimovic and Pichler (2006) and Bennouri and Falconieri (2006).

(institutional) investors; or b) institutional investors' preferences may not be linear. In the next section, we look at the impact of institutional investors with non-linear preferences on our results.

4 Non-Linear Preferences

The previous section suggested that the optimality of uniform pricing essentially depends on whether the seller can freely use the allocation rule to discriminate among investors. It is then natural to ask whether introducing risk aversion or some kind of concavity in institutional investors' preferences may also affect optimal pricing, to the extent that this reduces the seller's discretion in allocating shares. We therefore relax the risk-neutrality assumption in this section, and introduce concavity in the institutional investors' valuation function, which becomes:[17]

$$z(q_i, v(s)) = q_i(\alpha v(s) - \frac{\delta}{2} q_i). \tag{22}$$

When $\alpha \geq 1$, this new valuation function $z(\cdot, \cdot)$ is concave in quantity, satisfies Assumptions A1-A6 and produces a marginal valuation which is decreasing at a constant rate, δ.[18] The reason we consider this specification is that concavity indirectly restricts the discretion of the seller to allocate the shares, thereby making it easier to link the optimality of uniform pricing to the existence of restrictions on the allocation rule. Further, concavity in quantity may be interpreted as aversion to *inventory risk,* i.e. the risk associated with portfolio composition.[19] We also assume $\alpha > 1$, which means that institutional investors value the shares more than do retail investors for small quantities, but with a decreasing marginal valuation. In other words, institutional investors are very keen to get into the sale and obtain a positive quantity of shares but their marginal utility decreases as the allocation increases, possibly due to inventory risk. If there is no cash constraint on retail investors, the optimal allocation rule with $\alpha = 1$ is the

[17] Risk neutrality is the most common assumption in the IPO literature. Other papers that have, like us, assumed non-linear preferences are Stoughton and Zechner (1998) and Benveniste and Wilhelm (1997).

[18] Notice that the new valuation function does not describe standard risk-averse preferences, which are typically concave in wealth.

[19] This is well-known in the market microstructure or foreign exchange market literatures (see for instance O'Hara, 1995, and Lyons, 2003).

same as that under risk neutrality, i.e. the seller allocates the entire quantity to retail investors. In this case institutional investors' risk aversion does not matter.

We first analyze, in the next section, the optimal mechanism in the absence of constraints on retail investors in order to isolate the impact of institutional investors' non-linear preferences. We subsequently add the retail investors' constraint.

4.1 No Constraints on Retail Investors

The seller's optimization program in this case can be written as:

$$\begin{aligned}
&\max_{\{q_i\}_1^n} \int_\Omega \left\{ v(s) + \textstyle\sum_i \left[q_i(s) \left((\alpha - 1)\, v(s) - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) \right) \right] \right\} f(s) ds \\
&s.t: \\
&(i) \quad U^i(\underline{s}, \underline{s}) = 0 \\
&(ii) \quad E_{s_{-i}} \left[\frac{\partial q_i(s)}{\partial s_i} \right] \geq 0 \qquad \text{for all } i \text{ and all } s \\
&(iii) \quad q_i(s) \geq 0 \qquad \text{for all } i \text{ and all } s \\
&(iv) \quad \sum_i q_i(s) \leq 1 \qquad \text{for all } s.
\end{aligned} \tag{$\mathcal{P}2$}$$

The next proposition derives the optimal allocation rule.

Proposition 2 *When institutional investors have non-linear preferences and allocation to retail investors is not constrained, the optimal IPO is characterized by the following allocation rule: there exists a threshold value of the signals* $s_i^\circ = \frac{\alpha\overline{s} - (\alpha-1)v_{-i}}{2\alpha - 1}$, *with* $v_{-i} = \sum_{j \neq i} s_j$, *such that:*

- *If all the informed investors report signals below this threshold, all the shares are awarded to the uninformed investors; otherwise*

- *All the institutional investors reporting a signal below* s_i° *obtain nothing, whereas the others receive a positive quantity which is equal to*

  – $\widetilde{q}_i = \dfrac{(\alpha-1)\, nv(s) - \alpha(\overline{s} - s_i)}{n\delta}$, *if the issue is not oversubscribed, i.e.* $\sum_i \widetilde{q}_i \leq 1$, *with retail investors receiving the remaining shares;*

– $\widehat{q}_i < \widetilde{q}_i$ *if the issue is oversubscribed, with* $\widehat{q}_i(s) = \widetilde{q}_i(s) - \mathcal{Q}$*, where* $\mathcal{Q}$ *is the amount of shares by which they are rationed, which is given by* $\mathcal{Q} = \frac{\beta(s)}{\delta}$ *where* $\beta(s)$ *and* δ *are the Kuhn-Tucker multipliers. Retail investors obtain no shares.*

We can check that the threshold s_i° is decreasing in the parameter α. This offers an interesting interpretation of the above result. When institutional investors' preferences are such that $\alpha = 1$, the threshold s_i° equals $\overline{s}$ and we obtain the same results as in the case of risk neutrality. As α increases, the threshold decreases because the marginal valuation of the asset for an institutional investor at $q_i = 0$ is now $\alpha v(s)$ whereas the marginal valuation of retail investors is $v(s)$. As long as the threshold is below $\overline{s}$, institutional investors receive an information rent. However, because the issuer can always sell the shares to retail investors at the price of $v(s)$, he does not need to pay excess rents to the institutional investors. We show in the next proposition that this price can be uniform. Specifically, the institutional allocation rule is such that the optimal quantity $\widetilde{q}_i(s)$ is increasing in the signal s_i reported by the institutional investor. In other words, the seller rewards better information about the stock value (i.e. higher signals) with a larger quantity. This is in line with the existing literature on bookbuilding (Benveniste and Spindt (1989) and Cornelli and Goldreich (2001)), which shows that institutions receive more shares in "hot" issues, i.e. IPOs with strong pre-market demand. Notice also that we are able to endogenously derive the optimal rationing rule which requires that all institutional investors be rationed by the same quantity of shares. Thus, the rationing rule preserves the monotonicity of the quantity schedule, thereby allocating more shares to investors reporting higher signals.

We can however show, that despite institutional investors' risk-aversion, the optimal IPO can be implemented by an uniform pricing schedule as in the standard case. The result is stated in the next proposition.

Proposition 3 *When institutional investors have non-linear preferences and retail investors are not constrained, then there exists at least one uniform pricing rule that implements the optimal IPO mechanism.*

To give an idea of the reasoning behind the proof of this proposition, first recall that the optimal prices for institutional and retail investors must satisfy, respectively, Equations (15) and (16). The proof unfolds in two steps. The first step consists in proving a preliminary result, i.e. we show the existence of a uniform price schedule satisfying Equation (15) for all institutional investors. We next show that the same price schedule also solves the retail investors' participation constraint. It is worth noting that the optimal uniform price is not unique. In other words there may be more than one uniform price which implements the optimal mechanism.

This result shows that the assumption of non-linear preferences does not affect the optimality of uniform pricing. It does introduce an endogenous constraint on the allocation rule, which is however not tight enough to compromise the optimality of uniform pricing. Overall this suggests that institutional investors' preferences do not matter for optimal pricing, although they do of course affect optimal allocations. This issue will be explored in more detail in the next section.

Comparing our result to that of Benveniste and Wilhelm (1990), we note that, like us, they assume that retail investors are able to buy all the shares on sale, and that institutional investors also have a certain type of concave preferences. However, institutional investors' preferences are such that they are risk neutral up to a quantity $\overline{q}$ and infinitely risk averse above this threshold. This is essentially equivalent to constraining the seller to allocate them at most $\overline{q}$. This, in our opinion, explains why price discrimination is required in their case to elicit information from institutional investors and achieve optimality.[20]

As in the case of risk neutrality, the next subsection considers constrained retail investors. Intuitively, we expect to find that if the optimal IPO cannot be implemented with uniform pricing under risk-neutral preferences, this a fortiori will equally not be the case under non-linear preferences. Our results formalize this intuition.

[20] Brisley (2003) in a note shows that Benveniste and Wilhelm's result only holds when regular investors are, on average, *less* informed than retail investors. Otherwise, the uniform price restriction does not affect IPO proceeds.

4.2 Quantity Constraints

Let us now consider a quantity constraint as defined in Equation (19); the seller's optimization problem then becomes

$$\max_{\{q_i\}_1^n} \int_\Omega \left\{ v(s) + \sum_i \left[q_i(s) \left((\alpha - 1)\, v(s) - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n} (\overline{s} - s_i) \right) \right] \right\} f(s) ds$$

$$\begin{aligned}
&s.t: \\
&(i) \quad U^i(\underline{s}, \underline{s}) = 0 \\
&(ii) \quad E_{s_{-i}} \left[\frac{\partial q_i(s)}{\partial s_i} \right] \geq 0 \qquad \text{for all } i \text{ and all } s \\
&(iii) \quad q_i(s) \geq 0 \qquad \text{for all } i \text{ and all } s \\
&(iv) \quad \sum_i q_i(s) \leq 1 \qquad \text{for all } s. \\
&(v) \quad \sum_i q_i(s) \geq 1 - K \qquad \text{for all } s
\end{aligned} \tag{$\mathcal{P}3$}$$

The optimal IPO is then as follows:

Proposition 4 *When institutional investors have non-linear preferences and retail investors are quantity constrained, the optimal IPO is characterized by the following allocation and pricing rules:*

1. *(**Allocation Rule**) there exists a threshold value of the signals* $s_i^\circ = \frac{\alpha \overline{s} - (\alpha-1) v_{-i}}{2\alpha - 1}$, *with* $v_{-i} = \sum_{j \neq i} s_j$, *such that,*
   - *Informed investors reporting the lowest signal* $\underline{s}$ *receive no shares;*
   - *All institutional investors reporting a signal above the threshold obtain a positive quantity* $\widetilde{q}_i(s) = \dfrac{(\alpha - 1)\, nv(s) - \alpha\, (\overline{s} - s_i)}{n\delta}$, *provided that* $1 - K \leq \sum_i \widetilde{q}_i(s) \leq 1$ *with the remaining shares being allotted to retail investors.*
   - *If* $\sum_i \widetilde{q}_i(s) < 1 - K$, *then* K *shares are allocated to retail investors and all institutional investors with a signal above the lowest level,* $\underline{s}$, *obtain a positive quantity of shares* $\widetilde{q}_i^K(s) \neq \widetilde{q}_i(s)$;

- *In the case of oversubscription, all institutional investors with signals above the threshold are rationed, whereas retail investors receive nothing.*

2. *(**Pricing Rule**) The optimal pricing rule is such that*

   - *If the retail investors' allocation constraint is not binding, the optimal IPO can be implemented by uniform pricing;*
   - *Conversely, if the constraint binds, then a discriminatory pricing rule is optimal, i.e.* $p_i(v) = p_I$ *for all* i *with* $p_I \neq p_R$.

The allocation rule result is quite intuitive. In contrast to the "no quantity constraint" case, there are now more restrictions on the allocation rule, which forces the seller to allocate the informed investors more shares than would otherwise be optimal. Specifically, even institutional investors with relatively bad signals, i.e. signals below the threshold, now receive a positive quantity of shares.

The proof of the pricing rule is very similar to that in Proposition 3, and is therefore omitted. If the retail investors' constraint does not bind in equilibrium, we are back to the case of no constraints, where we know that a uniform price is optimal. However, discriminatory pricing becomes optimal when the constraint binds. Also, note that even in the case of a discriminatory price all of the institutional investors pay the same price. The reason for this is that institutional investors are already discriminated against by the quantity so there is no need to further discriminate by price. Therefore, uniform pricing is applied within each group of investors.

An alternative reading of this result is that, where parameter K to be determined endogeneously, it would be optimally set so that the constraint never binds.

5 Cash-constrained retail investors

The results of the previous sections suggest that the optimal pricing scheme crucially depends on whether retail investors are constrained or not. Conversely, institutional investors' preferences do not seem to play a major role. We can also think about different types of constraint. For instance, retail investors may want to invest a maximum amount of money in the IPO (for risk diversification purposes for instance) and consequently be subject to a cash constraint, i.e. they can afford/are willing to invest at most K in the sale:

$$q_R(s)p_R(s_i, s_{-i}) \leq K \tag{23}$$

We are aware that with a continuum of retail investors, it is less plausible to justify the existence of such a constraint at the aggregate level, even though the constraint holds individually. However, the purpose of our analysis is to investigate whether different kinds of allocation constraints on retail investors have different impact on the optimal pricing rule. In other words, we want to understand whether it is just the existence of such a constraint that matters or also its shape (the functional form). We expect this to be the case, as different forms of the retail investors' constraint will determine the tightness of the allocation rule. For instance, contrary to the maximum quantity constraint previously considered, the above *cash* constraint depends on both the number of shares purchased and their price, thereby providing, by construction, more flexibility to the issuer when it comes to IPO design, as he can rely on both price and quantity to achieve optimality. As before, we first analyze the retail investors' budget constraint when institutional investors are risk neutral, and then move to the case of non-linear preferences.

5.1 Risk neutrality

With risk-neutral institutional investors and cash-constrained retail investors, the seller's optimization program is similar to program ($\mathcal{P}1$), except that the last constraint, *(v)*, is replaced by the retail investors budget constraint defined by Equation (23).

$$\max_{\{q_i\}_1^n} \int_\Omega \left\{ v(s) - \sum_i \left[\frac{1}{n}(\overline{s} - s_i) q_i(s) \right] \right\} f(s) ds$$
$$s.t:$$
$$\begin{aligned}
&(i) \quad U_i(\underline{s}, \underline{s}) = 0 && \text{for all } i \\
&(ii) \quad E_{s_{-i}} \left[\frac{\partial q_i(s)}{\partial s_i} \right] \geq 0 && \text{for all } i \text{ and all } s_i \\
&(iii) \quad q_i(s) \geq 0 && \text{for all } i \text{ and all } s \\
&(iv) \quad \sum_i q_i(s) \leq 1 && \text{for all } s. \\
&(v) \quad \left(1 - \sum_i q_i(s)\right) p_R(s_i, s_{-i}) \leq K && \text{for all } s
\end{aligned} \qquad (\mathcal{P}4)$$

The optimal mechanism is described in the following proposition:

Proposition 5 *With risk neutral institutional investors and cash constrained retail investors, the optimal IPO is characterized as follows:*

1. *(**Allocation rule**) The issuer satisfies retail investors up to their cash constraint. The remaining shares are allocated to the institutional investor reporting the highest signal* $s_m = \max\{s_1, s_2, ..., s_n\}$*.*

2. *(**Pricing rule**) There exists at least one uniform price that implements the optimal IPO.*

The above results imply that, as before with linear preferences, retail investors have priority in the allocation of shares. In this case, though, the allocation rule depends on the price. We thus need to solve for the optimal pricing rule and then step back to obtain explicitly the equilibrium quantities. This suggests that the issuer has enough leeway to choose an allocation rule that can be supported by an optimal uniform price.

More generally, the cash constraint imposes fewer restrictions on the allocation rule than the quantity constraint. The flexibility gained by the issuer is enough to allow uniform pricing at the optimum. Again, we do not obtain a unique optimal price. We find that there may be more than one optimal uniform price, and that equilibria may exist with discriminatory pricing.

5.2 Non-Linear Preferences

We now again assume that institutional investors' preferences are non-linear and, specifically, that they correspond to the valuation function defined in Equation (22); we also assume that retail investors are subject to the cash constraint defined in Equation (23). The new optimization problem for the seller is

$$\begin{aligned}
&\max_{\{q_i\}_1^n} \int_\Omega \left\{ v(s) + \textstyle\sum_i \left[q_i(s) \left((\alpha - 1)\, v(s) - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) \right) \right] \right\} f(s) ds \\
&s.t: \\
&(i) \quad U^i(\underline{s}, \underline{s}) = 0 \\
&(ii) \quad E_{s_{-i}} \left[\frac{\partial q_i(s)}{\partial s_i} \right] \geq 0 \qquad \text{for all } i \text{ and all } s \\
&(iii) \quad q_i(s) \geq 0 \qquad \text{for all } i \text{ and all } s \\
&(iv) \quad \sum_i q_i(s) \leq 1 \qquad \text{for all } s. \\
&(v) \quad p_R(s) \left[1 - \sum_i q_i(s) \right] \leq K \qquad \text{for all } s
\end{aligned} \tag{$\mathcal{P}5$}$$

Contrary to the case without budget constraints for retail investors, here the price paid by retail investors appears explicitly in the optimization problem. However, writing the Kuhn-Tucker conditions for this problem (with the quantities allocated to institutional investors and $p_R(s)$ as the decision variables) reveals that the optimal allocation rule is the same as that without budget constraints and that prices for retail investors are set so that the budget constraint is satisfied. The Kuhn-Tucker conditions for quantities q_i are

$$\begin{cases}
(\alpha - 1)v(s) - \delta q_i - \frac{\alpha}{n}(\overline{s} - s_i) + \lambda_i(s) - \beta(s) + \gamma(s) p_R(s) = 0, \qquad \text{for all } i \\
\lambda_i(s) q_i(s) = 0 \\
\beta(s)[1 - \sum_i q_i(s)] = 0 \\
\gamma(s) \left[K - p_R(s) \left(1 - \sum_i q_i(s) \right) \right] = 0
\end{cases} \tag{24}$$

Since p_R is also a decision variable in this case, and as it appears only in the budget constraint in the optimization problem, the seller only has to choose p_R to satisfy the budget constraint.[21]

[21] Note that the budget constraint is not necessarily binding in equilibrium.

Relative to the unconstrained problem, the seller is thus restricted, as prices for retail investors must satisfy the budget constraints. The allocation rule, however, has the same structure as that in Proposition 2. The price that retail investors pay, when they obtain positive quantities, is restricted to satisfy the budget constraint. This is the case when all of the shares are allocated to retail investors or when the optimal quantity allocated to institutional investors (with a signal higher than the threshold ${s^\circ}_i(v_{-i})$) is less than the total quantity to be sold. To demonstrate that the optimal IPO may involve uniform pricing in the proof of Proposition 3 (with unconstrained retail investors), we rely on the seller's leeway in setting prices for retail investors when they obtain all of the shares. When retail investors are restrained by a budget constraint, this leeway diminishes, which may affect the probability that the optimal mechanism involves a uniform price.

To check whether there are cases with optimal uniform pricing under budget constraints we start by deriving the unique uniform price for institutional investors (which is the same as that in the no budget constraint case) and then see whether this price satisfies retail investors' budget constraints. If this is the case, we also show the existence of a price applicable to retail investors for the cases where $s \in \Omega^-$. In the proof of Proposition 3, we show that when there are no budget constraints, the seller has enough leeway in setting prices for retail investors to satisfy their participation constraints. When there are budget constraints, this leeway is reduced and for some parameter values there is no price that satisfies both the participation and budget constraints for the seller.[22]

This section's results have emphasised that the *shape* (functional form) of the retail investors' budget constraint is also important for the optimality of uniform pricing. This comes about because different functional forms imply different degrees of restrictions on the allocation rule, which is what ultimately determines whether discriminatory pricing is needed to implement the optimal pricing rule.

[22] Note that this reasoning also holds under the more general budget constraint $B(p_R(s), q_R(s)) \leq K$, where we only require that $B(\cdot, \cdot)$ increase in both arguments. In this more general case, the possibility of implementing the optimal mechanism with uniform price depends on the parameter values of the model.

6 Discussion

This paper has investigated the conditions under which the uniform pricing rule in IPOs is optimal. The issuer can potentially use both quantity and price discrimination to elicit information from informed investors and achieve optimality, where the latter here refers to the maximization of sale proceeds. Our findings show that as long as quantity discrimination is sufficiently unrestricted, the issuer does not further need price discrimination, so that the optimal IPO can be implemented with uniform pricing. The allocation rule may be restricted because retail investors are budget constrained and/or because institutional investors have non-linear preferences. We consider budget constraints of different forms which entail different degrees of tightness of the allocation rule. Our findings with respect to optimal pricing are summarized in the table below:

RI Constraint\II Preferences	*Linear*	*Non-Linear*
No constraints	UP	UP
Quantity Constraint	DP	DP
Cash Constraint	UP	Uncertain

RI = retail investors; II = institutional investors.
UP = uniform pricing; DP = discriminatory pricing.

In conclusion, we can say that the most important determinant of the optimal pricing rule is the budget constraint on retail investors and its *shape.* Institutional investors' preferences do not seem to matter although there is one case - cash constraints and non-linear preferences - which remains open because we cannot obtain a closed solution. Nonetheless, we suspect that even in this case for some range of parameter values a uniform price may be optimal.

Assuming non-linear preferences for institutional investors does not affect optimal pricing but it does affect optimal allocation. We show that, contrary to the case of risk neutrality, institutional investors with non-linear preferences will in equilibrium have priority in share allocation with retail investors being residual claimants. Furthermore, the number of shares allocated to institutional investors increases with their signal, i.e. a higher signal implies more shares.

The model produces a number of empirical implications:

- The most direct implication is that uniform pricing and discriminatory pricing will lead to the same IPO proceeds if the allocation rule is unrestricted. Seeing as discriminatory pricing is forbidden in most countries, a direct test of our results will only be possible in the few countries that have allowed discriminatory pricing in IPOs. For example, both Taiwan and Japan have used discriminatory IPO auctions (Jagannathan and Sherman, (2006)), i.e. pay-your-bid auctions, that could allow a test of our results. However, there is still no example of discriminatory pricing in bookbuilding.

- A second empirical implication predicts that greater underpricing should be observed in IPOs where there are tighter restrictions on the allocation rule, e.g. the requirement that a minimum quantity of shares be allocated to institutional investors, which translates into a maximum quantity constraint on retail investors. This also implies that the imposition of this kind of constraint on the IPO is detrimental to the overall efficiency of the selling mechanism and results in lower proceeds.

- Finally, what type of allocation constraint - cash or quantity - is more relevant in practice remains ultimately an empirical question that deserves further investigation. Whilst most of the IPO papers assume the existence of a quantity constraint on retail investors, no one, to our knowledge, provides sound empirical support to this assumption.[23] Answering this question is of importance for our results suggest that the efficiency of the offering mechanism is affected differently by different types of allocation constraints.

7 Appendix

Proof of Proposition 1. For the allocation part, note that the seller's objective function in program ($\mathcal{P}1$) is decreasing in the quantity allocated to institutional investors ($q_i(s)$). The cost of allocating a positive quantity to them is measured by $\int_{\Omega} \left\{ \sum_i \left[\frac{1}{n} (\overline{s} - s_i) q_i(s) \right] \right\} f(s) ds$ and is decreasing in the signal s_i. Consequently, the allocation rule maximizing the seller's revenues consists in allocating as much as possible to retail investors, i.e. up to their budget constraint,

[23] Derrien (2005) is the only paper to our knowledge that documents that, in some French IPOs, a fraction of the shares on sale is explicitly reserved to retail investors.

and any residual quantity to the institutional investor(s) having (and reporting) the highest announced signal, i.e. the agent with the signal $s_m = \max\{s_1, s_2, ..., s_n\}$. Since signals are continuously distributed only one agent announces s_m. Denote by $q(s)$ the quantity allocated to this agent.

For the pricing part, from the binding participation constraint for retail investors and the IICs for institutional investors which define prices, and considering the optimal allocation rule, we have the following equations:

$$\int_{\Omega_{-i}} p_i(s)q_i(s)f_{-i}(s_{-i})ds_{-i} = \int_{\Omega_{-i}} \left\{ v(s)q_i(s) - \frac{1}{n}\left[\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i\right]\right\} f_{-i}(s_{-i})ds_{-i} \quad \text{for all } s_i \text{ and all } i. \tag{25}$$

and

$$\int_{\Omega} p_R(s)f(s)ds = \int_{\Omega} v(s)f(s)ds \tag{26}$$

Now assume the existence of a uniform price, $p(s) = p_i(s) = p_R(s)$ for all s, which implements the optimal mechanism. Applying this to Equation (25), taking expectations over s_i and summing over n gives

$$\int_{\Omega} p(s)\left[\sum_i q_i(s)\right] f(s)ds = \int_{\Omega} \left\{ v(s)\left[\sum_i q_i(s)\right] - \left\{\sum_i \frac{1}{n}\left[\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i\right]\right\}\right\} f(s)ds. \tag{27}$$

Since $\sum_i q_i(s) = 1 - K$ and $\int_{\Omega} p(s)f(s)ds = \int_{\Omega} v(s)f(s)ds$, a necessary condition for a uniform price is

$$\int_{\Omega} \left\{\sum_i \frac{1}{n}\left[\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i\right]\right\} f(s)ds = 0$$

However, $q_i(s)$ is always non-negative and will sometimes be strictly positive. so this last equation cannot hold. This contradicts our assumption of the existence of a uniform pricing rule in this case and completes the proof of the proposition. ■

Proof of Proposition 2. We consider the relaxed problem, i.e. we drop the monotonicity constraint in program ($\mathcal{P}2$) and check it ex post. As such, the objective function becomes an ordinary maximand with the constraints defined at each point and can be maximized pointwise on Ω. The number of shares, $q_i(s)$, the seller must assign to investor i in order to elicit his information is given by the following maximization problem, for each $s \in \Omega$:

$$\max_{\{q_i\}_{i=1,..n}} \sum_i \left[q_i(s) \left((\alpha - 1)\, v(s) - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) \right) \right]$$

$$\begin{array}{l} s.t: \\ U_i(\underline{s}, \underline{s}) = 0 \qquad \text{for all } i \\ q_i(s) \geq 0 \quad \text{and} \quad \sum\limits_i q_i(s) \leq 1 \quad \text{for all } i \text{ and } s. \end{array}$$

The Kuhn-Tucker conditions for this maximization program are:

$$\begin{cases} (\alpha - 1)v(s) - \delta q_i - \frac{\alpha}{n}(\overline{s} - s_i) + \lambda_i(s) - \beta(s) = 0, & \text{for all } i \\ \lambda_i(s) q_i(s) = 0 \\ \beta(s)[1 - \sum\limits_i q_i(s)] = 0. \end{cases} \tag{28}$$

with λ_i and β being the Kuhn-Tucker multipliers associated to respectively the feasibility constraint and the FAC. Now denote the seller's objective function by $H(q, v(s))$, that is:

$$H(q, v(s)) = \sum_i \left[q_i(s) \left((\alpha - 1)\, v(s) - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) \right) \right] \tag{29}$$

with $q_i \in [0, 1]$ and $s \in \Omega = [\underline{s}, \overline{s}]^n$. This function is concave in q_i for all i, since $\frac{\partial^2 H}{\partial q_i^2} \leq 0$. Now let $s^\circ{}_i(v_{-i}) = \frac{\alpha \overline{s} - (\alpha - 1) v_{-i}}{(2\alpha - 1)}$ for all i and v_{-i} and define the following sets for all $s \in \Omega$

$$\begin{aligned} N^-(s) &= \{ i \in N \mid s_i \leq s^\circ{}_i(v_{-i}) \} \\ N^+(s) &= \{ i \in N \mid s_i > s^\circ{}_i(v_{-i}) \} \end{aligned}$$

We can easily show that for each $i \in N^-(s)$ it must hold that $q_i(s) = 0$. Then, for each $i \in N^+(s)$, define the quantity $\widetilde{q}_i$ as that which solves the equation below,

$$(\alpha - 1)v(s) - \delta \widetilde{q}_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) = 0 \tag{30}$$

By the concavity of H (with respect to q_i), $\widetilde{q}_i(s)$ is strictly positive (for each $i \in N^+(s)$). Now, if

- $\sum\limits_{i \in N^+(s)} \widetilde{q}_i(s) \leq 1$, then the quantity $\widetilde{q}_i(s)$ is the solution of our mechanism.[24]

[24] In this case, the equation defining $\widetilde{q}_i(s)$ is the FOC of our objective function H, since the Kuhn-Tucker multipliers, $\lambda_i(s)$ and $\beta(s)$ are both zero.

- $\sum_{i\in N^+(s)} \widetilde{q}_i(s) > 1$, which corresponds to *oversubscription* of the new shares, then the quantity $\widetilde{q}_i(s)$ cannot be optimal as it violates the FAC. The optimal quantities, which we denote by $\widehat{q}_i(s)$, are given by the solution of the following system of equations,

$$\begin{cases} \widehat{q}_i(s)[(\alpha-1)v(s) - \delta\widehat{q}_i(s) - \dfrac{\alpha}{n}(\overline{s}-s_i) - \beta(s)] = 0 \\ \displaystyle\sum_{i\in N^+(s)} \widehat{q}_i(s) = 1; \qquad \beta(s) > 0, \end{cases} \tag{31}$$

which implies that $\widehat{q}_i(s)$ is either zero or is positive and solves the following equation

$$(\alpha-1)v(s) - \delta\widehat{q}_i(s) - \frac{\alpha}{n}(\overline{s}-s_i) - \beta(s) = 0 \tag{32}$$

Clearly, in this case, all the shares are allocated to institutional investors. Retail investors receive nothing in equilibrium.

■

Proof of Proposition 3. We divide the proof into three steps. In the first step, we derive conditions for the existence of a uniform pricing rule among institutional investors. Then in step 2, we show the existence of a unique pricing rule satisfying these conditions. Finally, in step three, we show that the optimal uniform pricing rule may be applied to retail investors.

Step 1: The linear transfer for institutional investors must satisfy Equation (15). Consider the following price function

$$p_i^0(s) = \frac{z(q_i(s), v(s)) - \frac{1}{n}\left[\int_{\underline{s}}^{s_i} z_2(q_i(\widetilde{s}_i, s_{-i}), v(\widetilde{s}_i, s_{-i}))d\widetilde{s}_i\right]}{q_i(s)} \tag{33}$$

for each s_i and each $q_i(s)$ such that $q_i(s) \neq 0$.[25] Any price satisfying Equation (15) can also be written as $p_i(s) = p_i^0(s) + \Phi_i(s)$, where Φ_i satisfies the following equation

$$\int_{\Omega_{-i}} \Phi_i(s) q_i(s_i, s_{-i}) f_{-i}(s_{-i}) ds_{-i} = 0, \qquad \text{for all } i \text{ and } s_i. \tag{34}$$

It should be noted that, given the symmetry of our model, the existence of a uniform price for institutional investors implies that the marginal effects of changes in the private signal of

[25] Otherwise $p_i^0(s) = 0$.

different investors on the price are equal, that is $\frac{\partial p_i(s)}{\partial s_i} = \frac{\partial p_i(s)}{\partial s_j}$ for all i and j. Applying this uniformity condition to an admissible pricing function gives

$$\frac{\partial p_i^0(s)}{\partial s_i} + \frac{\partial \Phi_i(s)}{\partial s_i} = \frac{\partial p_i^0(s)}{\partial s_j} + \frac{\partial \Phi_i(s)}{\partial s_j}. \tag{35}$$

with,

$$\frac{\partial p_i^0(s)}{\partial s_i} = \frac{\partial q_i(s)}{\partial s_i} \frac{[-p_i^0(s) + v(s)]}{q_i(s)} \tag{36}$$

and

$$\frac{\partial p_i^0(s)}{\partial s_j} q_i(s) = \frac{\partial q_i(s)}{\partial s_j} \left[-p_i^0(s) + v(s)\right] + \frac{1}{n}\left[q_i(s) - \int_{\underline{s}}^{s_i} \frac{\partial q_i(\widetilde{s}_i, s_{-i})}{\partial s_j} d\widetilde{s}_i\right]. \tag{37}$$

Multiplying both sides of Equation (35) by $q_i(s)$ and using Equations (36) and (37) yields

$$\begin{aligned} &\left(-p_i^0(s) + \alpha v(s) - \delta q_i(s)\right)\left(\frac{\partial q_i(s)}{\partial s_i} - \frac{\partial q_i(s)}{\partial s_j}\right) \\ &-\frac{\alpha}{n}\left[q_i(s) - \int_{\underline{s}}^{s_i} \frac{\partial q_i(\widetilde{s}_i, s_{-i})}{\partial s_j} d\widetilde{s}_i\right] + \frac{\partial \Phi_i(s)}{\partial s_i} q_i(s) = \frac{\partial \Phi_i(s)}{\partial s_j} q_i(s). \end{aligned} \tag{38}$$

We can easily show from Proposition 2 that the optimal quantities allocated to institutional investors are such that

$$\frac{\partial q_i(s)}{\partial s_i} - \frac{\partial q_i(s)}{\partial s_j} = \frac{\alpha}{n\delta} \tag{39}$$

for each s and each i. This allows us to re-write Equation (38) as follows

$$\frac{\alpha}{n}\left\{\frac{[-p_i^0(s) + \alpha v(s) - \delta q_i(s)]}{\delta q_i(s)} - 1 + \frac{\displaystyle\int_{\underline{s}}^{s_i} \frac{\partial q_i(\widetilde{s}_i, s_{-i})}{\partial s_j} d\widetilde{s}_i}{q_i(s)}\right\} + \frac{\partial \Phi_i(s)}{\partial s_i} = \frac{\partial \Phi_i(s)}{\partial s_j}. \tag{40}$$

Replacing $p_i^0(s)$ by the value defined in Equation (33) yields

$$\frac{\alpha}{n}\left\{-\frac{3}{2} + \frac{\alpha}{n\delta} \frac{\displaystyle\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i}) d\widetilde{s}_i}{[q_i(s)]^2} + \frac{\displaystyle\int_{\underline{s}}^{s_i} \frac{\partial q_i(\widetilde{s}_i, s_{-i})}{\partial s_j} d\widetilde{s}_i}{q_i(s)}\right\} + \frac{\partial \Phi_i(s)}{\partial s_i} = \frac{\partial \Phi_i(s)}{\partial s_j}. \tag{41}$$

Replacing $\frac{\alpha}{n\delta}$ by $\frac{\partial q_i(s)}{\partial s_i} - \frac{\partial q_i(s)}{\partial s_j}$ we obain after some simple computations

$$\frac{\partial}{\partial s_i}\left[\Phi_i(s) - \frac{\alpha}{n} \frac{\displaystyle\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i}) d\widetilde{s}_i}{q_i(s)}\right] = \frac{\partial}{\partial s_j}\left[\Phi_i(s) - \frac{\alpha}{n} \frac{\displaystyle\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i}) d\widetilde{s}_i}{q_i(s)}\right] + \frac{\alpha}{2n}, \tag{42}$$

for each i and each j. This is a partial differential equation in $\Phi_i(s) - \frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i}{q_i(s)}$ whose generic solution is given by

$$\Phi_i(s) = \varphi(s_1 + ... + s_n) + \frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i}{q_i(s)} + \frac{\alpha}{2n}s_i, \quad \text{for all } i \tag{43}$$

where $\varphi(\cdot)$ is a twice-differentiable function defined on the set $[n\underline{s}, n\overline{s}] = n\Omega$.

Summing up, we have so far shown that the optimal mechanism may be implemented by a uniform price schedule if and only if

$$p_i(s) = p_i^0(s) + \varphi(s_1 + ... + s_n) + \frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i}{q_i(s)} + \frac{\alpha}{2n}s_i \tag{44}$$

where φ is a twice differentiable function satisfying the following integral equation

$$\int_{\Omega_{-i}} \left[\varphi(s_1 + ... + s_n) + \frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i}{q_i(s)} + \frac{\alpha}{2n}s_i\right] q_i(s_i, s_{-i})f_{-i}(s_{-i})ds_{-i} = 0. \tag{45}$$

or alternatively

$$\begin{gathered}\int_{\Omega_{-i}} \varphi(s_1 + ... + s_n)q_i(s_i, s_{-i})f_{-i}(s_{-i})ds_{-i} = \\ -\int_{\Omega_{-i}} \left[\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i, s_{-i})d\widetilde{s}_i}{q_i(s)} + \frac{\alpha}{2n}s_i\right] q_i(s_i, s_{-i})f_{-i}(s_{-i})ds_{-i} = g(s_i).\end{gathered} \tag{46}$$

Henceforth, proving the existence of a uniform price schedule is equivalent to proving the existence of this function φ.

STEP 2: We first show that Equation (46) can be written as a Volterra integral equation of the first kind.[26] Then, by simply applying the general properties of this kind of integral

[26] A Volterra integral equation of the the first kind is defined in the following way:

$$\int_{y_0}^{\tau(x)} f(x,y)h(x,y)dy = g(x)$$

in other words, one of the integral limits must depend on the variable x.

equation we can prove the existence and the *uniqueness* of the function φ. To do so, we first need to transform Equation (46) into a simple integral equation, i.e. with the support defined on $\mathbb{R}$. Notice that, since $q_i(s_i, s_{-i}) = 0$ when $s_i \leq s_i^0(v_{-i})$, the support of the integral Equation (46) is equal to $\Omega^0_{-i} = \{(s_1, s_2, .., s_{i-1}, s_{i+1}, .., s_n) \mid \sum_{j\neq i} s_j = v_{-i} \geqslant v^0_{-i}(s_i)\}$ where $v^0_{-i}(s_i)$ is defined as the inverse of $s_i^0(v_{-i})$, i.e.

$$v^0_{-i}(s_i) = \frac{\alpha\overline{s} - (2\alpha - 1)s_i}{\alpha - 1} \tag{47}$$

Also, from the definition of the optimal quantity, the LHS term of Equation (46) equals $\int_{\Omega^0_{-i}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})f_{-i}(s_{-i})ds_{-i}$. Now, by applying the Generalized Change Variable Theorem (GVCT) we can set $v_{-i} = \gamma_i(s_{-i})$ for each i, which finally implies that $\gamma_i(\Omega^0_{-i}) = [v^0_{-i}(s_i); (n-1)\overline{s}] \in \mathbb{R}$.[27] One key implication of the GVCT is that there exists a measure λ defined over $[v^0_{-i}(s_i); (n-1)\overline{s}]$ such that

$$\int_{\Omega^0_{-i}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})f_{-i}(s_{-i})ds_{-i} = \int_{v^0_{-i}(s_i)}^{(n-1)\overline{s}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})\lambda(dv_{-i}). \tag{48}$$

Last, by applying the Radon-Nikodým theorem[28], we are also able to prove the existence of a density function ρ associated with the measure λ such that

$$\int_{\Omega^0_{-i}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})f_{-i}(s_{-i})ds_{-i} = \int_{v^0_{-i}(s_i)}^{(n-1)\overline{s}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})\rho(v_{-i})dv_{-i}. \tag{49}$$

From the above result, the integral Equation (46) reduces to

$$\int_{v^0_{-i}(s_i)}^{(n-1)\overline{s}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})\rho(v_{-i})dv_{-i} = g(s_i) \tag{50}$$

This is a Volterra integral equation of the first kind which ensures that, as long as the function g is well behaved, a solution in φ always exists. This shows the existence of a unique uniform pricing rule for institutional investors. We denote such a pricing rule by $p_I(s)$ in the following.

Step 3: To complete the proof it remains to show that the uniform price for institutional investors also applies to retail investors. This is equivalent to showing that $p_I(s)$ satisfies the retail investors' participation constraint. This problem is relevant only in the cases for which

[27] See Dunford and Schwartz (1988, 3rd Ed.), chapter 3, lemma 8, page 182.

[28] See, for example, Dunford and Schwartz (1988, 3rd Ed.), chapter 3, theorem 2, page 176.

both retail and institutional investors receive positive amounts of shares at the optimum. We start by defining the following sets:

- $\Omega^- = \{s \mid q_i = 0 \text{ for all } i\}$, i.e. all the quantity is distributed to retail investors;
- $\Omega \backslash \Omega^- = \{s \mid q_i(s) \neq 0 \quad \text{for at least one } i\}$

Recall that the retail investors' participation constraint can be written as follows

$$\int_\Omega p_R(s) q_R(s) f(s) ds = \int_\Omega v(s) \left(1 - \sum_i q_i(s)\right) f(s) ds \tag{51}$$

Proving that uniform pricing applies to all investors is then equivalent to demonstrating the existence of a price function $p_R(s)$ solving the above integral equation and such that:

$$p_R(s) = \begin{cases} p^-(s) & \text{for all } s \in \Omega^- \\ p_I(s) & \text{for all } s \in \Omega \backslash \Omega^- \end{cases} \tag{52}$$

which requires that retail investors are charged different prices depending on whether they receive the whole quantity or not. The problem then boils down to proving the existence of a price $p^-(s)$ such that

$$\int_{\Omega^-} p^-(s) f(s) ds + \int_{\Omega \backslash \Omega^-} p_I(s) \left(1 - \sum_i q_i(s)\right) f(s) ds = \int_\Omega v(s) \left(1 - \sum_i q_i(s)\right) f(s) ds \tag{53}$$

or equivalently

$$\int_{\Omega^-} p^-(s) f(s) ds = \int_\Omega v(s) \left(1 - \sum_i q_i(s)\right) f(s) ds - \int_{\Omega \backslash \Omega^-} p_I(s) \left(1 - \sum_i q_i(s)\right) f(s) ds \tag{54}$$

We then prove the following:

Lemma 1 *A price function $p^-(s)$ as defined in Equation (54) exists if and only if it satisfies the following equation*

$$\int_{\Omega^-} p^-(s) f(s) ds = \int_\Omega \{v(s) + H(q, v(s))\} f(s) ds - \int_{\Omega \backslash \Omega^-} p_I(s) f(s) ds. \tag{55}$$

where the function $H(q, v(s))$ defines the seller's payoff at the optimum (see the proof of Proposition 2).

Proof of Lemma 1:

$\Longrightarrow$ Suppose there exists a price $p^-(s)$ which satisfies the participation constraint of retail investors. This implies that the seller's expected payoff at the optimum is given by

$$\int_{\Omega^-} p^-(s)f(s)ds + \int_{\Omega\backslash\Omega^-} p_I(s)f(s)ds = \int_{\Omega}\{v(s) + H(q, v(s))\}f(s)ds \tag{56}$$

due to the uniform pricing rule and the fact that all of the shares are always sold.

$\Longleftarrow$ Taking the expectation of Equation (15) over s_i and summing over i gives

$$\int_\Omega \sum_i p_i(s)q_i(s)f(s)ds = \int_\Omega \sum_i \left\{ z(q_i(s), v(s)) - \frac{1}{n}\left[\int_{\underline{s}}^{s_i} z_2(q_i(\widetilde{s}_i, s_{-i}), v(\widetilde{s}_i, s_{-i}))d\widetilde{s}_i\right]\right\} f(s)ds \tag{57}$$

using the fact that all institutional investors pay the same price and adding $-\int_\Omega p_I(s)f(s)ds$ and $\int_\Omega v(s)\left(1-\sum_i q_i(s)\right)f(s)ds$ to both sides yields the following

$$\begin{gathered}\int_\Omega p(s)\left(1-\sum_i q_i(s)\right)f(s)ds - \int_\Omega v(s)\left(1-\sum_i q_i(s)\right)f(s)ds = \\ \int_\Omega p(s)f(s)ds - \int_\Omega \{v(s) + H(q, v(s))\}f(s)ds.\end{gathered} \tag{58}$$

If the uniform price exists then the RHS of the above equation is equal to zero which immediately implies that the retail investors' participation constraint, on the LHS of the equation, holds. This ends the proof of Lemma 1 ∎

The right-hand side of Equation (55) does not depend on the agents' signals and so the integral equation defined over $p^-(s)$ has many solutions. This proves that the seller could find an optimal pricing function where a uniform price is applied to all investors. Note finally that the seller may choose among different pricing rules that may be applied for retail investors when s belongs to Ω^-. ∎

Proof of Proposition 4. In the proof of Proposition 2, we defined ${s^\circ}_i(v_{-i}) = \frac{\alpha\overline{s}-(\alpha-1)v_{-i}}{(2\alpha-1)}$. Define the following sets for all s

$$\begin{aligned} N^-(s) &= \{i \in N \mid s_i \le {s^\circ}_i(v_{-i})\} \\ N^+(s) &= \{i \in N \mid s_i > {s^\circ}_i(v_{-i})\} \\ \Omega^+(s) &= \{i \in N \mid s_i \neq \underline{s}\} \end{aligned}$$

and let $\varpi^+ = Card(\Omega^+(s))$. The solution of the relaxed problem is very similar to that of Proposition 2. Pointwise maximization leads to the following Kuhn-Tucker conditions:

$$\begin{cases} (\alpha-1)v(s) - \delta q_i - \dfrac{\alpha}{n}(\overline{s}-s_i) + \lambda_i(s) - \beta(s) + \gamma(s) = 0, & \text{for all } i \\ \lambda_i(s)q_i(s) = 0 \\ \beta(s)[1-\sum\limits_i q_i(s)] = 0 \\ \gamma(s)[1-\sum\limits_i q_i(s) - K] = 0 \end{cases} \tag{59}$$

where λ_i, β and γ are the Kuhn-Tucker multipliers associated to the three remaining constraints. Note that β and γ cannot be different from zero at the same time. If we consider that $\gamma = 0$, then we obtain the same problem as in the case without budget constraints which proves the first, second and fourth points in the allocation part of the proposition, and the first point in the pricing rule part. For the third point of the allocation rule, suppose that $\sum\limits_{i \in N^+(s)} \widetilde{q}_i(s) \leq 1-K$. In this case $\beta(s) = 0$ and $\lambda_i(s)$ should be equal to zero for all i, otherwise some shares remain unsold. In this case, retail investors receive K shares and the optimal allocation rule for informed investors is that satisfying the following condition:

$$(\alpha-1)v(s) - \delta q_i - \frac{\alpha}{n}(\overline{s}-s_i) + \gamma(s) = 0 \tag{60}$$

for all $i \in \Omega^+(s)$, with $\gamma(s)$ solving the equation below:

$$\gamma(s) = \frac{\delta(1-K)}{\varpi^+} + \frac{\alpha}{n}\overline{s} - \frac{\alpha}{n\varpi^+}\sum_{j \in \Omega^+(s)} s_j - (\alpha-1)v(s) \tag{61}$$

By replacing $\gamma(s)$ in the previous FOC, we finally obtain the optimal quantity for informed investors

$$\widetilde{q}_i^K(s) = \frac{1}{\delta\varpi^+}\left\{\delta(1-K) + \frac{\alpha}{n}(\varpi^+-1)s_i - \frac{\alpha}{n}\sum_{j \in \Omega^+(s),\ j \neq i} s_j\right\}. \tag{62}$$

For the second point in the pricing rule, we proceed exactly as in Proposition 3. We begin by deriving, in a first step, the conditions for the existence of a uniform pricing rule among institutional investors. Then, in a second step, we show the existence of a unique pricing rule satisfying these conditions. Finally, in a third step we show that such a unique uniform pricing rule cannot be optimally applied to retail investors. Since the proof is exactly the same as that for Proposition 3, we omit it. ■

Proof of Proposition 5. For the allocation part we use the same argument based on the facts that the objective function is decreasing in the quantities allocated to institutional investors and that informational rents are decreasing in s_i, which gives the result.

For the pricing rule, note that the uniform price schedule must satisfy retail investors' participation constraints and budget constraints. This yields the following conditions

$$\begin{aligned} \int_{\Omega} v(s)\,(1-q(s))\,f(s)ds &= K \\ p(s)\,(1-q(s)) &= K \qquad \text{for all } s \end{aligned} \tag{63}$$

where p, the unitary price paid by the institutional agent, satisfies Equation (15). The substitution of p from the retail investors' budget constraint in Equation (15) gives the following condition

$$\int_{[\underline{s},s_m]^{n-1}} \left\{ \left[\frac{K}{(1-q(s))}\right] q(s) + \frac{1}{n}\left[\int_{\underline{s}}^{s_m} q(\widetilde{s}_m, s_{-m})d\widetilde{s}_m\right]\right\} f_{-m}(s_{-m})ds_{-m} = 0. \tag{64}$$

Note that in Equation (64), integration is only considered for the values of s_{-m} for which $q(s)$ assigned to agent m is positive. Proving the existence of a uniform price function is then equivalent to proving the existence of a function $q(s)$ that is the solution to equations (64) and (63). Finally denote by $z(s_{-m})$ a function such that:

$$\int_{[\underline{s},s_m]^{n-1}} z(s_{-m})f_{-m}(s_{-m})ds_{-m} = 0.$$

We can thus re-write Equation (64) as follows

$$\left[\frac{K}{(1-q(s))}\right] q(s) + \frac{1}{n}\left[\int_{\underline{s}}^{s_m} q(\widetilde{s}_m, s_{-m})d\widetilde{s}_m\right] = z(s_{-m}) \tag{65}$$

Denoting $X(s_m, s_{-m}) = \int_{\underline{s}}^{s_m} q(\widetilde{s}_m, s_{-m})d\widetilde{s}_m$,[29] we can also write Equation (65) as follows

$$[K - \frac{1}{n}X(s_m, s_{-m}) + z(s_{-m})]X'(s_m, s_{-m}) + \frac{1}{n}X(s_m, s_{-m}) - z(s_{-m}) = 0 \tag{66}$$

[29] More precisely this is $X(s_m, s_{-m}) = \int_{s_l}^{s_m} q(\widetilde{s}_m, s_{-m})ds_m$ where s_l is the highest signal in s_{-m}. To keep notation simple, we rewrite the integral as indicated, using the fact that $q(\widetilde{s}_m, s_{-m}) = 0$ for some values of $\widetilde{s}_m$ in the interval $[\underline{s}, s_m]$.

where $X'(s_m, s_{-m})) = \frac{\partial X(s_m, s_{-m})}{\partial s_m} = q(s_m, s_{-m})$.

We have so far shown that the existence of a uniform price for all investors is equivalent to proving the existence of a function $X(s_m, s_{-m})$ that solves the non-linear first-order differential equation presented in Equation (66) with Equation (63) as a final condition. This differential equation has at least one solution for each chosen function $z(\cdot)$. This gives the seller enough leeway to construct the desired uniform price consistent with the optimal mechanism. ■

References

[1] Aggarwal, R., 2003, "Allocation of Initial Public Offerings and Flipping Activity", *Journal of Financial Economics,* 68 (1): 11-135.

[2] Aggarwal R., N. Prabhala and M. Puri, 2002, "Institutional Allocations in Initial Public Offerings: Empirical Evidence", *Journal of Finance*, 57: 1421-42.

[3] Allen F. and G. Faulhaber, 1989, "Signaling by Underpricing in the IPO Market", *Journal of Financial Economics,* 23 (2): 303-323.

[4] Beatty R. and J. Ritter, 1986, "Investment Banking, Reputation, and the Underpricing of Initial Public Offerings", *Journal of Financial Economics*, 15: 213-232.

[5] Bennouri M. and S. Falconieri, 2006, "Optimal Auctions with Asymmetrically Informed Bidders", *Economic Theory*, 28: 585-602.

[6] Benveniste L.M. and P.A. Spindt, 1989, "How Investment Bankers Determine the Offer Price and the Allocation of New Issues", *Journal of Financial Economics*, 24: 213-323.

[7] Benveniste L.M., W. Busaba and W.J. Wilhelm, 1996, "Price Stabilization as a Bonding Mechanism in New Equity Issues", *Journal of Financial Economics*, 42: 223-255.

[8] Benveniste L.M. and W.J. Wilhelm, 1997, "Initial Public Offerings: Going by the Book", *Journal of Applied Corporate Finance*, 10 (1): 98-108.

[9] Benveniste, L.M. and W.J. Wilhelm, 1990, "A Comparative Analysis of IPO Proceeds under Alternative Regulatory Environments", *Journal of Financial Economics,* 28: 173-207.

[10] Biais B. and A.M. Faugeron-Crouzet, 2002, "IPO auctions: English, Dutch, ... French and Internet", *Journal of Financial Intermediation,* 11: 9-36.

[11] Brisley, Neil, 2003, "A Comparative Analysis of IPO Proceeds under Alternative Regulatory Environments: A Comment", *Economics Bulletin*, 7: 1-7

[12] Bulow J. and P. Klemperer, 1996, "Auctions vs. Negotiations", *American Economic Review*, 86: 180-194.

[13] Bulow J. and P. Klemperer, 2002, "Prices and the Winner's Curse", *Rand Journal of Economics*, 33 (1): 1-21.

[14] Cornelli F. and D. Goldreich, 2001, "Bookbuilding and Strategic Allocation", *Journal of Finance*, 56: 2337-69.

[15] Derrien, F., 2005, "IPO Pricing in "Hot" Market Conditions: Who Leaves Money on the Table?", *Journal of Finance,* 60(1): 487-521.

[16] Dunford and J.T. Schwartz, 1988, "*Linear Operators: Part I: General Theory*", John Wiley and Sons, New York, Third Edition.

[17] Fudenberg D. and J. Tirole, 1991, "*Game Theory*", MIT Press.

[18] Hanley K.W. and Wilhlem W.J., 1996, "Evidence on the Strategic Allocation of Initial Public Offerings', *Journal of Financial Economics* 37: 239–257.

[19] Jagannathan R. and A.E. Sherman, 2006, "Why Do IPO Auctions Fail?", Working Paper.

[20] Jenkinson T. and A. Ljungqvist, 2001, "*Going Public: The Theory and Evidence of How Companies Raise Equity Finance*", Oxford University Press.

[21] Lyons R., 2003, "*The Microstructure Approach to Exchange Rates*", MIT Press.

[22] Maksimovic V. and P. Pichler, 2006 "Structuring the Initial Offering: Who to Sell To and How to Do It", *Review of Finance,* 10: 1-35.

[23] Malakhov, A., 2006, "The Role of Uninformed Buyers in Optimal IPO Mechanisms", Available at SSRN: http://ssrn.com/abstract=687167.

[24] Mello, S. A. and J. E. Parsons, 1998, "Going Public and the Ownership Structure of the Firm", *Journal of Financial Economics*, 49: 79-109.

[25] O'Hara M., 1995, "*Market Microstructure*", Blackwell Publishers.

[26] Ritter J. , 1987, "The Costs of Going Public", *Journal of Financial Economics*, 19: 269-281.

[27] Ritter J. and I. Welch, 2002, "A Review of IPO Activity, Pricing, and Allocations", *Journal of Finance,* 57 (4): 1795-1828.

[28] Rock K, 1986, "Why New Issues Are Underpriced", *Journal of Financial Economics*, 15: 197-212.

[29] Stoughton N. and J. Zechner, 1998, "IPO-mechanism, Monitoring and Ownership Structure", *Journal of Financial Economics*, 49: 45-77.