TRAFFIC EQUILIBRIUM

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FROM TRAFFIC EQUILIBRIUM TO NETWORK DESIGN

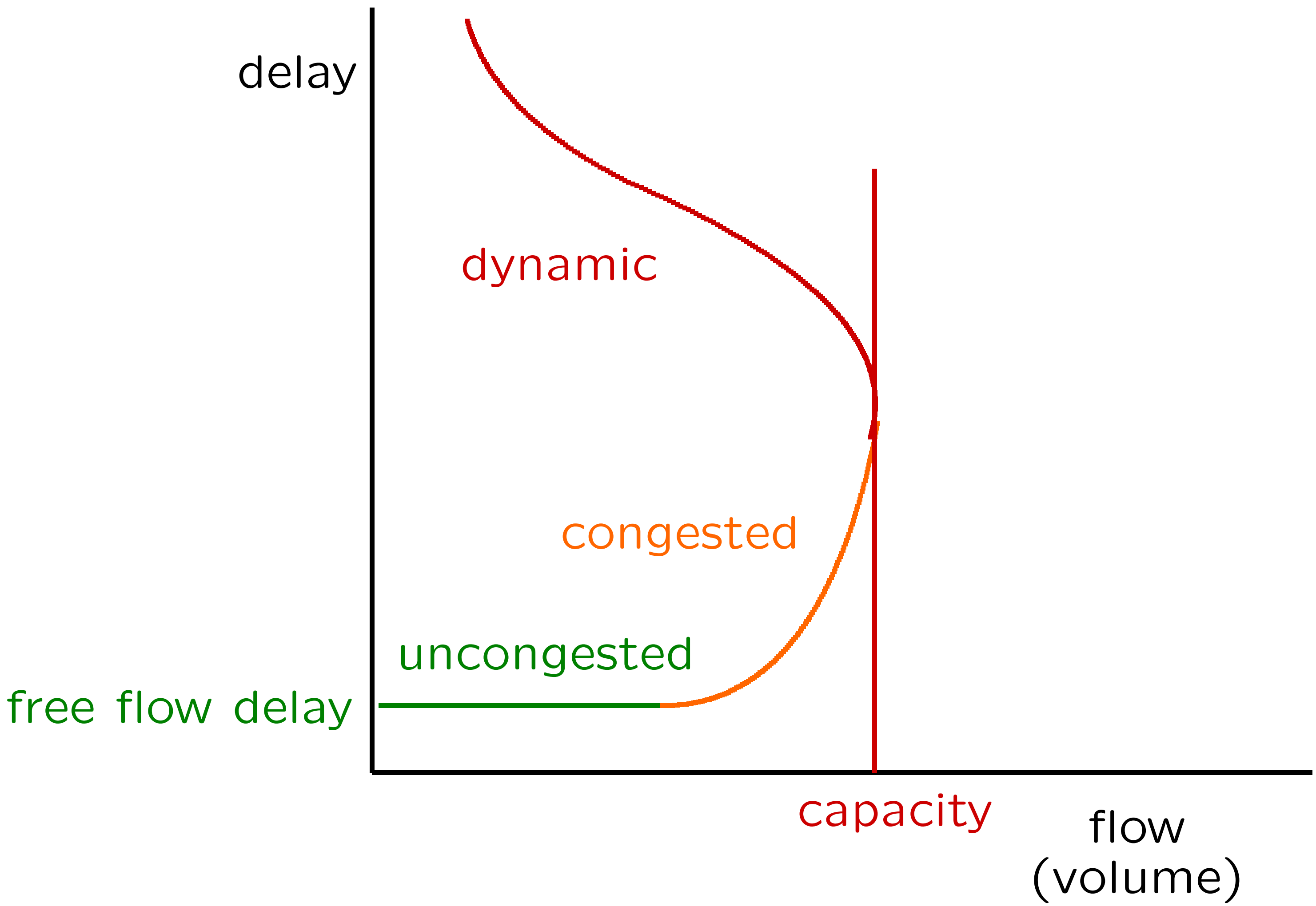
EQUILIBRIUM: Predictive model of traffic flows based on behavioral principles and formulated as a variational inequality or fixed point problem.

USEFULNESS: Assessing network performance associated with designs or scenarios. Leads to bilevel programs that explicitly involve user reaction to network modifications.

Equilibria arise in a variety of shades and colours:

- deterministic (macroscopic, steady-state)
- stochastic
- dynamic (macroscopic, mesoscopic, microscopic)
- multimodal
- multiclass
- transit
- strategic

VOLUME-DELAY CURVES

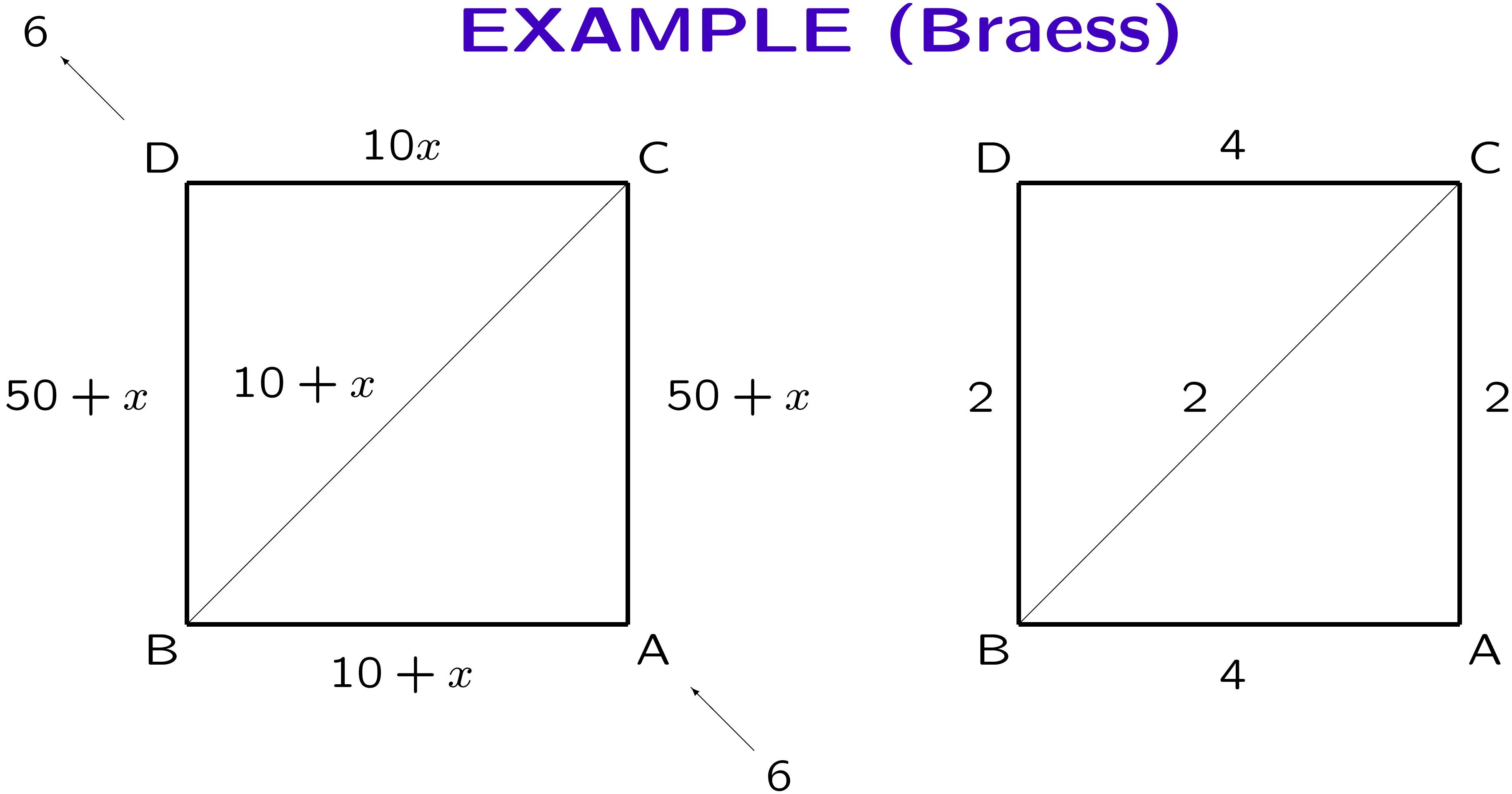


EQUILIBRIUM PRINCIPLE (Wardrop)

USED PATHS HAVE EQUAL DELAYS

UNUSED PATHS HAVE HIGHER DELAYS

EXAMPLE (Braess)



$x^*_{ABD} = x^*_{ACD} = x^*_{ABCD} = 2$

(common delay = 92)

checking equilibrium is easy :-)

finding an equilibrium is difficult :-(

NOTATION

x_p : flow on path $p \in P$ $\quad x = (x_p)_{p \in P}$

v_a : flow on arc $a \in A$ $\quad v = (v_a)_{a \in A}$

$C_a(v)$: delay on arc $a \in A$

$F_p(x)$: delay on path $p \in P$

X : set of feasible flows $\quad \{x \geq 0 : \sum_{p \in P} x_p = D\}$

$$
\begin{aligned}
v_a &= \sum_{p \ni a} x_p \\
v &= Ax \\
F_p(x) &= \sum_{a \in p} C_a(v) \\
&= A^t F(Ax)
\end{aligned}
$$

A : arc-path incidence matrix

FORMULATION

$$F_p(x) = \min_{p'} F_{p'}(x) \qquad \forall p \in P$$

$$F_{p'}(x) > F_p(x) \Rightarrow x_p = 0 \qquad \forall p, p' \in P$$

$\exists u : \forall p \in P$

$$x_p > 0 \quad \Rightarrow \quad F_p(x) = u$$

$$F_p(x) > u \quad \Rightarrow \quad x_p = 0$$

COMPLEMENTARITY FORMULATION

Let $e = (1, \ldots, 1)^t$

$$\begin{array}{rcl} x & \in & X \\ F(x) & \geq & ue \\ F(x) - ue & \perp & x \end{array}$$

VARIATIONAL INEQUALITY

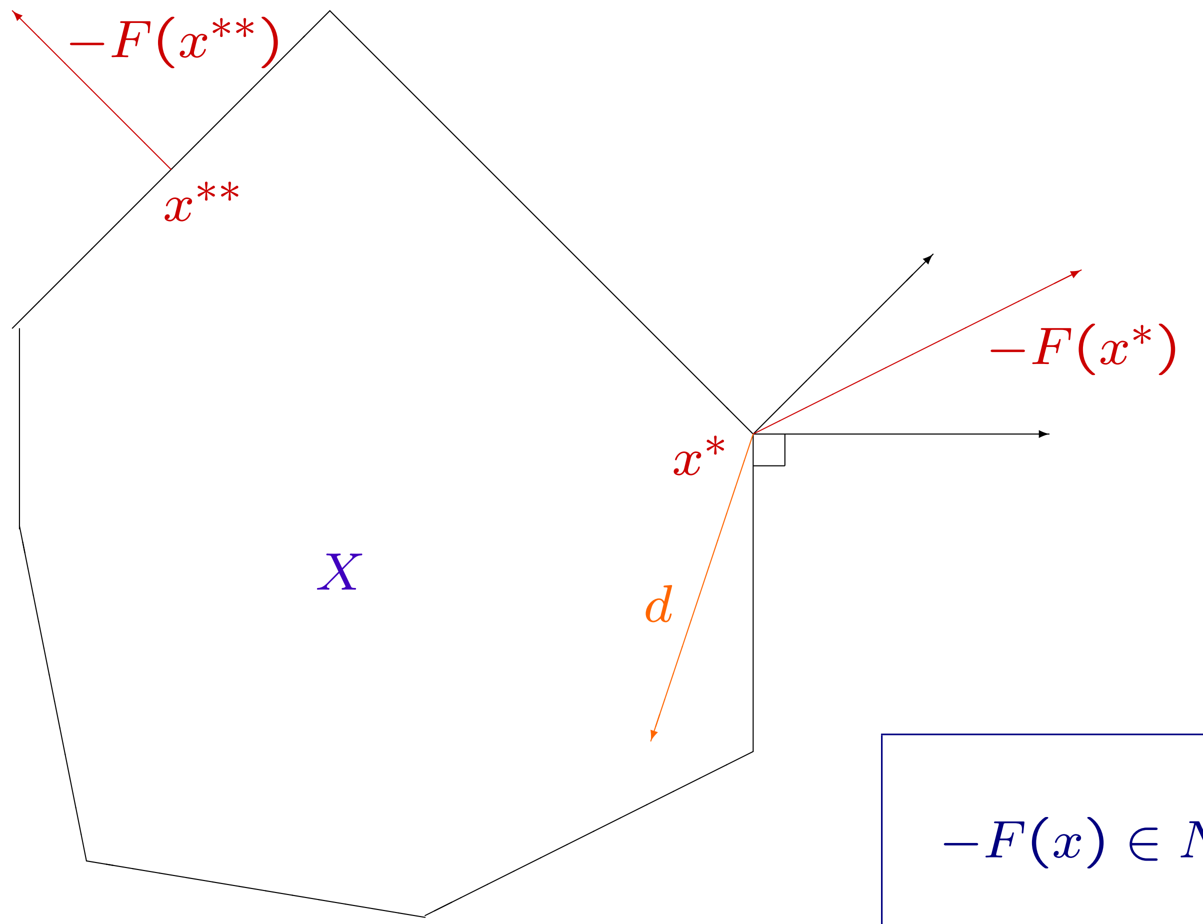
$$x \in X$$

$$\langle F(x), x - y \rangle \leq 0 \qquad \forall y \in X$$

If F is the gradient of some function f, then the above reduces to the first-order necessary optimality conditions of the mathematical program

$$\min_{x \in X} f(x)$$

GEOMETRY



$$-F(x) \in N_X(x)$$

$-F(x)$ makes an obtuse angle with all feasible directions if and only if it lies in the normal cone of X at x

THEORETICAL RESULTS

Monotonicity of a mapping H:

ordinary:	$\langle H(x) - H(y), x - y \rangle \geq 0$	$\forall x \in X$
strict:	$\langle H(x) - H(y), x - y \rangle > 0$	$\forall x \in X$
strong:	$\langle H(x) - H(y), x - y \rangle \geq \beta \|x - y\|^2$	$\forall x \in X$

- existence: follows from Brouwer's fixed point theorem
- uniqueness: achieved in link flows under strict monotonicity of $C(v)$ but not in path flows

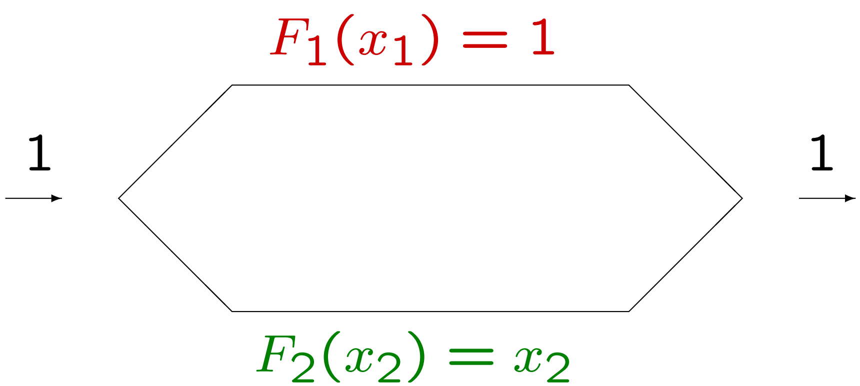
A Wardrop equilibrium is NOT a <u>non-atomic</u> Nash equilibrium.

THE PRICE OF ANARCHY

$$\rho = \max \frac{\langle F(x^*), x^* \rangle}{\langle F(x), x \rangle}$$

$$= \frac{(p+1)^{1+1/p}}{(p+1)^{1+1/p} - p} \quad \in \quad \Theta\left(\frac{p}{\ln p}\right)$$

(polynomial case)



$x^* = (0, 1)$

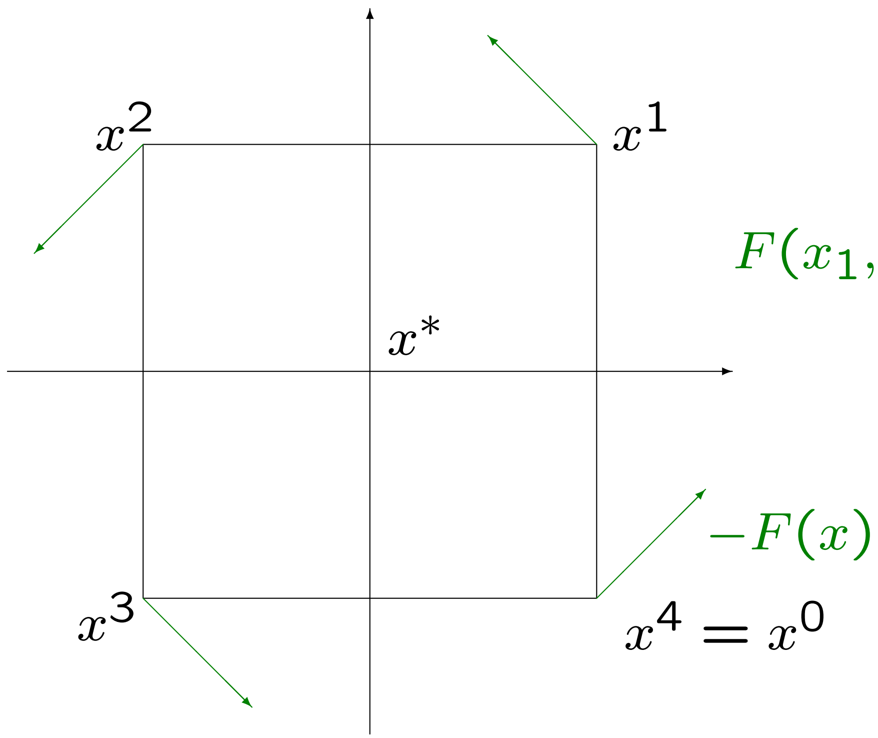
$x = (\frac{1}{2}, \frac{1}{2})$

$\rho = (\frac{1}{2} + \frac{1}{2})/(\frac{1}{2} + \frac{1}{4}) = 4/3$

affine case ($p = 1$)

SOLUTION ALGORITHMS

- integrable case: Frank-Wolfe
- projection: $x^{+} = \text{proj}_X(x - \gamma F(x))$
- optimization: $\min_x \text{ gap}(x) = \max_{y \in X} \langle F(x), x - y \rangle$

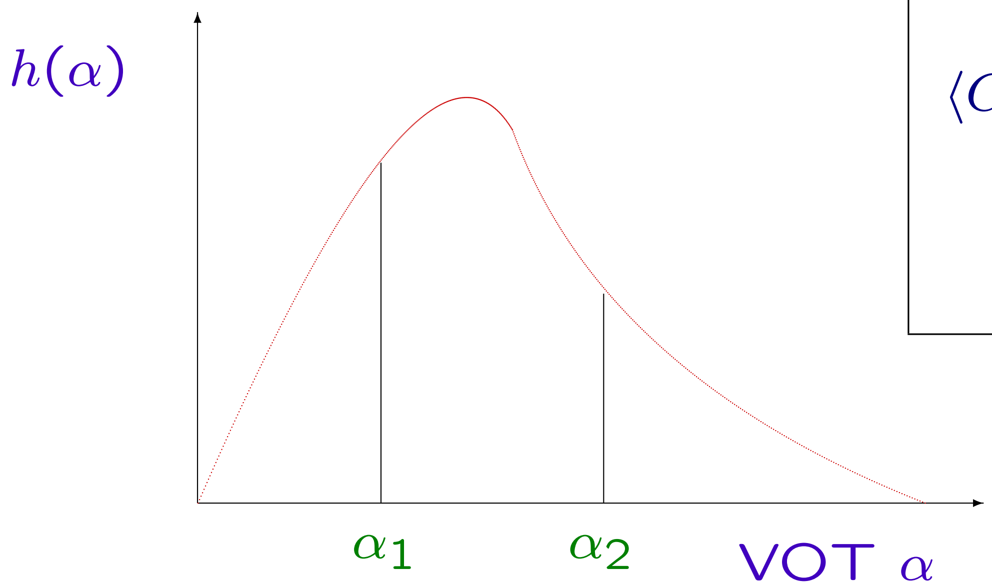


$F(x_1, x_2) = \frac{1}{2}(x_2, -x_1)^t$

pseudo FW: cycling occurs

MULTICLASS (BI-CRITERION) MODEL

$C(x)$:	cost mapping	$F(x)$:	delay mapping
α:	value of time (user α)	$C(x)+\alpha F(x)$:	disutility
$h(\alpha)$:	pdf of VOT	$X(\alpha)$:	$X(\alpha)=h(\alpha)X$



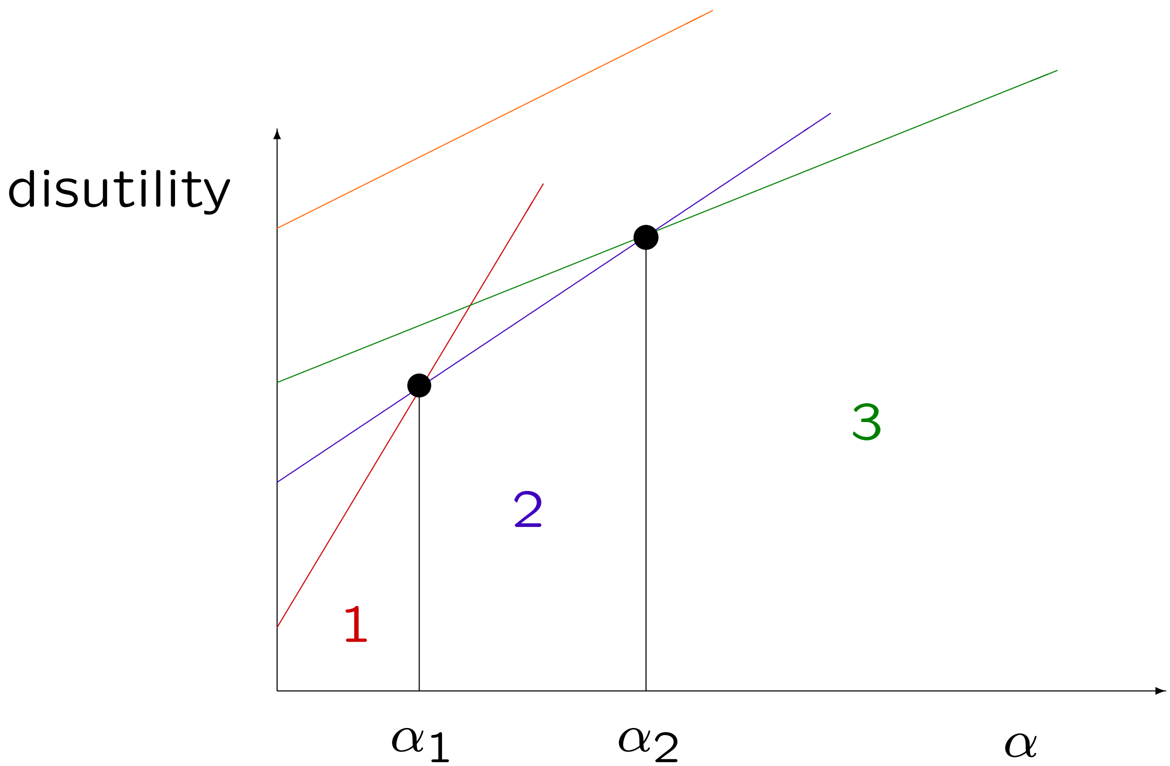
$$x(\alpha) \in X(\alpha)$$

$$\langle C(x)+\alpha F(x), x(\alpha)-y(\alpha)\rangle \leq 0$$

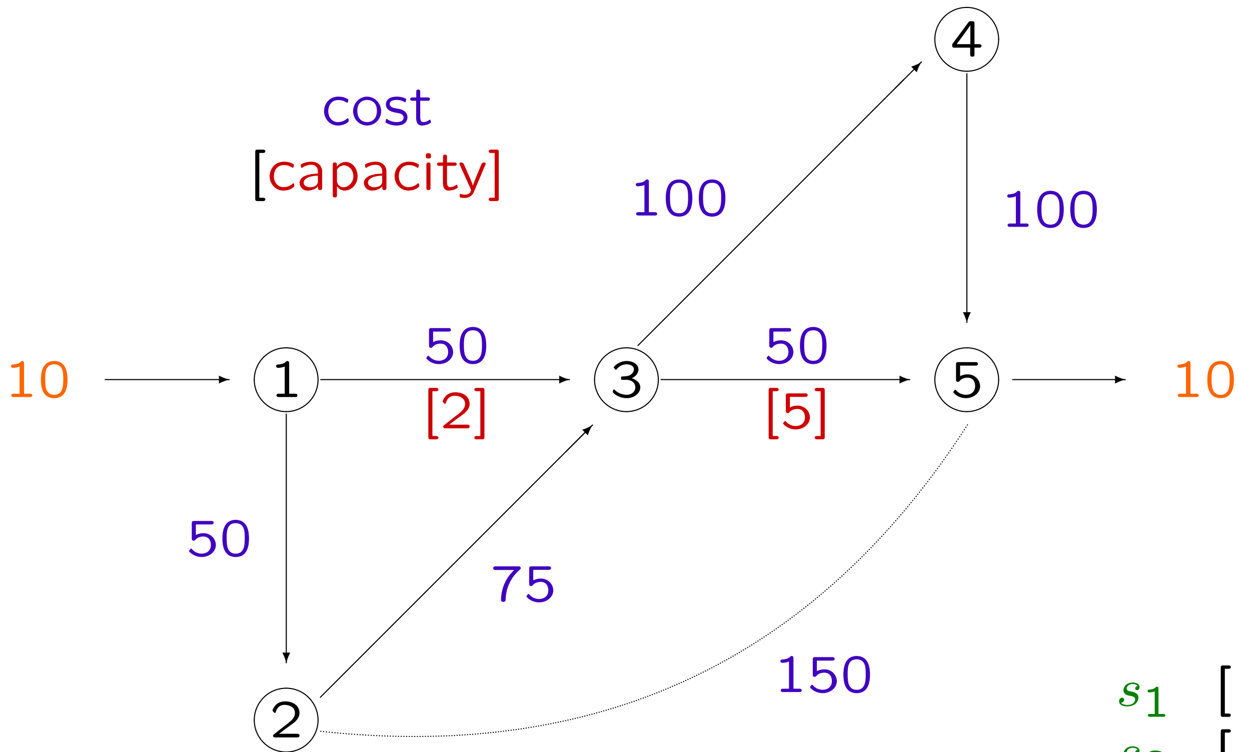
$$\forall y(\alpha) \in X(\alpha)$$

SOLUTION ALGORITHM

- Reduces to a parametric LP.
- Under favorable conditions, solution is unique!
- As easy to solve as a single-class problem!



STRATEGIC MODEL



	1	2	3	4
s_1	[3,2]	[3]	[5,4]	[5]
s_2	[3,2]	[5]	[5,4]	[5]
s_3	[2]	[3]	[5,4]	[5]
s_4	[2]	[5]	[ ]	[ ]
s_5	[3,2]	[3]	[5]	[5]
s_7	[3,2]	[3]	[4,5]	[5]

Equilibrium: $x^{s_1} = x^{s_2} = 5$

(average cost: 185)

Situation improves if
one removes arc (2,3)!

OPTIMIZING w.r.t. EQUILIBRIUM

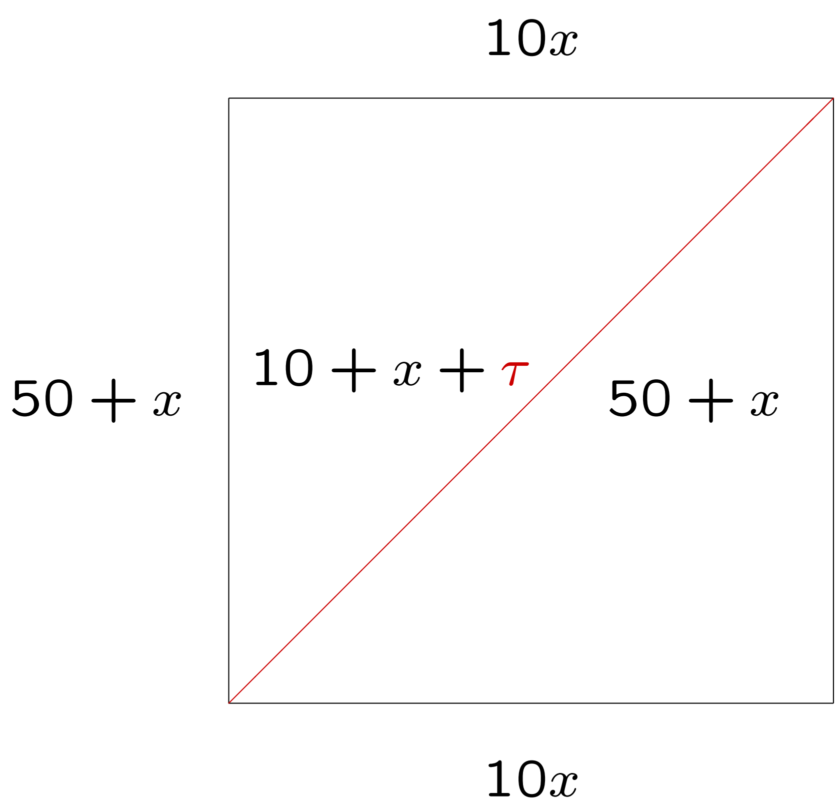
Let $\overline{x}$ solve $VI(G, X)$. It is easy to make $\overline{x}$ a Wardrop flow by setting toll vector τ at $G(\overline{x}) - F(\overline{x})$.

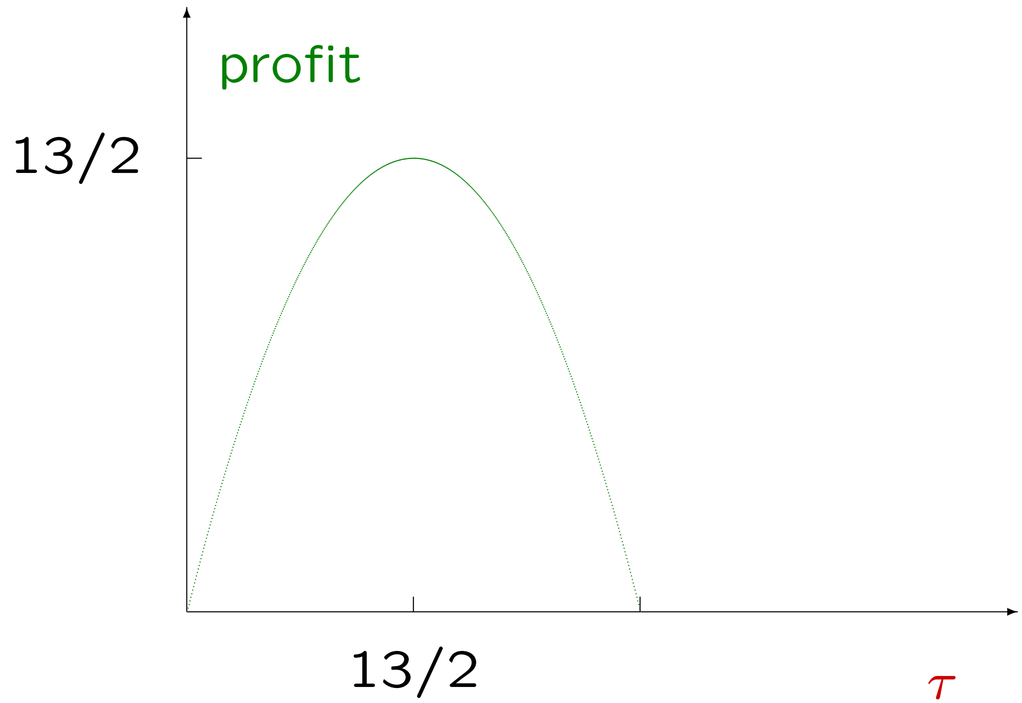
Marginal cost pricing

$$\min_{x \in X} \quad \langle F(x), x \rangle \qquad \equiv \qquad VI(F(x) + F'(x)x, X)$$

$$\tau = F'(x^{SO})x^{SO}$$

TOLLS FOR RAISING REVENUES





Cost has decreased from 92 to 87.5 !

FURTHER RESEARCH

- Dynamic traffic assignment
  - steady-state models;
  - transient models (incidents);
  - realistic?
- Network design
  - toll policies;
  - revenue management.